

On the Arithmetic Fundamental Lemma in the minuscule case

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ABSTRACT

The arithmetic fundamental lemma conjecture of the third author connects the derivative of an orbital integral on a symmetric space with an intersection number on a formal moduli space of p -divisible groups of Picard type. It arises in the relative trace formula approach to the arithmetic Gan-Gross-Prasad conjecture. We prove this conjecture in the minuscule case.

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1. Introduction

In this introduction, we first formulate (a variant of) the fundamental lemma conjecture (FL) of Jacquet-Rallis [7] and the arithmetic fundamental lemma conjecture (AFL) of the third author [19]. Then we state our main result, which is a confirmation of the second conjecture in arbitrary dimension under restrictive conditions.

Let p be an odd prime. Let F be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_F$, uniformizer π and residue field k with q elements. Let E be an unramified quadratic extension, with ring of integers $\mathcal{O}_E$, and residue field k' . We denote the non-trivial element in $\text{Gal}(E/F)$ by σ or by $a \mapsto \bar{a}$. Also, we denote by $\eta = \eta_{E/F}$ the quadratic character of $F^\times$ corresponding to E/F .

Let $n \geq 1$. Let $v = (0, 0, \dots, 0, 1) \in F^n$. We denote by F^{n-1} the subspace of vectors in F^n

2010 Mathematics Subject Classification primary 11G18, 14G17; secondary 22E55

Keywords: Arithmetic Gan-Gross-Prasad conjecture, Arithmetic Fundamental Lemma, Rapoport-Zink spaces, special cycles

Research of Rapoport and Terstiege partially supported by SFB/TR 45 “Periods, Moduli Spaces and Arithmetic of Algebraic Varieties” of the DFG. Research of Zhang partially supported by NSF grant DMS 1204365.

with trivial last entry. We have a canonical inclusion $GL_{n-1} \hookrightarrow GL_n$ of algebraic groups over F . An element $g \in GL_n(E)$ is called regular semi-simple (with respect to the action of $GL_{n-1}(E)$ by conjugation) if both the vectors $(g^i v)_{i=0, \dots, n-1}$ and the vectors $({}^t v g^i)_{i=0, \dots, n-1}$ are linearly independent. This property is equivalent to the condition that the stabilizer $\text{Stab}_{GL_{n-1}}(g)$ is trivial, and that the orbit of g under GL_{n-1} is Zariski closed in GL_n , cf. [13, Theorem 6.1]¹. To $g \in GL_n(E)$ we associate the following numerical invariants: the coefficients of the characteristic polynomial $\text{char}_g(T) \in E[T]$, and the $n-1$ elements ${}^t v g^i v \in E$ for $i = 1, \dots, n-1$. Then two regular semi-simple elements are conjugate under an element of $GL_{n-1}(E)$ if and only if they have the same invariants, cf. [19].

Let

$$S_n(F) = \{s \in GL_n(E) \mid s\sigma(s) = 1\}. \quad (1.1)$$

Then $GL_{n-1}(F)$ acts on $S_n(F)$, and two elements in $S_n(F)$ which are regular semi-simple (as elements of $GL_n(E)$) are conjugate under $GL_{n-1}(E)$ if and only if they are conjugate under $GL_{n-1}(F)$.

Let $J \in \text{Herm}_{n-1}(E/F)$ be a hermitian matrix of size $n-1$. It defines a hermitian form on E^{n-1} . We obtain a hermitian form $J \oplus 1$ of size n , which corresponds to extending the hermitian form to E^n by adding an orthogonal vector u of length 1. We obtain an inclusion of unitary groups

$$U(J)(F) \hookrightarrow U(J \oplus 1)(F),$$

and therefore an action of $U(J)(F)$ on $U(J \oplus 1)(F)$ by conjugation. We consider $U(J \oplus 1)(F)$ as a subset of $GL_n(E)$ in the obvious way by sending u to v , and E^{n-1} to the subspace of vectors with trivial last entry. We call an element $g \in U(J \oplus 1)(F)$ regular semi-simple if it is regular semi-simple as an element of $GL_n(E)$. Two regular semi-simple elements $\gamma \in S_n(F)$ and $g \in U(J \oplus 1)(F)$ are said to match if they are conjugate under $GL_{n-1}(E)$ (when considered as elements of $GL_n(E)$), or, equivalently, if they have the same invariants. This property only depends on the orbits of γ under $GL_{n-1}(F)$, resp. g under $U(J)(F)$. This matching condition defines a bijection between orbit spaces [19], Lemma 2.3,

$$[U(J_0 \oplus 1)(F)_{\text{rs}}] \sqcup [U(J_1 \oplus 1)(F)_{\text{rs}}] \simeq [S_n(F)_{\text{rs}}]. \quad (1.2)$$

Here J_0 denotes the split hermitian form, and J_1 the non-split hermitian form, i.e., the discriminant of J_0 has even valuation, and the discriminant of J_1 has odd valuation.

For $\gamma \in S_n(F)_{\text{rs}}$, and $f \in C_c^\infty(S_n(F))$, consider the weighted orbital integral

$$O(\gamma, f) = \int_{GL_{n-1}(F)} f(h^{-1}\gamma h) \eta(\det h) dh, \quad (1.3)$$

where we normalize the measure so that $GL_{n-1}(\mathcal{O}_F)$ has measure 1. Similarly, for any $g \in U(J_0 \oplus 1)(F)_{\text{rs}}$, and $f \in C_c^\infty(U(J_0 \oplus 1)(F))$, we form the orbital integral

$$O(g, f) = \int_{U(J_0)(F)} f(h^{-1}gh) dh, \quad (1.4)$$

where we normalize the measure so that the stabilizer K' of a self-dual lattice Λ' in E^{n-1} has measure 1. Let K be the stabilizer of the self-dual lattice $\Lambda = \Lambda' \oplus \mathcal{O}_E u$.

The FL is now the following statement (for the ‘‘Lie algebra’’ version see [7]).

¹In [13], the Lie algebra version is considered. But it is easy to deduce the group version from the Lie algebra version. Moreover, what is called ‘‘regular semi-simple’’ here is called ‘‘regular’’ in [13].

CONJECTURE 1.1. For $\gamma \in S_n(F)_{\text{rs}}$,

$$O(\gamma, 1_{S_n(\mathcal{O}_F)}) = \begin{cases} \omega(\gamma)O(g, 1_K) & \text{if } \gamma \text{ matches } g \in U(J_0 \oplus 1)(F)_{\text{rs}}, \\ 0 & \text{if } \gamma \text{ matches no } g \in U(J_0 \oplus 1)(F)_{\text{rs}}. \end{cases}$$

Here the sign $\omega(\gamma)$ is given by

$$\omega(\gamma) = (-1)^{v(\det(\gamma^i v)_{i=0, \dots, n-1})}. \quad (1.5)$$

Both orbital integrals appearing in the conjecture count certain $\mathcal{O}_E$ -lattices in E^n . Let $L = L_g$ be the lattice generated by the vectors $u, gu, \dots, g^{n-1}u$, where we recall the vector u of length one from above. Then the first clause of the above identity can be written as

$$\omega(\gamma) \sum_{\{\Lambda | L \subset \Lambda \subset L^*, g\Lambda = \Lambda, \Lambda^\tau = \Lambda\}} (-1)^{\ell(\Lambda/L)} = \omega(\gamma) \sum_{\{\Lambda | L \subset \Lambda \subset L^*, g\Lambda = \Lambda, \Lambda^* = \Lambda\}} 1.$$

Here τ is the antilinear involution on E^n , depending on g , which sends $g^i u$ to $g^{-i} u$ for $i = 0, \dots, n-1$. Also, for any lattice Λ , we denote by Λ^* the lattice of elements of E^n which pair integrally with all elements of Λ (*dual lattice*).

The equal characteristic analogue of FL was proved by Z. Yun, for $p > n$; J. Gordon deduced FL in the p -adic case, for p large enough (but unspecified), cf. [18].

Now we come to the AFL conjecture. For $\gamma \in S_n(F)_{\text{rs}}$, and $f \in C_c^\infty(S_n(F))$, and $s \in \mathbb{C}$, let

$$O(\gamma, f, s) = \int_{GL_{n-1}(F)} f(h^{-1}\gamma h) \eta(\det h) |\det h|^s dh,$$

and introduce

$$O'(\gamma, 1_{S_n(\mathcal{O}_F)}) = \frac{d}{ds} O(\gamma, 1_{S_n(\mathcal{O}_F)}, s) \Big|_{s=0}. \quad (1.6)$$

Then the conjecture is as follows.

CONJECTURE 1.2. For $\gamma \in S_n(F)_{\text{rs}}$ which matches $g \in U(J_1 \oplus 1)(F)_{\text{rs}}$,

$$O'(\gamma, 1_{S_n(\mathcal{O}_F)}) = -\omega(\gamma) \langle \Delta(\mathcal{N}_{n-1}), (\text{id} \times g) \Delta(\mathcal{N}_{n-1}) \rangle.$$

On the RHS appears the arithmetic intersection product of two formal subschemes inside the formal scheme $\mathcal{N}_{n-1} \times_{\text{Spf } \mathcal{O}_{\check{F}}} \mathcal{N}_n$. Here $\mathcal{N}_n$ denotes the moduli space over the ring of integers in the completion of the maximal unramified extension $\check{F}$ of F of formal $\mathcal{O}_F$ -modules of height n with $\mathcal{O}_E$ -action of signature $(1, n-1)$ and with principal polarization compatible with the involution σ on $\mathcal{O}_E$. ($\mathcal{N}_n$ is a special case of an *RZ-space* [14].) Similarly for $\mathcal{N}_{n-1}$, which is naturally embedded in $\mathcal{N}_n$. The element $g \in U(J_1 \oplus 1)(F)$ acts on $\mathcal{N}_n$ in a natural way. Then $\Delta(\mathcal{N}_{n-1})$ and $(\text{id} \times g) \Delta(\mathcal{N}_{n-1})$ are two formal schemes of formal dimension $n-1$, contained in the formal scheme $\mathcal{N}_{n-1} \times_{\text{Spf } \mathcal{O}_{\check{F}}} \mathcal{N}_n$ of formal dimension $2(n-1)$, i.e., we are in a situation of middle dimension intersection. We refer to [19] for the precise definition of $\mathcal{N}_n$, and for the definition of the intersection product, and the proof of the fact that the RHS is a finite quantity (cf. also §§2–4 below).

In the following, we fix $n \geq 2$ and denote $\mathcal{N}_n$ simply by $\mathcal{N}$, and $\mathcal{N}_{n-1}$ by $\mathcal{M}$.

As before, the LHS can be expressed in a combinatorial way, as

$$\omega(\gamma) \log q \sum_{\{\Lambda | L \subset \Lambda \subset L^*, g\Lambda = \Lambda, \Lambda^\tau = \Lambda\}} (-1)^{\ell(\Lambda/L)} \ell(\Lambda/L).$$

In the case that the intersection of the formal schemes $\Delta(\mathcal{M})$ and $(\text{id} \times g)\Delta(\mathcal{M})$ is proper, i.e., is a set of isolated points, and using the Bruhat-Tits stratification of $\mathcal{N}_{\text{red}}$ [17], the RHS can also be written as a sum over lattices, as

$$-\omega(\gamma)\log q \sum_{\{\Lambda \mid L \subset \Lambda \subset L^*, g\Lambda = \Lambda, \pi\Lambda \subset \Lambda^* \subset \Lambda\}} \text{mult}(\Lambda).$$

Here the number $\text{mult}(\Lambda)$ is the intersection multiplicity of $\Delta(\mathcal{M})$ and $(\text{id}_{\mathcal{M}} \times g)\Delta(\mathcal{M})$ along the stratum $\mathcal{V}(\Lambda)^{\circ}$.

We note that this conjecture holds true for $n = 2, n = 3$, by results of the third author [19]. In these cases the intersection appearing above is automatically proper.

We now come to the description of the results of this paper, which are valid for any n but with strong restrictions on g . Let

$$(\mathbb{Z}^n)_+ = \{(r_1, \dots, r_n) \in \mathbb{Z}^n \mid r_1 \geq \dots \geq r_n\}.$$

Let $\text{inv}(g) = (r_1, \dots, r_n) \in (\mathbb{Z}^n)_+$ be the unique element such that L_g^* has a basis $e_1, \dots, e_n$ such that $\pi^{r_1}e_1, \dots, \pi^{r_n}e_n$ is a basis of L_g . Note that $r_n = 0$, and that $\sum_i r_i$ is odd. It turns out that the ‘bigger’ $\text{inv}(g)$ is, the more difficult it is to prove the identity in AFL. From this point of view we treat here the simplest non-trivial case.

THEOREM 1.3. *Let $g \in U(J_1 \oplus 1)(F)_{\text{rs}}$.*

(i) *The underlying reduced scheme of the intersection $\Delta(\mathcal{M}) \cap (\text{id}_{\mathcal{M}} \times g)\Delta(\mathcal{M})$ has a stratification by Deligne-Lusztig varieties (we refer to §6 for the precise description of which Deligne-Lusztig varieties can occur).*

From now on assume that $\text{inv}(g)$ is minuscule, i.e., that $\text{inv}(g) = (1^{(m)}, 0^{(n-m)})$, for some $m \geq 1$. Then:

(ii) *The intersection of $\Delta(\mathcal{M})$ and $(\text{id}_{\mathcal{M}} \times g)\Delta(\mathcal{M})$ is proper. Furthermore, the arithmetic intersection product $\langle \Delta(\mathcal{M}), (\text{id} \times g)\Delta(\mathcal{M}) \rangle$ is equal to*

$$\log q \sum_{x \in (\Delta(\mathcal{M}) \cap (\text{id} \times g)\Delta(\mathcal{M}))(\bar{k})} \ell(\mathcal{O}_{\Delta(\mathcal{M}) \cap (\text{id} \times g)\Delta(\mathcal{M}), x}),$$

i.e., there are no higher Tor-terms.

(iii) *The intersection of $\Delta(\mathcal{M})$ and $(\text{id}_{\mathcal{M}} \times g)\Delta(\mathcal{M})$ is concentrated in the special fiber, i.e., the uniformizer π annihilates its structure sheaf.*

(iv) *The AFL identity holds, provided that $n \leq 2p - 2$. Furthermore, in this case the lengths of the local rings appearing in (ii) are all identical.*

Assertion (i) is proved in section 6, and (ii) follows from Propositions 4.2 and 6.1. Assertion (iii) follows from Theorem 9.4, and assertion (iv) follows from Propositions 8.1, 8.2, and 9.1. That the lengths of all local rings are identical follows from our explicit determination of these lengths, although we think that there should be an *a priori* proof, without the restriction on p .

In fact, we will prove the assertions above only in the case when $F = \mathbb{Q}_p$, because in this case we can refer to [16] and [17] for the structure of $\mathcal{N}_n$, and also to [9] for some global results. However, there is no doubt that the results should generalize to arbitrary p -adic fields.

There is a fundamental difference between the seemingly very similar combinatorial descriptions of both sides in the FL and in the AFL. Whereas in the FL there is a rather simple criterion

to decide when both sides of the identity are non-zero, the corresponding question for the AFL seems very subtle in general. However, in the case of a minuscule element g , we give a simple criterion in terms of the induced automorphism of the k' -vector space L_g^*/L_g to decide when the two sides of the AFL identity are non-zero, cf. §8.

There is some relation between the AFL problem and the problem of intersecting *special divisors* considered in [8]. Indeed, the intersection $\Delta(\mathcal{M}) \cap (\mathrm{id}_{\mathcal{M}} \times g)\Delta(\mathcal{M})$ is contained in the intersection of the special divisors (in the sense of [8]) $\mathcal{Z}(g^i u)$, for $i = 0, \dots, n-1$. Then point (iii) of Theorem 1.3 is a consequence of the following theorem, which is of independent interest.

THEOREM 1.4. *Consider the intersection of special divisors $\mathcal{Z}(x_1), \dots, \mathcal{Z}(x_n)$ on $\mathcal{N}_n$, where the fundamental matrix (in the sense of [8]) is equivalent to the diagonal matrix $\mathrm{diag}(\pi^{(m)}, 1^{(n-m)})$. Then this intersection is concentrated in the special fiber, and is in fact equal to a closed Bruhat-Tits stratum of type m of $(\mathcal{N}_n)_{\mathrm{red}}$.*

Again, we prove this only in the case $F = \mathbb{Q}_p$. We view this theorem as a confirmation of the following conjecture in a special case.

CONJECTURE 1.5. *Consider the intersection of special divisors $\mathcal{Z}(x_1), \dots, \mathcal{Z}(x_n)$ on $\mathcal{N}_n$, where the fundamental matrix is equivalent to the diagonal matrix $\mathrm{diag}(\pi^{r_1}, \pi^{r_2}, \dots, \pi^{r_n})$ with $r_1 \geq r_2 \geq \dots \geq r_n$. Then π^{r_1} annihilates the structure sheaf of $\mathcal{Z}(x_1) \cap \mathcal{Z}(x_2) \dots \cap \mathcal{Z}(x_n)$.*

The lay-out of the paper is as follows. In sections 2 and 3 we recall some facts about the formal moduli spaces $\mathcal{N}_n$ and the geometry of their underlying schemes. In section 4 we explain the intersection product appearing on the RHS of Conjecture 1.2. In sections 5 and 6 we address the problem of determining the underlying point set of the intersection. More precisely, we write in section 5 this intersection $\Delta(\mathcal{M}) \cap (\mathrm{id}_{\mathcal{M}} \times g)\Delta(\mathcal{M})$ as a disjoint union over Bruhat-Tits strata of certain fixed point sets in each stratum. The determination of the individual fixed point sets then becomes a problem in Deligne-Lusztig theory that is discussed in section 6. In particular, we give a criterion for when this fixed point set is finite. In section 7 we explain the statements of the FL and the AFL, and show that these conjectures can be interpreted as elementary counting expressions of lattices, as mentioned above. In the rest of the paper we concentrate on the minuscule case. In section 8 we determine the cardinality of the intersection $\Delta(\mathcal{M}) \cap (\mathrm{id}_{\mathcal{M}} \times g)\Delta(\mathcal{M})$ and calculate the LHS of Conjecture 1.2, which turn out to be amusing combinatorial exercises. In section 9 we reduce the calculation of the length of the local ring at each point of this intersection to Theorem 9.4, alias Theorem 1.4 above, and Theorem 9.5. These theorems are then proved in sections 10 and 11. Here the main tool is Zink's theory of displays of formal groups.

We conclude this introduction with a few remarks and questions. One remark is that we find it striking that the intersection $\Delta(\mathcal{M}) \cap (\mathrm{id}_{\mathcal{M}} \times g)\Delta(\mathcal{M})$ may be a discrete point set, but not consist entirely of *superspecial* points. This is in contrast to what occurs, e.g., in [8], or [19]. A question that seems very interesting to us is to clarify the relationship between the regular semi-simplicity of an element $g \in U(J_1 \oplus 1)(F)$ and the finiteness of the intersection $\Delta(\mathcal{M}) \cap (\mathrm{id}_{\mathcal{M}} \times g)\Delta(\mathcal{M})$: it is easy to see that there are regular semi-simple elements g such that the length of this intersection is not finite. It would be very interesting to characterize those regular semi-simple elements with corresponding proper intersection, in analogy with the corresponding characterization in [8] of the cases when the intersection of special divisors is finite. For instance, if g is not regular semi-simple, is the intersection $\Delta(\mathcal{M}) \cap (\mathrm{id}_{\mathcal{M}} \times g)\Delta(\mathcal{M})$ of infinite length?

We thank J.-B. Bost, B. Gross, X. He, S. Kudla, G. Lusztig, P. Scholze, J.-L. Waldspurger,

X. Zhu, and Th. Zink for helpful discussions. We also thank the referee for his/her careful reading of the paper.

Parts of this work were done during research stays of the first author at Harvard University, of the first two authors at the Erwin Schrödinger Institute in Vienna, and of the third author at the Morningside Center of Mathematics and MSC of Tsinghua University at Beijing. We thank these institutions for their hospitality.

Notation Throughout the paper, we make the blanket assumption that p is odd. We fix an algebraic closure $\bar{k} = \mathbb{F}$ of k , and denote by σ the relative Frobenius in $\text{Gal}(\mathbb{F}/k)$. We identify k' with the quadratic extension of k in $\mathbb{F}$.

2. The set up

Fix $n \geq 1$. In the following, $\mathcal{N}_n$ is the formal moduli space of p -divisible groups of unitary type of signature $(1, n-1)$, that parametrizes tuples $(X, \iota, \lambda, \rho)$, where the quasi-isogeny ρ is of height zero, cf. [8]. Here is what we mean.

Let $\check{F}$ be the completion of the maximal unramified extension of F , with ring of integers $\mathcal{O}_{\check{F}}$ and residue field $\bar{k}$. We denote by $\text{Nilp} = \text{Nilp}_{\mathcal{O}_{\check{F}}}$ the category of $\mathcal{O}_{\check{F}}$ -schemes such that locally π is a nilpotent element in the structure sheaf. We consider triples (X, ι, λ) where X is a formal $\mathcal{O}_F$ -module of height $2n$, and $\iota : \mathcal{O}_E \rightarrow \text{End}(X)$ is an action of $\mathcal{O}_E$ on X with Kottwitz condition of signature $(1, n-1)$, and where λ is a principal polarization whose associated Rosati involution induces the automorphism σ on $\mathcal{O}_E$. There is a unique such triple $(\mathbb{X}, \iota, \lambda)$ over $\bar{k}$ such that $\mathbb{X}$ is supersingular, up to $\mathcal{O}_E$ -linear isogeny preserving the polarizations up to a scalar. We also write $\mathbb{X}_n$ when we want to stress the dependence on n . Then $\mathcal{N}_n$ represents the functor which to $S \in \text{Nilp}$ associates the set of isomorphism classes of quadruples $(X, \iota, \lambda, \rho)$, where (X, ι, λ) is a triple as above over S , and where ρ is an $\mathcal{O}_E$ -linear quasi-isogeny $\rho : X \times_S \bar{S} \rightarrow \mathbb{X}_n \times_{\text{Spec } \bar{k}} \bar{S}$ of height zero, which carries the polarization on $\mathbb{X}_n$ into one which differs locally by an element in $\mathcal{O}_F^\times$ from $\lambda \times_S \bar{S}$. Here $\bar{S} = S \times_{\text{Spec } \mathcal{O}_{\check{F}}} \text{Spec } \bar{k}$.

The functor $\mathcal{N}_1$ is representable by $\text{Spf } \mathcal{O}_{\check{F}}$, with universal object (Y, ι_0, λ_0) . We denote by $(\bar{Y}, \bar{\iota}_0, \bar{\lambda}_0)$ the same formal $\mathcal{O}_F$ -module as Y , but where $\bar{\iota}_0$ is obtained from ι_0 by pre-composing with σ . We may (and will) assume that for the framing objects for $\mathcal{N}_{n-1}$, resp. $\mathcal{N}_n$ we have the relation

$$\mathbb{X}_n = \mathbb{X}_{n-1} \times (\bar{Y} \times_{\text{Spec } \mathcal{O}_{\check{F}}} \text{Spec } \bar{k}).$$

For fixed $n \geq 2$, we abbreviate $\mathcal{N}_{n-1}$ into $\mathcal{M}$ and $\mathcal{N}_n$ into $\mathcal{N}$. We define the embedding

$$\delta : \mathcal{M} \hookrightarrow \mathcal{N}, \tag{2.1}$$

via

$$\delta((X, \iota, \lambda, \rho)) = (X \times \bar{Y}, \iota \times \bar{\iota}_0, \lambda \times \bar{\lambda}_0, \rho \times \text{id}).$$

Let

$$G = G_n = \{g \in \text{End}_E^0(\mathbb{X}_n) \mid gg^\dagger = 1\}.$$

Here $\dagger$ is the Rosati involution induced by λ . Then G acts on $\mathcal{N}_n$, by changing ρ into $g \circ \rho$.

3. The Bruhat-Tits stratification

We recall some basic structure of the reduced part of the formal schemes of the last section, especially the Bruhat-Tits stratification, comp. [8, 16, 17]. This applies both to $\mathcal{M} = \mathcal{N}_{n-1}$ and to $\mathcal{N} = \mathcal{N}_n$. Let us explain the case of $\mathcal{M}$.

We identify E with the invariants of σ^2 in $\check{F}$. Let C_{n-1} be the hermitian space of dimension $n-1$ with hermitian form isomorphic to $\text{diag}(1, \dots, 1, p)$ (this differs by the factor p from the form in [16]). Recall the concept of a *vertex lattice* in C_{n-1} : this is a lattice Λ with $\pi\Lambda \subset \Lambda^* \subset \Lambda$, cf. [8]. Here, as elsewhere in the paper, Λ^* denotes the dual lattice, consisting of elements in the ambient vector space which pair integrally with all elements of Λ . The *type* of Λ is the dimension of the k' -vector space Λ/Λ^* .

We denote by τ the automorphism $\text{id} \otimes \sigma^2$ of $C_{n-1} \otimes_E \check{F}$. We extend the hermitian form on C_{n-1} to a sesqui-linear form on $C_{n-1} \otimes_E \check{F}$ by

$$(x \otimes c, y \otimes c') = c\sigma(c') \cdot (x, y), \quad x, y \in C_{n-1}; \quad c, c' \in \check{F}.$$

The set $\mathcal{M}(\mathbb{F})$ can be identified with the set of lattices $A \subset C_{n-1} \otimes_E \check{F}$ such that

$$A^* \subset^1 A \subset \pi^{-1}A^*,$$

where the notation “ $\subset^1$ ” means that the quotient A/A^* is a $\bar{k}$ -vector space of dimension 1.

Recall [8] that to a lattice $A \in \mathcal{M}(\mathbb{F})$, there is associated a vertex lattice $\Lambda = \Lambda(A)$ in C_{n-1} via the following rule:

$$\Lambda(A) \otimes_{\mathcal{O}_E} \mathcal{O}_{\check{F}} = \sum_0^d \tau^i A \text{ is } \tau\text{-stable, for some } d. \quad (3.1)$$

Then $\Lambda(A) \otimes_{\mathcal{O}_E} \mathcal{O}_{\check{F}}$ is the smallest τ -invariant lattice containing A . Dually, $\Lambda(A)^* \otimes_{\mathcal{O}_E} \mathcal{O}_{\check{F}}$ is the largest τ -invariant lattice contained in A^* . The *type* of A is the integer $t = t(A) = 2d + 1$, where d is minimal. Equivalently, it is the type $t(\Lambda)$ of $\Lambda = \Lambda(A)$.

For a given vertex lattice Λ , the lattices A with $\Lambda = \Lambda(A)$ form the open Bruhat-Tits stratum $\mathcal{V}_\Lambda(\mathbb{F})^o$ associated to Λ . The closed Bruhat-Tits stratum associated to Λ is given by

$$\mathcal{V}_\Lambda(\mathbb{F}) = \{A \in \mathcal{M}(\mathbb{F}) \mid A \subset \Lambda \otimes_{\mathcal{O}_E} \mathcal{O}_{\check{F}}\}.$$

It turns out that these strata are in fact the $\mathbb{F}$ -points of algebraic subvarieties of $\mathcal{M}_{\text{red}}$, cf. [17]. More precisely, for any vertex Λ , $\mathcal{V}_\Lambda(\mathbb{F})$ is the set of $\mathbb{F}$ -points of a closed irreducible subvariety $\mathcal{V}_\Lambda$ of $\mathcal{M}_{\text{red}}$ which is smooth of dimension $\frac{1}{2}(t(\Lambda) - 1)$, the inclusions $\mathcal{V}_{\Lambda'}(\mathbb{F}) \subset \mathcal{V}_\Lambda(\mathbb{F})$ for $\Lambda' \subset \Lambda$ are induced by closed embeddings of algebraic varieties over $\mathbb{F}$, and the open stratum $\mathcal{V}_\Lambda(\mathbb{F})^o$ is the set of $\mathbb{F}$ -points of the open subvariety of $\mathcal{V}_\Lambda$ obtained by removing all $\mathcal{V}_{\Lambda'}$ for $\Lambda' \subsetneq \Lambda$.

Let $C_n = C_{n-1} \oplus Eu$ with $(u, u) = 1$. We again extend the pairing to a sesqui-linear pairing on $C_n \otimes_E \check{F}$. Then the preceding explanations apply to $\mathcal{N}$ instead of $\mathcal{M}$, and in particular

$$\mathcal{N}(\mathbb{F}) = \{B \mid B \text{ a lattice in } C_n \otimes_E \check{F} \text{ with } B^* \subset^1 B \subset \pi^{-1}B^*\},$$

and again there is a Bruhat-Tits stratification, this time parametrized by vertex lattices in C_n . The relation between $\mathcal{M}$ and $\mathcal{N}$ is given by the following lemma.

LEMMA 3.1. *The injection $\delta : \mathcal{M}(\mathbb{F}) \rightarrow \mathcal{N}(\mathbb{F})$ induced on $\mathbb{F}$ -points by (2.1) is given by $A \mapsto B = A \oplus \mathcal{O}_{\check{F}}u$. Furthermore,*

$$\Lambda(B) = \Lambda(A) \oplus \mathcal{O}_{Eu}.$$

In particular, the types of B and A are the same.

Proof. The first assertions follow easily from the identification of $\mathcal{N}(\bar{k})$ in terms of lattices. The last assertion is obvious since u is a unimodular vector. $\square$

COROLLARY 3.2. *The morphism δ is compatible with the Bruhat-Tits stratifications of $\mathcal{M}_{\text{red}}$, resp. $\mathcal{N}_{\text{red}}$, in the sense that the stratum of $\mathcal{M}_{\text{red}}$ corresponding to the vertex lattice Λ in C_{n-1} is mapped to the stratum of $\mathcal{N}_{\text{red}}$ corresponding to the vertex lattice $\Lambda \oplus \mathcal{O}_E u$ in C_n . $\square$*

4. An intersection problem

The morphism δ induces a closed embedding of formal schemes,

$$\Delta : \mathcal{M} \rightarrow \mathcal{M} \times_{\text{Spf } \mathcal{O}_{\bar{F}}} \mathcal{N}, \quad (4.1)$$

with components $\text{id}_{\mathcal{M}}$ and δ . Let $g \in G$. Then g induces an automorphism $g : \mathcal{N} \rightarrow \mathcal{N}$. We denote by $\mathcal{N}^g$ the fixed point locus, defined to be the intersection in $\mathcal{N} \times_{\text{Spf } \mathcal{O}_{\bar{F}}} \mathcal{N}$,

$$\mathcal{N}^g = \Delta_{\mathcal{N}} \cap \Gamma_g.$$

Here $\Delta_{\mathcal{N}} \subset \mathcal{N} \times_{\text{Spf } \mathcal{O}_{\bar{F}}} \mathcal{N}$ is the diagonal of $\mathcal{N}$, and Γ_g is the graph of g .

DEFINITION 4.1. An element $g \in G$ is called regular semi-simple, if the matrix in $M_n(E)$

$$(g^i u, g^k u), i = 0, \dots, n-1; k = 0, \dots, n-1,$$

is non-singular. Equivalently, the vectors $g^i u, i = 0, \dots, n-1$, form a basis of C_n .

Here we have identified the group $G = G_n$ with the unitary group of C_n as explained in §2.2 of [19].

This definition coincides with the definition of regular semi-simplicity in the introduction. Indeed, we may identify C_n with the hermitian space (E^n, J) for $J = J_1 \oplus 1$ such that u is mapped to $v = (0, 0, \dots, 0, 1)$. If the vectors $g^i v, i = 0, \dots, n-1$, form a basis of C_n , then so do the vectors ${}^t g^i v = J^{-1} \bar{g}^i J u, i = 0, \dots, n-1$ since $J u = u$. Hence ${}^t v g^i, i = 0, \dots, n-1$, also form a basis of E^n .

PROPOSITION 4.2. (i) *There is an equality of formal schemes over $\text{Spf } \mathcal{O}_{\bar{F}}$*

$$\delta(\mathcal{M}) \cap \mathcal{N}^g = \Delta(\mathcal{M}) \cap (\text{id}_{\mathcal{M}} \times g) \Delta(\mathcal{M}).$$

Now let $F = \mathbb{Q}_p$.

(ii) *If g is regular semi-simple, then this formal scheme is a scheme (i.e., any ideal of definition is nilpotent) with underlying reduced subscheme proper over $\text{Spec } \mathbb{F}$.*

(iii) *If g is regular semi-simple and $(\delta(\mathcal{M}) \cap \mathcal{N}^g)(\mathbb{F})$ is finite, then*

$$\mathcal{O}_{\Delta(\mathcal{M}) \cap (\text{id}_{\mathcal{M}} \times g) \Delta(\mathcal{M})} = \mathcal{O}_{\Delta(\mathcal{M})} \otimes^{\mathbb{L}} \mathcal{O}_{(\text{id}_{\mathcal{M}} \times g) \Delta(\mathcal{M})},$$

i.e., the sheaf on the LHS represents the object on the RHS in the derived category.

Proof. The first assertion follows by checking the equality on S -valued points, for $S \in \text{Nilp}$, where it is a tautology. For the second assertion, we refer to [19], Lemma 2.8.

For the third assertion, first note that if the intersection has a finite number of points, it is an artinian scheme. Now both $\Delta(\mathcal{M})$ and $\mathcal{M} \times_{\text{Spf } \mathcal{O}_{\bar{F}}} \mathcal{N}$ are regular formal schemes of dimension $n-1$, resp. $2(n-1)$, and therefore locally $\Delta(\mathcal{M})$ is the intersection of $n-1$ regular divisors in $\mathcal{M} \times_{\text{Spf } \mathcal{O}_{\bar{F}}} \mathcal{N}$. The same applies to $(\text{id}_{\mathcal{M}} \times g) \Delta(\mathcal{M})$. Hence, if $\Delta(\mathcal{M}) \cap (\text{id}_{\mathcal{M}} \times g) \Delta(\mathcal{M})$ is

discrete, the intersection of the $2(n-1)$ regular divisors is proper. Hence there are no higher Tor-terms, and the assertion follows, comp. [15]. $\square$

REMARK 4.3. The hypothesis $F = \mathbb{Q}_p$ for (ii) and (iii) is made because the proof of [19], Lemma 2.8 makes use of global methods. In fact, the proof uses a globalization of the special divisors $\mathcal{Z}(x)$ of [8]. The assertions should be true for arbitrary F .

It follows from (ii) that the Euler-Poincaré characteristic of $\mathcal{O}_{\Delta(\mathcal{M})} \otimes^{\mathbb{L}} \mathcal{O}_{(\text{id}_{\mathcal{M}} \times g)\Delta(\mathcal{M})}$ is finite. The *arithmetic intersection number* is defined to be

$$\langle \Delta(\mathcal{M}), (\text{id}_{\mathcal{M}} \times g)\Delta(\mathcal{M}) \rangle = \chi(\mathcal{O}_{\Delta(\mathcal{M})} \otimes^{\mathbb{L}} \mathcal{O}_{(\text{id}_{\mathcal{M}} \times g)\Delta(\mathcal{M})}) \log q. \quad (4.2)$$

REMARK 4.4. In case the intersection $\delta(\mathcal{M}) \cap \mathcal{N}^g$ is discrete, it follows that locally at a point of intersection the fixed point locus $\mathcal{N}^g$ is purely formally one-dimensional: indeed, in this case the formal scheme $\mathcal{N}^g$ intersects properly the formal divisor $\mathcal{M}$ of $\mathcal{N}$.

REMARK 4.5. In case the intersection $\delta(\mathcal{M}) \cap \mathcal{N}^g$ is discrete, its underlying set is stratified by the Bruhat-Tits stratification of $\mathcal{M}$. We define in this case for a vertex lattice Λ in C_{n-1}

$$\text{mult}(\Lambda) = \sum_{x \in \mathcal{V}(\Lambda)^o(\mathbb{F})} \ell(\mathcal{O}_{\Delta(\mathcal{M}) \cap (\text{id}_{\mathcal{M}} \times g)\Delta(\mathcal{M}), x}). \quad (4.3)$$

The total arithmetic intersection number (4.2) is in this case given by a finite sum

$$\langle \Delta(\mathcal{M}), (\text{id}_{\mathcal{M}} \times g)\Delta(\mathcal{M}) \rangle = \log q \sum_{\Lambda} \text{mult}(\Lambda).$$

Our next task will be to analyze which vertex lattices Λ contribute effectively to this sum, and to understand the set of points in $\mathcal{V}(\Lambda)^o \cap \mathcal{N}^g$.

5. Description of $(\delta(\mathcal{M}) \cap \mathcal{N}^g)(\mathbb{F})$

Let $g \in G$. It is clear that

$$\mathcal{N}^g(\mathbb{F}) = \{B \in \mathcal{N}(\mathbb{F}) \mid g(B) \subset B\} = \{B \in \mathcal{N}(\mathbb{F}) \mid g(B) = B\}. \quad (5.1)$$

LEMMA 5.1. *If $B \in \mathcal{V}_{\Lambda}(\mathbb{F})^o$ is stable under g , then $g(\Lambda) = \Lambda$. In particular, if $\mathcal{N}^g(\mathbb{F}) \neq \emptyset$, then the characteristic polynomial $\text{char}_g(T)$ of g has integral coefficients, i.e., $\text{char}_g(T) \in \mathcal{O}_E[T]$.*

Proof. Obvious, since g and τ commute. $\square$

To a regular semi-simple element $g \in G$ we associate the $\mathcal{O}_E$ -lattice L_g in C_n generated by $g^i u$ (an $\mathcal{O}_E$ -lattice, since $u, gu, \dots, g^{n-1}u$ form a basis of C_n).

LEMMA 5.2. *Let $g \in G$ be regular semisimple. Then for any $B = A \oplus \mathcal{O}_{\check{F}} u$ stable under g , we have*

$$L_g \subset \Lambda(B)^* \subset \Lambda(B) \subset L_g^*.$$

In particular, the invariants $(g^i u, g^k u)$ take values in $\mathcal{O}_E$.

Conversely, if $B \in \mathcal{N}(\mathbb{F})$ contains L_g , then B is of the form $B = A \oplus \mathcal{O}_{\check{F}} u$, for a unique $A \in \mathcal{M}(\mathbb{F})$.

Proof. Recall that $\Lambda(B)^*$ is the largest τ -invariant lattice contained in B^* . Since B is of the form $B = A \oplus \mathcal{O}_{\check{F}} u$, it follows that $u \in B^*$. Since $gB = B$, we also have $gB^* = B^*$. Hence $g^i u \in B^*$

for all $i \geq 0$. Hence $L_g \subset B^*$, and therefore $L_g \subset \Lambda(B)^*$ by the maximality of $\Lambda(B)^*$. The other inclusion is obtained by taking duals.

For the converse, note that the inclusion $L_g \subset B^*$ implies that $u \in B^*$. Since u is a unimodular vector, setting $A = u^\perp$, we obtain $B = A \oplus \mathcal{O}_{\tilde{F}} u$. $\square$

By definition, $L = L_g$ is a g -cyclic lattice, i.e., there exists $v \in L$ such that L is generated over $\mathcal{O}_E$ by $\{g^i v \mid 0 \leq i \leq n-1\}$.

LEMMA 5.3. *Let $g \in G$ and let L be a g -cyclic lattice with $gL = L$. Then L^* is also a g -cyclic lattice with $gL^* = L^*$.*

Proof. Let L be generated by $v_i = g^i v$, where $i = 0, \dots, n-1$. Then the v_i form a basis of C_n . Let v'_i be the dual basis, i.e.,

$$(v_i, v'_j) = \delta_{ij}.$$

Then L^* is the $\mathcal{O}_E$ -span of $\{v'_i \mid i = 0, \dots, n-1\}$. Let $v' = v'_0$. We claim that v' is a cyclic generator of L^* , i.e., that the elements $w_i = g^i v'$ for $i = 0, \dots, n-1$ generate L^* as an $\mathcal{O}_E$ -module. It is clear that $w_i \in L^*$ for all i .

Claim: $v'_j - w_j$ is a $\mathcal{O}_E$ -linear combination of $w_0, \dots, w_{j-1}$, or equivalently, $v'_j - w_j$ is a $\mathcal{O}_E$ -linear combination of $v'_0, \dots, v'_{j-1}$ (i.e., the matrix representing the base change from v'_i to w_i is a unipotent upper triangular matrix with integral entries).

It is clear that this claim implies the lemma. Now for any vector w , we have $w = \sum_i (w, v_i) v'_i$. Hence the claim is equivalent to

$$(v'_j - w_j, v_i) = \begin{cases} \in \mathcal{O}_E & \text{if } i < j, \\ 0 & \text{if } i \geq j. \end{cases} \quad (5.2)$$

Now for $i \geq j$, we have

$$(w_j, v_i) = (g^j v', g^i v) = (v', g^{i-j} v) = \delta_{ij} = (v'_j, v_i).$$

This proves the second clause in (5.2). The first clause is trivial since $v'_j - w_j \in L^*$. $\square$

DEFINITION 5.4. Let L be a lattice in C_n with $L \subset L^*$. Then set

$$\text{Vert}(L) = \{\Lambda \mid \Lambda \text{ vertex lattice with } L \subset \Lambda^* \subset \Lambda \subset L^*\},$$

cf. [8]. If $g \in G$ with $gL = L$, then g acts on $\text{Vert}(L)$, and we set

$$\text{Vert}^g(L) = \{\Lambda \in \text{Vert}(L) \mid g\Lambda = \Lambda\}.$$

Note that by Lemma 5.2, the assumption on L is satisfied for $L = L_g$, if $\delta(\mathcal{M}) \cap \mathcal{N}^g \neq \emptyset$. We may summarize Lemma 5.2 as follows.

COROLLARY 5.5. (i) *If g is regular semi-simple, there is an equality of sets*

$$(\delta(\mathcal{M}) \cap \mathcal{N}^g)(\mathbb{F}) = \{B \in \mathcal{N}(\mathbb{F}) \mid L_g \otimes_{\mathcal{O}_E} \mathcal{O}_{\tilde{F}} \subset B^* \subset B \subset L_g^* \otimes_{\mathcal{O}_E} \mathcal{O}_{\tilde{F}}, gB = B\}.$$

(ii) *There is an equality of $\bar{k}$ -varieties*

$$(\delta(\mathcal{M}) \cap \mathcal{N}^g)_{\text{red}} = \bigcup_{\Lambda \in \text{Vert}^g(L_g)} (\mathcal{V}(\Lambda)^o)^{g_\Lambda}.$$

Here (ii) makes use of Proposition 4.2, and the algebraicity of the Bruhat-Tits stratification. The action of g on $\mathcal{V}(\Lambda)^o$ is induced by the automorphism g_Λ on Λ/Λ^* induced by g .

6. Fixed point set in a stratum

We next analyze the fixed point variety of g_Λ on $\mathcal{V}(\Lambda)^\circ$. Since $\mathcal{V}(\Lambda)^\circ$ is a Deligne-Lusztig variety [17], this can be considered as a general question on Deligne-Lusztig varieties (called “DL-varieties” below for brevity). Accordingly, we use notation that is standard in this context, e.g., [4].

PROPOSITION 6.1. *Let Λ be a vertex lattice in C_n with $g_\Lambda = \Lambda$, and denote by $\bar{g} = g_\Lambda \in \mathrm{U}(V)(\mathbb{F}_p)$ the automorphism of the hermitian space $V = V_\Lambda = \Lambda/\Lambda^*$ over k' induced by g_Λ .*

- (i) *If $(\mathcal{V}(\Lambda)^\circ)^{\bar{g}}$ is non-empty, then g_Λ is semi-simple and contained in a Coxeter type maximal torus.*
- (ii) *If $(\mathcal{V}(\Lambda)^\circ)^{\bar{g}}$ is a non-empty finite set, then g_Λ is a regular elliptic element contained in a Coxeter type maximal torus. Furthermore, in this case the cardinality of $(\mathcal{V}(\Lambda)^\circ)^{\bar{g}}$ is given by the type of Λ .*

Proof. This follows from the following lemmas on DL-varieties. □

We first recall that an element w in the Weyl group W is called *elliptic*, if the following equivalent properties are satisfied:

- (i) The torus T_w of type w is elliptic, i.e., T_w/Z is anisotropic (i.e., $X^*(T_w/Z)^F = (0)$).
- (ii) T_w is not contained in a proper F -stable parabolic subgroup.
- (iii) 1 is not an eigenvalue of $w \cdot F_*$.
- (iv) The F -conjugacy class of w contains no element in a proper F -stable parabolic subgroup of W .

Here F_* denotes the action of Frobenius on $X_*(T)$, where T is a maximal torus contained in a Borel subgroup (if G is split, then F_* is trivial; for a unitary group in n variables with standard basis for the hermitian space, F_* acts through the longest element in S_n). Note that any Coxeter element in the sense of Lusztig [10] is elliptic.

(The equivalence of (i) and (iii) follows from [3], Proposition 3.2.2. The equivalence of (iii) and (iv) follows from [6], Lemma 7.2. The equivalence of (i) and (ii) is easy².)

The DL-varieties appearing in [17] are associated to unitary groups in an odd number of variables and standard Coxeter elements. More precisely, let V be a hermitian vector space over $\mathbb{F}_{q^2}$ of dimension $n = 2d + 1$. We choose the basis $e_1, \dots, e_n$ in such a way that under the hermitian pairing e_i pairs trivially with e_j , unless $i + j = n + 1$, and we identify W with the symmetric group S_n . Then the DL-variety of interest is associated to the cyclic permutation $w = (d+1, d+2, \dots, n)$. The DL-varieties associated to different Coxeter elements all differ at most by a power of Frobenius [10], Prop. 1.10; in particular, they are all universally homeomorphic. The DL-variety X_w associated to the Coxeter element $w = (1, 2, \dots, d+1)$ is the variety of complete flags $\mathcal{F}_\bullet$ such that

$$\mathcal{F}_{n-i}^\perp \subset \mathcal{F}_{i+1}, \mathcal{F}_{n-i}^\perp \neq \mathcal{F}_i, (1 \leq i \leq d); \mathcal{F}_i = \mathcal{F}_{n-i}^\perp, (d+1 \leq i \leq n-1).$$

Let $\tau = \sigma^2$. Then X_w can also be identified with the variety of complete selfdual flags $\mathcal{F}_\bullet$ of V such that

$$\mathcal{F}_i + \tau(\mathcal{F}_i) = \mathcal{F}_{i+1}, i = 1, \dots, d.$$

² We thank X. He for pointing out these references.

In other words, X_w is the variety of complete isotropic flags $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}_d$ of V such that

$$\mathcal{F}_1 \neq \tau(\mathcal{F}_1) \subset \mathcal{F}_2, \mathcal{F}_2 \neq \tau(\mathcal{F}_2) \subset \mathcal{F}_3, \dots, \mathcal{F}_{d-1} \neq \tau(\mathcal{F}_{d-1}) \subset \mathcal{F}_d, \mathcal{F}_d \neq \tau(\mathcal{F}_d).$$

Hence we can identify X_w with the set of $\ell \in \mathbb{P}(V)$ such that

$$(\ell, \ell) = (\ell, \tau(\ell)) = \dots = (\ell, \tau^{d-1}(\ell)) = 0; (\ell, \tau^d(\ell)) \neq 0.$$

This DL-variety is defined over $\mathbb{F}_{q^2}$.

LEMMA 6.2. (Lusztig [11, 5.9]) *Let X_w be a DL-variety, where w is elliptic and of minimal length in its F -conjugacy class. Let $s \in G(\mathbb{F}_q)$. If the fixed point set X_w^s is non-empty, then s is semi-simple.* $\square$

Proof. (Lusztig) In the case of a unitary group in an odd number of variables and the standard Coxeter element, this can be easily seen as follows. In this case, as explained above, X_w can be viewed as a subset of projective space, by associating to a complete flag its one-dimensional component $\ell \subset V \otimes_{\mathbb{F}_{q^2}} \mathbb{F}$. Now assume that ℓ is fixed under s . Then so are $\tau(\ell), \tau^2(\ell), \dots$. But if $\ell \in X_w$, then $\ell, \tau(\ell), \dots, \tau^{n-1}(\ell)$ form a basis of $V \otimes_{\mathbb{F}_{q^2}} \mathbb{F}$, cf. [12], Prop. 26, (i). Hence s is a diagonal element wrt this basis. $\square$

LEMMA 6.3. *Let X_w be a DL-variety, and let $s \in G(\mathbb{F}_q)$ be a semi-simple element. Then the fixed point set X_w^s is non-empty if and only if s is conjugate under $G(\mathbb{F}_q)$ to an element in $T(\mathbb{F}_q)$ for a maximal torus T of type w .*

Proof. This follows immediately from the formula (4.7.1) for X_w^s in [4], Prop. 4.7. $\square$

REMARK 6.4. We know from [4] that the fixed point set is a finite disjoint sum of DL-varieties, for various groups and various Weyl group elements. Let us spell out which DL-varieties occur in the case of interest to us, namely the unitary group of odd size $n = 2d + 1$, and when $w = (1, 2, \dots, d + 1)$ is the Coxeter element as above. Now in this case the maximal torus T of type w is given by

$$T(\mathbb{F}_q) = \text{Ker} \left(\text{Nm}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^n}} : \mathbb{F}_{q^{2n}}^\times \rightarrow \mathbb{F}_{q^n}^\times \right).$$

We may identify the hermitian space V with $\mathbb{F}_{q^{2n}}$, equipped with the hermitian form $(x, y) \mapsto \text{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^2}}(\sigma^n(x)y)$. Now $s \in T(\mathbb{F}_q)$. Hence s generates a subfield $\mathbb{F}_q(s)$ of $\mathbb{F}_{q^{2n}}$. Let $\mathbb{F}_q(s) = \mathbb{F}_{q^h}$. Then $h|2n$. If h is odd, then the norm equation for s gives $s^2 = 1$, hence $h = 1$ and $s = \pm 1$, and s acts trivially on X_w . If $h = 2k$ is even, then $k|n$. In this case, we may identify the hermitian space V with $\mathbb{F}_{q^{2n}}$, equipped with the hermitian form $(x, y) \mapsto \text{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^2}}(\text{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^{2k}}}(\sigma^n(x)y))$. Then the centralizer $Z^0(s)$ can be identified with

$$Z^0(s) = \text{Res}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(U_h),$$

where U_h is the unitary group for the hermitian form $\text{Tr}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^{2k}}}(\sigma^n(x)y)$ on $\mathbb{F}_{q^{2n}}$, and the maximal torus can be identified with the restriction of scalars of the maximal torus $\text{Ker Nm}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^n}}$ of U_h . In this case the corresponding DL-variety is simply the DL-variety of dimension $\frac{1}{2}(\frac{n}{k} - 1)$ associated to the Coxeter torus in a unitary group of odd size $\frac{n}{k}$ over $\mathbb{F}_{q^k}$. And the fixed point set X_w^s is a disjoint sum of isomorphic copies of this DL-variety.

LEMMA 6.5. *Let w be elliptic, and $s \in G(\mathbb{F}_q)$ semi-simple. If X_w^s has only finitely many elements, then s is regular, and conversely.*

Proof. We use the formula (4.7.2) for X_w^s in [4], Prop. 4.7., which presents X_w^s as a disjoint union of varieties which are DL-varieties for $Z^0(s)$, of the form $X_{T' \subset B'}$. However, T' is of the same type as T , hence is elliptic. On the other hand, if the fixed point set is finite, then $\dim X_{T' \subset B'} = 0$. This implies that $T = Z^0(s)$, which is precisely the claim. The converse is obvious, because a regular element has only finitely many fixed points in the flag variety. $\square$

LEMMA 6.6. *Let $s \in G(\mathbb{F}_q)$ be regular and contained in a maximal torus T of type w . Then the number of fixed points of s in X_w is equal to the cardinality of the F -centralizer of w in W .*

Proof. We use the formula (4.7.1) in [4], Prop. 4.7. It shows that the cardinality of X_w^s is equal to the cardinality of $N(\mathbb{F}_q)/T(\mathbb{F}_q)$, where N denotes the normalizer of T . However $N(\mathbb{F}_q)/T(\mathbb{F}_q)$ can be identified with the fixed points under the action of Frobenius on N/T . After identifying N/T with W , this action is via $x \mapsto wF(x)w^{-1}$. Hence the fixed points are identified with the F -centralizer of w . $\square$

LEMMA 6.7. *Let G be the unitary group in an odd number n of variables. Then*

- (i) *The F -centralizer of a Coxeter element w has n elements.*
- (ii) *Let $s \in T(\mathbb{F}_q)$ be a regular element in a Coxeter torus. Then all points in X_w^s are conjugate under $\text{Gal}(\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^2})$, and in fact, this Galois group acts simply transitively on the fixed points.*

Proof. By Lemma 6.6, the second assertion implies the first one, since $\text{Gal}(\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^2})$ has n elements. Now we may identify T with $\text{Ker}(\text{Nm}_{\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^n}})$, and the hermitian space V with $\mathbb{F}_{q^{2n}}$, cf. above. Then the set of fixed points of a regular $s \in T(\mathbb{F}_q)$ in $\mathbb{P}(V)$ is just the set of eigenlines of $\mathbb{F}_{q^{2n}}^\times$ in $V \otimes_{\mathbb{F}_{q^2}} \mathbb{F}$. These all lie in X_w , and this implies the assertion. $\square$

At this point all statements of Proposition 6.1 are proved. We also note the following consequence.

COROLLARY 6.8. *Let $\Lambda \in \text{Vert}^g(L)$ such that $(\mathcal{V}(\Lambda)^o)^{g_\Lambda}$ is finite. Then there is no $\Lambda' \in \text{Vert}^g(L)$, with Λ' strictly contained in Λ .*

Proof. Indeed, Λ' would correspond to a proper parabolic in $U(V_\Lambda)(\mathbb{F}_p)$; but g_Λ is not contained in a proper parabolic by Proposition 6.1, and hence cannot fix Λ'^*/Λ^* . $\square$

7. Statement of the AFL

Let C'_n be a hermitian space of dimension n with discriminant of *even* valuation, and equipped with a vector u of norm one. We fix a self-dual lattice L_0 in C'_n such that $u \in L_0$. We denote by K the stabilizer of L_0 in $U(C'_n)(F)$. We define, for $g \in U(C'_n)(F)$ regular semi-simple,

$$O(g, 1_K) = \int_{U(u^\perp)(F)} 1_K(h^{-1}gh)dh, \quad (7.1)$$

where the Haar measure is normalized by $\text{vol}(K \cap U(u^\perp)(F)) = 1$. Here $u^\perp$ denotes the orthogonal complement of u in C'_n .

We now denote by C either C_n or C'_n . For $g \in U(C)(F)$ regular semi-simple, we denote by $L = L_g$ the lattice in C generated by $g^i u, i = 0, 1, \dots, n-1$. We define an involution τ on $C = L_g \otimes_{\mathcal{O}_E} E$ (depending on g) by requiring that $(a \cdot g^i u)^\tau = \bar{a} \cdot g^{-i} u$ for $a \in E$ and $i = 0, 1, \dots, n-1$.

LEMMA 7.1. *Let $g \in U(C)(F)$ be regular semisimple. Then*

$$O(g, 1_K) = \sum_{\{\Lambda \mid L \subset \Lambda \subset L^*, g\Lambda = \Lambda, \Lambda^* = \Lambda\}} 1.$$

Proof. The orbital integral (7.1) counts the number of self-dual lattices λ in $u^\perp$ such that $\lambda \oplus \mathcal{O}_E u$ is fixed by g . To show the equality, it suffices to show that any lattice Λ occurring on the RHS splits as a direct sum $\lambda \oplus \mathcal{O}_E u$ for a self-dual lattice λ . But since $(u, u) = 1$ it follows that $\Lambda = (\Lambda \cap u^\perp) \oplus \mathcal{O}_E u$, where $\lambda = \Lambda \cap u^\perp$ is self-dual. $\square$

Now let S_n be the variety over F whose F -points are

$$S_n(F) = \{s \in GL_n(E) \mid s\bar{s} = 1\}$$

In fact, S_n is defined over $\mathcal{O}_F$. For $\gamma \in S_n(F)$, recall that its invariants are the characteristic polynomial $\text{char}_\gamma(T) \in E[T]$ and the $n-1$ elements $v\gamma^i v$, $i = 1, 2, \dots, n-1$ of E , for v the row vector $(0, \dots, 0, 1)$.

For $\gamma \in S_n(F)$ regular semi-simple and $s \in \mathbb{C}$, we consider

$$O(\gamma, 1_{S_n(\mathcal{O}_F)}, s) = \int_{GL_{n-1}(F)} 1_{S_n(\mathcal{O}_F)}(h^{-1}\gamma h) \eta(\det h) |\det h|^s dh,$$

where the Haar measure on $GL_{n-1}(F)$ is normalized by $\text{vol}(GL_{n-1}(\mathcal{O}_F)) = 1$. This is a polynomial in $\mathbb{Z}[q^s, q^{-s}]$, comp. Lemma 7.2 below.

We will simply denote the value at $s = 0$ by $O(\gamma, 1_{S_n(\mathcal{O}_F)})$; it is given by

$$O(\gamma, 1_{S_n(\mathcal{O}_F)}) = \int_{GL_{n-1}(F)} 1_{S_n(\mathcal{O}_F)}(h^{-1}\gamma h) \eta(\det h) dh. \quad (7.2)$$

We also introduce the first derivative at $s = 0$:

$$O'(\gamma, 1_{S_n(\mathcal{O}_F)}) = \frac{d}{ds} O(\gamma, 1_{S_n(\mathcal{O}_F)}, s) \Big|_{s=0}. \quad (7.3)$$

For a regular semisimple $\gamma \in S_n(F)$, we define $\ell(\gamma) = v(\det(\gamma^i v)) \in \mathbb{Z}$, where $(\gamma^i v)$ is the matrix $(v, \gamma v, \dots, \gamma^{n-1} v)$. And we define a sign

$$\omega(\gamma) = (-1)^{\ell(\gamma)} \in \{\pm 1\}.$$

Now let $g \in U(C)(F)$ match γ , i.e., have the same invariants as γ . Then, with $L = L_g$, we define the set of lattices in C

$$M = \{\Lambda \mid L \subset \Lambda \subset L^*, g\Lambda = \Lambda, \Lambda^\tau = \Lambda\}$$

and its subsets indexed by $i \in \mathbb{Z}$,

$$M_i = \{\Lambda \mid \Lambda \in M, \ell(\Lambda/L) = i\}.$$

Here $\ell(\Lambda/L)$ is the length of the $\mathcal{O}_E$ -module Λ/L .

LEMMA 7.2. *Let $GL_{n-1}(F)_i$ be the open subset of $GL_{n-1}(F)$ consisting of h with $v(\det h) = i$. Then*

$$\int_{GL_{n-1}(F)_i} 1_{S_n(\mathcal{O}_F)}(h^{-1}\gamma h) dh = |M_{i-\ell(\gamma)}|.$$

Proof. Consider the row vector space F^n , with F^{n-1} as a natural subspace (vectors with zero last entry). We also consider $v = (0, \dots, 0, 1)$ as an vector in F^n . Consider the set of lattices

$$\mathfrak{m} := \{\lambda \subset F^{n-1} \mid \gamma(\Lambda) = \Lambda, \text{ where } \Lambda = (\lambda \otimes \mathcal{O}_E) \oplus \mathcal{O}_E v\}$$

and the subsets

$$\mathfrak{m}_i := \{\lambda \in \mathfrak{m} \mid \ell(\lambda/\lambda_0) = i\}, \quad \lambda_0 = \mathcal{O}_F^{n-1}.$$

Here the length is defined by $\ell(\lambda/\mathcal{O}_F^{n-1}) := \ell(\lambda/A) - \ell(\mathcal{O}_F^{n-1}/A)$ for any lattice $A \subset \lambda \cap \mathcal{O}_F^{n-1}$. It is obvious that the LHS in Lemma 7.2 is given by the cardinality $|\mathfrak{m}_i|$.

Denote by σ the Galois conjugation on E^n . Define a hermitian form on E^n by requiring that

$$(\gamma^i v, \gamma^j v) := v \gamma^{i-j} t v.$$

Let $\mathcal{L} = \mathcal{L}_\gamma$ be the $\mathcal{O}_E$ -lattice in E^n generated by $\gamma^i v, i = 0, 1, \dots, n-1$. Denote by $\mathcal{L}^*$ the dual of $\mathcal{L}$, i.e.,

$$\mathcal{L}^* = \{x \in E^n \mid (x, \mathcal{L}) \subset \mathcal{O}_E\}.$$

Now we introduce the set of lattices

$$\mathfrak{m}' := \{\Lambda \subset E^n \mid \mathcal{L} \subset \Lambda \subset \mathcal{L}^*, \gamma\Lambda = \Lambda, \Lambda^\sigma = \Lambda\},$$

and

$$\mathfrak{m}'_i := \{\Lambda \in \mathfrak{m}' \mid \ell(\Lambda/\mathcal{L}) = i\}.$$

We claim that the map $\lambda \mapsto \Lambda := (\lambda \otimes \mathcal{O}_E) \oplus \mathcal{O}_E v$ defines a bijection between $\mathfrak{m}$ and $\mathfrak{m}'$. First of all, such Λ do lie in $\mathfrak{m}'$. Indeed, we only need to verify that $\Lambda \subset \mathcal{L}^*$ or, equivalently, $(\Lambda, \gamma^i v) \in \mathcal{O}_E$ for all i . This follows from $\gamma\Lambda = \Lambda$ and $(\Lambda, v) \in \mathcal{O}_E$. Now we only need to show the surjectivity of the map. Similarly to the unitary case, any $\Lambda \in \mathfrak{m}'$ is a direct sum $(\Lambda \cap E^{n-1}) \oplus \mathcal{O}_E v$. Obviously $\Lambda \cap E^{n-1}$ is also invariant under the Galois conjugation on E^{n-1} . So we may find a lattice $\lambda \subset F^{n-1}$ such that $\lambda \otimes \mathcal{O}_E = \Lambda \cap E^{n-1}$. This proves the surjectivity.

We claim that the set $\mathfrak{m}_i$ is sent to $\mathfrak{m}'_{i-\ell(\gamma)}$. Clearly we have $\ell(\lambda/\lambda_0) = \ell(\Lambda/\mathcal{O}_E^n)$ under this map. Hence the claim follows, since the length of $\mathcal{L}$ over the image of $\mathcal{O}_E^n$ is obviously given by $\ell(\gamma)$.

To finish the proof, we need to exhibit a bijection from $\mathfrak{m}'$ to M that sends $\mathfrak{m}'_i$ to M_i . Since $(g^i u, u) = v \gamma^i t v$ for all i , the map $\gamma^i v \mapsto g^i u$ defines an isometry between $\mathcal{L} = \mathcal{L}_\gamma$ and $L = L_g$. Moreover, the involution σ on E^n maps $\gamma^i v$ to $\bar{\gamma}^i v = \gamma^{-i} v$. Therefore σ transfers to the involution τ on $L \otimes E$. Clearly this map sends $\mathfrak{m}'_i$ to M_i , since it sends $\mathcal{L}$ to L .

□

COROLLARY 7.3. *Let $\gamma \in S_n(F)$ be regular semisimple and match $g \in U(C)(F)$.*

(1) *If $C = C'_n$, then*

$$O(\gamma, 1_{S_n(\mathcal{O}_F)}) = \omega(\gamma) \sum_{\{\Lambda \mid L \subset \Lambda \subset L^*, g\Lambda = \Lambda, \Lambda^\tau = \Lambda\}} (-1)^{\ell(\Lambda/L)}.$$

(2) *If $C = C_n$, then $O(\gamma, 1_{S_n(\mathcal{O}_F)}) = 0$ and*

$$O'(\gamma, 1_{S_n(\mathcal{O}_F)}) = -\omega(\gamma) \log q \sum_{\{\Lambda \mid L \subset \Lambda \subset L^*, g\Lambda = \Lambda, \Lambda^\tau = \Lambda\}} (-1)^{\ell(\Lambda/L)} \ell(\Lambda/L).$$

Proof. By Lemma 7.2, we have

$$O(\gamma, 1_{S_n(\mathcal{O}_F)}, s) = \sum_{i \in \mathbb{Z}} (-1)^i |M_{i-\ell(\gamma)}| q^{-is}.$$

Or equivalently

$$O(\gamma, 1_{S_n(\mathcal{O}_F)}, s) = (-1)^{\ell(\gamma)} \sum_{i \in \mathbb{Z}} (-1)^i |M_i| q^{-(i+\ell(\gamma))s}.$$

This shows that

$$O(\gamma, 1_{S_n(\mathcal{O}_F)}) = (-1)^{\ell(\gamma)} \sum_{i \in \mathbb{Z}} (-1)^i |M_i|.$$

In particular, if we set $C = C'_n$, the first identity is proved.

Now let $C = C_n$. The map $\Lambda \mapsto \Lambda^*$ defines an involution on M and sends M_i to M_{r-i} where r is the length of L^*/L , which is odd. This shows that $O(\gamma, 1_{S_n(\mathcal{O}_F)}) = 0$. We now take the first derivative

$$\begin{aligned} O'(\gamma, 1_{S_n(\mathcal{O}_F)}, 0) &= -(-1)^{\ell(\gamma)} \log q \sum_{i \in \mathbb{Z}} (-1)^i |M_i| (i + \ell(\gamma)) \\ &= -\omega(\gamma) \log q \sum_{i \in \mathbb{Z}} (-1)^i i |M_i|. \end{aligned}$$

This completes the proof. $\square$

Using Lemma 7.1 and Corollary 7.3 above, the statement of the FL (cf. Introduction) is the following identity for $g \in U(C'_n)(F)$ regular semi-simple:

$$\sum_{\{\Lambda | L \subset \Lambda \subset L^*, g\Lambda = \Lambda, \Lambda^\tau = \Lambda\}} (-1)^{\ell(\Lambda/L)} = \sum_{\{\Lambda | L \subset \Lambda \subset L^*, g\Lambda = \Lambda, \Lambda^* = \Lambda\}} 1. \quad (7.4)$$

Now, in the special case that the intersection of $\Delta(\mathcal{M})$ and $(\text{id}_{\mathcal{M}} \times g)\Delta(\mathcal{M})$ is discrete, the statement of the AFL (cf. Introduction) is as follows.

CONJECTURE 7.4. *Let $g \in U(C_n)(F)$ be regular semi-simple. Assume that $(\delta(\mathcal{M}) \cap \mathcal{N}^g)(\mathbb{F})$ is finite. Then*

$$\sum_{\{\Lambda | L \subset \Lambda \subset L^*, g\Lambda = \Lambda, \pi\Lambda \subset \Lambda^* \subset \Lambda\}} \text{mult}(\Lambda) = - \sum_{\{\Lambda | L \subset \Lambda \subset L^*, g\Lambda = \Lambda, \Lambda^\tau = \Lambda\}} (-1)^{\ell(\Lambda/L)} \ell(\Lambda/L).$$

Here the number $\text{mult}(\Lambda)$ is the intersection multiplicity of $\Delta(\mathcal{M})$ and $(\text{id}_{\mathcal{M}} \times g)\Delta(\mathcal{M})$ along the stratum $\mathcal{V}(\Lambda)^o$, cf. (4.3).

8. The minuscule case

In this section we assume that $V = L^*/L$ is killed by π . We thus consider it as a vector space over $k' = \mathbb{F}_{q^2}$, the residue field of $\mathcal{O}_E$. Denote by r its dimension. Then r is an odd integer between 1 and $n-1$. Then the hermitian form on L naturally induces a non-degenerate hermitian form on V . (It is obtained as follows: For $\bar{x}, \bar{y} \in V$ with representatives $x, y \in L^*$, the value $(\bar{x}, \bar{y})_V$ of the hermitian form on V is the image modulo (π) of $\pi \cdot (x, y) \in \mathcal{O}_E$, where $(\ , \)$ denotes the form on $L \otimes E$.) We denote the corresponding unitary group by $U(V)$ and consider it as an algebraic group defined over $\mathbb{F}_q$. As g defines automorphisms of both L and L^* , it induces an automorphism $\bar{g} \in U(V)$. Then via the map $\Lambda \mapsto \Lambda^*/L$ the set of vertices $\Lambda \in \text{Vert}^g(L)$ is in natural bijection with the set of $\bar{g}$ -invariant k' -subspaces W of V such that W is totally isotropic with respect to

the hermitian form on V . We thus define $\text{Vert}^{\bar{g}}(V)$ to be the set of such W . And we write $\mathcal{V}_W$ for the closed Bruhat-Tits stratum $\mathcal{V}_\Lambda$ that corresponds to $\Lambda \in \text{Vert}^g(L)$ in the sense of Vollaard's paper [16], cf. also [17], and we call *type* of W the type of Λ , i.e., the dimension of $W^\perp/W$. The open stratum $\mathcal{V}_W^\circ$ can then be identified with the Deligne-Lusztig variety associated to a Coxeter torus of $U(W^\perp/W)$, cf. [16].

We will consider the characteristic polynomial $P_{\bar{g}|V}(T) := \det(T - \bar{g}|V) \in k'[T]$ of degree r . Since, by Lemma 5.3, L^* is g -cyclic, V is $\bar{g}$ -cyclic. This is equivalent with the regularity of $\bar{g}$ as an endomorphism of V . In particular, its characteristic polynomial is equal to its minimal polynomial. Let

$$P_{\bar{g}} = \prod_{i=1}^{\ell} P_i^{a_i} \quad (8.1)$$

be the decomposition into irreducible monic polynomials.

If $P(T) = T^d + b_1 T^{d-1} + \dots + b_d \in k'[T]$, $b_d \neq 0$, we set

$$P^*(T) = \bar{b}_d^{-1} T^d \overline{P}(T^{-1}),$$

where the bar denotes the Galois conjugate on k' . Since $\bar{g} \in U(V)$, we have $P_{\bar{g}} = P_{\bar{g}}^* = \prod_i P_i^{*a_i}$, and hence we have an involution τ of $\{1, 2, \dots, \ell\}$ such that $P_i^* = P_{\tau(i)}$ and $a_{\tau(i)} = a_i$, cf. [1]. Note that since V has odd dimension, the degree of $P_{\bar{g}}$ is odd, and hence there exists at least one index i with $\tau(i) = i$ and such that a_i is odd.

PROPOSITION 8.1. (i) *The set $(\delta(\mathcal{M}) \cap \mathcal{N}^g)(\mathbb{F})$ is non-empty if and only if there exists a unique $i_0 \in \{1, 2, \dots, \ell\}$ such that $\tau(i_0) = i_0$ and such that a_{i_0} is odd. Then the set $(\delta(\mathcal{M}) \cap \mathcal{N}^g)(\mathbb{F})$ is finite.*

(ii) *If $(\delta(\mathcal{M}) \cap \mathcal{N}^g)(\mathbb{F})$ is non-empty (hence finite), these points lie on some strata $\mathcal{V}_W^\circ$, all of the same type $\deg P_{i_0}$ for the unique i_0 in part (i). And the cardinality of $(\delta(\mathcal{M}) \cap \mathcal{N}^g)(\mathbb{F})$ is given by*

$$\prod_{\{i,j\}, j=\tau(i) \neq i} (1 + a_i) \cdot \deg P_{i_0}.$$

Proof. If $(\delta(\mathcal{M}) \cap \mathcal{N}^g)(\mathbb{F})$ is non-empty, then there exists $W \in \text{Vert}^{\bar{g}}(V)$. Then $W^\perp$ is $\bar{g}$ -invariant and the hermitian form allows us to identify W with the dual of $V/W^\perp$. This yields a filtration

$$0 \subset W \subset W^\perp \subset V$$

and a decomposition

$$P_{\bar{g}|V} = P_{\bar{g}|W} \cdot P_{\bar{g}|W^\perp/W} \cdot P_{\bar{g}|V/W^\perp}, \quad (8.2)$$

with the property

$$P_{\bar{g}|W^\perp/W} = P_{\bar{g}|W^\perp/W}^*, \quad P_{\bar{g}|W} = P_{\bar{g}|V/W^\perp}^*. \quad (8.3)$$

Note that the fixed point set $\mathcal{V}_W^{\circ, \bar{g}}$ is non-empty if and only if $\bar{g}|(W^\perp/W)$ lies in a Coxeter torus, i.e., (because we are dealing here with a unitary group in an *odd* number of variables), if and only if $\bar{g}|(W^\perp/W)$ generates inside $\text{End}(W^\perp/W)$ a subfield of $\mathbb{F}_{q^{2r}}$. Since $\bar{g}$ is a regular endomorphism, so is the induced endomorphism on $W^\perp/W$. Hence the fixed point set $\mathcal{V}_W^{\circ, \bar{g}}$ is non-empty if and only if $P_{\bar{g}|W^\perp/W}$ is an irreducible polynomial. This irreducible polynomial has to be of the form P_{i_0} with $\tau(i_0) = i_0$. Moreover if $P_i | P_{\bar{g}|W}$, then $P_i^* | P_{\bar{g}|V/W^\perp}$. This shows that a_{i_0} is odd and that

for every $j \neq i_0$, either $\tau(j) \neq j$ or $\tau(j) = j$ and a_j is even. This shows the “only if” part of (i). Moreover, the type of W , i.e., the dimension of $W^\perp/W$, is equal to the degree of P_{i_0} .

We now assume that there exists a unique i_0 such that $\tau(i_0) = i_0$, and with a_{i_0} odd. To show the “if” part of (i) and part (ii), it suffices to prove the formula of cardinality and that the strata $\mathcal{V}_W^o$ have the desired type.

The decomposition (8.1) induces a decomposition as a direct sum of generalized eigenspaces

$$V = \bigoplus_{i=1}^{\ell} V_i, \quad V_i := \text{Ker } P_i^{a_i}(\bar{g}).$$

For $v_i \in V_i, v_j \in V_j$, there is some non-zero constant c such that (cf. [1])

$$0 = \langle P_i^{a_i}(\bar{g})v_i, v_j \rangle = c \langle v_i, P_i^{*a_i}(\bar{g}^{-1})v_j \rangle = c \langle v_i, P_i^{*a_i}(\bar{g})\bar{g}^{-s_i}v_j \rangle,$$

where $s_i = a_i \deg P_i$. Then we have two cases:

- If $\tau(i) = i$, by the above equation we see that V_i is orthogonal to $\bigoplus_{j \neq i} V_j$ and the restriction of the Hermitian form to V_i is non-degenerate.
- If $\tau(i) = j \neq i$, then $V_i \oplus V_j$ is orthogonal to $V_k, k \neq i, j$. And the restriction of the hermitian form to $V_i \oplus V_j$ is non-degenerate, both V_i and V_j being totally isotropic subspaces.

Consider the decomposition

$$W = \bigoplus_i W_i, \quad W_i := W \cap V_i.$$

Then each W_i is invariant under $\bar{g}$ and totally isotropic in each V_i . By the regularity of $\bar{g}$, we may list all $\bar{g}$ -invariant subspaces in V_i : for each $m = 0, 1, \dots, a_i$: there is precisely one invariant subspace (denoted by $V_{i,m}$) of dimension $m \cdot \deg P_i$ and these exhaust all invariant subspaces of V_i . Moreover $V_{i,m} = \text{Ker } P_i^m(\bar{g})$. Let now $i = i_0$. The proof of the “only if” part of (i) shows that $W_{i_0} = V_{i_0, \frac{a_{i_0}-1}{2}}$. We also know that $W' := \bigoplus_{i \neq i_0} W_i$ must be maximal totally isotropic in $V' := \bigoplus_{i \neq i_0} V_i$. Now suppose that $i \neq i_0$. We have two cases.

- If $\tau(i) = i$, W_i must be a maximal totally isotropic subspace of V_i . Hence $W_i = V_{i, a_i/2}$ is unique (note that a_i must be even by the assumption of the uniqueness of i_0).
- If $\tau(i) = j \neq i$, then $W_i \oplus W_j$ must be a maximal totally isotropic subspace of $V_i \oplus V_j$. Therefore W_j is uniquely determined by W_i and we can take $W_i = V_{i,m}$ for $m = 0, 1, \dots, a_i$. We thus have precisely $a_i + 1 = a_j + 1$ number of choices.

In summary we have shown that the cardinality of the set of W with $\mathcal{V}_W^{o, \bar{g}}$ non-empty is

$$\prod_{\{i,j\}, j=\tau(i) \neq i} (1 + a_i) \cdot \prod_{i \neq i_0, \tau(i)=i} 1.$$

Moreover, this also shows that the type of all such $\mathcal{V}_W^o$ is the same, namely $\deg P_{i_0}$. And by Lemma 6.7, for each W , the cardinality of $\mathcal{V}_W^{o, \bar{g}}$ is precisely $\dim W^\perp/W = \deg P_{i_0}$. We conclude that the cardinality of $(\delta(\mathcal{M}) \cap \mathcal{N}^g)(\mathbb{F})$ is

$$\sum_{W \in \text{Vert}^{\bar{g}}(V)} |\mathcal{V}_W^{o, \bar{g}}| = \prod_{\{i,j\}, j=\tau(i) \neq i} (1 + a_i) \cdot \deg P_{i_0}.$$

□

We now calculate the derivative of the orbital integral using Lemma 7.1.

PROPOSITION 8.2. *Let g be as above. Let $\gamma \in S_n(F)$ match g . Then $O'(\gamma, 1_{S_n(\mathcal{O}_F)}) = 0$ unless there is a unique i_0 such that $\tau(i_0) = i_0$ and with a_{i_0} odd, in which case*

$$O'(\gamma, 1_{S_n(\mathcal{O}_F)}) = -\omega(\gamma) \log q \prod_{\{i,j\}, j=\tau(i) \neq i} (1 + a_i) \cdot \deg P_{i_0} \cdot \frac{a_{i_0} + 1}{2}.$$

Proof. By mapping Λ to Λ/L , the set of lattices

$$\{\Lambda \mid L \subset \Lambda \subset L^*, g\Lambda = \Lambda, \Lambda^\tau = \Lambda\}$$

is in bijective correspondence with the set of subspaces

$$\mathcal{W} := \{W \mid W \subset V, \bar{g}W = W, W^{\bar{\tau}} = W\},$$

where $\bar{\tau}$ is the involution on V induced by the restriction of the involution τ to L^* . We need to describe this involution. By definition, the involution τ on C has the property that

$$(x^\tau, y^\tau) = (y, x), \quad (g\tau)^2 = 1$$

The induced involution $\bar{\tau}$ on V inherits the same properties. In particular, for a polynomial $P \in k'[T]$, we have

$$P(\bar{g})\bar{\tau} = \bar{\tau}\bar{P}(\bar{g}^{-1}).$$

In particular, $\bar{\tau}$ maps $V_{i,m} = \text{Ker } P_i^m(\bar{g})$ to $V_{\tau(i),m}$ and this is the reason we use the same notation τ to denote the involution on the index set $\{1, 2, \dots, \ell\}$. And clearly we have $\ell(\Lambda/L) = \dim_{k'} \Lambda/L$.

We consider the decomposition $V = \bigoplus_i V_i$ and $W = \bigoplus_i W_i$, where $W_i = W \cap V_i$.

First we assume there is a unique $i = i_0$ such that $\tau(i) = i$ and with a_i odd. According to $W_{i_0} = W \cap V_{i_0}$, we write $\mathcal{W} = \coprod_{m=0}^{a_{i_0}} \mathcal{W}_m$ as a disjoint union where $\mathcal{W}_m$ consists of $W \in \mathcal{W}$ such that $W_{i_0} = V_{i_0,m}$. Since $V_{i_0,m}$ is invariant under the involution $\bar{\tau}$, it is clear that the map $W \mapsto W \oplus V_{i_0,m}$ defines a bijection between $\mathcal{W}_0$ and $\mathcal{W}_m$. Therefore we may write the sum of Lemma 7.1

$$\sum_{W \in \mathcal{W}} (-1)^{\dim W} \dim W = \sum_{W \in \mathcal{W}_0} (-1)^{\dim W} \sum_{m=0}^{a_{i_0}} (-1)^{\dim V_{i_0,m}} (\dim W + \dim V_{i_0,m}).$$

Note that $a_{i_0} \cdot \deg P_{i_0}$ is odd. Therefore the inner sum simplifies to

$$-\frac{a_{i_0} + 1}{2} \deg P_{i_0},$$

which is independent of $W \in \mathcal{W}_0$.

We now compute the sum $\sum_{W \in \mathcal{W}_0} (-1)^{\dim W}$. As before we still use the notation $V' = \bigoplus_{i \neq i_0} V_i$. Note that $\mathcal{W}_0 = \{W \subset V' \mid W \in \mathcal{W}\}$. Then for $W \in \mathcal{W}_0$ we have the decomposition $W = \bigoplus_{i \neq i_0} W_i$. Similar to the proof of the previous proposition, we have two cases for $i \neq i_0$:

- If $\tau(i) = i$, then a_i is even and there are $a_i + 1$ choices of $W_i = V_{i,m}$ for $m = 0, 1, \dots, a_i$.
- If $\tau(i) = j \neq i$, then $W_j = \bar{\tau}W_i$ and there are $a_i + 1$ choices of $W_i = V_{i,m}$ for $m = 0, 1, \dots, a_i = a_j$.

Hence $\sum_{W \in \mathcal{W}_0} (-1)^{\dim W}$ is equal to the product of

$$\sum_{m=0}^{a_i} (-1)^{2\dim V_{i,m}} = a_i + 1$$

for each pair (i, j) with $j = \tau(i) \neq i$, and

$$\sum_{m=0}^{a_i} (-1)^{\dim V_{i,m}} = 1$$

for $i \neq i_0$ with $\tau(i) = i$ (and since a_i is even). This proves the formula when there is a unique i such that $\tau(i) = i$ and a_i odd.

Now suppose that there are at least two such i 's. We claim that then $O'(\gamma, 1_{S_n(\mathcal{O}_F)}) = 0$. The same argument as above shows that $\sum_{W \in \mathcal{W}_0} (-1)^{\dim W}$ is a product which has a factor of the form

$$\sum_{m=0}^{a_i} (-1)^{\dim V_{i,m}} = 0,$$

when $i \neq i_0$ with $\tau(i) = i$ and a_i odd! This shows that $O'(\gamma, 1_{S_n(\mathcal{O}_F)}) = 0$ and hence completes the proof. $\square$

9. Calculation of length

We continue to assume $\pi \cdot (L^*/L) = (0)$, and keep all other notation from the previous section. In particular, we assume that $(\delta(\mathcal{M}) \cap \mathcal{N}^g)(\mathbb{F})$ is non-empty.

A point of $(\delta(\mathcal{M}) \cap \mathcal{N}^g)(\mathbb{F})$ corresponds to a lattice B in $C_n \otimes_E \check{F}$, occurring in a chain of inclusions of lattices,

$$L \otimes_{\mathcal{O}_E} \mathcal{O}_{\check{F}} \subset \Lambda^* \otimes_{\mathcal{O}_E} \mathcal{O}_{\check{F}} \subset B^* \subset^1 B \subset \Lambda \otimes_{\mathcal{O}_E} \mathcal{O}_{\check{F}} \subset L^* \otimes_{\mathcal{O}_E} \mathcal{O}_{\check{F}}, \quad (9.1)$$

or also to a subspace U of $V \otimes_{k'} \bar{k}$, occurring in a chain of inclusions of vector spaces over $\bar{k}$, all of which are invariant under $\bar{g}$,

$$(0) \subset W \otimes_{k'} \bar{k} \subset U^\perp \subset^1 U \subset W^\perp \otimes_{k'} \bar{k} \subset V \otimes_{k'} \bar{k}. \quad (9.2)$$

Let λ be the eigenvalue of $\bar{g}|(U/U^\perp)$. Then by the regularity of $\bar{g}|U$, there exists a unique Jordan block to λ in U . The size of this Jordan block is of the form $c + 1$, where c is the size of the Jordan block of $\bar{g}|U^\perp$. The size of the Jordan block of $\bar{g}$ to λ is equal to the exponent a_{i_0} with which the irreducible polynomial $P = P_{i_0}$ occurs in $P_{\bar{g}}$ and is equal to $a_{i_0} = 2c + 1$. To see this, consider the decomposition $\prod P_i^{a_i}$ of the characteristic polynomial of $\bar{g}$ into irreducible factors. Over $\bar{k}$ only λ is a zero of P_{i_0} . Since P_{i_0} is an irreducible polynomial over a finite field, P_{i_0} has only simple zeros in $\bar{k}$. Since the minimal polynomial of $\bar{g}$ equals the characteristic polynomial of $\bar{g}$ it follows that the size of the (unique) Jordan block of the eigenvalue λ is the multiplicity of the zero λ in the characteristic polynomial, and this is a_{i_0} . Now we use the chain of inclusions (9.2) and the formulas (8.2) and (8.3) to conclude that $a_{i_0} = 2c + 1$.

PROPOSITION 9.1. *Assume $F = \mathbb{Q}_p$. Suppose that $n \leq 2p - 2$. The length of the local ring of $\delta(\mathcal{M}) \cap \mathcal{N}^g$ at the point $[B] \in (\delta(\mathcal{M}) \cap \mathcal{N}^g)(\mathbb{F})$ corresponding to B is equal to $c + 1 = \frac{1}{2}(a_{i_0} + 1)$.*

We note that this proposition, together with Propositions 8.1 and 8.2, proves assertion (iv) of Theorem 1.3.

We first determine the tangent space of $(\mathcal{M} \cap \mathcal{N}^g) \otimes \mathbb{F}$ at the point $[B]$ corresponding to B .

LEMMA 9.2. *The tangent space of $(\mathcal{M} \cap \mathcal{N}^g) \otimes \mathbb{F}$ at $[B]$ is a one-dimensional subspace of the tangent space of $\mathcal{M} \otimes \mathbb{F}$ if $c \geq 1$. If $c = 0$, the tangent space is trivial.*

Proof. For this, we have to first recall how one associates the lattice B to a point $(X, \iota, \lambda, \rho)$ of $\mathcal{N}_n(\mathbb{F})$. Let $M(X)$ be the Dieudonné module of X . Then ι induces a $\mathbb{Z}/2$ -grading of $M(X)$, and λ induces an alternating form $\langle \cdot, \cdot \rangle$ on $M(X)$. Furthermore, ρ induces an identification of $M(X)_0 \otimes_{\mathcal{O}_{\check{F}}} \check{F}$ with $C_n \otimes_E \check{F}$ such that $\text{id}_{C_n} \otimes \sigma^2$ corresponds to πV^{-2} , and the extended form on $C_n \otimes_E \check{F}$ to $x, y \mapsto \pi^{-1} \delta^{-1} \langle x, \pi V^{-1} y \rangle$. Here δ is a fixed element of $\mathcal{O}_E^\times$ with $\bar{\delta} = -\delta$. We then have $B = M(X)_0$, and $B^* = \pi V^{-1} M(X)_1$.

Now the tangent space is given by $\text{Hom}_{[B]}(\text{Spec } \mathbb{F}[\epsilon], \mathcal{N}_n)$, where the index $[B]$ indicates that only morphisms are considered whose image point is $[B]$. By Grothendieck-Messing theory, the tangent space $\mathcal{T}_{\mathcal{N}_n, [B]}$ is equal to

$$\text{Hom}_{[B]}(\text{Spec } \mathbb{F}[\epsilon], \mathcal{N}_n) = \text{Hom}_{0, \text{isot}}(VM(X)/\pi M(X), M(X)/VM(X)),$$

where the index indicates that only homomorphisms are considered which respect the $\mathbb{Z}/2$ -grading and the alternating form. Hence we have

$$\begin{aligned} \mathcal{T}_{\mathcal{N}_n, [B]} &= \text{Hom}(VM(X)_0/\pi M(X)_1, M(X)_1/VM(X)_0) \\ &\simeq \text{Hom}(M(X)_0/\pi V^{-1} M(X)_1, V^{-1} M(X)_1/M(X)_0) \\ &\simeq \text{Hom}(B/B^*, B^*/\pi B). \end{aligned}$$

Similarly, if $[B] \in \mathcal{N}^g$, the tangent space to $\mathcal{N}^g$ at $[B]$ is given by $\text{Hom}_g(B/B^*, B^*/\pi B)$, where the index indicates that only g -equivariant homomorphisms are considered. And if $[B] \in \mathcal{M}$, then

$$B = A \oplus \mathcal{O}_{\check{F}} u \text{ and } B^* = A^* \oplus \mathcal{O}_{\check{F}} u,$$

and

$$B/B^* = A/A^* \text{ and } B^*/\pi B = A^*/\pi A \oplus \mathbb{F} \bar{u}.$$

Then the tangent space to $\delta(\mathcal{M})$ at $[B]$ is equal to the subspace of $\text{Hom}(B/B^*, B^*/\pi B)$ consisting of homomorphisms whose image is contained in $A^*/\pi A$, and the tangent space to $\delta(\mathcal{M}) \cap \mathcal{N}^g$ at $[B]$ is given by the subspace of $\text{Hom}_g(B/B^*, B^*/\pi B)$ of elements which factor through $A^*/\pi A$. In other words, this tangent space is identified with the intersection of the eigenspace $(B^*/\pi B)(\lambda)$ to λ in $B^*/\pi B$ with $A^*/\pi A$.

We now show that this intersection, denoted by $\mathcal{T}$, has dimension one if $c \geq 1$ and zero if $c = 0$. Since $(B, B^*) \subset \mathcal{O}_{\check{F}}$ and $B^* \subset B$, we have an induced sesqui-linear pairing on $B^*/\pi B$ valued in $\mathbb{F}$. We still denote this pairing by $(\cdot, \cdot)$. As $u \in B^*$, we may denote by $\bar{u}$ its image in the $\mathbb{F}$ -vector space $B^*/\pi B$. The map $\bar{b} \mapsto (\bar{b}, \bar{u})$ defines an $\mathbb{F}$ -linear functional denoted by ℓ_u on $B^*/\pi B$. Using this, we may identify $A^*/\pi A$ with the kernel of ℓ_u . If $\bar{b} \in \mathcal{T}$, it is an eigenvector of $\bar{g}$ with eigenvalue λ , and $(\bar{b}, \bar{u}) = 0$. Since $\bar{b}$ is eigenvector of $\bar{g}$ we have

$$\bar{g}^{-1} \bar{b} = \lambda^{-1} \bar{b},$$

and hence

$$(\bar{g}^{-1} \bar{b}, u) = \lambda^{-1} (\bar{b}, \bar{u}) = 0, \text{ i.e., } (\bar{b}, \bar{g} \bar{u}) = 0.$$

Similarly, $(\bar{b}, \bar{g}^i \bar{u}) = 0$ for all $i \in \mathbb{Z}$. This implies that $(b, L) = 0 \pmod{\pi}$, where $b \in B^*$ is any lifting of $\bar{b} \in B^*/\pi B$. Equivalently we have $(b/\pi, L) \in \mathcal{O}_{\check{F}}$, and hence $b \in \pi L^* \otimes_{\mathcal{O}_E} \mathcal{O}_{\check{F}}$. We note the following sequence of inclusions,

$$\pi B \subset \pi L^* \otimes_{\mathcal{O}_E} \mathcal{O}_{\check{F}} \subset L \otimes_{\mathcal{O}_E} \mathcal{O}_{\check{F}} \subset \Lambda^* \otimes_{\mathcal{O}_E} \mathcal{O}_{\check{F}} \subset B^* \subset^1 B. \quad (9.3)$$

We have proved that $\mathcal{T}$ is a subspace of $X := (\pi L^* \otimes_{\mathcal{O}_E} \mathcal{O}_{\check{F}})/\pi B$. And, in fact, $\mathcal{T}$ is precisely the λ -eigenspace in X . Now X is obviously isomorphic to $Y := (L^* \otimes_{\mathcal{O}_E} \mathcal{O}_{\check{F}})/B$ as $\bar{g}$ -modules,

and hence is a $\bar{g}$ -cyclic $\mathbb{F}$ -vector space. It is easy to see that the λ -eigenspace $Y(\lambda)$ of Y is one-dimensional when $c \geq 1$ and zero if $c = 0$. $\square$

It follows from the preceding lemma that the completed local ring $R = \widehat{\mathcal{O}}_{\mathcal{M} \cap \mathcal{N}^g, [B]}$ is an $\mathcal{O}_{\check{F}}$ -algebra of the form

$$R = \mathcal{O}_{\check{F}}[[t]]/I \text{ if } c \geq 1, \text{ resp. } R = \mathcal{O}_{\check{F}}/I \text{ if } c = 0, \quad (9.4)$$

where I is an ideal in $\mathcal{O}_{\check{F}}[[t]]$, resp. in $\mathcal{O}_{\check{F}}$. Therefore Proposition 9.1 follows from the following proposition.

PROPOSITION 9.3. *Assume $F = \mathbb{Q}_p$. Suppose that $n \leq 2p - 2$. Then $I = (\pi, t^{c+1})$.*

The fact that $\pi \in I$ follows from the relation to the special divisors of [8]. Recall that to any non-zero element $x \in C_n$, there is associated the special divisor $\mathcal{Z}(x)$ of $\mathcal{N}_n$, cf. [8], Lemma 3.9. It is the closed formal subscheme of $\mathcal{N}$ with S -valued points

$$\{(X, \iota, \lambda, \rho) \mid \text{the composed quasi-homomorphism } \bar{Y} \times_{\mathcal{O}_{\check{F}}} \bar{S} \xrightarrow{x} \mathbb{X} \times_{\mathbb{F}} \bar{S} \xrightarrow{\varrho_X^{-1}} X \times_S \bar{S} \\ \text{lifts to an } \mathcal{O}_E\text{-linear homomorphism } \bar{Y} \times_{\mathcal{O}_{\check{F}}} S \rightarrow X\}.$$

Here we have identified C_n with $\text{Hom}_{\mathcal{O}_E}(\bar{Y} \times_{\mathcal{O}_{\check{F}}} \mathbb{F}, \mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ as explained in [8], Lemma 3.9. The elements of $\text{Hom}_{\mathcal{O}_E}(\bar{Y} \times_{\mathcal{O}_{\check{F}}} \mathbb{F}, \mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ are called *special homomorphisms*, cf. [8]. The special divisor $\mathcal{Z}(x)$ is a relative divisor, with set of $\mathbb{F}$ -points equal to

$$\mathcal{Z}(x)(\mathbb{F}) = \{B \in \mathcal{N}_n(\mathbb{F}) \mid x \in B^*\}. \quad (9.5)$$

Similarly, if $\mathbf{x} = [x_1, \dots, x_m] \in (C_n)^m$, then $\mathcal{Z}(\mathbf{x}) = \mathcal{Z}(x_1) \cap \dots \cap \mathcal{Z}(x_m)$ has $\mathbb{F}$ -points equal to

$$\mathcal{Z}(\mathbf{x})(\mathbb{F}) = \{B \in \mathcal{N}_n(\mathbb{F}) \mid \{x_1, \dots, x_m\} \subset B^*\}.$$

Now let $g \in G_n$ be regular semi-simple. Then

$$\delta(\mathcal{M}) \cap \mathcal{N}^g \subset \mathcal{Z}(u, gu, \dots, g^{n-1}u). \quad (9.6)$$

Indeed, $\delta(\mathcal{M})$ can be identified with $\mathcal{Z}(u)$, comp. [8], Lemma 5.2. Hence the assertion follows by the g -invariance of the LHS in (9.6).

Note that the fundamental matrix of $(u, gu, \dots, g^{n-1}u)$ in the sense of [8] is equivalent to the diagonal matrix $\pi^{\text{inv}(g)}$. Therefore we may apply the following theorem.

THEOREM 9.4. *Assume $F = \mathbb{Q}_p$. Let $\mathbf{x} = [x_1, \dots, x_n] \in (C_n)^n$ with fundamental matrix $T(\mathbf{x})$ equivalent to π^μ , where $\mu = (1^{(m)}, 0^{(n-m)})$ is minuscule. Then*

$$\mathcal{Z}(x_1, x_2, \dots, x_n) \subset \mathcal{N}_n \otimes_{\mathcal{O}_{\check{F}}} \bar{k}.$$

The proof is given in §10. Assuming this theorem, we may write

$$R = \mathbb{F}[[t]]/\bar{I}, \quad (9.7)$$

with an ideal $\bar{I} \subset \mathbb{F}[[t]]$. Note that at this point, the case $c = 0$ is proved completely (in particular, in this case the restriction $n \leq 2p - 2$ is not needed). The general case follows from the following theorem which together with Theorem 9.4 implies Propositions 9.3 and 9.1.

THEOREM 9.5. *Assume $F = \mathbb{Q}_p$. If $n \leq 2p - 2$, the ideal $\bar{I} \subset \mathbb{F}[[t]]$ equals (t^{c+1}) .*

The proof is given in §11.

COROLLARY 9.6. Assume $F = \mathbb{Q}_p$. Let $n \leq 2p - 2$. Assume that g is regular semisimple and that L_g^*/L_g is killed by π . Then the contribution of $\Lambda \in \text{Vert}^g(L_g)$ to the intersection multiplicity, if non-zero, is equal to

$$\text{mult}(\Lambda) = \deg P_{i_0} \frac{a_{i_0} + 1}{2}.$$

Of course, we are using here the notation of Proposition 8.1. Now all assertions of Theorem 1.3 are proved.

10. Proof of Theorem 9.4

In this section, we assume $F = \mathbb{Q}_p$. Accordingly, we write $\mathbb{Q}_{p^2}$ for E , and W for $\mathcal{O}_{\tilde{F}}$. In the terminology of [8], we will prove the following theorem.

THEOREM 10.1. Let $j_1, \dots, j_n$ be special homomorphisms such that the corresponding fundamental matrix $T(j_1, \dots, j_n)$ is equivalent to a diagonal matrix of the form $\text{diag}(p, \dots, p, 1, \dots, 1)$ (where p occurs say m times and 1 occurs $n - m$ times). Let $\mathcal{Z} = \bigcap_{i=1, \dots, n} \mathcal{Z}(j_i) \subseteq \mathcal{N}$. Then $\mathcal{Z}$ is an integral scheme. In particular, $p \cdot \mathcal{O}_{\mathcal{Z}} = 0$. In fact, $\mathcal{Z}$ is equal to $\mathcal{V}(\Lambda)$ for some vertex lattice Λ in C_n of type m .

REMARK 10.2. We point out that this theorem gives a modular interpretation of the closure $\mathcal{V}(\Lambda)$ of the Deligne-Lusztig variety $\mathcal{V}(\Lambda)^o$. Here $\Lambda = \langle x_1, \dots, x_n \rangle^*$ is the dual of the lattice generated by the elements $x_1, \dots, x_n$ of C_n corresponding to $j_1, \dots, j_n$.

First we remark that we may replace n by m , cf. [8], proof of Lemma 5.2. Hence we may assume that $T(j_1, \dots, j_n)$ is equivalent to the diagonal matrix $\text{diag}(p, \dots, p)$.

We use the following simple fact.

LEMMA 10.3. Let $\mathcal{O}$ be a complete discrete valuation ring, with uniformizer π and algebraically closed residue field k . Let $\mathcal{Y}$ be a (formal) scheme locally (formally) of finite type over $\text{Spf } \mathcal{O}$ and such that its special fiber $\mathcal{Y}_k$ is regular. Suppose that there does not exist a $\mathcal{O}/(\pi^2)$ -valued point of $\mathcal{Y}$. Then $\pi \cdot \mathcal{O}_{\mathcal{Y}} = 0$.

Proof. Suppose the claim is false. Then there exists a k -valued point x of $\mathcal{Y}$ such that $\pi \neq 0$ in $\mathcal{O}_{\mathcal{Y}, x}$. We show that under this assumption there is an $\mathcal{O}/(\pi^2)$ -valued point of $\mathcal{Y}$ with underlying k -valued point x . Locally around x , the (formal) scheme $\mathcal{Y}$ is a closed (formal) subscheme of a (formal) scheme $\mathcal{X}$ which is locally of finite type over $\mathcal{O}$ and smooth over $\mathcal{O}$. Let $R = \hat{\mathcal{O}}_{\mathcal{X}, x}$. We may identify R with $\mathcal{O}$ or with $\mathcal{O}[[x_1, \dots, x_N]]$ for some $N \geq 1$. In the first case there is nothing to do, so we assume the second case. Let I be the ideal of $\mathcal{Y}$ in R . Thus we assume that $\pi \notin I$. It is enough to construct an $\mathcal{O}$ -linear homomorphism $\psi : R/I \rightarrow \mathcal{O}/(\pi^2)$. Let $\mathfrak{m}$ be the maximal ideal of R and let $\mathfrak{m}'$ be the maximal ideal of $R/(\pi)$. Let $l = \dim \mathcal{X}_k - \dim \mathcal{Y}_k$. Since $\mathcal{Y}_k$ is regular, we find l distinct elements $q_1, \dots, q_l \in I$ such that the images of $q_1, \dots, q_l$ in $R/(\pi)$ generate the ideal of $\mathcal{Y}_k$ in $R/(\pi)$ and such that the images of $q_1, \dots, q_l$ in $\mathfrak{m}'/\mathfrak{m}'^2$ are linearly independent. We extend the q_i to a system of generators $q_1, \dots, q_r$ of I . For $i \leq l$, let $y_i = q_i$. We find elements $y_{l+1}, \dots, y_N \in \mathfrak{m}$ such that the images of $y_1, \dots, y_N$ in $\mathfrak{m}'/\mathfrak{m}'^2$ form a basis of $\mathfrak{m}'/\mathfrak{m}'^2$. Thus $\mathcal{O}[[x_1, \dots, x_N]] = \mathcal{O}[[y_1, \dots, y_N]]$. Now we consider the $\mathcal{O}$ -linear homomorphism $\phi : \mathcal{O}[[y_1, \dots, y_N]] \rightarrow \mathcal{O}$ given by $y_i \mapsto \pi^2$ for all i .

Claim The image of the ideal I under ϕ is (π^2) .

The ideal $\phi(I)$ is generated by $\phi(q_1), \dots, \phi(q_r)$. By definition we have $\phi(q_i) = \pi^2$ for $i \leq l$. Now assume that $i > l$. Then $q_i = \sum_{k \leq l} c_k q_k + \pi \cdot z$ for suitable elements $c_1, \dots, c_l, z \in R$ (depending on i). Since we assume $\pi \notin I$, it follows that z is not a unit, hence $z \in \mathfrak{m} = (\pi, y_1, \dots, y_N)$. Hence $\phi(z) \in (\pi)$ and $\phi(q_i) \in (\pi^2)$. This confirms the claim.

Using the claim it follows that ϕ induces a $\mathcal{O}$ -linear homomorphism $\psi : R/I \rightarrow \mathcal{O}/(\pi^2)$ yielding an $\mathcal{O}/(\pi^2)$ -valued point of $\mathcal{Y}$. But this contradicts our assumption. Hence $\pi \cdot \mathcal{O}_{\mathcal{Y}} = 0$. $\square$

REMARK 10.4. Taking into account Grothendieck's infinitesimal characterization of smoothness, the previous lemma gives a purely infinitesimal *sufficient condition* for a (formal) $\mathcal{O}$ -scheme to be a (formal) k -scheme.

We will prove Theorem 10.1 by showing in Proposition 10.5 and Corollary 10.8 that $\mathcal{Z}$ satisfies the hypotheses of the previous lemma.

For any (formal) W -scheme S , we denote by S_p its special fiber.

PROPOSITION 10.5. *Let $j_1, \dots, j_n$ and $\mathcal{Z}$ be as in the theorem. Then $\mathcal{Z}$ does not have a $W/(p^2)$ -valued point.*

Proof. We may assume that $j_1, \dots, j_n$ all have valuation 1 and are all perpendicular to each other. We also first assume that $n > 1$. Suppose there was a $W/(p^2)$ -valued point ϱ of $\mathcal{Z}$. Let M be the Dieudonné module of the underlying $\mathbb{F}$ -valued point. Let $M_{W/(p^2)} = M \otimes_W W/(p^2)$. We obtain a Hodge filtration $\mathcal{F} \hookrightarrow M_{W/(p^2)}$ corresponding to ϱ , and lifting the Hodge filtration of the underlying $\mathbb{F}$ -valued point. From the $\mathbb{Z}_{p^2}$ -action we get a decomposition $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$, where $\mathcal{F}_0$ is free of rank of rank $n-1$ and $\mathcal{F}_1$ is free of rank 1. Let $x_i = j_i(\bar{1}_0) \in M_0$, where we are using the notation of [8]. We denote the image of x_i in $M_{W/(p^2)}$ by $\bar{x}_i$. Then it follows that $\bar{x}_i \in \mathcal{F}_0$. Let $\bar{f}_1, \dots, \bar{f}_{n-1}$ be a basis of $\mathcal{F}_0$. Let $f_i \in M_0$ be a lift of $\bar{f}_i$, and choose $f_n \in M_0$ such that $f_1, \dots, f_n$ is a basis of M_0 . Let $\hat{x}_i$ be the image of x_i in the span of $f_1, \dots, f_{n-1}$ (viewed as a quotient of M_0). Then $\hat{x}_1, \dots, \hat{x}_n$ are linearly dependent, i.e., $\sum_i c_i \hat{x}_i = 0$ for suitable $c_i \in W$, which are not all zero. We may assume that the valuation of c_n is minimal among the valuations of the c_i . Dividing by $-c_n$ we may therefore assume that $\hat{x}_n = \sum_{i < n} c_i \hat{x}_i$. Therefore $x_n = \sum_{i < n} c_i x_i + c f_n$ for some $c \in W$. Since the image of $x_n - \sum_{i < n} c_i x_i = c f_n$ in $M_{W/(p^2)}$ lies in $\mathcal{F}_0$, it follows that c is divisible by p^2 .

Now for any $i \neq n$ we have $0 = \{x_n, x_i\} = c_i \{x_i, x_i\} + c \{f_n, x_i\} = c_i p + c \{f_n, x_i\}$. Here $\{, \}$ is the hermitian form on $C = (M_0 \otimes \mathbb{Q})^{V^{-1}F}$ as in §3. Since $x_i \in M_0^*$, it follows that $\{f_n, x_i\}$ is integral. Since further c is divisible by p^2 , it follows that c_i is divisible by p . It follows that $x_n/p \in M_0^*$ so that j_n has valuation bigger than 1, a contradiction which completes the proof in the case $n > 1$.

Finally, we observe that this reasoning also works for $n = 1$, since it shows that in this case $x_n = c f_n$, where c is divisible by p^2 so that j_n cannot have valuation 1. $\square$

LEMMA 10.6. *Let n be odd. Let x be a $\mathbb{F}$ -valued point of $\mathcal{N}$. The following conditions are equivalent.*

- (i) x lies on only one irreducible component of $\mathcal{N}_{\text{red}}$.
- (ii) No special cycle of valuation 0 passes through x .
- (iii) The Dieudonné module modulo p of x is of type $\mathbb{B}(n)$, in the sense of [17], §3 (cf. also the beginning of the proof of Theorem 10.7 below).

We remark that (ii) and (iii) both imply that n is odd. A point satisfying (ii) is called *super-general*.

Proof. The equivalence of (ii) and (iii) follows from [2], Proposition 3.6 and Lemma 4.1.

Next we prove the equivalence of (i) and (ii). The point x lies on two irreducible components if and only if there are two vertex lattices Λ_1 and Λ'_1 of type n in C_n such that x is a $\mathbb{F}$ -valued point of the corresponding irreducible components $\mathcal{V}(\Lambda_1)$ and $\mathcal{V}(\Lambda'_1)$. This is equivalent to the statement that $x \in \mathcal{V}(\Lambda)(\mathbb{F})$ for some vertex lattice Λ of type $t < n$. (Given Λ_1 and Λ'_1 define Λ as $\Lambda_1 \cap \Lambda'_1$.) We claim that for any vertex lattice Λ of type $t < n$ there is a special homomorphism j of valuation 0 with $\mathcal{V}(\Lambda) \subseteq \mathcal{Z}(j)$. To see this, note that Λ has an orthogonal basis $e_1, \dots, e_n$ such that $\{e_i, e_i\} = 1/p$ for $i \leq t$ and $\{e_i, e_i\} = 1$ for $i > t$. Let j be the special homomorphism with $j(\bar{1}_0) = e_{t+1}$. Then for the hermitian form $h(\cdot, \cdot)$ on the space $\mathbb{V}$ of special homomorphisms, we have $h(j, j) = 1$ and $y = j(\bar{1}_0) \in \Lambda^*$. This shows the claim, which implies (ii) $\implies$ (i). For the reverse implication, assume that Λ is a vertex lattice such that Λ^* contains a vector y with $\{y, y\} = 1$. Then Λ cannot be of type n . This shows that $\mathcal{V}(\Lambda)$ cannot be contained in special divisor of valuation 0. $\square$

THEOREM 10.7. *Let $n \geq 3$. Let x be a super-general $\mathbb{F}$ -valued point of $\mathcal{N}$. Then the following statements hold.*

(i) *For any special homomorphism j with $x \in \mathcal{Z}(j)(\mathbb{F})$ and $x \notin \mathcal{Z}(j/p)(\mathbb{F})$, the special fiber $\mathcal{Z}(j)_p$ is regular at x .*

(ii) *Let $j_1, \dots, j_n$ be a basis of the $\mathbb{Z}_{p^2}$ -module of special homomorphisms j with $x \in \mathcal{Z}(j)(\mathbb{F})$. Then the intersection $\bigcap \mathcal{Z}(j_i)_p$ is regular at x .*

Proof. It is enough to show the claims of the theorem in $\hat{\mathcal{O}}_{\mathcal{N}_p, x}$ instead of $\mathcal{O}_{\mathcal{N}_p, x}$. Let (X, ι, λ) be the p -divisible group with its $\mathbb{Z}_{p^2}$ -action and its p -principal polarization corresponding to x , and let M be its Dieudonné module. Let $M_p = M_{p,0} \oplus M_{p,1}$ be the reduction mod p of M . Since we assume that no special cycle of valuation 0 passes through x , it follows that M_p is isomorphic to $\mathbb{B}(n)$ and that n is odd. Here we are using the notation of [17], §3.1. This means that we find bases $\bar{e}_1, \dots, \bar{e}_n$ of $M_{p,0}$ and $\bar{f}_1, \dots, \bar{f}_n$ of $M_{p,1}$ such that $V(\bar{f}_i) = (-1)^i \bar{e}_{i+1}$ for $i < n$, $V(\bar{e}_n) = \bar{f}_1$, $F(\bar{f}_i) = (-1)^i \bar{e}_{i-1}$ for $i \geq 3$, $F(\bar{f}_2) = -\bar{e}_1$, $F(\bar{e}_1) = \bar{f}_n$ and for the induced alternating form we have $\langle \bar{e}_i, \bar{f}_j \rangle = \varepsilon_i \delta_{ij}$, where $\varepsilon_i = 1$ for $i = 1$ and $\varepsilon_i = -1$ for $i > 1$.

We find lifts $e_i \in M_0$ of $\bar{e}_i$ and lifts $f_i \in M_1$ of $\bar{f}_i$ such that still $\langle e_i, f_j \rangle = \varepsilon_i \delta_{ij}$. Denote by T the W -span of $e_1, f_2, f_3, \dots, f_n$ and by L the W -span of $f_1, e_2, e_3, \dots, e_n$. Then

$$M = L \oplus T, \quad VM = L \oplus pT.$$

Let $h_1 = e_1, h_2 = f_2, \dots, h_n = f_n, h_{n+1} = f_1, h_{n+2} = e_2, \dots, h_{2n} = e_n$. Define the matrix (α_{ij}) by

$$Fh_j = \sum_i \alpha_{ij} h_i \text{ for } j = 1, \dots, n,$$

$$V^{-1}h_j = \sum_i \alpha_{ij} h_i \text{ for } j = n+1, \dots, 2n.$$

Since we know the action of F resp. V on the $\bar{e}_i$ and $\bar{f}_i$, we can conclude that $V^{-1}(f_1) = e_n + \sum_{i < n} x_i e_i + pe$ for suitable $x_i \in W$ and $e \in M_0$. Similarly for $i \geq 2$ we have $V^{-1}(e_i) = (-1)^{i-1} f_{i-1} + y_i f_n + pg_i$ for suitable $y_i \in W$ and $g_i \in M_1$.

Thus (α_{ij}) is of the form

$$(\alpha_{ij}) = \left(\begin{array}{c|cccccc} & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ \hline 1 & & & & & & \\ \hline & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ \hline & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ \hline & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ \hline & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ \hline \end{array} \right) + pD,$$

where D has entries in W and also maps M_0 to M_1 and M_1 to M_0 . (The vertical and horizontal lines divide the first matrix into four $n \times n$ matrices, and only non-zero entries are displayed.) It follows (see [20], p. 48) that the universal deformation of X over $\mathbb{F}[[t_{11}, \dots, t_{nn}]]$ corresponds to the display $(L \oplus T) \otimes W(\mathbb{F}[[t_{11}, \dots, t_{nn}]])$ with matrix $(\alpha_{ij})^{\text{univ}}$ (wrt. the basis $h_1, \dots, h_{2n}$ and with entries in $W(\mathbb{F}[[t_{11}, \dots, t_{nn}]])$ given by

$$(\alpha_{ij})^{\text{univ}} = \begin{pmatrix} 1 & [t_{11}] & \dots & [t_{1n}] \\ & \ddots & \vdots & \ddots \\ & & 1 & [t_{n1}] \\ & & & \ddots \\ & & & & 1 \end{pmatrix} \cdot (\alpha_{ij}).$$

Here $[t]$ denotes the Teichmüller representative of t . Now let $A' = W[[t_{11}, \dots, t_{nn}]]$ and let $R' = \mathbb{F}[[t_{11}, \dots, t_{nn}]]$. We extend the Frobenius σ on W to A' by setting $\sigma(t_{ij}) = t_{ij}^p$. Let R be the completed universal deformation ring (in the special fiber) of X , together with its $\mathbb{Z}_{p^2}$ -action and its p -principal polarization. Then R is a quotient of R' by an ideal J . Using the fact that $(\alpha_{ij})^{\text{univ}}$ has to respect the $\mathbb{Z}/2$ -grading, it is easy to see that the ideal describing the deformation of the $\mathbb{Z}_{p^2}$ -action is $(t_{11}, t_{ij})_{i,j \neq 1}$. Using this, it is easy to see that $J = ((t_{11}, t_{ij})_{i,j \neq 1}, (t_{1i} - t_{i1})_{i \leq n})$. (Compare also [5], p. 231.) Thus we may identify R with the ring $\mathbb{F}[[t_2, \dots, t_n]]$, where t_i corresponds to the image of t_{1i} in R'/J . We also define $A = W[[t_2, \dots, t_n]]$. For any $m \in \mathbb{N}$, denote by $\mathfrak{a}_m$ resp. $\mathfrak{r}_m$ the ideal in A resp. in R generated by the monomials $t_2^{a_2} \dots t_n^{a_n}$, where $a_i \geq 0$ and $\sum a_i = m$. Hence $\mathfrak{r}_m = \mathfrak{m}^m$, where $\mathfrak{m}$ denotes the maximal ideal of R . Let $A_m = A/\mathfrak{a}_m$ and $R_m = R/\mathfrak{r}_m$. Then A' is a frame for R' , resp. A is a frame for R , resp. A_m is a frame for R_m . (See [21] for the definition of frames.)

For an A' - R' -window $(M', M'_1, \Phi', \Phi'_1)$, let $M_1'^\sigma = A' \otimes_{A', \sigma} M'_1$ and denote by $\Psi' : M_1'^\sigma \rightarrow M'$ the linearization of Φ'_1 . It is an isomorphism of A' -modules. Denote by $\alpha' : M'_1 \rightarrow M_1'^\sigma$ the composition of the inclusion map $M'_1 \hookrightarrow M'$ followed by Ψ'^{-1} . In this way, the category of formal p -divisible groups over R' becomes equivalent to the category of pairs (M'_1, α') consisting of a free A' -module of finite rank and an A' -linear injective homomorphism $\alpha' : M'_1 \rightarrow M_1'^\sigma$ such that $\text{Coker } \alpha'$ is a free R' -module, and satisfying the *nilpotence condition* [20]. Since we will

only consider deformations of formal p -divisible groups, the nilpotence condition will be fulfilled automatically, and we will ignore it, comp. also [8], section 8. A corresponding description holds for the category of formal p -divisible groups over R resp. R_m .

In the sequel we are using notation that is customary in Zink's theory. The notation M_1 conflicts with its usage when taking the degree-1-component of M under the $\mathbb{Z}/2$ -grading. Henceforward we will write M^1 for the degree-1-component.

Let $(\beta_{ij})^{\text{univ}}$ be the matrix over A' which is obtained from $(\alpha_{ij})^{\text{univ}}$ by replacing the $[t_i]$ by t_i and by multiplying the last n rows by p . We consider the A' - R' -window $(M', M'_1, \Phi', \Phi'_1)$ given by $M' = M \otimes A'$, $M'_1 = VM \otimes A'$, $\Phi' = (\beta_{ij})^{\text{univ}} \sigma$, $\Phi'_1 = \frac{1}{p} \cdot \Phi'$, where the matrix of Φ' is described in the basis $h_1, \dots, h_{2n}$. The corresponding display is the universal display described above (easy to see using the procedure described on p.2 of [21]). Hence $(M', M'_1, \Phi', \Phi'_1)$ is the universal window. Using this and the form of the ideal J given above, one checks that the map $\alpha : M_1 \rightarrow M_1^\sigma$ corresponding to the A - R window of the universal deformation of (X, ι, λ) (which is the base change of (M'_1, α')) can be written as follows (using the bases $pe_1, pf_2, \dots, pf_n$, $f_1, e_2, \dots, e_n$ resp. $p(1 \otimes e_1), p(1 \otimes f_2), \dots, p(1 \otimes f_n), 1 \otimes f_1, 1 \otimes e_2, \dots, 1 \otimes e_n$)

$$\tilde{\alpha} = \left(\begin{array}{cccccc|cccccc} -py_3 & py_4 & \dots & -py_n & p & -t_n + y_2 + \sum_{i \geq 3} (-1)^{i-1} y_i t_{i-1} & & & & & \\ -p & & & & & & t_2 & t_3 & \dots & t_{n-1} & x_1 + t_n & \\ & & & & & & -1 & & & & x_2 & \\ & & & & & & & 1 & & & x_3 & \\ & & & & & & & & \ddots & & \vdots & \\ & & & & & & & & & -1 & x_n & \\ \hline & & & & & & & & & & 1 & \\ & p & & & & & -1 & & & & & \\ & & -p & & & & -t_2 & & & & & \\ & & & \ddots & & & t_3 & & & & & \\ & & & & p & & \vdots & & & & & \\ & & & & & & -t_{n-1} & & & & & \end{array} \right) + B.$$

Here $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, where B_{ij} has size $n \times n$, and B_{11} and B_{21} have entries in p^2A and B_{12} and B_{22} have entries in pA . Rewriting this in the bases $pe_1, e_2, \dots, e_n, f_1, pf_2, \dots, pf_n$ resp. $p(1 \otimes e_1), 1 \otimes e_2, \dots, 1 \otimes e_n, 1 \otimes f_1, p(1 \otimes f_2), \dots, p(1 \otimes f_n)$ we obtain a block matrix $\alpha = \begin{pmatrix} \tilde{U} & U \end{pmatrix}$, where U is of the form

$$U = \left(\begin{array}{cccccc} -t_n + y_2 + \sum_{i \geq 3} (-1)^{i-1} y_i t_{i-1} & -py_3 & py_4 & \dots & -py_n & p \\ -1 & & & & & \\ -t_2 & p & & & & \\ t_3 & & -p & & & \\ \vdots & & & \ddots & & \\ -t_{n-1} & & & & p & \end{array} \right) + pB_U,$$

where B_U has entries in A and, in the last $n-1$ rows, even has entries in pA , and $\tilde{U}$ is of the

form

$$\tilde{U} = \begin{pmatrix} & & & & & 1 \\ -p & t_2 & t_3 & \dots & t_{n-1} & x_1 + t_n \\ & -1 & & & & x_2 \\ & & 1 & & & -x_3 \\ & & & \ddots & & \vdots \\ & & & & -1 & x_{n-1} \end{pmatrix} + pB_{\tilde{U}},$$

where $B_{\tilde{U}}$ has entries in A and in the first row even has entries in pA .

The corresponding universal p -divisible groups over R_m correspond to the pairs $(M_1(m), \alpha(m))$ obtained by base change from (M_1, α) .

Consider the p -divisible group $\bar{\mathbb{Y}}$ with its Dieudonné module $\bar{\mathbb{M}} = W\bar{\mathbb{I}}_0 \oplus W\bar{\mathbb{I}}_1$. Let $\bar{M}_1 = W\bar{\mathbb{I}}_0 \oplus Wp\bar{\mathbb{I}}_1$ and let $n_0 = \bar{\mathbb{I}}_0$ and $n_1 = p\bar{\mathbb{I}}_1$. Then $\bar{\mathbb{Y}}$ corresponds to the pair $(\bar{M}_1, \beta)$ where $\beta(n_0) = 1 \otimes n_1$ and $\beta(n_1) = -p \otimes n_0$. By base change $W \rightarrow A$ resp. $W \rightarrow A_m$ we obtain pairs $(\bar{M}_1R, \beta)$ resp. $(\bar{M}_1R_m, \beta)$ corresponding to the constant p -divisible group $\bar{\mathbb{Y}}$ over R resp. R_m . We denote the matrix of β by S , hence

$$S = \begin{pmatrix} 0 & -p \\ 1 & 0 \end{pmatrix}.$$

Now let j be as in the statement of the theorem, i.e. $x \in \mathcal{Z}(j)(\mathbb{F})$ but $x \notin \mathcal{Z}(j/p)(\mathbb{F})$. We want to investigate the ideal in R describing the maximal deformation of the homomorphism j , and its image in R_m . We will determine explicitly the image of this ideal in R_p .

The map j corresponds to a map $j(1) : \bar{M}_1 \rightarrow M_1(1)$ such that the following diagram commutes,

$$\begin{array}{ccc} \bar{M}_1 & \xrightarrow{\beta} & \bar{M}_1^\sigma \\ j(1) \downarrow & & \downarrow \sigma(j(1)) \\ M_1(1) & \xrightarrow{\alpha(1)} & M_1(1)^\sigma. \end{array}$$

Then j lifts over R_m if and only if there is a lift $j(m)$ of $j(1)$ such that the following diagram commutes,

$$\begin{array}{ccc} \bar{M}_{1R_m} & \xrightarrow{\beta} & \bar{M}_{1R_m}^\sigma \\ j(m) \downarrow & & \downarrow \sigma(j(m)) \\ M_1(m) & \xrightarrow{\alpha(m)} & M_1(m)^\sigma. \end{array}$$

We write $j(\bar{\mathbb{I}}_0) = a_1 \cdot pe_1 + a_2 \cdot e_2 + \dots + a_n \cdot e_n$. We also write $j(1) = (X(1), Y(1))$. Then $X(1)$ can be written in the above basis as

$$X(1) = \begin{pmatrix} a_1 & 0 \\ \vdots & \vdots \\ a_n & 0 \end{pmatrix}.$$

Similarly we write $j(p\bar{1}_1) = b_1 \cdot f_1 + b_2 \cdot pf_2 + \dots + b_n \cdot pf_n$ and

$$Y(1) = \begin{pmatrix} 0 & b_1 \\ \vdots & \vdots \\ 0 & b_n \end{pmatrix}.$$

Since j commutes with the Frobenius operator, we have $j(p\bar{1}_1) = FjF^{-1}(p\bar{1}_1) = Fj(\bar{1}_0)$. Using the matrix (α_{ij}) we see that

$$b_1 = -pa_2^\sigma + p^2s_1, \quad b_i = (-1)^i a_{i+1}^\sigma + ps_i \text{ for } 2 \leq i \leq n-1, \text{ and } b_n = a_1^\sigma + \sum_{i \geq 2} a_i^\sigma y_i + ps_n,$$

for suitable elements $s_i \in W$.

Similarly, exploiting the relation $j(\bar{1}_0) = FjF^{-1}(\bar{1}_0) = -\frac{1}{p}Fj(p\bar{1}_1)$, we obtain the system of equations

$$a_1 = -b_1^\sigma x_1/p - b_2^\sigma/p + r_1, \quad a_i = -b_1^\sigma x_i + (-1)^i b_{i+1}^\sigma + pr_i \text{ for } 2 \leq i \leq n-1, \text{ and } a_n = b_1^\sigma + pr_n,$$

for suitable elements $r_i \in W$. The equations $b_1 = -pa_2^\sigma + p^2s_1$ and $a_1 = -b_1^\sigma x_1/p - b_2^\sigma/p + r_1$ show that b_1 and b_2 are divisible by p . However, not all b_i are divisible by p . Indeed, if they were, then, because of $b_i = (-1)^i a_{i+1}^\sigma + pr_i$ for $2 \leq i \leq n-1$, the elements $a_3, \dots, a_n$ would also be divisible by p . Using $a_2 = -b_1^\sigma x_2 + b_3^\sigma + pr_2$, we see that then also a_2 would be divisible by p and, using $b_n = a_1^\sigma + \sum_{i \geq 2} a_i^\sigma y_i + ps_n$, it finally would follow that also a_1 is divisible by p . However, this would contradict our assumption that $x \in \mathcal{Z}(j)(\mathbb{F}) \setminus \mathcal{Z}(j/p)(\mathbb{F})$.

We are looking for liftings $X(m)$ of $X(1)$ and $Y(m)$ of $Y(1)$ over A_m such that

$$UY(m) = \sigma(X(m))S \quad \text{and} \quad \tilde{U}X(m) = \sigma(Y(m))S. \quad (10.1)$$

Suppose $m = p^l$, where $l \geq 1$, and suppose we have found liftings $X(p^{l-1})$ and $Y(p^{l-1})$ satisfying (10.1). For any choice of liftings $X(p^l)$ and $Y(p^l)$ of $X(p^{l-1})$ and $Y(p^{l-1})$, the matrices $\sigma(X(p^l))$, resp. $\sigma(Y(p^l))$ are equal to $\sigma(X(p^{l-1}))$ resp. $\sigma(Y(p^{l-1}))$, interpreted as matrices over A_{p^l} . Hence there are liftings $X(p^l)$ and $Y(p^l)$ satisfying (10.1) if and only if the matrices

$$U^{-1}\sigma(X(p^{l-1}))S \quad \text{and} \quad \tilde{U}^{-1}\sigma(Y(p^{l-1}))S$$

are integral, and in this case

$$X(p^l) = \tilde{U}^{-1}\sigma(Y(p^{l-1}))S \quad \text{and} \quad Y(p^l) = U^{-1}\sigma(X(p^{l-1}))S.$$

Define now inductively matrices $X_{\mathbb{Q}}(p^l)$ and $Y_{\mathbb{Q}}(p^l)$ over $A_{p^l} \otimes_{\mathbb{Z}} \mathbb{Q}$ as follows: $X_{\mathbb{Q}}(1) = X(1)$ and $Y_{\mathbb{Q}}(1) = Y(1)$ and

$$X_{\mathbb{Q}}(p^{l+1}) = \tilde{U}^{-1}\sigma(Y(p^l))S \quad \text{and} \quad Y_{\mathbb{Q}}(p^{l+1}) = U^{-1}\sigma(X(p^l))S.$$

(Again $\sigma(X_{\mathbb{Q}}(p^l))$ and $\sigma(Y_{\mathbb{Q}}(p^l))$ are well defined over $A_{p^{l+1}} \otimes_{\mathbb{Z}} \mathbb{Q}$.) It is easy to see that $Y_{\mathbb{Q}}(p)$ is integral.

Let $\bar{b}_i$ denote the image of b_i in $\mathbb{F}$. Let $D = \{i \geq 3 \mid \bar{b}_i \neq 0\}$. (This set is not empty as we saw above.) Using the form of the matrix $\tilde{U}$ we easily see that $X_{\mathbb{Q}}(p)$ is of the form

$$X_{\mathbb{Q}}(p) = \begin{pmatrix} \frac{1}{p} \sum_{i \in D} (-1)^i b_i^\sigma t_{i-1} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} + A(p),$$

where $A(p)$ is integral. We claim that the equation of $\mathcal{Z}(j)_p$ in $R/\mathfrak{m}^p$ is $\sum_{i \in D} (-1)^i \bar{b}_i^\sigma t_{i-1} = 0$. Let $t = \sum_{i \in D} (-1)^i \bar{b}_i^\sigma t_{i-1}$ and lift t to an element $\tilde{t} \in A$ using Teichmüller lifts of the coefficients of t . Then we claim that $\tilde{A} := A/((t_2, \dots, t_n)^p + \tilde{t} \cdot A)$ is a frame for $R/(\mathfrak{m}^p + t \cdot R)$. Since $\tilde{A}$ is isomorphic to $W[[X_1, \dots, X_{n-2}]]/(X_1, \dots, X_{n-2})^p$, it is torsion free as an abelian group. Let σ be the endomorphism on $\tilde{A}$ which extends the Frobenius on W by sending (the images of) the t_i to 0. Then σ induces the Frobenius on $R/(\mathfrak{m}^p + t \cdot R)$. The ideal $p \cdot \tilde{A}$ is in an obvious way equipped with a pd-structure. Using this frame, the same calculation as above shows that j lifts over $R/(\mathfrak{m}^p + t \cdot R)$. Since $\mathcal{Z}(j)_p$ is a divisor ([8], Proposition 3.5), it follows that we can write the equation for $\mathcal{Z}(j)_p$ in $R/\mathfrak{m}^p$ in the form $t \cdot s = 0$. We have to show that s is a unit. Assume s is not a unit. Then it follows that j lifts over $R/\mathfrak{m}^2$. For this ring we have the obvious frame $A/(t_2, \dots, t_n)^2$. Again the same calculation as above shows that j does not lift over $R/\mathfrak{m}^2$ because $\sum_{i \in D} (-1)^i \bar{b}_i^\sigma t_{i-1}$ is not divisible by p in $A/((t_2, \dots, t_n)^2)$. Thus the equation of $\mathcal{Z}(j)_p$ in $R/\mathfrak{m}^p$ is indeed $\sum_{i \in D} (-1)^i \bar{b}_i^\sigma t_{i-1} = 0$. Claim i) of the theorem follows.

Now we come to claim ii). Let $\mathfrak{n}$ be the maximal ideal in $\hat{\mathcal{O}}_{\mathcal{Z}, x}$. We need to show $\dim \mathfrak{n}/\mathfrak{n}^2 = \dim \mathcal{Z}_p$. Since x is super-general, it lies on a unique irreducible component of $\mathcal{N}_{\text{red}}$, of the form $\mathcal{V}(\Lambda)$, where Λ is a vertex lattice of type n , cf. [8], §4. Furthermore, $x \in \mathcal{Z}(j)(\mathbb{F})$ if and only if $j(\bar{1}_0) \in \Lambda^* = p\Lambda$. By Lemma 10.6, $\text{ord}_p(h(j_i, j_i)) \geq 1$ for all i , and by the results of [8], §4, the dimension of $\mathcal{Z}(j_i)_{\text{red}}$ is $(n-1)/2$ at x for all i . Hence $\mathcal{Z}_{\text{red}} = \mathcal{V}(\Lambda)$ locally at x , and has dimension $(n-1)/2$. We will show that $\dim \mathfrak{n}/\mathfrak{n}^2 = (n-1)/2$, which will prove that $\mathcal{Z}_p = \mathcal{Z}_{\text{red}}$ at x , and will finish the proof.

We saw above that the equation of $\mathcal{Z}(j)$ in $\mathfrak{m}/\mathfrak{m}^2$ is a linear equation of the form

$$\sum_{i \geq 1} (-1)^i \bar{b}_i(j)^\sigma t_{i-1} = 0,$$

where the coefficients $\bar{b}_i(j)$ arise by expressing $j(p\bar{1}_1)$ in terms of a specific basis of the $\mathbb{F}$ -vector space VM^0/pVM^0 with $\bar{b}_1(j) = \bar{b}_2(j) = 0$. We have to see that the rank of this system of linear equations, as $j(\bar{1}_0)$ varies through $p\Lambda$, is equal to $(n-1)/2$.

However, as j varies, the elements $j(p\bar{1}_1)$ generate the W -lattice $pV(\Lambda \otimes_{\mathbb{Z}_{p^2}} W)$ inside VM^0 , and the dimension of $pV(\Lambda \otimes_{\mathbb{Z}_{p^2}} W)/pVM^0$ is equal to $(n-1)/2$. $\square$

COROLLARY 10.8. *Let $j_1, \dots, j_n$ and $\mathcal{Z}$ be as in Theorem 10.1. Then the special fiber $\mathcal{Z}_p$ of $\mathcal{Z}$ is regular.*

Proof. We use induction on m (notation as in Theorem 10.1). We observe that m is always odd. For $m = 1$ there is nothing to do, since for $n = 1$ we have $\mathcal{N}_p \cong \text{Spec } \mathbb{F}$. If x is super-general, the assertion follows from Theorem 10.7.

Now assume that x is not super-general. By Lemma 10.6, there is a special cycle $\mathcal{Z}(j_0)$ of valuation 0 passing through x . We consider the $\mathbb{Z}_{p^2}$ -submodule J of $\mathbb{V}$ generated by $j_0, j_1, \dots, j_n$ and we define $\mathcal{Z}(J) = \cap_{i=0, \dots, n} \mathcal{Z}(j_i)$. (Recall that $\mathbb{V}$ is the $\mathbb{Q}_{p^2}$ -space of special homomorphisms, with hermitian form $h(\cdot, \cdot)$.)

Claim *There is an orthogonal $\mathbb{Z}_{p^2}$ -basis $b_1, \dots, b_n$ of J with $h(b_i, b_i) \in \{1, p\}$ for all i .*

We denote by U the $\mathbb{Z}_{p^2}$ -submodule of $\mathbb{V}$ generated by $j_1, \dots, j_n$, so that $U \subseteq J \subseteq \mathbb{V}$. Both U and J are free $\mathbb{Z}_{p^2}$ -modules of rank n . Let $U^\vee$ (resp. $J^\vee$) be the set of $j \in \mathbb{V}$ with $h(j, c) \in \mathbb{Z}_{p^2}$ for all $c \in U$ (resp. all $c \in J$). It follows that $pU^\vee = U$. Let $c_1, \dots, c_n$ be an orthogonal basis of J , and denote by α_i the valuation of $h(c_i, c_i)$. Assume now that $\alpha_i \geq 2$ for some i . Then it

follows that $p^{-1}c_i \in pJ^\vee \subseteq pU^\vee = U$. Hence $c_i \in pU \subseteq pJ$. But an element of pJ cannot be a member of a basis of J . Hence all c_i have valuation 0 or 1. Thus the claim follows.

The number of b_i which have valuation 0 is positive because there is an element of valuation 0 in J (e.g. j_0). Hence by the induction hypothesis $\mathcal{Z}(J)_p$ is regular. We know that the dimension of $\mathcal{Z}_p$ is the dimension of the supersingular locus of $\mathcal{N}_p$. Therefore the dimension of $\mathcal{Z}(J)_p = \mathcal{Z}_p \cap \mathcal{Z}(j_0)_p$ is smaller than the dimension of $\mathcal{Z}_p$. Since $\mathcal{Z}(J)_p$ is regular, it follows that $\mathcal{Z}_p$ is regular at x . $\square$

REMARK 10.9. Consider the isogeny $\alpha : \overline{Y}^n \rightarrow X$ defined by $(j_1, \dots, j_n)$. The kernel of α is a finite flat group scheme G of rank p^m , of type $(p, p, \dots, p)$ and equipped with an action of $\mathbb{F}_{p^2}$. As Zink pointed out, if $m = 1$, such a group scheme can only exist over a base Y with $p \cdot \mathcal{O}_Y = 0$. (He uses Oort-Tate theory to show this.) We do not know whether Theorem 10.1 can be seen from this angle in the general case.

11. Proof of Theorem 9.5

Choose a W -basis of B as follows. Choose $e_0, e_1, \dots, e_c \in B$ such that $e_1, \dots, e_c$ project to vectors in B^*/pB and e_0 is in $B \setminus B^*$, and such that the images of these vectors in B/L span the Jordan block relative to the eigenvalue λ of $\bar{g}$ in $U/U^\perp$. Next let l be the minimal integer ≥ 0 such that $g^l u \in pB$. For $l > i \geq 0$ denote by e_{c+l+i} the element $g^i u \in L$. Finally, we complete this to a basis by lifting vectors which project to Jordan blocks other than λ . These last vectors we call $e_{c+l+1}, \dots, e_{n-1}$. We therefore obtain the following identities modulo L ,

$$ge_0 \equiv \lambda e_0 + e_1, ge_1 \equiv \lambda e_1 + e_2, \dots, ge_{c-1} \equiv \lambda e_{c-1} + e_c, ge_c \equiv \lambda e_c. \quad (11.1)$$

If $m > c + l$ and $e_m, \dots, e_{m'}$ give rise to a Jordan block of g in B/L to an eigenvalue μ , then

$$ge_m \equiv \mu e_m + e_{m+1}, \dots, ge_{m'-1} \equiv \mu e_{m'-1} + e_{m'}, ge_{m'} \equiv \mu e_{m'}. \quad (11.2)$$

By perhaps changing the e_i by adding a suitable element of L , we may (and will) assume that these congruences also hold modulo pB .

The vectors e_i form a W -basis of $M_0 = B$, where M is the Dieudonné module of X , the p -divisible group belonging to B . Let $f_0, \dots, f_{n-1}$ be a basis of M_1 such that $\langle e_i, f_j \rangle = \delta_{ij}$. Denote by T the W -span of $e_0, f_1, f_2, \dots, f_{n-1}$ and by L' the W -span of $f_0, e_1, e_2, \dots, e_{n-1}$. (We only write L' instead of the usual notion L since the letter L is already used.) Then

$$M = L' \oplus T, \quad VM = L' \oplus pT.$$

Let $h_1 = e_0, h_2 = f_1, \dots, h_n = f_{n-1}, h_{n+1} = f_0, h_{n+2} = e_1, \dots, h_{2n} = e_{n-1}$. Define the matrix (α_{ij}) by

$$Fh_j = \sum_i \alpha_{ij} h_i \text{ for } j = 1, \dots, n,$$

$$V^{-1}h_j = \sum_i \alpha_{ij} h_i \text{ for } j = n+1, \dots, 2n.$$

It follows (see [20], p. 48) that the universal deformation of X over $\mathbb{F}[[t_{11}, \dots, t_{nn}]]$ corresponds to the display $(L' \oplus T) \otimes W(\mathbb{F}[[t_{11}, \dots, t_{nn}]])$ with matrix $(\alpha_{ij})^{\text{univ}}$ (wrt. the basis $h_1, \dots, h_{2n}$ and

with entries in $W(\mathbb{F}[[t_{11}, \dots, t_{nn}]])$ given by

$$(\alpha_{ij})^{\text{univ}} = \begin{pmatrix} 1 & & [t_{11}] & \dots & [t_{1n}] \\ & \ddots & \vdots & \ddots & \vdots \\ & & 1 & [t_{n1}] & \dots & [t_{nn}] \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \cdot (\alpha_{ij}).$$

Here the $[t_{ij}]$ denote the Teichmüller representatives of the t_{ij} . Now let $A' = W[[t_{11}, \dots, t_{nn}]]$ and let $R' = \mathbb{F}[[t_{11}, \dots, t_{nn}]]$. We extend the Frobenius σ on W to A' putting $\sigma(t_{ij}) = t_{ij}^p$. Let R be the completed universal deformation ring (in the special fiber) of X together with the $\mathbb{Z}_p$ -action and the p -principal polarization. Then R is a quotient of R' by an ideal $\mathfrak{J}$. Using the fact that $(\alpha_{ij})^{\text{univ}}$ has to respect the $\mathbb{Z}/2$ grading, it is easy to see that the ideal describing the deformation of the $\mathbb{Z}_p$ -action is $(t_{11}, t_{ij})_{i,j \neq 1}$. Using this, it is easy to see that $\mathfrak{J} = ((t_{11}, t_{ij})_{i,j \neq 1}, (t_{1i} - t_{i1})_{i \leq n})$. (Compare also [5], p. 231.) Thus we may identify R with the ring $\mathbb{F}[[t_1, \dots, t_{n-1}]]$, where t_i corresponds to the image of t_{1i+1} in $R'/\mathfrak{J}$. We also define $A = W[[t_1, \dots, t_{n-1}]]$.

Let $(\beta_{ij})^{\text{univ}}$ be the matrix over A' which is obtained from $(\alpha_{ij})^{\text{univ}}$ by replacing the $[t_i]$ by t_i and by multiplying the last n rows by p . We consider the $A' - R'$ window (M', M'_1, Φ') given by $M' = M \otimes A'$, $M'_1 = VM \otimes A'$, $\Phi' = (\beta_{ij})^{\text{univ}} \sigma$, where the matrix of Φ' is described in the basis $h_1, \dots, h_{2n}$. (We consider the h_i as elements in M' , and they form a basis of M' ; similarly $ph_1, \dots, ph_n, h_{n+1}, \dots, h_{2n}$ form a basis of M'_1 .) The corresponding display is the universal display described above (easy to see using the procedure described on p.2 of [21]). Hence we call (M', M'_1, Φ') the universal window.

For an element $f = \sum a_{k_1, \dots, k_{n-1}} t_1^{k_1} \dots t_{n-1}^{k_{n-1}} \in R$ we denote by $\tilde{f} \in A$ the element $\tilde{f} = \sum \tilde{a}_{k_1, \dots, k_{n-1}} t_1^{k_1} \dots t_{n-1}^{k_{n-1}}$, where $\tilde{a}_{k_1, \dots, k_{n-1}}$ is the Teichmüller lift of $a_{k_1, \dots, k_{n-1}}$. Thus $\tilde{f}$ is a lift of f .

Let $\mathfrak{m}$ be the maximal ideal of R . In the sequel, we call an ideal $J \subseteq \mathfrak{m}$ of R *admissible* if R/J is isomorphic to $\mathbb{F}[T]/(T^l)$ for some l with $1 \leq l \leq p$. (In particular, J contains $\mathfrak{m}^p$.)

Let J be admissible. We now construct a frame for R/J . If $l = 1$ (i.e. $J = \mathfrak{m}$) then there is nothing to do since W is a frame for $\mathbb{F}$. Thus we may assume that $l \geq 2$. Let $\mathfrak{m}_J$ be the maximal ideal of R/J . The map $R \rightarrow R/J$ induces a surjective linear map of $\mathbb{F}$ -vector spaces

$$\phi : \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{m}_J/\mathfrak{m}_J^2.$$

Since $R/J \cong \mathbb{F}[T]/(T^l)$, the dimension of $\mathfrak{m}_J/\mathfrak{m}_J^2$ is 1. Let $\bar{X}_1 \in \mathfrak{m}/\mathfrak{m}^2$ be an element that is not in the kernel of ϕ . We can extend $\bar{X}_1$ to a basis $\bar{X}_1, \dots, \bar{X}_{n-1}$ of $\mathfrak{m}/\mathfrak{m}^2$ such that $\bar{X}_2, \dots, \bar{X}_{n-1}$ are in the kernel of ϕ . Let $X_1 \in \mathfrak{m}$ be any lift of $\bar{X}_1$. We find lifts $X_2, \dots, X_{n-1}$ of $\bar{X}_2, \dots, \bar{X}_{n-1}$ which are all contained in J . It follows that $J = (X_1^l, X_2, \dots, X_{n-1})$. Let $\tilde{X}_i \in A$ be the lifts of the X_i as explained above. Then it follows that $R = \mathbb{F}[[X_1, \dots, X_{n-1}]]$ and $A = W[[\tilde{X}_1, \dots, \tilde{X}_{n-1}]]$. Define now $\tilde{J} = ((\tilde{X}_1)^l, \tilde{X}_2, \dots, \tilde{X}_{n-1})$. Then $(A/\tilde{J})/(p) = R/J$, and $A/\tilde{J}$ is torsion free as an abelian group. The endomorphism on A extending the Frobenius on W by sending $t_i \mapsto t_i^p$ induces an endomorphism on $A/\tilde{J}$ sending the images of the t_i to 0 since we assume that $l \leq p$ and hence $(t_1, \dots, t_{n-1})^p = (\tilde{X}_1^p, \dots, \tilde{X}_{n-1}^p) \subseteq (\tilde{X}_1^p, \tilde{X}_2, \dots, \tilde{X}_{n-1}) \subseteq \tilde{J}$. Since furthermore the ideal $p \cdot A/\tilde{J}$ in $A/\tilde{J}$ is obviously equipped with a pd-structure, it follows that indeed $A/\tilde{J}$ is a frame for R/J .

Let I be the ideal of $\mathcal{M} \cap \mathcal{Z}(g)$ in R . By Lemma 9.2, $R/I \cong \mathbb{F}[T]/(T^l)$ for some $l \geq 1$. Thus, the ideal $I + \mathfrak{m}^p$ is the smallest admissible ideal J such that $\text{Spec}(R/J) \subseteq \mathcal{M} \cap \mathcal{Z}(g)$.

Let $J \subseteq R$ be an admissible ideal such that $I + \mathfrak{m}^p \subseteq J$.

By base change from the universal window (M', M'_1, Φ') we obtain a window $(M^{(J)}, M_1^{(J)}, \Phi^{(J)})$ over R/J . We have a $\mathbb{Z}/2$ -grading³ $M^{(J)} = M^{(J),0} \oplus M^{(J),1}$ and $M_1^{(J)} = M_1^{(J),0} \oplus M_1^{(J),1}$. We denote by $G^{(J)}$ the matrix of g wrt. the basis of $M^{(J)}$ coming from the above basis of M' , and by $G_1^{(J)}$ the matrix of g wrt. the basis of $M_1^{(J)}$ coming from the above basis of M'_1 . Denote by $\tilde{\Phi}$ the matrix of Φ wrt. this basis of $M^{(J)}$. Then $G^{(J)}$ and $G_1^{(J)}$ are integral. Since g commutes with Φ , we have

$$G^{(J)} = \tilde{\Phi} \sigma(G^{(J)}) \tilde{\Phi}^{-1} = \tilde{\Phi} \sigma(G^{(\mathfrak{m})}) \tilde{\Phi}^{-1}.$$

Here we use that since $\mathfrak{m}^p \subseteq J$, we have $\sigma(G^{(J)}) = \sigma(G^{(\mathfrak{m})})$, where we view $G^{(\mathfrak{m})}$ (i.e. the matrix of g over W corresponding to the $\mathbb{F}$ -valued point) as a matrix with entries in $A/\bar{J}$. Similarly,

$$G_1^{(J)} = \begin{pmatrix} p^{-1}1_n & \\ & 1_n \end{pmatrix} \tilde{\Phi} \begin{pmatrix} p^{-1}1_n & \\ & 1_n \end{pmatrix}^{-1} \sigma(G_1^{(J)}) \begin{pmatrix} p^{-1}1_n & \\ & 1_n \end{pmatrix} \tilde{\Phi}^{-1} \begin{pmatrix} p^{-1}1_n & \\ & 1_n \end{pmatrix}^{-1}.$$

Again we observe that $\sigma(G_1^{(J)}) = \sigma(G_1^{(\mathfrak{m})})$. Let

$$\tau = \begin{pmatrix} 0 & t_1 & \dots & t_{n-1} \\ t_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ t_{n-1} & 0 & \dots & 0 \end{pmatrix}.$$

Then

$$\tilde{\Phi} = \begin{pmatrix} 1_n & \tau \\ 0 & 1_n \end{pmatrix} (\alpha_{ij}) \begin{pmatrix} 1_n & \\ & p \cdot 1_n \end{pmatrix}. \quad (11.3)$$

If $S \neq 0$ is a quotient of R by an admissible ideal and if (M^S, M_1^S, Φ^S) denotes the corresponding window obtained by base change from the universal one, let $g^S : M^S \otimes \mathbb{Q} \rightarrow M^S \otimes \mathbb{Q}$ be the map which is induced by the map $g^{(\mathfrak{m}^p)} : M^{(\mathfrak{m}^p)} \otimes \mathbb{Q} \rightarrow M^{(\mathfrak{m}^p)} \otimes \mathbb{Q}$ which in turn lifts g and commutes with $\Phi^{(\mathfrak{m}^p)}$. (Here $(M^{(\mathfrak{m}^p)}, M_1^{(\mathfrak{m}^p)}, \Phi^{(\mathfrak{m}^p)})$ is the window over $R/\mathfrak{m}^p$ obtained by base change from the universal one, where we use the obvious frame $A/(t_1, \dots, t_{n-1})^p$ for $R/\mathfrak{m}^p$.) The map $g^{(\mathfrak{m}^p)}$ is given by the matrix $G^{(\mathfrak{m}^p)}$ of $g^{(\mathfrak{m}^p)}$ with respect to the above basis of $M^{(\mathfrak{m}^p)} \otimes \mathbb{Q}$, i.e. $G^{(\mathfrak{m}^p)} = \tilde{\Phi} \sigma(G^{(\mathfrak{m}^p)}) \tilde{\Phi}^{-1} = \tilde{\Phi} \sigma(G^{(\mathfrak{m})}) \tilde{\Phi}^{-1}$, where again by abuse of notation we write $G^{(\mathfrak{m})}$ for an arbitrary lift of $G^{(\mathfrak{m})}$ over $A/(t_1, \dots, t_{n-1})^p$ and $\tilde{\Phi}$ is the matrix of $\Phi^{(\mathfrak{m}^p)}$ wrt. the above basis of $M^{(\mathfrak{m}^p)} \otimes \mathbb{Q}$. We claim that g lifts over S if and only if g^S maps $M^{S,0}$ into $M^{S,0}$ and $M_1^{S,0}$ into $M_1^{S,0}$. It is obvious that these conditions are necessary. Suppose they are fulfilled.

Since g is unitary, $\langle x, y \rangle = \langle gx, gy \rangle$ for all $x, y \in M$. In other words, for the adjoint $g^\dagger$ of g we have $g^\dagger = g^{-1}$. It is obvious (from the above formulas for the matrices of the lifts of g) that the map $p^2 g$ lifts to a map $\tilde{g}_1$ over $R/\mathfrak{m}^p$. By rigidity $\tilde{g}_1^\dagger = p^4 \tilde{g}_1^{-1}$. Let $\langle \cdot, \cdot \rangle_p$ be the alternating form on $M^{(\mathfrak{m}^p)} \otimes_{\mathbb{Z}} \mathbb{Q}$. Then it follows that $\langle p^{-2} \tilde{g} x, y \rangle_p = \langle x, (p^{-2} \tilde{g})^{-1} y \rangle_p$. Suppose now that g^S maps $M^{S,0}$ into $M^{S,0}$ and $M_1^{S,0}$ into $M_1^{S,0}$. Denoting by $\langle \cdot, \cdot \rangle_S$ the alternating form on M^S , this means that $\langle g^S e_i, f_j \rangle_S$ is integral for all i, j and $\langle g^S e_i, f_1 \rangle_S$ is an integral multiple of p for all i . This implies that $\langle e_i, (g^S)^{-1} f_j \rangle_S$ is integral for all i, j and $\langle e_i, (g^S)^{-1} f_1 \rangle_S$ is an integral multiple of p for all i . Since the determinant of g (restricted to an endomorphism of $M^1 \otimes_{\mathbb{Z}} \mathbb{Q}$) is a unit in W , also the determinant of g^S (restricted to an endomorphism of $M^{S,1} \otimes_{\mathbb{Z}} \mathbb{Q}$) is a unit in S . Hence

³As in the previous section, we now write the grading index as an upper index, to avoid a conflict of notation with Zink's theory.

it follows that also $\langle e_i, (g^S)f_j \rangle_S$ integral for all i, j and $\langle e_i, (g^S)f_1 \rangle_S$ is an integral multiple of p for all i . This shows that g^S maps $M^{S,1}$ into $M^{S,1}$ and $M_1^{S,1}$ into $M_1^{S,1}$, confirming the claim.

Let $J \subseteq R$ again be an admissible ideal such that $I + \mathfrak{m}^p \subseteq J$. We compute:

$$\begin{aligned} G_1^{(J)} &= \begin{pmatrix} p^{-1}1_n & \\ & 1_n \end{pmatrix} \tilde{\Phi} \begin{pmatrix} p^{-1}1_n & \\ & 1_n \end{pmatrix}^{-1} \sigma(G_1^{(\mathfrak{m})}) \begin{pmatrix} p^{-1}1_n & \\ & 1_n \end{pmatrix} \tilde{\Phi}^{-1} \begin{pmatrix} p^{-1}1_n & \\ & 1_n \end{pmatrix}^{-1} \\ &= \begin{pmatrix} p^{-1}1_n & \\ & 1_n \end{pmatrix} \tilde{\Phi} \sigma(G^{(\mathfrak{m})}) \tilde{\Phi}^{-1} \begin{pmatrix} p^{-1}1_n & \\ & 1_n \end{pmatrix}^{-1}. \end{aligned}$$

Using that $G^{(\mathfrak{m})} = (\alpha_{ij}) \begin{pmatrix} 1_n & 0 \\ 0 & p1_n \end{pmatrix} \sigma(G^{(\mathfrak{m})}) \begin{pmatrix} 1_n & 0 \\ 0 & p1_n \end{pmatrix}^{-1} (\alpha_{ij})^{-1}$ and the equation (11.3), we obtain

$$G_1^{(J)} = \begin{pmatrix} p^{-1}1_n & \\ & 1_n \end{pmatrix} \begin{pmatrix} 1_n & \tau \\ 0 & 1_n \end{pmatrix} G^{(\mathfrak{m})} \begin{pmatrix} 1_n & -\tau \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} p^{-1}1_n & \\ & 1_n \end{pmatrix}^{-1}.$$

This matrix respects the $\mathbb{Z}/2$ -grading and it is integral since $I + \mathfrak{m}^p \subseteq J$. We consider the block matrix of $g = g^{R/\mathfrak{m}}$ according to the $\mathbb{Z}/2$ -grading of M , and denote by H the ‘left upper block’ describing the endomorphism of M^0 induced by g wrt the basis $e_0, \dots, e_{n-1}$. Let $H^{(J)}$ the upper left block of the matrix obtained from $G^{(J)}$ by base change to the basis $e_0, \dots, e_{n-1}, f_0, \dots, f_{n-1}$. Similarly, let $H_1^{(J)}$ be the upper left block of the matrix obtained from $G_1^{(J)}$ by base change to the basis $pe_0, e_1, \dots, e_{n-1}, f_0, pf_1, \dots, pf_{n-1}$. Then $H_1^{(J)}$ is given by

$$H_1^{(J)} = \begin{pmatrix} p^{-1} & \\ & 1_{n-1} \end{pmatrix} \begin{pmatrix} 1 & t_1 & \dots & t_{n-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} H \begin{pmatrix} 1 & -t_1 & \dots & -t_{n-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} p^{-1} & \\ & 1_{n-1} \end{pmatrix}^{-1}.$$

Note that we only want to consider deformations which factor through $\delta(\mathcal{M})$. In terms of the parameters $t_1, \dots, t_{n-1}$, this condition just says that $t_{c+1} = 0$.

The matrix H is up to multiples of p given by the description of the action of g on the e_i in the beginning of the proof.

We want to find the conditions that the matrix $H_1^{(J)}$ is integral. For this it is enough to check when the entries in the first line are integral (the coefficients of pe_0 in the images of the basis vectors). Using the equations (11.1) (and calculating modulo integral elements, i.e., modulo elements of R), the first $c+l$ entries are

$$\begin{aligned} &\lambda + t_1, (-t_1^2 + t_2)/p, (-t_1t_2 + t_3)/p, \dots, (-t_1t_{c-1} + t_c)/p, (-t_1t_c)/p, \\ &(-(\lambda + t_1)t_{c+1} + t_{c+2})/p, \dots, (-(\lambda + t_1)t_{c+l-1} + t_{c+l})/p, (-(\lambda + t_1)t_{c+l})/p. \end{aligned}$$

Using $t_{c+1} = 0$, this shows that in R/J

$$t_1t_i = t_{i+1}, \forall i \leq c-1, t_1t_c = 0, \text{ and } t_{c+2} = \dots = t_{c+l} = 0.$$

If $e_m, \dots, e_{m'}$ span another Jordan block (mod L) to an eigenvalue μ , then by equations (11.1) and (11.2), the entries with index between m and m' are (modulo integral elements)

$$(-\lambda t_m - t_1t_m + \mu t_m + t_{m+1})/p, \dots, (-\lambda t_{m'-1} - t_1t_{m'-1} + \mu t_{m'-1} + t_{m'})/p, t_{m'}(-\lambda + \mu - t_1)/p.$$

Since $\lambda \neq \mu$ (modulo p), the expression $(-\lambda + \mu - t_1)$ is a unit, hence we obtain $t_{m'} = 0$ in R/J .

Inductively we obtain that $t_m = \dots = t_{m'-1} = t_{m'} = 0$ in R/J . Therefore

$$(t_1 t_i - t_{i+1} \text{ for } i \leq c-1, t_1 t_c, t_i \text{ for } i \geq c+1) \subseteq J.$$

Note that there do not occur further conditions from the integrality of

$$H^{(J)} = \begin{pmatrix} 1 & t_1 & \dots & t_{n-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} H \begin{pmatrix} 1 & -t_1 & \dots & -t_{n-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

We claim that $J_0 := (t_1 t_i - t_{i+1} \text{ for } i \leq c-1, t_1 t_c, t_i \text{ for } i \geq c+1) + \mathfrak{m}^p = I + \mathfrak{m}^p$. Obviously J_0 is admissible. If we choose $S = R/J_0$ then the same calculation as above shows that g lifts over S . Since $t_{c+1} \in J_0$ it follows that $\text{Spec}(S) \subseteq \mathcal{M} \cap \mathcal{Z}(g)$. It follows that indeed $I + \mathfrak{m}^p = J_0$.

Since $2c+1 \leq n$ and since we assume $n \leq 2p-2$, we have $c+1 < p$. Hence the claim of Theorem 9.5 follows from the next lemma. $\square$

LEMMA 11.1. *Let $r > s > 0$ be integers. Consider the ideal J in $\mathbb{F}[[X_1, \dots, X_n]]$, where*

$$J = (X_1^s, X_2, \dots, X_n).$$

Let I be an ideal in $\mathbb{F}[[X_1, \dots, X_n]]$ such that

$$I + \mathfrak{m}^r = J,$$

where $\mathfrak{m}$ denotes the maximal ideal. Then $I = J$.

Proof. We proceed in several steps.

Step 1. Consider the projection $\mathbb{F}[[X_1, \dots, X_n]] \rightarrow \mathbb{F}[[X_2, \dots, X_n]]$, obtained by dividing out by (X_1) . Let $\bar{I}$, resp. $\bar{J}$, be the image of I , resp. J , and let $\bar{\mathfrak{m}}$ be the maximal ideal of $\mathbb{F}[[X_2, \dots, X_n]]$. Then $\bar{I} = \bar{J} = \bar{\mathfrak{m}}$ by Nakayama's Lemma.

Step 2. Let $b \in J$. Then b is congruent modulo I to an element of $\mathfrak{m}^r$. Writing this latter element as a sum of monomials in $X_1, \dots, X_n$, and using step 1, we see that b is congruent modulo I to an element in the ideal (X_1^r) . Hence it suffices to prove that $X_1^r \in I$.

Step 3. We claim that in fact $X_1^s \in I$. We will show that $X_1^s \in I + \mathfrak{m}^{kr}$ for all k , which will prove the claim.

We proceed by induction on k , the case $k = 1$ holding true by hypothesis. Assume that $X_1^s \in I + \mathfrak{m}^{kr}$. Hence we are assuming that X_1^s is congruent modulo I to an element of $\mathfrak{m}^{kr}$. Writing this element of $\mathfrak{m}^{kr}$ as a sum of monomials in $X_1, \dots, X_n$, we subdivide this sum into

- a sum of monomials, where the exponent of X_1 is $\geq s$,
- a sum of monomials, where the exponent of X_1 is $< s$.

In the first sum, we extract the factor X_1^s ; since $kr > s$, the remainder lies in $\mathfrak{m}$. Bringing this first sum to the left hand side, we see that this expression differs from X_1^s by a unit. Hence we may disregard the first sum.

In the second sum, the total degree in $X_2, \dots, X_n$ of each monomial is strictly larger than $kr - s$, i.e., is at least $(k-1)r + 2$. Now $X_2, \dots, X_n$ are congruent modulo I to elements in $\mathfrak{m}^r$ and we may replace each $X_2, \dots, X_n$ by an element in $\mathfrak{m}^r$. Then each summand lies in $\mathfrak{m}^{((k-1)r+2)r} \subset \mathfrak{m}^{(k+1)r}$, which concludes the induction step. $\square$

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ON THE ARITHMETIC FUNDAMENTAL LEMMA IN THE MINUSCULE CASE

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