GRADUATE SEMINAR ON APPLIED LOGIC (S4A6) SS 2023

Applications of Tame Geometry

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Time and Place. Fridays 12.15-14, Kleiner Hörsaal

Organizational Meeting. Friday February 10th, 12 PM, on Zoom

https://uni-bonn.zoom.us/j/69610521460?pwd=OUo5RUlDSHJKbW05MlZRb3RsZGV1UT09

If you want to give a talk, send me (Philipp Hieronymi) an email by 27.2.2023 indicating which talks are interested in. If possible, list at least three topics.

Abstract. In this seminar, we cover some of the striking recent applications of o-minimality in number theory and arithmetic geometry. The main goal is to understand how o-minimality is used in the recent proof of the André-Oort conjecture, a vast generalization of the Manin-Mumford conjecture, by Pila, Shankar and Tsimerman [16] following work of Binyamini, Schmidt and Yafaev [5]. We outline the general strategy, usually referred to as the Pila-Zannier strategy, in the simple case of tori, and, step by step, consider the more general cases of abelian varieties and Shimura varieties. There are three key ingredients needed in all these settings: (1) definability of certain transcendental maps in some o-minimal structure, (2) functional transcendence a la Ax-Schanuel, and (3) lower bound for certain Galois orbits. We focus on the first two of these, as they are maybe a little bit closer to the material covered in previous (logic) courses. However, if there are participants with the right background knowledge, we can also have talks on the third ingredient in the generality of Shimura varieties (or at least $\mathcal{A}_g$).

This seminar is very much inspired by (and often follows) the program of the following three reading seminars/lecture series:

1. GAUS AG-Seminar – SoSe 2022, https://www.uni-frankfurt.de/118874390.pdf
2. O-minimality and Ax-Schanuel Reading Seminar, Harvard 2022, https://people.math.harvard.edu/~tayou/program.pdf
3. Felix Klein lectures, Bonn 2019, https://drive.google.com/file/d/1KNXtWekGk1mAyJki-xT6bnbr9ofjdZGs/view

In contrast to these, we assume knowledge of o-minimal structures, in particular cell decomposition and the Pila-Wilkie theorem. We also focus more on how o-minimality is used, and are not (necessarily) planning to prove all theorems (in particular, André-Oort) in the greatest generality.

In addition to giving a talk, each participant (who wants to get credit) is required to submit a 4 or more page summary of their topic prepared using latex. The plan is to compile these into in a proceedings volume of the seminar.

Prerequisites. This seminar is designed for students who have taken *V4A8 - Advanced Mathematical Logic II* in Wintersemester 2022/23. This means that we try to keep the assumptions on background knowledge from arithmetic geometry as minimal as possible. Students with no (or little) background in o-minimality, but experience in arithmetic geometry (for example with Shimura varieties) are also very welcome! Results from o-minimality can be black-boxed quite effectively, and we have several talks that would benefit tremendously from background knowledge in arithmetic geometry.

Talks.

(0) **O-minimality** (if necessary, by Philipp Hieronymi)
Recall the main theorems from o-minimality covered in previous courses, in particular monotonicty theorem, cell decomposition and the Pila-Wilkie theorem.

(1) **Extensions of the Pila-Wilkie theorem** (1 talk)
Recall the Pila-Wilkie theorem, including the necessary notation. Then prove the block version of the counting theorem [4, Theorem 8.4] and the extension of Pila-Wilkie to counting algebraic points [4, Theorem 8.5].
References: Bhardwaj and van den Dries [4], Pila [15]

(2) **Counting Lattice Points and O-Minimal Structures** (1-2 talks, optional)
Motivate and prove [3, Theorem 1.3]. Assume knowledge Section 3 of [3], and black box Section 2 of [3] if necessary. Focus on the use of o-minimality.
References: Barroero and Widmer [3]

(3) **Pila-Zannier strategy and Mann's theorem** (1 talk)
Present a proof of Mann's theorem [18, Theorem 2.1] using the Pila-Wilkie strategy. For a more detailed argument, see Marker [9, Case of Tori] or Valle Thiele [22, Chapter 1].
References: Scanlon [18], Marker [9], Valle Thiele [22]

(4) **An introduction to abelian varieties and $\mathcal{A}_g$** (1-2 talks)
Introduce the theory of abelian varieties and the moduli space $\mathcal{A}_g$ of g-dimensional, principally polarized abelian varieties. Follow (for example) [11, Sections 2-6]. Do not attempt to give proofs, but rather cover all definitions and theorems necessary for the next talk.
References: Orr [11]

(5) **An o-minimal proof of Manin-Mumford** (1 talk)
Follow [19, Section 5.1] to give a proof of Manin-Mumford over a number field. Black box Ax's theorem (but state it in the appropriate generality) and Masser's theorem, but otherwise try to give full details.
References: Scanlon [19], Pila-Zannier [17]

(6) **Shimura varieties: a gentle introduction** (1-2 talks)
Explain the action of $\mathrm{SL}_2(\mathbb{R})$ on the upper-half plane $\mathbb{H}$ and introduce the modular curve $X = \mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$. Explain that it may be rewritten as the double quotient $X = \mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ (see [8, Example 2.2.7]). Explain how a (connected) Shimura variety may be considered as a generalization of such a curve by replacing SL_2 with a semisimple Lie group G, the group $\mathrm{SO}(2)$ with a compact maximal subgroup of G, and $\mathrm{SL}_2(\mathbb{Z})$ with an arithmetic subgroup, so that the quotient G/K is a Hermitian symmetric domain. Provide examples of such symmetric domains in the case of $G = Sp_{2g}(\mathbb{R})$ following [8, Subsections 3.1.1, 3.1.2], and of $G = \mathrm{SO}_{n,2}(\mathbb{R})$ following [8, Subsections 3.4.1, 3.4.2]. Show $\mathcal{A}_g$ is a connected Shimura variety [8, Subsections 3.1.5]. Throughout, focus on the examples with

as much details as possible.
References: Lan [8]

(7) **Shimura varieties and o-minimality** (1 talk)
Show that the uniformization map for a Shimura variety is definable in some o-minimal structure if we restrict it to some semi-algebraic fundamental set. First introduce Siegel sets [1, Subsetions 2.1-2.2], and then prove first part of [1, Theorem 1.1(1)].
References: Bakker-Klingler-Tsimerman [1], Fresán [7, Section 3]

(8) **O-minimal Chow's theorem** (1-2 talks)
State the classical Chow Theorem. Prove the affine version of o-minimal Chow following [2, Section 1.3]. Introduce definable topological spaces [2, Section 2] and the definabilization functor [2, Definition 2.1.5]. Then cover basic definable analytic spaces [2, Section 2.2] and definable analytic spaces [2, Section 2.3], and introduce the analytification functor. State the general version of o-minimal Chow [2, Corollary 3.4.4] and prove it using a finite definable affine cover.
References: Bakker [2], Peterzil-Starchenko [12, 13, 14]

(9) **A proof of the classical Ax-Schanuel using O-minimality** (1 talk)
Explain the equivalence of the different formulations of Ax–Schanuel [20, Theorems 1.1-1.3], show how to deduce Ax-Schanuel-Weierstrass, and then give the proof from [20, Section 2].
References: Tsimerman [20]

(10) **Sketch of the proof of Ax–Schanuel for pure Shimura varieties** (1 talk)
State Ax–Schanuel [10, Theorem 1.1, Theorem 1.2] and give the proof following [10, Section 4].
References: Mok, Pila, Tsimerman [10]

(11) **Pila-Zannier strategy for André-Oort** (1-2 talks)
State the André-Oort conjecture of Shimura varieties and outline the Pila-Zannier strategy for proving André-Oort following [6] or [21, Section 6].
References: Daw [6]

(12) **Galois bounds and the height bound conjecture** (1 talk)
State the height bound conjecture [5, Conjecture 2], and state the point counting theorem [5, Theorem 3]. Prove [5, Theorem 1] (assuming [5, Theorem 3]).
References: Binyamini, Schmidt, Yafaev [5]

References

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