A Historical Survey of the Model Theoretic Universe

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1 Introduction

Model theory explores the relationships between formal theories—collections of sentences in a formal language—and the mathematical structures, or models, in which these statements hold true. Examples of such theories include algebraically closed fields, infinite sets, linear orders, Boolean algebras, the random graph, and ZFC. Given the vast array of possible theories, one might initially perceive them as a "formless magma", prompting the question: Is there any underlying organization? Since its inception in the early twentieth century, model theory has gradually merged its syntactical origins in mathematical logic with the semantics of the structures it studies. This evolution has led to the development of appropriate notions and properties that not only answer this question affirmatively but also provide a comprehensive classification of all first-order theories.

This achievement represents the culmination of extensive contributions from many mathematicians, each working on key areas that have shaped the development of model theory. In the 1970s, Saharon Shelah, through his work on stability theory, formalized the Classification Program and unified these diverse efforts into a coherent framework, establishing it as the central focus of model theory. Bruno Poizat highlighted the significance of this transformation in his review of John T. Baldwin's book *Classification Theory* [Bal88a]:

> "Perhaps the editor found it too sensitive to use a more ambitious title, as it might convey a sense of arrogance or imply an intention to exclude those working in more traditional areas... However, those of us unconcerned with such scruples will restore to this book the only title it truly deserves: Model Theory." [Poi90]

Since the 1980s, model theorists have made significant progress in advancing the Classification Program. Various virtuous properties of theories beyond stability, such as simplicity, distality, and o-minimality, are now considered important dividing lines for classification, and numerous theories have been positioned within this framework. Notably, many of the theories under investigation, such as valued fields and differentially closed fields, are also central to algebraic geometry, and bidirectional results—linking model theory with algebraic geometry and vice versa—are effectively bridging the historical gap between logic and "classical mathematics".

Since 2013, Gabriel Conant, from the University of Illinois at Chicago, has been visualizing these advancements in what he calls the *Map of the Universe.* This map not only depicts the various implications among these fundamental properties leading to possible classes of theories, but also places specific theories within each class and provides references for the papers from which these results are derived. As such, it serves as both a crucial point of reference for understanding these intricate relationships and a foundation for the present work.

In the *Map of the Universe* the red dots represent individual theories while the blue dividing lines delineate the boundaries between different classes of theories.[1] The current version is as follows:

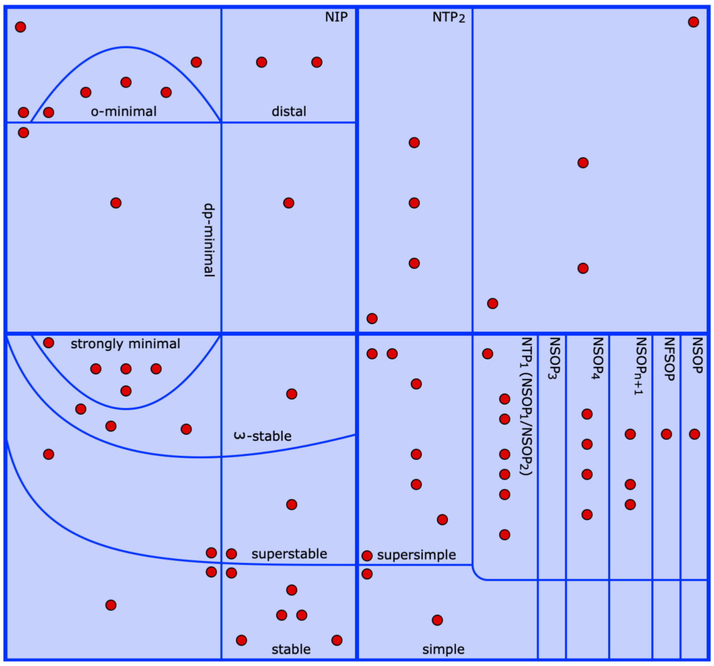

Figure 1: Gabriel Conant's *Map of the Universe*.

The aim of this work is to offer a comprehensive survey of the historical developments that have shaped model theory from its inception to the present, with a particular focus on the evolution of classification theory—specifically, the emergence of the various *Building Blocks* that form the *Map of the Universe*. By providing context for the properties and theories represented in this map, this work aims to serve as a self-contained, accessible introduction to the subject.

The work is divided into four main parts that trace the historical development of model theory. However, the results are presented using contemporary terminology, so foundational notions like language, theory, and model are introduced earlier in the narrative than when they were formally established. To address this, a preliminary section titled "What is a Model?" is included, providing answers to this question and introducing the essential concepts needed to understand the subsequent historical progression.

Chapter 2 starts in the late 19th century, covering the prehistory of model theory and marking the inception of the field with Löwenheim's Theorem in 1915. It explores foundational contributions by Hilbert, Tarski, and Gödel in the 1920s and 1930s, focusing on concepts like completeness and decidability, and exam-

[1] An interactive version of this diagram is available on Conant's website: `https://www.forkinganddividing.com`

ines key developments from the 1950s and 1960s, including Łoś's ultraproducts, Fraïssé's back-and-forth methods, and Robinson's use of diagrams.

Chapter 3 addresses the emergence of the concept of the space of types and its significance in understanding the definable sets of a structure. It examines the work of Vaught and Ehrenfeucht on constructing desirable models and discusses Ryll-Nardzewski's findings on the number of models a theory can have. This progression leads to Morley's Categoricity Theorem in 1965, and its implications for the development of geometric model theory.

Chapter 4 highlights Shelah's contributions, especially his work on stable theories and the non-forking independence relation, which serves as a tool for assigning dimensions to models. It outlines Shelah's stability hierarchy, represented in the bottom left quadrant of Conant's map, and situates this within his broader classification program. The chapter also examines the Main Gap Theorem, which establishes a fundamental dichotomy between theories and serves as the culmination of his classification efforts.

Chapter 5 slightly deviates from the linear historical narrative to survey Conant's map beyond stability, drawing attention to the most remarkable results in each class. It explores simple theories as part of the broader non strict order property (NsOP) class in the bottom right quadrant, and examines o-minimal theories as a representative of the non independence property (NIP) class in the top left quadrant.

A wealth of excellent literature addresses these topics, particularly those in Chapters 1, 2, and 3. This work has greatly benefited from classical, detailed textbooks such as *Model Theory: An Introduction* by Marker [Mar03] and *Fundamentals of Stability Theory* by Baldwin [Bal88b], as well as comprehensive philosophical works like *Philosophy and Model Theory* by Button and Walsh [BW18] and *Model Theory and the Philosophy of Mathematical Practice* by Baldwin [Bal18].

However, this work aims to bridge the gap between highly specialized textbooks and broader philosophical expositions, targeting readers who are interested but not yet specialists. On the one hand, the work has a formal mathematical style, presenting definitions and theorems rigorously, and ensuring that concepts build logically on each other. On the other hand, it provides a narrated explanation, focusing on understanding ideas rather than on exhaustive proofs.

Due to the scope and limitations of this project, certain topics are necessarily omitted and this selection reflects a personal preference. Some of the results that will not be covered are abstract elementary classes, regular cardinals, finite model theory, nonstandard analysis or infinitary logic. Specifically, the theories discussed will always be assumed to be in a countable first-order language unless explicitly stated otherwise.

A basic familiarity with the syntax and semantics of first-order logic, along with some experience in formulating mathematical statements within this framework, may be helpful, though a brief description is provided in the section *What* *is a Model?*. Familiarity with basic set theory and combinatorics will also be beneficial, but key concepts are introduced as needed. While model theory intersects with various areas of mathematics, the focus here is primarily on familiar theories arising from algebraic geometry, such as algebraically closed fields, and on intuitive axiomatizations like the random graph. Detailed axioms defining these theories are not explored; for those specifics, readers are referred to classical textbooks like *Model Theory: An Introduction* by Marker.

Finally, as an implicit contribution to the philosophy of mathematics, this book aims to reflect Lakatos's view that mathematical knowledge evolves through a dialectical process [Lak76]. It also aims to echo the significant paradigm shift that model theory underwent in the 1960s, as highlighted by Baldwin [Bal19], which aligns with a specific interpretation of Kuhn's theory of paradigm shifts [Kuh62].

This work was inspired by Christian d'Elbée's course, "Axiomatic Theory of Independence Relations in Model Theory", held during the summer semester of 2023 at the University of Bonn. I am deeply grateful to him for introducing me to this captivating field. I also wish to express my gratitude to my advisor, Philipp Hieronymi, for his reliable support and encouragement throughout this unconventional project. I owe special thanks to my high school teacher, Carlos Usón, whose inspiring lessons ignited my passion for mathematics. I had also the pleasure of discussing specific topics with Daniel Ibaibarriaga, whose knowledge and generosity were invaluable to this work. Lastly, my warmest thanks go to Nil Rodellas for his unconditional trust and to my family for their unwavering love.

1.1 What is a Model?

A formal definition of the syntax and semantics of first-order logic can often be more tedious than illuminating. Instead, a brief overview is provided here, with examples and historical context to help clarify the ideas. For those interested in a more detailed exploration of these concepts, standard texts in logic or model theory, such as those by Marker [Mar03] and Poizat [Poi00], are recommended.

Baldwin conveys the essence of model-theoretic methods:

> "Model theory is the activity of a 'self-conscious' mathematician. This mathematician distinguishes an object language (syntax) and a class of structures for this language and 'definable' subsets of those structures (semantics)." [Bal10]

In terms of syntax, *first-order logic* contains several *logical symbols*: equality $=$, a sequence of variables v_i, the logical connectives $\wedge, \neg, \vee$ and the quantifiers $\forall, \exists$, which can only range over elements of the model. A *language* $\mathcal{L}$ for first-order logic consists of a collection of relation symbols $\mathcal{R}$, function symbols $\mathcal{F}$ and constant symbols $\mathcal{C}$ that is appropriate for the area of mathematics being formalized. Each function and relation symbol has an associated integer, n_f or n_R, indicating that f is a function of n_f variables and R is an n_R-ary relation. For example, the language of rings $\mathcal{L}_r = \{+, -, \cdot, 0, 1\}$ includes binary function symbols $+$, $-$, $\cdot$, and constants 0, 1. On top of this, the set of $\mathcal{L}$-terms is built up inductively from constants and variables using the function symbols of the language.

An *atomic formula* is an expression of the form $R(t_1, \ldots, t_n)$, where R is a relation symbol with n arguments, and the t_i are terms, or an equation of the form $t_i = t_j$. The set of $\mathcal{L}$-*formulas* is the smallest set of formulas containing all atomic formulas and closed under the Boolean operations and quantification over individuals. Examples of formulas are $x_1 = 0 \vee x_1 > 0$ or $\exists x_2\, (x_2 \cdot x_2 = x_1)$. Particularly important are $\mathcal{L}$-*sentences*, which are formulas in which no variable x appears outside the scope of the quantifier $\exists x$ or $\forall x$, in other words, no variable is *free.* Lastly, a $\mathcal{L}$-*theory* T simply refers to a set of $\mathcal{L}$-sentences.

As an example, consider the language $\mathcal{L} = \{<\}$, where $<$ is a binary relation symbol. The theory of linear orders can be axiomatized by the following $\mathcal{L}$-sentences: $\forall x\, \neg(x < x)$, $\forall x \forall y \forall z\, ((x < y \wedge y < z) \to x < z)$, $\forall x \forall y\, (x < y \vee x = y \vee y < x)$. It is possible to define an *extension* of this theory by adding the sentence $\forall x \forall y\, (x < y \to \exists z\, (x < z \wedge z < y))$ and thus get the theory of dense linear orders, or DLO.

In terms of semantics, an $\mathcal{L}$-*structure* $\mathcal{M}$ consists of a nonempty domain M and an interpretation of the symbols in $\mathcal{L}$. An interpretation consist on the following: a function from M^n into M for each function symbol with n arguments in the language; a subset of M^n for each n-ary relation symbols, and lastly an element from M for each constant symbol. The cardinality of $\mathcal{M}$ is the cardinality of its domain M. For instance, in the language $\mathcal{L} = \{\cdot, e\}$, where $\cdot$ is a binary function symbol and e is a constant symbol, an $\mathcal{L}$-structure could be $\mathcal{M} = (\mathbb{R}, \cdot, 1)$, where the domain is $\mathbb{R}$, $\cdot$ is interpreted as real multiplication, and e is interpreted as 1. Given two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$, an *$\mathcal{L}$-embedding* η is a one-to-one map in the domains $\eta : M \to N$ that preserves the interpretation of all of the symbols of $\mathcal{L}$. If it is bijective, it is called an *$\mathcal{L}$-isomorphism*.

The central notion for defining a model is that of *truth in a structure*, which is also defined inductively. If there are no quantifiers, a formula is true in the structure $\mathcal{M}$ if the interpretation of the terms lies in the relation which is the interpretation of the formula. The truth of Boolean combinations and of quantified formulas is defined in the natural way. For example, $\mathcal{M} \models (\exists x)\varphi(x)$ if for some $a \in M$, $\mathcal{M} \models \varphi(a)$. In this case, $\varphi(a)$ is said to be true in $\mathcal{M}$, or that $\mathcal{M}$ satisfies $\varphi(a)$.

With these elements established, it is now time to answer the main question of this section:

Definition 1.1.1. Let T be an $\mathcal{L}$-theory and let $\mathcal{M}$ be an $\mathcal{L}$ -structure. The structure $\mathcal{M}$ is a *model* of T (or $\mathcal{M}$ models T) if $\mathcal{M} \models \sigma$ for all sentences $\sigma \in T$.

As is evident, the notion of a model is intricate, and its historical development is equally nuanced. However, drawing on insights from Hodges [Hod00; Hod93], several events deserve mention. The practice of using of a set of statements to define a class of structures dates back to the late 19th century, notably in the foundations of geometry, as exemplified by Hilbert's *Grundlagen der Geometrie* [Hil99]. In 1903, Russell likely provided the first clear account of how such statements define a class [Rus03]. However, until the mid-1950s, the distinction between variables with changing interpretations and constants with fixed interpretations remained ambiguous. The specific emergence and evolution of the notions of satisfaction and truth in a structure are explored in Chapter 2.3.

2 An Increasing Role of Semantics in Logic

2.1 The Algebra of Logic Tradition and Löwenheim's Theorem

The origins of Model Theory can be traced back to the algebra of logic tradition in the second half of the 19th century, which aimed to develop an explicit algebraic system to uncover the underlying mathematical structure of logic [BL21]. This tradition finds its roots in George Peacock's work on symbolical algebra, which shifted the emphasis from the meaning of symbols and signs to the laws of operation. An important aspect of his conception of symbolical algebra was that "whilst remaining a science of undefined symbols and signs, an interpretation of algebraic symbols and signs could follow, even if it did not precede, algebraic manipulation" [Pyc81].

Using this same idea in the realm of logic, George Boole founded the algebra of logic tradition. In the 1850s, logic was still largely based on Aristotle's syllogistic framework, but Boole sought to develop a mathematical treatment of logic. His goal was to provide classical logic with a systematic foundation and extend its scope by assigning precise meanings to logical symbols, similarly to the methods of symbolical algebra. As Boole noted, this new algebraic approach represented "a proper 'science of reasoning', and not a 'mnemonic art' like traditional syllogistics" [GB97]. He first presented this framework in *The Mathematical Analysis of Logic* [Boo47] and later refined it in his more famous work, *An Investigation of the Laws of Thought* [Boo54].

According to Peckhaus, Boole founded his methods on the concept of classes, groups of things to which a name or description applies, and was driven by the desire to retain as much of the standard algebraic formalism as possible [Pec09]. Indeed, a central aspect of his work was the use of algebraic formulas to represent logical relations. For example, Boole defined addition as the union of sets. However, he imposed a crucial restriction: the expression $x+y$ (representing the class containing elements from either x or y) was valid only when $xy = 0$, indicating that the intersection of x and y is empty. According to Schlimm, this restriction was intended to prevent the equation $x + x = x$, which would conflict with the common algebraic principles of natural numbers [Sch85].

Similarly, Boole associate multiplication to the intersection of sets, which resulted in his idempotent law, $x^2 = x$. While this equation is true when variables represent classes, it does not hold for natural numbers, except when $x = 0$ or $x = 1$. This observation led Boole to focus on a new system that used only the numbers 0 and 1. One of the key applications of this system was a method for facilitating logical reasoning by eliminating certain class symbols in equations. This led Boole to formulate his Elimination Theorem for the calculus of classes.

The theorem proposed that by substituting class symbols with 0 and 1 in all possible combinations, multiplying the resulting expressions, and setting the final product to 0, one could effectively eliminate the class symbol from the equation.

Although Boole's commitment to drawing analogies between algebra and logic was instrumental in developing his calculus of classes, it also limited his ability to create a system that met other desirable criteria. Subsequent logicians, such as William Stanley Jevons, sought to move beyond this analogy to advance the calculus of logic. In 1864, Jevons retained the use of equations as the fundamental form of logical statements, but shifted the focus from classes associated with quantity to predicates related to quality [Jev64]. This shift involved replacing Boole's partial operation of addition with the modern one, which also accounts for non-disjoint classes and supports the law $x + x = x$. According to Peckhaus, moving away from Boole's close analogy to mathematical notation was indeed the defining feature of Jevons' logic, and marked a considerable step forward in the calculus of logic [Pec09].

In this context, Charles Peirce, along with his student O. H. Mitchell, introduced several innovative ideas. He defined a binary relation called "subsumption", which could be interpreted in various ways, such as a subclass relation or implication, moving beyond the use of equality as the sole primitive symbol. He also incorporated unrestricted unions (Σ) and intersections (Π) into Boole's framework [Pei97], laying the groundwork for modern quantifier notation [Bra00]. Later, Peirce developed a "general algebra" centered around quantifiers, where he made the first distinction between first-order and second-order logic [Bad04].

Peirce's framework, especially his concept of subsumption, significantly influenced Ernst Schröder's work. In his three-volume *Vorlesungen über die Algebra der Logik* [Sch90], Schröder provided the first comprehensive axiomatization of the calculus of classes and expanded the theory of relations. His goal was to build a general algebraic theory with broad applications, centered around the algebra of logic. As Brady observed [Bra00], this work introduced abstract lattice theory and Dedekind's theory of chains, but its most notable contribution to model theory was the revival of Boole's Elimination Theorem. Schröder stated:

> "Getting a handle on the consequences of any premises, or at least the fastest methods for obtaining these consequences, seems to me to be the noblest, if not the ultimate goal of mathematics and logic." [Sch90]

Schröder approached this goal as follows. Given a premise $\phi(x, y)$ involving classes x and y, the aim is to derive $\psi(y)$. If it is possible to find a witness relation $\varphi_0(y)$ such that $\exists x\, \phi(x, y) \iff \varphi_0(y)$, then the task reduces to deriving $\psi(y)$ from the seemingly simpler statement $\varphi_0(y)$. After exploring this approach in several specific cases, Schröder recognized the significance of finding such witness relations and recommended this topic as an important area of research, which became known as the *Elimination Problem.*

In approaching this problem, Schröder also embraced Peirce's distinction between first-order and second-order logic. Specifically, he realized that higher-order statements could describe complex, uncountable structures—such as the second-order axioms for the real numbers. However, it remained unclear whether certain first-order sentences could only have uncountable models. These questions, and particularly the general framework of the algebra of logic as described in Schröder's work, became the most advanced form of mathematical logic at the time [Bra00].

Despite this, the emphasis in mathematical logic began to shift towards type theory with the publication of *Principia Mathematica* [RW10]. Nevertheless, in 1915, Leopold Löwenheim renewed interest in the algebra of logic tradition with his publication *On Possibilities in the Calculus of Relatives* [Löw67], where he answered Schröder's question negatively.

In modern terms, Löwenheim's theorem states that if a first-order sentence is true in every finite model but not in every model, then it can be falsified in a model with elements from a countably infinite domain. However, it is important to note that the contemporary formulation—if a first-order sentence φ has a model $\mathcal{M}$, then it also has a countable model $\mathcal{N}$—tacitly relies on Gödel's completeness theorem [Göd29] and is attributed to Malcev in 1936 [Mal71b].

In his works [Sko67a; Sko67b], Thoralf Skolem extended Löwenheim's result to cover not just individual sentences but any countable set of sentences. More significantly, he demonstrated that the countable model $\mathcal{N}$ could always be selected as a submodel of the original model $\mathcal{M}$, or, in modern terms, as an elementary submodel of $\mathcal{M}$ (see Definition 2.4.2). This joint result, known as the Downward Löwenheim-Skolem Theorem, remains one of the most significant and widely used theorems in model theory to this day.

Theorem 2.1.1 (Downward Löwenheim-Skolem Theorem). *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let κ be an infinite cardinal such that $|\mathcal{L}| \leq \kappa \leq |M|$. Then there exists an elementary substructure of $\mathcal{M}$ of cardinality κ.*

Skolem's proof built on the work of Löwenheim, who himself drew from Schröder's concept of witness relations. In his approach, Skolem expanded the language of a theory T by introducing a function $f_\varphi(x)$ for each formula $\varphi(x)$ of the form $\exists y \psi(x, y)$; these functions are now known as Skolem functions. He then added the corresponding axioms $\forall x\, ((\exists y)\, \psi(x, y) \to \psi(x, f_\varphi(x))]$ to T, resulting in what is now called the Skolemization of T. With this modified theory, Skolem demonstrated that it was enough to prove the theorem for sentences in this specific form. This strategy laid the groundwork for quantifier elimination, which would soon became a central tool in model theory.

Beyond its proof, Löwenheim's theorem was groundbreaking because it was the first to establish a meaningful link between a first-order sentence, a syntactic construct, and its models, a semantic construction. However, in the 19th century, logic was still primarily treated as a syntactic discipline, and the distinction between syntax and semantics was not clearly defined. As a result, the prevailing view, shaped by van Heijenoort [Hei67a], was that Löwenheim had presented a flawed argument.

Badesa [Bad04], through a detailed analysis of Löwenheim's paper and its historical context, argues that Löwenheim did indeed provide a complete and correct proof of the theorem. He believes that Löwenheim's work marked the beginning of "the logic community's gradual understanding of the modern distinction between syntax and semantics, that is, between systems of symbolic expressions and the meanings that can be assigned to them" [Avi06]. Given the importance of this distinction, Badesa considers the publication of Löwenheim's *On Possibilities in the Calculus of Relatives* as the birth of Model Theory.

2.2 Completeness, Compactness & Decidability

A pivotal aspect of model theory is its emphasis on first-order logic. Although Peirce first recognized the distinction between first- and second-order logic, and Löwenheim's theorem revealed a key property in first-order logic, neither fully explored its practical significance. The breakthrough came with David Hilbert, who, inspired by the *Principia Mathematica* [RW10], set out to develop his own axiomatic systems for various logics, including first-order logic. In his 1917 lectures [Hil17], Hilbert re-discovered it and introduced new metamathematical techniques, distinguishing the syntactic presentation of a formal system from its interpretation in a given domain. Hilbert also emphasized the importance of addressing fundamental questions about completeness, consistency, and decidability, recognizing the value of metamathematics in tackling these issues.

Even so, throughout the 1920s, first-order logic was mostly regarded as just a component of type theory. It was not until 1928, with the publication of *Grundzüge der Theoretischen Logik* by Hilbert and Ackermann [HA28], that first-order logic was clearly established as a distinct system [Ewa19]. In this influential work, they demonstrated how to prove the completeness of propositional calculus [Ewa19] and highlighted the question of whether first-order logic "is complete in the sense that from it all logical formulas that are correct for each domain of individuals can be derived" [Hei67b, p. 48].

This was proved by Kurt Gödel in 1929, whose completeness theorem states that if a formula is logically valid, then it is provable. The extension of this result to uncountable theories was first achieved by Anatoly Malcev [Mal71a], and later, independently, by Leon Henkin [Hen49], who is often credited with the stronger form of the completeness theorem:

Theorem 2.2.1 (Completeness of First-Order Logic)**.** *Let T be an $\mathcal{L}$-theory and φ an $\mathcal{L}$-sentence, then $T \vDash \varphi$ if and only if $T \vdash \varphi$. In particular, if T is consistent, then T has a model.*

Gödel's proof closely followed the methods of Löwenheim and Skolem, requiring the expansion of the language with additional relation symbols of all arities. In contrast, Henkin's took a different approach by first expanding the theory to a maximal syntactically consistent set, thus requiring the addition only of constants to the given language $\mathcal{L}$. He added sufficient constants so that they served as witnesses of existential formulas. This ensured that for any $\mathcal{L}$-

formula $\varphi(v)$ with one free variable v, there exists a constant $c \in \mathcal{L}$ such that $T \models (\exists v\, \varphi(v)) \to \varphi(c)$. This construction is known as the Henkinization of T. Like Skolemization, it acknowledges quantifiers, Henkinization using constants and Skolemization using Skolem functions. Beyond the future impact of this construction, Baldwin notes that Henkin's approach is noteworthy for its focus on working within the given vocabulary, marking a important step towards the modern conception of model theory [Bal18].

A key consequence of the completeness theorem, recognized by Gödel just a year later, is the compactness theorem [Göd86]. The compactness theorem arises from the observation that any proof is finite and thus relies on only a finite subset of assumptions. Consequently, if a theory T is contradictory, some finite $T_0 \subseteq T$ must also yield the contradiction. While this provides a syntactic proof, in subsequent years, more model-theoretic proofs emerged, the most notable being Łos's using ultrafilters (see Definition 2.4.7) [Łoś55]).

Theorem 2.2.2 (Compactness Theorem)**.** *A theory T is is satisfiable if and only if every finite subset of T is satisfiable.*

The compactness theorem is one of the most important results in model theory, primarily because it serves as a powerful tool for constructing models. For example, if an $\mathcal{L}$-theory T has an infinite model $\mathcal{M}$, building a model $\mathcal{N}$ of cardinality κ, can be done by expanding $\mathcal{L}$ by adding κ constants. A theory T' can be then formed by adding the sentences $c_i \neq c_j$ for each $i < j < \kappa$ to T. By the compactness theorem, there exists a model of T' and, *a fortiori*, a model of T with cardinality at least κ.

The earliest written proof of this result was given by Malcev [Mal71a], but Robert Vaught [Vau86] notes that Tarski first identified it during his 1927–28 seminar. Because of its parallel to the downward version, it is often called the Upward Löwenheim–Skolem–Tarski Theorem. Despite Tarski's central role, his name is frequently omitted, while Skolem's remains tied to the theorem. Ironically, as Hodges [Hod00] points out, Skolem himself found this attribution troubling, as he considered uncountable sets fictional and the result meaningless.

Theorem 2.2.3 (Upward Löwenheim-Skolem-Tarski Theorem)**.** *If T is a countable first-order theory with a model $\mathcal{M}$, then T also has a model in every uncountable cardinality.*

The naming of the theorem can still be justified due to its deeper connection to compactness, from which it is derived. While compactness formally follows from Gödel's completeness theorem, some aspects of Gödel's result were arguably implicit in Skolem's earlier work. However, neither Skolem nor his contemporaries fully realized this link, as the metamathematical framework needed to articulate it had yet to be developed. Gödel wrote:

> "The Completeness Theorem, mathematically, is indeed an almost trivial consequence of Skolem 1923. However, the fact is that, at that time, nobody (including Skolem himself) drew this conclusion neither from Skolem 1923 nor, as I did, from similar considerations

of his own... This blindness (or prejudice, or whatever you may call it) of logicians is indeed surprising. But I think the explanation is not hard to find. It lies in the widespread lack, at that time, of the required epistemological attitude toward metamathematics and toward non-finitary reasoning." [Fef+03]

The absence of a mature metamathematical perspective in the 1920s partly explains this "blindness", but Dreben and van Heijenoort suggest that much of the attention at the time was actually focused on the quest for decidability [DH86]. Hilbert, in particular, made the decision problem a key part of his efforts to establish a formal foundation for mathematics. He believed that every mathematical problem had a solution, meaning it should always be possible to determine whether any given mathematical statement is true or false. In 1921 [Beh22], Heinrich Behmann first introduced the term "Entscheidungsproblem" for the decision problem, and by 1928, Hilbert and Ackermann emphasized its importance, identifying it as one of the central challenges in mathematical logic [HA28].

Definition 2.2.4. Let T be an $\mathcal{L}$-theory. The theory T is said to be *decidable* if there is an effective procedure that takes as input an $\mathcal{L}$-sentence σ and returns the truth value of $T \models \sigma$.

During this period, the concept of decidability was already beginning to take shape in the work of Löwenheim and Skolem, although they considered their results to be purely algebraic, as the notion was embedded within their early efforts in quantifier elimination [Bal18]. Similarly, Langford established the decidability of dense and discrete linear orders without explicitly recognizing it, as his focus was on using Löwenheim and Skolem's methods to identify complete sets of axioms for these theories.

Starting in 1927, Alfred Tarski took a crucial step by recognizing the potential of quantifier elimination as a general method for solving traditional mathematical problems [MZ15]. Originally, quantifier elimination involved simplifying a formula by removing its quantifiers and reducing it to a Boolean combination of simpler, sometimes quantifier-free formulas, with the same truth assignments as the original formula. Over time, this technique evolved to studying entire theories by applying this process to every statement within a theory. In the best cases, it led to the following property [Sho67]:

Definition 2.2.5. Let T be an $\mathcal{L}$-theory. The theory T is said to have *quantifier elimination* if for every $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$, there exists a quantifier-free $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$ such that $T \models \varphi \leftrightarrow \psi$.

Tarski and his students explored quantifier elimination across various mathematical theories, discovering that in certain cases, quantifiers could be eliminated after slightly extending the language. Mojżesz Presburger, one of Tarski's students, applied this strategy to study the first-order theory of the integers with the standard arithmetic operations in the language $\mathcal{L} = \{+, -, 0, 1\}$. He enriched the language by introducing an infinite collection of predicates P_n, each representing divisibility by n for every natural number. Although quantifier elimination was not feasible within the original language $\mathcal{L}$, Presburger's extension enabled the elimination of quantifiers in the new language $\mathcal{L}'$ [Pre30].

It is always possible to find an extended language that allows quantifier elimination, but such constructions are usually impractical as the resulting quantifier-free sets can become more complex. In Presburger's case, however, the extended language $\mathcal{L}'$ remains simple enough that these definable sets (see Definition 2.2.6) are still easy to describe, allowing for a clear proof of the theory's completeness. Furthermore, Presburger noted that a similar method could be applied to the ordered group of integers in the language $\mathcal{L} = \{+, -, <, 0, 1\}$, a theory nowadays known as Presburger arithmetic.

As hinted earlier, the significance of these quantifier elimination results becomes clearer when examining their implications. Removing quantifiers yields an axiomatisation of the set of all first-order sentences true in the structure. This, in turn, enables the development of an algorithm to test the truth of any statement within the structure, thereby establishing the theory's decidability. However, this crucial connection was not initially apparent, and it was only in 1940 that Tarski explicitly highlighted this relationship.

> "It is possible to defend the standpoint that in all cases in which a theory is tested with respect to its completeness, the essence of the problem is not in the mere proof of completeness, but in giving a decision procedure (or in the demonstration that it is impossible to give such a procedure)." [Tar67]

Although the concept of decidability remained somewhat obscure for many years, an important aspect of quantifier elimination was its ability to clarify the definable sets within a structure. Essentially, it aimed to address *the* core question: what sets are definable in a given structure $\mathcal{A}$ using formulas from a specific formal language $\mathcal{L}$? This question has remained central. Indeed, Marker describes model theory as "a branch of mathematical logic where we study mathematical structures by considering the first-order sentences true in those structures and the sets definable by first-order formulas" [Mar03].

Although E. Schröder initiated the study of first-order definable sets and relations [Sch90], it was Tarski who fully advanced this framework and formalised its definition [Tar83a].

Definition 2.2.6. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. A subset $X \subseteq M^n$ is *definable* if and only if there is an $\mathcal{L}$-formula $\varphi(v_1, \ldots, v_n, w_1, \ldots, w_m)$ and $b \in M^m$ such that $X = \{a \in M^n : \mathcal{M} \models \varphi(a, b)\}$. The subset X is *A-definable* or *definable over A* if there is a formula $\psi(v, w_1, \ldots, w_l)$ and $b \in A^l$ such that $\psi(v, b)$ defines X.

Tarski introduced definable sets in his analysis of the real field, leading to one of the most significant results of that period [Tar31]. The theory of real numbers does not have quantifier elimination in the language of rings $\mathcal{L}_r = \{+, -, \cdot, 0, 1\}$. Instead, Tarski considered the real numbers expanded language of ordered rings $\mathcal{L}_{or} = \mathcal{L}_r \cup \{<\}$. Since the order relation $x < y$ can be expressed in the real field using the formula $\exists z\, (z \neq 0 \wedge x + z^2 = y)$ any subset of $\mathbb{R}^n$ definable with an $\mathcal{L}_{or}$-formula is already definable using an $\mathcal{L}_r$-formula. In the language $\mathcal{L}_{or}$, Tarski proved quantifier elimination for the theory of the ordered real field, and as a consequence, established its completeness and decidability.

A key result of Tarski's quantifier elimination, which holds lasting importance for contemporary model theory (see Definition 5.3.1), is that the definable subsets in $\mathbb{R}$ are exactly the unions of finitely many sets, each being either a singleton or an open interval with endpoints in the field or $\pm\infty$. These sets are now known as semialgebraic sets:

Definition 2.2.7. We say a subset of $\mathbb{R}^n$ is *semialgebraic* if it can be expressed as a finite union of sets of the form $\{a \in \mathbb{R}^n : p(a) = 0, q_1(a) > 0, \ldots, q_m(a) > 0\}$ where $p, q_1, \ldots, q_m \in \mathbb{R}[X_1, \ldots, X_n]$.

Semialgebraic sets have several notable properties, such as being closed under Boolean operations (finite unions, intersections, and complements) and under Cartesian products. The most crucial property, however, is that semialgebraic sets defined by polynomial equations and inequalities in $(n+1)$-dimensional space can be projected down to n-dimensional space while preserving their definability using polynomial identities and inequalities. This property is formalized in the Tarski–Seidenberg Theorem [Sei74; Tar31]:

Theorem 2.2.8 (Tarski–Seidenberg)**.** *Let $X \subseteq \mathbb{R}^{n+1}$ be a semialgebraic set and let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection map onto the first n-coordinates. Then $\pi(X) \subseteq \mathbb{R}^n$ is semialgebraic.*

In his work on the real field, Tarski also identified a set T of sentences that axiomatizes the real field, meaning a first-order sentence is true in the field if and only if it is provable from T [Tar31]. It was later realized that T precisely constitutes the axioms defining real closed fields, so these results extend immediately to every real closed field. This made evident that quantifier elimination provides a crucial framework for understanding first-order definable relations across a wide range of important mathematical structures.

2.3 Truth in a Structure

Tarski's most significant contribution is arguably his formalization of the concept of truth, a project he began in the late 1920s influenced by Whitehead and Russell [RW10]. Although he was deeply engaged with the philosophical debates on truth, he also saw that the prevailing mathematical approaches were equally insufficient. Despite the progress, Tarski felt that a purely mathematical connection between sentences and structures was still missing. He expressed his concern, stating: "I shall not make use of any semantical concept if I am not able previously to reduce it to other concepts" [Tar33]. Motivated by both mathematical and philosophical considerations, Tarski set out to define truth in a way that relied entirely on set theory and syntax [Hod00]. His work on truth was first published in Polish in 1933 [Tar33] and was later translated as *On the Concept of Truth in Formal Languages* [Tar83a]. Vaught described Tarski's achievements:

> "Since the notion 'φ is true in $\mathcal{M}$' is highly intuitive, it had been possible to go even as far as the completeness theorem by treating truth essentially as an undefined notion. But no one had made an analysis of truth, not even of exactly what is involved in treating it in the way just mentioned. [...] In 1933, Tarski made the needed analysis of truth. For one thing, he discussed just what axioms are needed if truth is taken (as above) to be undefined. But his major contribution was to show that the notion 'φ is true in $\mathcal{M}$' can simply be defined inside ordinary mathematics." [Vau74]

When considering a language $\mathcal{L}$ within a consistent metalanguage $\mathcal{L}'$, Tarski reached two key conclusions about defining truth, depending on the relationship between $\mathcal{L}$ and $\mathcal{L}'$ [Hod22]. First, if $\mathcal{L}$ and $\mathcal{L}'$ are identical, any attempt to define truth within $\mathcal{L}$ fails because it leads to contradictions. This is the essence of Tarski's undefinability theorem, which shows that truth in $\mathcal{L}$ cannot be defined from within the language itself and requires a higher-level metalanguage [Cie15]. The second scenario occurs when $\mathcal{L}'$ contains second-order logic and the Peano Arithmetic axioms [KP80]. In this case, truth for $\mathcal{L}$ can be defined in $\mathcal{L}'$ by encoding syntactic notions through number theory, where formulas correspond to specific numerical properties.

Tarski's construction of truth in this second scenario begins by first addressing the broader concept of satisfaction. To define satisfaction, Tarski considers a language $\mathcal{L}$ with precisely defined syntax, with two levels of symbols: constants, which carry fixed meanings, and variables, which lack inherent meaning. However, it is possible to assign an object to each variable and evaluate whether a given formula ϕ in $\mathcal{L}$ holds true when each variable represents its assigned object. If A is an allowed assignment of objects to variables and A makes ϕ true, then A (or $\mathcal{A}$) is said to satisfy or to be a model of ϕ, and ϕ is said to be true in $\mathcal{A}$.

With this definition of satisfaction, Tarski was able to express truth when ϕ is a formal sentence whose non-logical symbols are interpreted within a fixed structure $\mathcal{M}$. He did so by constructing a metamathematical formula $\varphi(\mathcal{M}, \phi)$ in $\mathcal{L}'$, using only higher-order logic, syntax, and the symbols of $\mathcal{L}$, which expressed: "the sentence ϕ is true in the structure $\mathcal{M}$". However, this approach didn't fully capture a model-theoretic definition that could define truth consistently across varying structures $\mathcal{M}$. The issue arises because, alongside constants and variables, model-theoretic languages use a third kind of symbols, such as quantifiers, which only take on meaning when applied to a specific structure [Hod85].

Hodges notes that Tarski's first clear model-theoretic definition of truth in a structure appeared in his 1957 joint paper with Robert Vaught, *Arithmetical Extensions of Relational Systems* [TV57]. In this paper, they resolved the earlier issue by replacing the non-logical symbols in ϕ with variables $\bar{x}$ and then applying Tarski's original truth definition to express that $\mathcal{M}$ satisfies the new formula $\phi(\bar{x})$, expressing the truth of ϕ in $\mathcal{M}$, as required.

The significance of this model-theoretic definition of truth is clear, as it continues to serve as the standard in modern model theory. Tarski's 1933 definition was also profoundly impactful in its time, forming the basis for much of his subsequent work in metamathematics, especially his paper *On the Concept of Logical Consequence* [Tar83b]. In this paper, he argued that a conclusion follows logically from its premises if and only if every model of the premises is also a model of the conclusion, thereby redefining the notion of validity. These contributions, along with major advances in completeness and decidability, made the period from 1915 to 1935, in words of Vaught, extraordinary [Vau74].

2.4 Equivalence & Mappings of Structures

In the 1950s, model theory shifted its focus from examining the properties of various logics to the systematic study of first-order theories [Bal18]. The primary aim became understanding how the syntactic properties of a theory relate to the nature of its models and, conversely, how structural features of models determine whether they satisfy the same set of sentences. This perspective, influenced by contemporary developments such as the rise of category theory, emphasized various notions of equivalence between structures and the study of mappings between them [Hod00].

The central notion of equivalence between structures, attributed to Tarski, is that of *elementary equivalence*. Tarski first introduced the idea of elementary equivalence as part of his early work on the axiomatization of the field of real numbers during the 1930s [Tar35]. However, it was not until the 1950s that he formalized the modern definition of elementary equivalence that is still used today [Tar52].

Definition 2.4.1. Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are said to be *elementary equivalent*, denoted by $\mathcal{M} \equiv \mathcal{N}$ if $\mathcal{M} \models \phi$ if and only if $\mathcal{N} \models \phi$ for all $\mathcal{L}$-sentences ϕ.

Another description of elementary equivalence of $\mathcal{M}$ and $\mathcal{N}$ is by writing $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$, where $\mathrm{Th}(\mathcal{M})$ denotes the set of all sentences of $\mathcal{L}$ that hold in the $\mathcal{L}$-structure $\mathcal{M}$, also referred to as the *complete theory* of $\mathcal{M}$. Moreover, elementary equivalence is a weakening of isomorphism of structures, underscoring the expressive power of first-order languages to capture core properties of structures while disregarding specific details [Ewa19].

To illustrate this definition, consider the real numbers $\mathbb{R}$ and the rational numbers $\mathbb{Q}$ each in the language $\mathcal{L} = \{<\}$. As both satisfy the theory of dense linear orders and this theory is complete any two of its models satisfy exactly the same set of sentences, and therefore are elementarily equivalent. However, the two structures are not isomorphic as they differ in cardinality.

A few years later, Tarski and Vaught formalized the stronger concept of an elementary extension, where $\mathcal{M}$ is a substructure of $\mathcal{N}$, and introduced the broader notion of an elementary embedding [TV57]. Notably, if there is an elementary embedding from $\mathcal{M}$ to $\mathcal{N}$, then $\mathcal{M}$ and $\mathcal{N}$ are elementary equivalent. This concept naturally leads to the definition of the appropriate category for first-order model theory: the class of models of a complete first-order theory with elementary embeddings as the morphisms [MV62].

Definition 2.4.2. Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures, and let $f : A \to \mathcal{N}$ be such that $A \subseteq \mathcal{M}$. The function f is *elementary* if for every $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and every $a_1, \ldots, a_n \in A$: $\mathcal{M} \models \varphi(a_1, \ldots, a_n)$ if and only if $\mathcal{N} \models \varphi(f(a_1), \ldots, f(a_n))$.

If $A = M$, f is called an *elementary embedding.* If $\mathcal{M}$ is a substructure of $\mathcal{N}$, $\mathcal{M}$ is called an *elementary substructure* of $\mathcal{N}$, written $\mathcal{M} \preccurlyeq \mathcal{N}$, if the inclusion map is elementary, in this case $\mathcal{N}$ is called an *elementary extension* of $\mathcal{M}$.

Recognizing that embeddings in algebraically closed fields of the same characteristic are elementary, Abraham Robinson focused his work on these fields [Rob52]. To explore it further, Robinson introduced the concept of *elementary diagrams*, comprehensive collections of first-order sentences that describe a structure.[1] By associating each element of a structure with an individual constant, Robinson showed that a structure $\mathcal{M}$ satisfies the elementary diagram of another structure $\mathcal{N}$ if and only if $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$.

Definition 2.4.3. Let $\mathcal{L}$ be a language and let $\mathcal{M}$ be an $\mathcal{L}$-structure. For $A \subseteq M$, denote by $\mathcal{L}_A$ the language $\mathcal{L} \cup \{c_a : a \in A\}$; that is, the language obtained from $\mathcal{L}$ by adding constant symbols for every $a \in A$. The elementary diagram of A in $\mathcal{M}$ is the $\mathcal{L}_A$-theory $\{\varphi(c_{a_1}, \ldots, c_{a_n}) : \mathcal{M} \models \varphi(a_1, \ldots, a_n)\}$.

Robinson used elementary diagrams to construct embeddings, proving that the first-order theory of algebraically closed fields with a given characteristic is complete. Although Tarski had already established this result using quantifier elimination[2] [Tar98], Robinson's method stood out for its innovative approach intertwining model theory with algebraic results. He combined the use of mappings between structures with the algebraic fact that two algebraically closed fields with the same characteristic and transcendence degree are isomorphic. Building on this work, Robinson introduced the notion of model completion [Mac77].

Definition 2.4.4. Let T be an $\mathcal{L}$-theory. The theory T is *model complete* if for all $\mathcal{M}, \mathcal{N} \models T$, if $\mathcal{M}$ is a substructure of $\mathcal{N}$, then $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$.

According to Baldwin, Robinson's identification of model-complete theories together with his focus on the complete theories of significant algebraic structures [Rob56] was an early instance of studying classes of theories [Bal18]. Another major contribution of Robinson, was his criterion for quantifier elimination and model completeness, from which follows that if a theory T has quantifier elimination, then T is model complete.

[1] In [Hod93] Hodges suggests that the concept of diagrams can be traced back to Wittgenstein's Tractatus Logico-Philosophicus [Wit21], where he states: "Die Angabe aller wahren Elementarsatze beschreibt die Welt vollstandig. Die Welt ist vollstandig beschrieben durch die Angaben aller Elementarsatze plus der Angabe, welche von ihnen wahr und welche falsch sind."

[2] Bruno Poizat [Poi00] notes that this result can be viewed as a modern articulation of the ancient method of solving systems of polynomial equations and inequalities by successively eliminating unknowns. Thus, this technique dates back to Babylonian mathematics and was further refined during the Chinese Middle Ages [Hoe77].

Theorem 2.4.5. *Let T be an $\mathcal{L}$-theory. T is model complete if and only if for every $\mathcal{L}$-formula φ, there is a universal $\mathcal{L}$-formula ψ such that $T \models \varphi \leftrightarrow \psi$.*

This result was particularly important because it simplifies the analysis of definable subsets within a model of T. One of the earliest applications of these ideas was the development of a test for completeness by Robert L. Vaught in 1954 [Vau54], which was independently discovered by Jerzy Łoś the same year [Łoś54].

Theorem 2.4.6 (Łoś–Vaught Test)**.** *Let T be a countable theory with only one model (up to isomorphism) of cardinality λ for some infinite cardinal λ and has no finite models. Then T is complete.*

A year later, Łoś published his influential work on ultraproducts, offering a different approach to understanding the equivalence of structures [Łoś55]. However, ultraproducts had been used sporadically prior to that time, notably in [Sko34], where an ultraproduct was employed to construct a nonstandard model of arithmetic, laying the groundwork for Robinson's nonstandard analysis [Rob61].

Łoś' construction involved taking the product $\mathcal{M} = \prod_{i \in I} \mathcal{N}_i$ of structures in a fixed language $\mathcal{L}$, alongside an ultrafilter $\mathcal{D}$ on I (a maximal filter on the powerset $\mathcal{P}(I)$). In this setup, function and constant symbols are defined based on the indices where they hold. Similarly, two elements of the product are considered equal if, and only if, the set of indices where they agree belongs to $\mathcal{D}$.

Definition 2.4.7. Let $\mathcal{L}$ be a language and suppose that I is an infinite set. Suppose that $\mathcal{M}_i$ is an $\mathcal{L}$-structure for each $i \in I$. Let $\mathcal{D}$ be an ultrafilter on I. Define a new structure $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / \mathcal{D}$, which is called the *ultraproduct* of the $\mathcal{M}_i$ using $\mathcal{D}$. Define a relation $\sim$ on

$$X = \prod_{i \in I} M_i = \{f : I \to \bigcup_{i \in I} M_i \mid f(i) \in M_i \text{ for all } i \in I\}$$

by $f \sim g$ if and only if $\{i \in I \mid f(i) = g(i)\} \in \mathcal{D}$. If all $\mathcal{M}_i$ are the same, $\mathcal{M}$ is called an *ultrapower*.

Łoś provided a key result characterizing how a first-order formula $\varphi(x)$ is satisfied in an ultraproduct $\mathcal{M}$ based on its satisfaction in the corresponding $\mathcal{N}_i$. This result, known as the fundamental theorem of ultraproducts, influenced Keisler's work on ultrapowers [Kei61], which eventually led to Shelah's proof that two structures are elementarily equivalent if and only if they have isomorphic ultrapowers [She71a].

Theorem 2.4.8 (Łoś's Theorem)**.** *Let $\varphi(x_1, \ldots, x_m)$ be an $\mathcal{L}$-formula and let $g_1, \ldots, g_m \in \prod_{i \in I} \mathcal{M}_i$. Then $\mathcal{M} \vDash \varphi([g_1]_\mathcal{D}, \ldots, [g_m]_\mathcal{D})$ if and only if $\{i \in I : M_i \vDash \varphi(g_1(i), \ldots, g_m(i))\} \in \mathcal{D}$.*

An alternative characterization elementary equivalence was introduced by Roland Fraïssé [Fra56]. Rather than focusing on formulas, Fraïssé's approach relies on comparing two structures, $\mathcal{M}$ and $\mathcal{N}$, through mappings between them. However, this idea became widely known when Andrzej Ehrenfeucht reframed it as a game [Ehr61], which is now called the Ehrenfeucht–Fraïssé back-and-forth game.

In Ehrenfeucht's version [Mar03], two players take turns in a game $G_k(\mathcal{M},\mathcal{N})$, which lasts for k rounds and is used to compare the structures $\mathcal{M}$ and $\mathcal{N}$. In each round, Player I starts by either picking an element $m_i \in M$, challenging Player II to match it by choosing $n_i \in N$, or picking $n_i \in N$, challenging Player II to respond with $m_i \in M$. Player II loses if, at any point in the game, the elements chosen from one structure satisfy a quantifier-free formula that isn't satisfied by the corresponding elements in the other structure. Otherwise, Player II has a strategy that allows her to continue playing for at least k stages, and this happens if and only if the countable structures $\mathcal{M}$ and $\mathcal{N}$ agree on all sentences of quantifier rank at most k [Hod00]. Extending this to every finite k gives a full characterization of elementary equivalence.

Theorem 2.4.9. *Let $\mathcal{L}$ be a finite language without function symbols, and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. Then, $\mathcal{M} \equiv \mathcal{N}$ if and only if the second player has a winning strategy in $G_n(\mathcal{M},\mathcal{N})$ for all n.*

Aside from his work on equivalence, Fraïssé is best known for his 1954 construction of structures [Fra54]. He pointed out that finite linear orderings can be thought of as approximations of the ordering of the rationals. Building on this idea, he developed a method for constructing structures from these finite approximations, which is similar to the concept of a direct limit in category theory. One of the key ideas he highlighted was the amalgamation property, which ensures that combining elements from different structures is done in a consistent and compatible way.

Definition 2.4.10. We say that T has the *amalgamation property* if and only if whenever $\mathcal{M}_0$, $\mathcal{M}_1$, and $\mathcal{M}_2$ are models of T and $f_i : \mathcal{M}_0 \to \mathcal{M}_i$ are embeddings, there exists $\mathcal{N} \models T$ and $g_i : \mathcal{M}_i \to \mathcal{N}$ such that $f_1 \circ g_1 = f_2 \circ g_2$. The merging structure $\mathcal{N}$ can be seen as the push-out in the following diagram.

$$\begin{tikzcd}
\mathcal{M}_0 \arrow[r, "f_1"] \arrow[d, "f_2"] & \mathcal{M}_1 \arrow[d, dashed, "g_1"] \\
\mathcal{M}_2 \arrow[r, dashed, "g_2"] & \mathcal{N}
\end{tikzcd}$$

Referring to this diagram, in 1993, Hodges wrote:

> "Everything that has happened in model theory during the last thirty years has confirmed how important this diagram is. To see why, imagine some structure $\mathcal{N}$ and a small part $\mathcal{M}_0$ of $\mathcal{N}$, and ask, 'How does $\mathcal{N}$ sit around $\mathcal{M}_0$?' For the answer, we need to know how $\mathcal{M}_0$ can be extended within $\mathcal{N}$ to structures $\mathcal{M}_1$, $\mathcal{M}_2$, etc. But then we also need to know how any two of these extensions $\mathcal{M}_1$, $\mathcal{M}_2$ of $\mathcal{M}_0$ are related to each other inside $\mathcal{N}$: what formulas relate the elements of $\mathcal{M}_1$ to the elements of $\mathcal{M}_2$? An amalgam of $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{M}_0$ answers this last question." [Hod93]

The events that Hodges refers to here mainly branch into two directions. One is to use amalgamation to exploring possible extensions of the base structure and classifying how these extensions can be amalgamated. In favorable cases, this, *non-trivially*, leads to a structural understanding of all models of a theory (see Chapter 4.3). The second use, which Fraïssé originally worked on, involves taking small structures, extending them, and then amalgamating these extensions to form a larger structure with significant properties.

Theorem 2.4.11 (Fraïssé's Theorem). *Let $\mathcal{L}$ be a countable language, and let $\mathcal{K}$ be a countable set of finitely generated structures that have the hereditary property,*[3] *the joint embedding property,*[4] *and the amalgamation property. Then there exists a countable structure $\mathcal{N}$, known as Fraïssé limit or Fraïssé construction, which is both universal and homogeneous.*

These two properties are very significant. They imply that $\mathcal{N}$ is a model in which all the structures in $\mathcal{K}$ can be embedded (universal), and any isomorphism between finite substructures extends to an isomorphism of $\mathcal{N}$ (homegeneous). This idea of a homogeneous-universal structure was extended by Jónsson in [Jón56; Jón60], removing the limitation to finite and countable structures in Fraïssé's construction.

Definition 2.4.12. Let κ be an infinite cardinal. A model $\mathcal{M} \models T$ is κ-homogeneous if, whenever $A \subseteq M$ with $|A| < \kappa$, and $f : A \to M$ is a partial elementary map, and $a \in M$, there exists $f' \supseteq f$ such that $f' : A \cup \{a\} \to \mathcal{M}$ is a partial elementary map. The model $\mathcal{M}$ is homogeneous if it is $|M|$-homogeneous.

Definition 2.4.13. A model $\mathcal{M} \models T$ is κ-universal if for all $\mathcal{N} \models T$ with $|N| < \kappa$, there is an elementary embedding of $\mathcal{N}$ into $\mathcal{M}$. The model $\mathcal{M}$ is universal if it is $|M|^+$-universal.

Ten years later, this construction would become crucial for stability theory and, in turn, for model theory as a whole, but its significance wasn't immediately recognized. In fact, after the great achievements discussed, by the late 1950s, first-order theory was considered largely understood. The focus shifted to infinitary languages, aiming to extend the techniques developed for first-order languages. Daniel Lascar described the situation as "a moment of pause, as if the machinery, ready to run, didn't know which direction to take." [Las98] Hodges also recalls two anecdotes:

> "In about 1970 a Polish logician reported that a senior colleague of his had advised him not to publish a textbook on first-order model theory, because the subject was dead. And in 1966 David Park, who had just completed a PhD in first-order model theory with Hartley Rogers at MIT, visited the research group in Oxford and urged us to get out of first-order model theory because it no longer had any interesting questions." [Hod00]

At this point Michael Morley appeared in the scene, causing what has been called the second birth of Model Theory [Poi00].

[3] The hereditary property property states that any finitely generated substructure of a structure in $\mathcal{K}$, is isomorphic to a structure in $\mathcal{K}$.

[4] The joint embedding property states that for any two structures in $\mathcal{K}$, there exists a third structure in $\mathcal{K}$ into which both of the given structures can be embedded.

3 The Second Birth of Model Theory

3.1 Realizing and Omitting Types

In model theory, a common approach to studying a first-order theory is to analyze the complexity of the Boolean algebras formed by the definable sets in a models. Equivalently, one can analyse the complexity of their Stone duals, which are known as type spaces. Types encapsulate the relationships within a mathematical structure $\mathcal{M}$, addressing questions such as whether one part of $\mathcal{M}$ can remain small while another expands. In essence, a type is a collection of formulas that describe potential or actual elements within a structure, and is comparable to minimal polynomials in field theory.

The notion of a type emerged in the 1950s through the merging of logic and topology. As Schlimm explains, this unification was achieved through successive analogies: of propositional logic with algebra by Boole, of Boolean algebras with rings by Stone, and of deductive systems with Boolean algebra by Tarski [Sch85]. This historical progression established a fruitful interplay between algebraic and topological perspectives, a dynamic that remains at the heart of modern model theory.

Consider a set V of n propositional variables. From this set, one can construct a Boolean algebra $\mathcal{B}$ that includes these variables along with the constants 0 and 1 (representing truth values) and logical operators [Boo47]. Each truth assignment corresponds to an ultrafilter on the set of formulas—that is, the collection of all formulas true under that assignment. Consequently, the Boolean algebra $\mathcal{B}$ can be associated with a topological space of ultrafilters, denoted $S_n(\mathcal{B})$, known as the *n-th Stone space* of $\mathcal{B}$.

In 1936, Stone proved that the map sending each element $b \in \mathcal{B}$ to the set of all ultrafilters containing b is an isomorphism [Sto36]. This result implies that every Boolean algebra $\mathcal{B}$ is isomorphic to the algebra of subsets of its Stone space $S_n(\mathcal{B})$, and thus Boolean algebras can be represented by Stone spaces. Moreover, the natural topology on $S_n(\mathcal{B})$, with basis sets of the form $U_b = \{p \in S_n(\mathcal{B}) \mid b \in p\}$ for $b \in \mathcal{B}$, implies that the image of each b is a clopen set.

Lindenbaum and Tarski generalized this result from propositional logic to first-order logic [Tar35]. For a first-order theory T, they considered the set of $\mathcal{L}$-formulas with free variables among $v_1, \ldots, v_n$ and partitioned them into equivalence classes where two formulas φ and ψ are equivalent if $T \models \varphi \leftrightarrow \psi$. The set of these equivalence classes, equipped with the algebraic operations corresponding to the connectives, forms the *Lindenbaum-Tarski algebra* $\mathcal{B}_n(T)$ of the theory T, where the n represents the number of variables.

Using the reasoning above, the Stone space of the Lindenbaum-Tarski algebra of T is denoted as $\mathcal{S}_n(\mathcal{B}_n(T))$, which is usually abbreviated to $\mathcal{S}_n(T)$ and referred to as the n-th Stone space of T. In this context, the clopen sets are of the form: $[\varphi] = \{p \in \mathcal{S}_n(T) \mid p \models \varphi\}$ where $p \models \varphi$ means that the type p satisfies the formula φ. The elements p of $\mathcal{S}_n(T)$ are called complete n-types. A particularly noteworthy case is the 0-th Stone space $\mathcal{S}_0(T)$, which, up to theory equivalence, corresponds to the set of all complete $\mathcal{L}$-theories extending T.

Although this topological perspective is valuable, the concept of a type can also be defined directly in terms of formulas:

Definition 3.1.1. Let T be an $\mathcal{L}$-theory, and let $p(x_1, \ldots, x_n)$ be a set of $\mathcal{L}$-formulas. This set $p(x_1, \ldots, x_n)$ is called a type of T if $T \cup p(x_1, \ldots, x_n)$ is satisfiable. A n-type $p(x_1, \ldots, x_n)$ is called complete if, for every $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$, either $\varphi \in p$ or $\neg\varphi \in p$, and the set of all complete n-types is denoted by $\mathcal{S}_n(T)$. Sometimes $S(T)$ is used to denote $\bigcup_{n<\omega} S_n(T)$ (sometimes, equivalently, $S_1(T)$).

To say that $T \cup p(x_1, \ldots, x_n)$ is satisfiable means that there exists a structure $\mathcal{M}$ and a tuple $a \in \mathcal{M}^n$ such that $\mathcal{M} \models T$ and $\mathcal{M} \models \varphi(a)$ for every formula $\varphi(x_1, \ldots, x_n) \in p$.

Building on this idea, the notion of types of a theory can be extended to define types of a theory T within a specific model $\mathcal{M} \models T$ over a subset $A \subseteq M$. This extension makes it possible to identify which elements of $\mathcal{M}$ can be described using formulas with parameters from A. To formalize this, the language $\mathcal{L}$ is expanded to $\mathcal{L}_A$, where $\mathcal{L}_A$ includes additional constant symbols for each $a \in A$. In this expanded language, $\mathcal{M}$ is naturally viewed as an $\mathcal{L}_A$-structure by interpreting these new symbols as their corresponding elements in A.

Definition 3.1.2. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$, and let $p(x_1, \ldots, x_n)$ be a set of $\mathcal{L}_A$-formulas. The set $p(x_1, \ldots, x_n)$ is called an n-type of $\mathcal{M}$ over A if for all $k \in \mathbb{N}$ and $\varphi_1, \ldots, \varphi_k \in p$, there exists $a \in \mathcal{M}^k$ such that $\mathcal{M} \models \bigwedge_{i=1}^k \varphi_i(a)$. Let $S_n^{\mathcal{M}}(A)$ be the set of all complete n-types.

For complete theories T, the core correspondence is given by $\mathcal{S}_n(T) = \mathcal{S}_n^{\mathcal{M}}(\emptyset)$.

To illustrate this definition, consider the structure $\mathbb{Q}$ in the language $\mathcal{L} = \{<\}$ and let $A \subseteq \mathbb{Q}$ be the set of natural numbers. Define $p(x) = \{x > 1, x > 2, x > 3, \ldots\}$. By the Compactness Theorem, $p(v) \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable and $p(x)$ is a 1-type of $\mathbb{Q}$ over $\mathbb{N}$. Similarly, let $q(x) = \{\varphi(x) \in \mathcal{L}_A : \mathbb{Q} \models \varphi\left(\frac{2}{3}\right)\}$. For example, $x < 1$ belongs to $q(x)$, whereas $x > 2$ does not. For any $\mathcal{L}_A$-formula $\psi(x)$, either $\mathbb{Q} \models \psi\left(\frac{2}{3}\right)$ or $\mathbb{Q} \models \neg\psi\left(\frac{2}{3}\right)$, making $q(x)$ a complete 1-type.

This construction generalizes to characterize specific elements and complete types in arbitrary structures. Given an $\mathcal{L}$-structure $\mathcal{M}$, $A \subseteq M$, and $a = (a_1, \ldots, a_n) \in M^n$, the *type of a over A* is defined as: $\mathrm{tp}^{\mathcal{M}}(a/A) = \{\varphi(v_1, \ldots, v_n) \in \mathcal{L}_A : \mathcal{M} \models \varphi(a_1, \ldots, a_n)\}$. This set is a complete n-type. When $A = \emptyset$, it is simply written as $\mathrm{tp}^{\mathcal{M}}(a)$. The model $\mathcal{M}$ is omitted when clear.

An important property of a type is whether or not it is realized:

Definition 3.1.3. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$, and let $p(x_1, \ldots, x_n)$ be a type over A. A tuple $a \in M^n$ *realizes* p if $\mathcal{M} \models \varphi(a)$ for every $\varphi \in p$. In this case, p is said to be *realized* in $\mathcal{M}$. If no such a exists, p is *omitted* in $\mathcal{M}$.

By the Compactness Theorem, any type can be realized in an elementary extension of $\mathcal{M}$. Notably, if $\mathcal{N}$ is an elementary extension of $\mathcal{M}$, then $\mathrm{Th}_A(\mathcal{M}) = \mathrm{Th}_A(\mathcal{N})$, which implies $S_n^{\mathcal{M}}(A) = S_n^{\mathcal{N}}(A)$. However, a more insightful approach is to study models that realize as many types as possible themselves, a property known as saturation. This concept was first introduced by Vaught in its weaker form of ω-saturation [Vau61].

Definition 3.1.4. Let κ be an infinite cardinal and let $\mathcal{M}$ be an $\mathcal{L}$-structure. An $\mathcal{L}$-structure $\mathcal{M}$ is called κ-saturated if every type $p(x_1, \ldots, x_n)$ over A is realized in $\mathcal{M}$ for all $A \subseteq M$ with $|A| < \kappa$ and for every $n \in \mathbb{N}$. The structure $\mathcal{M}$ is considered *saturated* if it is $|M|$-saturated.

As an example, consider the theory of algebraically closed fields of characteristic 0 in the language $\mathcal{L} = \{+, \cdot, 0, 1\}$. The structure of the algebraic closure of $\mathbb{Q}$, $\mathbb{Q}_{\mathrm{alg}}$, is not $\aleph_0$-saturated, as it does not realize the type $p(x) = \{f(x) \neq 0 : f(x) \in \mathbb{Z}[x]\}$, which describes a transcendental element. In contrast, the structure $\mathbb{C}$, being $\aleph_1$-saturated, realizes all such types, illustrating the richer structure of a saturated model.

In the same work [Vau61], Vaught established an important equivalence between the space of types $S_n(T)$ and the concept of saturation, establishing a milestone in modern model theory.

Theorem 3.1.5. *Let T be a complete $\mathcal{L}$-theory with infinite models. Then T has a countable saturated model if and only if $|S_n(T)| \leq \aleph_0$ for all $n \in \mathbb{N}$.*

Morley and Vaught [MV62] extended these ideas by adapting Fraïssé and Jónsson's notion of universal-homogeneous structures to focus on models of a complete theory. They showed that this shift amounts to requiring that every type over a subset of size less than κ be realized in a model $\mathcal{M}$, leading to the broader notion of saturation. This approach bridged the gap between semantic and syntactic perspectives, demonstrating that the homogeneous-universal property—an algebraic trait of a class of models—corresponds to saturation, a model-theoretic feature of a single structure and its complete theory.

Theorem 3.1.6. *A structure $\mathcal{M}$ is saturated if and only if it is homogeneous-universal for the class of models of Th($\mathcal{M}$) with cardinality less than $|M|$ under the relation of elementary substructure. Moreover, any two saturated, elementary equivalent structures $\mathcal{M} \equiv \mathcal{N}$ of the same cardinality are isomorphic $\mathcal{M} \cong \mathcal{N}$.*

Because of its homogeneity and universality, a saturated model of a complete theory T is often viewed as the "most typical" model of T and serves as a convenient working structure. However, not every theory T guarantees the existence of a saturated model. Nevertheless, for any $\kappa \geq |\mathcal{L}|$, and for any structure $\mathcal{M}$ with $|M| \leq 2^\kappa$, there exists a κ^+-saturated elementary extension $\mathfrak{C} \succ \mathcal{M}$. This κ^+-saturated model can be constructed by realizing types and taking unions of elementary chains.[1]

[1] An elementary chain is a sequence of L-structures $(M_i : i \in I)$ indexed by a linear order

About a decade later, Shelah [She78] built on this concept by introducing the *universal domain*, now commonly known as the *monster model*, a term coined by John Baldwin. This model, denoted by $\mathfrak{C} \models T$, is κ-saturated, κ-universal, and κ-homogeneous for a *very* large cardinal κ. Thefore, the monster model facilitates the study of all models of a complete first-order theory T by treating them as elementary submodels of $\mathfrak{C}$.

The use of the monster model significantly simplifies arguments. For instance, instead of stating "let $N \succ M$ contain a realization $a \models p(x)$", it suffices to say "let $a \models p(x)$ within $\mathfrak{C}$". Additionally, any consistent type over a small set A, where $|A| < \kappa$, is realized in $\mathfrak{C}$. This eliminates the dependency on a specific model $\mathcal{M}$ containing $A \subseteq M$ when analyzing types over A. Many proofs also become more elegant with automorphism arguments, where solutions to a type $p \in S_n(A)$ are treated as orbits of the automorphism group of $\mathfrak{C}$ that fix A.

The monster model has become a foundational tool in model theory, representing a shift in perspective comparable to the changes in algebraic geometry during the 1970s. It functions as an "equivariant Grothendieck universe" for the definable sets of a first-order theory T, analogous to Weil's universal domain, which serves as a monster model for the theory of algebraically closed fields. As Hrushovski observes [Hru02], its role is to illuminate the geometric structure of a theory while minimizing dependence on specific algebraic constructs.

While saturated models aim to realize the maximum number of types, unsaturated models that omit specific types are equally important. By deliberately excluding certain types, these models avoid "problematic" conditions that might otherwise complicate the theory. One significant class of types in this context is the isolated types:

Definition 3.1.7. An n-type $p(x_1, \dots, x_n)$ of $\mathcal{M}$ over A is called *isolated* if there is an $\mathcal{L}_A$-formula $\varphi(x_1, \dots, x_n)$ such that for every $\mathcal{L}_A$-formula $\psi(x_1, \dots, x_n)$ in p, $\mathcal{M} \models \varphi \to \psi$.

An isolated type can only be omitted if its isolating formula is not witnessed in the model. In particular, if the theory is complete, then every isolated type is realized. For countable languages, this is also a sufficient condition. This result, known as the Omitting Types Theorem, was established by Robert Vaught [Vau61], who credited Andrzej Ehrenfeucht, and built on earlier work by Leon Henkin [Hen54] and Steven Orey [Ore56]. It can easily be generalized to allow the omission of countably many types simultaneously.

Theorem 3.1.8. *Let T an $\mathcal{L}$-theory. Suppose X is a countable collection of non-isolated types over $\emptyset$, where a type p is considered isolated in $S_n(T)$ if $\{p\} = [\varphi]$ for some formula φ. Then, there exists a countable model $\mathcal{M} \models T$ that omits all types $p \in X$.*

This theorem provides a powerful tool for constructing controlled models. In

$(I, <)$, where $M_i \subseteq M_j$ for $i < j$, and $M_i \prec M_j$ (elementarily embedded) for $i < j$. In [TV57] Tarski and Vaught proved that unions of elementary chains are elementary extensions of each $(M_i : i \in I)$.

particular, it allows the construction of small models of a complete theory that retain its essential features. In model-theoretic terms, such models are minimal in the sense that for any other model of the theory, there exists an elementary embedding from the smaller model into it. Vaught coined the term *prime models* for such structures [Vau61]. While saturated models realize as many types as possible, a prime model realizes only the necessary types, containing only the essential features of the theory.

Definition 3.1.9. An $\mathcal{M} \models T$ is called a *prime model* of T if, for any $\mathcal{N} \models T$, there exists an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$.

An example of this is found in the theory of algebraically closed fields of characteristic 0. The field $\mathbb{Q}_{\text{alg}}$, the algebraic closure of $\mathbb{Q}$, serves as a prime model of ACF_0. For any model $\mathcal{M} \models \text{ACF}_0$, there exists an elementary embedding of $\mathbb{Q}_{\text{alg}}$ into $\mathcal{M}$, highlighting how prime models capture the essential algebraic properties of the theory. Because ACF_0 is model complete, this embedding is always elementary.

Vaught further developed the theory of prime models, proving their uniqueness and establishing necessary and sufficient conditions for their existence in countable complete theories.

Theorem 3.1.10. *Let T is a complete $\mathcal{L}$-theory with infinite models and $A \subseteq M \models T$ is countable. If $|S_n(T)| < 2^{\aleph_0}$, then T has a prime model. Moreover, if $\mathcal{M}$ and $\mathcal{N}$ are prime models of T, then $\mathcal{M} \cong \mathcal{N}$.*

In retrospect, these concepts arising in the 1960s around the space of types, largely due to Vaught, will become cornerstones of the field. Morley will showcase the importance of the topology of Stone spaces and pioneer the sophisticated use of saturation, aided by the new notion of a prime model over potentially infinite sets. Moreover, as foreshadowed, Shelah will establish the monster model as the new standard framework in model theory.

3.2 Indiscernibles & Combinatorics

In the 1950s, Andrzej Ehrenfeucht and Andrzej Mostowski introduced a method of using linearly ordered sets to analyze and construct mathematical structures. Central to their approach was the concept of an indiscernible set—a collection of elements within a structure that cannot be distinguished from each other by any first-order formula. This concept is closely related to Fraïssé's earlier examination of *chainable relations* [Fra54], in which elements form an indiscernible set when considering quantifier-free formulas.

Definition 3.2.1. Let I be an infinite set and suppose that $X = \{x_i : i \in I\}$ is a set of distinct elements of M. The set X is called an *indiscernible set* (or totally indiscernible) if whenever $i_1, \ldots, i_m$ and $j_1, \ldots, j_m$ are two sequences of m distinct elements of I, then $M \models \varphi(x_{i_1}, \ldots, x_{i_m}) \leftrightarrow \varphi(x_{j_1}, \ldots, x_{j_m})$ for any L-formula φ with m free variables.

To illustrate this definition, consider an algebraically closed field $\mathcal{F}$ with infinite transcendence degree. Let $\{x_1, x_2, \ldots\}$ be an infinite set of algebraically independent elements in $\mathcal{F}$. For any two finite sequences of indices $i_1, \ldots, i_m$ and $j_1, \ldots, j_m$, there exists an automorphism of $\mathcal{F}$ mapping each x_{i_k} to x_{j_k}. This symmetry implies that $\{x_1, x_2, \ldots\}$ forms an infinite set of indiscernibles.

However, not all structures admit infinite sets of indiscernibles. For instance, in an infinite linear order, any two distinct elements a and b satisfy exactly one of the relations $a < b$ or $b < a$, inherently distinguishing them. As a result, no indiscernible set of even two elements can exist. To address this, Ehrenfeucht and Mostowski introduced the weaker notion order indiscernibles, where the order of the elements is taken into account.

Definition 3.2.2. Let $(I, <)$ be an ordered set, and let $(x_i : i \in I)$ be a sequence of distinct elements of M. The set $(x_i : i \in I)$ is a sequence of *(order) indiscernibles* if whenever $i_1 < i_2 < \cdots < i_m$ and $j_1 < \cdots < j_m$ are two increasing sequences from I, then $M \models \varphi(x_{i_1}, \ldots, x_{i_m}) \leftrightarrow \varphi(x_{j_1}, \ldots, x_{j_m})$.

In 1956 [EM79], Ehrenfeucht and Mostowski introduced a method for constructing structures around sequences of order indiscernibles. Their method allowed the properties of these linearly ordered sets to dictate the characteristics of the resulting models. This approach relied heavily on combinatorial set theory, using results from partition calculus, such as Ramsey's Theorem [Ram30] and the Erdős–Rado Theorem [ER56], which later became standard tools in model theory.

Theorem 3.2.3. *1. Ramsey's Theorem: $\omega \to (\omega)^n_k$.*

2. Erdős–Rado Theorem: $\beth_{\mu^+} \to (\mu^+)^{n+1}_{\mu}$, where $\beth$ denotes the beth function.[2]

For a set κ and a natural number n, let $[\kappa]^n$ be the set of all n-element subsets of κ. The notation $\kappa \to (\lambda)^n_\mu$ means that for any function $f : [\kappa]^n \to \mu$, there is a subset $A \subseteq \kappa$ with $|A| = \lambda$ such that f is constant on $[A]^n$.

These theorems extend the fundamental combinatorial principle that in large sets, certain patterns or regularities are inevitable. Specifically, given a sufficiently large collection of objects and a finite set of properties, some objects will unavoidably be indistinguishable based on those properties. This means that, under the right conditions, elements within a structure cannot be distinguished from one another using only a finite set of properties. By leveraging these combinatorial principles, in particular, Ramsey's Theorem, Ehrenfeucht and Mostowski demonstrated that it is always possible to construct models containing infinite sequences of order indiscernibles.

Theorem 3.2.4. *Let T be an $\mathcal{L}$-theory infinite models. For any ordered set $(I, <)$, there exists a model $\mathcal{M} \models T$ containing a sequence $(x_i : i \in I)$ of order indiscernibles.*

[2] The beth function $\beth$ is defined for an ordinal α and a cardinal μ as follows. Start by setting $\beth_0(\mu) = \mu$. For successor ordinals, when $\alpha = \beta + 1$, define $\beth_\alpha(\mu) = 2^{\beth_\beta(\mu)}$. If α is a limit ordinal, then $\beth_\alpha(\mu)$ is defined as the supremum of all previous beth numbers, that is, $\beth_\alpha(\mu) = \sup_{\beta<\alpha} \beth_\beta(\mu)$. Additionally, define $\beth_\alpha$ to be $\beth_\alpha(\aleph_0)$.

Furthermore, if $\mathcal{M}$ is the closure of a set X under Skolem functions (as can always be arranged), then $\mathcal{M}$ is called an Ehrenfeucht–Mostowski model of T. Additionally, by choosing the sets $(X, <)$ and $(X', <')$ sufficiently different, it is often possible to ensure that the Ehrenfeucht–Mostowski models constructed from these sets are not isomorphic. From this, Ehrenfeucht observed that if a theory defines an infinite linear ordering on n-tuples of elements, it must have a large number of non-isomorphic models of the same cardinality [Ehr61].

Theorem 3.2.5. *Let T be an $\mathcal{L}$-theory with infinite models. Suppose that for some $n < \omega$, the Stone space $S_n(T)$ is uncountable. Then, for every infinite cardinal λ, T has at least 2^ω models of cardinality λ which are pairwise non-isomorphic.*

This illustrates how the properties of an ordered set I can be intricately mirrored within models of the theory T, providing a powerful method for constructing models with desired characteristics. Looking ahead, Ehrenfeucht–Mostowski models will become pivotal in model theory as Morley demonstrates their flexibility to either realize or omit types, Moreover, this last result regarding non-isomorphic models will be central in Shelah's classification program.

3.3 Countable Categoricity

During the 1960s, the study of models exhibiting distinct properties, such as saturated, prime, or Ehrenfeucht–Mostowski models, sparked a renewed interest in understanding the isomorphism classes of these models across various cardinalities. However, this focus on isomorphic models dates significantly further back. In 1904, in his work on geometry, Oswald Veblen introduced the concept of *categoricity*, defining it as a property of a system of axioms that "is sufficient for the complete determination of a class of objects or elements" [Veb04]. In contemporary terms, this describes a theory that has exactly one model up to isomorphism.

With first-order logic as the standard framework, however, achieving categoricity for a theory with infinite models was impossible. The Löwenheim–Skolem theorem (Theorems 2.1.1 and 2.2.3) establishes that any first-order theory with at least one infinite model must have models of every infinite cardinality, ensuring the existence of non-isomorphic models of different sizes. This limitation is particularly striking because many foundational theories, such as arithmetic, analysis, and set theory, possess infinite models and, consequently, fail to be categorical.[3]

In response to this limitation, Robert Vaught [Vau54] and Jerzy Łoś [Łoś54] independently introduced the weaker notion of κ-categoricity in 1954, which served as a significant refinement and provided a key criterion for understanding completeness in theories with infinite models.

[3] The most straightforward way to achieve a categorical theory of arithmetic is to shift from first-order logic to second-order logic. Dedekind famously established the categoricity of second-order Peano Arithmetic in [Ded88].

Definition 3.3.1. Let κ be an infinite cardinal and let T be an $\mathcal{L}$-theory. The theory T is called *κ-categorical* if it has models of cardinality κ and every two models of T of cardinality κ are $\mathcal{L}$-isomorphic. T is categorical if it is κ-categorical for every κ.

Theorem 3.3.2 (Łos-Vaught Test)**.** *Let T be an $\mathcal{L}$-theory with no finite models. If T is κ-categorical for some infinite cardinal κ with $\kappa \geq |L|$, then T is complete.*

Early results on κ-categoricity focused on countable models, specifically on κ-categoricity for $\kappa = \aleph_0$.[4] For instance, Georg Cantor demonstrated that the complete first-order theory of the rationals as a linear order has exactly one countable model up to isomorphism [Can95], making it ω-categorical. This result follows immediately from Fraïssé's work (see Theorem 2.4.9), provided it is established that any two models $\mathcal{M}$ and $\mathcal{N}$ admit a back-and-forth system. While the back-and-forth method is sometimes attributed to Cantor, his original proof employed a different approach. According to Silver [Sil94], the earliest known instance of the back-and-forth proof appears in the work of Felix Hausdorff [Hau14].

The first significant result in the study of countable models was presented by Łoś [Łoś54], who established a criterion for determining when a theory reaches the maximum possible number of countable models. His approach used the cardinality of the theory's type space $S(T)$, making it one of the earliest results to link the semantic properties of a theory with the size of its type space.

Theorem 3.3.3. *For any complete theory T, if $S(T)$ is uncountable, then T has $2^{\aleph_0}$ countable models.*

At the opposite end of Łoś's theorem, when $S(T)$ is finite, Erwin Engeler [Eng59], Lars Svenonius [Sve59], and Czesław Ryll-Nardzewski [Ryl59] independently showed that this finiteness condition characterizes ω-categoricity. Although each made similar contributions, the result is often credited to Ryll-Nardzewski due to the historical impact of his method. Ryll-Nardzewski reinterpreted Henkin and Orey's omitting types theorem, originally stated in terms of ω-consistency without explicit reference to types, and presented it as a semantic criterion based on n-types, with whom to describe points in models. Thus, Łoś and Vaught (see Theorem 3.1.5, which built on these findings) and Ryll-Nardzewski were the originators of the semantic use of types.

Theorem 3.3.4. *A countable complete theory T is ω-categorical if and only if $S(T)$ is finite for every natural number n.*

Beyond this equivalence, the full version of the theorem has several other equivalences and provides a precise characterization of ω-categoricity. For instance, Lars Svenonius showed that models of ω-categorical theories are exactly those structures whose automorphism groups have finitely many orbits on n-element subsets for each finite n.

Beyond ω-categorical theories, Vaught aimed to explore the range of possible

[4] Historically, $\aleph_0$-categoricity has been referred to as ω-categoricity, so the symbols ω and $\aleph_0$ are used interchangeably in this context.

numbers of countable models for a given theory. In his 1961 paper, *Denumerable Models of Complete Theories*, Vaught defined the countable *spectrum function* $I(T, \aleph_0)$, which measures the number of nonisomorphic models of a complete theory T with cardinality $\aleph_0$. In his initial results, Vaught showed that a complete theory cannot have exactly two countable models, i.e., $I(T, \aleph_0) \neq 2$. He formulated the following conjecture:

Conjecture 3.3.5 (Vaught's Conjecture)**.** Can it be proved, without using the continuum hypothesis, that there exists a complete theory with exactly $\aleph_1$ nonisomorphic countable models, meaning $\aleph_0 < I(T, \aleph_0) < 2^{\aleph_0}$?

Vaught conjectured that there is no such theory. In 1970, Morley made significant progress on this conjecture by proving the following result [Mor70]:

Theorem 3.3.6. *Let T be a complete theory. Then, $I(T, \aleph_0)$ is either finite, $\aleph_0$, $\aleph_1$, or $2^{\aleph_0}$.*

Morley's theorem essentially resolved the conjecture, except in cases where the continuum hypothesis fails. In such scenarios, Vaught's Conjecture remains open for arbitrary theories. However, it has been confirmed for several significant classes, including ω-stable theories [SHM84] (see Definition 3.4.5), o-minimal theories [May88] (see Definition 5.3.1), theories of linear orders with unary predicates [Mil81], and theories of trees [Ste78].

3.4 Uncountable Categoricity

Uncountable categoricity refers to κ-categoricity for cardinals $\kappa > \aleph_0$. While it may seem that the set of uncountable cardinals for which a theory is κ-categorical could vary arbitrarily, many natural theories have exactly one model up to isomorphism for any uncountable cardinality. Take for instance the theory ACF_p of algebraically closed fields of characteristic p, where p is a prime or zero. It is a well known result in algebra that two algebraically closed fields are isomorphic if and only if they have the same characteristic and the same transcendence degree over their prime field. Thus, since for any $\kappa > \aleph_0$, any algebraically closed field with transcendence degree κ has cardinality κ, the theory is immediately κ-categorical for every uncountable κ. Inspired by such examples, in 1954, Łoś proposed the following conjecture [Łoś54]:

Conjecture 3.4.1. A complete theory that is categorical in one uncountable cardinality is categorical in all uncountable cardinalities .

As Hodges notes, this was an extraordinarily timely conjecture for two main reasons [Hod00]. First, the necessary tools to address this question were just beginning to emerge. Second, the conjecture was notable because it sought to characterize the behavior of *all* uncountable models of a theory, which would require the formulation of a structure theorem to explain the construction of any model within that theory, an objective that subsequently became central in model theory.

In 1965, in *Categoricity in Power*, Michael Morley proved Łoś's conjecture.

Theorem 3.4.2 (Morley's Categoricity Theorem). *Let T be a complete theory in a countable language with infinite models. If T is κ-categorical for some uncountable κ, then T is κ'-categorical for every uncountable κ', i.e., $I(T,\kappa)=1$ if and only if $I(T,\kappa')=1$ for every uncountable κ and κ'.*

Morley's theorem was not only remarkable for its unification of the categoricity property but also for introducing several innovative techniques in its proof. One of the most influential contributions of his proof emerged from his collaboration with Marshall Stone, where Morley identified the importance of type space topology in studying the structural properties of theories. By refining the Cantor–Bendixson rank on the Stone space, Morley developed a new topological rank for types, now called Morley rank. This rank assigns an ordinal value to complete types over arbitrary parameter sets A, extending beyond the earlier work of Łoś and Ryll-Nardzewski, which did not work over parameters. Morley's rank for formulas is defined as follows:

Definition 3.4.3. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, let $\varphi(x)$ be an $\mathcal{L}_{\mathcal{M}}$-formula, and let α be an ordinal. Let $\varphi(\mathcal{M})$ denote $\{x \in M^n : \mathcal{M} \models \varphi(x)\}$. Define $\mathrm{RM}_{\mathcal{M}}(\varphi) \geq \alpha$ recursively as follows:

1. $\mathrm{RM}_{\mathcal{M}}(\varphi) \geq 0$ if and only if $\varphi(\mathcal{M})$ is non-empty.
2. If $\alpha = \beta + 1$ for some ordinal β, then $\mathrm{RM}_{\mathcal{M}}(\varphi) \geq \alpha$ if and only if there exists a family $(\psi_i(x))_{i\in\mathbb{N}}$ of $\mathcal{L}_M$-formulas such that for all $i,j \in \mathbb{N}$:
   (a) $\mathrm{RM}_{\mathcal{M}}(\psi_i) \geq \beta$,
   (b) $\psi_i(\mathcal{M}) \cap \psi_j(\mathcal{M}) = \emptyset$ whenever $i \neq j$,
   (c) $\psi_i(\mathcal{M}) \subseteq \varphi(\mathcal{M})$.
3. If α is a limit ordinal, then: $\mathrm{RM}_{\mathcal{M}}(\varphi) \geq \alpha$ if and only if $\mathrm{RM}_{\mathcal{M}}(\varphi) \geq \beta$ for all ordinals $\beta < \alpha$.

If $\varphi(\mathcal{M})$ is non-empty, define the *Morley rank of φ in $\mathcal{M}$*, denoted by $\mathrm{RM}_{\mathcal{M}}(\varphi)$, as the maximal ordinal α such that $\mathrm{RM}_{\mathcal{M}}(\varphi) \geq \alpha$, if such an ordinal exists, and ∞ otherwise. If $\varphi(\mathcal{M})$ is empty, set $\mathrm{RM}_{\mathcal{M}}(\varphi) = -1$.[5]

The primary lasting significance of Morley rank was that it provided a notion of "dimension" for definable sets, serving as a model-theoretic analogue of the concept of dimension in linear algebra.

Definition 3.4.4. Suppose that $\mathcal{M} \models T$ and $X \subseteq \mathcal{M}^n$ is defined by the $\mathcal{L}_{\mathcal{M}}$-formula $\varphi(v)$. Define $\mathrm{RM}(X)$, the Morley rank of X, to be $\mathrm{RM}(\varphi)$.

In particular, Morley rank has several basic properties that make it suitable as a notion of dimension. For instance, if $\mathcal{M}$ is $\aleph_0$-saturated and $X \subseteq \mathcal{M}^n$ is definable, then $\mathrm{RM}(X) \geq \alpha + 1$ if and only if there exist pairwise disjoint definable subsets $Y_1, Y_2, \ldots$ of X with Morley rank at least α. Moreover, if X and Y are definable subsets of $\mathcal{M}^n$, then $X \subseteq Y$ implies $\mathrm{RM}(X) \leq \mathrm{RM}(Y)$. Similarly, $\mathrm{RM}(X \cup Y)$ is equal to the maximum of $\mathrm{RM}(X)$ and $\mathrm{RM}(Y)$.

[5] By working in the monster model $\mathfrak{C}$, Morley rank does not depend on the model from which the parameters are taken. It is sufficient to write $\mathrm{RM}(\varphi)$.

Another key innovation in Morley's proof is the introduction of totally transcendental theories. A theory T is defined as *totally transcendental* if, in any model $\mathcal{M}$, every formula has a finite Morley rank.[6] Morley showed that this topological condition imposes restrictions on the cardinality of the type space for such theories. His result established a foundational connection between Morley rank and stability. While Morley did not explicitly define the notion of κ-stability, the concept was soon introduced by Frederick Rowbottom in his work on the uncountable analog of Morley's theorem, and soon became the standard terminology [Row64].

Definition 3.4.5. Let κ be an infinite cardinal. A theory is κ-stable if $|S_n^{\mathcal{M}}(A)| \leq \kappa$ for all models $\mathcal{M}$ and for every set $A \subseteq M$ of size at most κ.[7]

The central form of κ-stability in Morley's work is ω-stability, where $|A| \leq \omega$ implies that $|S_n^{\mathcal{M}}(A)| \leq \omega$. One of Morley's primary contributions was showing that not only are ω-stable theories totally transcendental, but the converse is also true for theories in countable languages. Because this work focuses on countable languages, the main discussion will center on ω-stable theories, though many of the subsequent results extend naturally to uncountable languages as well. As Morley's aim was to study κ-categorical theories, he established that ω-stability serves as a broader generalization of this property:

Theorem 3.4.6. *Let T be a complete theory in a countable language with infinite models, and let $\kappa \geq \aleph_1$. If T is κ-categorical, then T is ω-stable.*

This theorem highlights that, much like $\aleph_0$-categorical theories, the number of complete types in an $\aleph_1$-categorical theory is tightly constrained, though not necessarily finite. Moreover, Morley demonstrated that ω-stability represents the strongest form of κ-stability:

Theorem 3.4.7. *Let T be a complete theory in a countable language. If T is ω-stable, then T is κ-stable for all infinite cardinals κ.*

To achieve these results, Morley refined several important techniques. He showed that indiscernibles, particularly Ehrenfeucht–Mostowski models, could be used to construct a large model that realize only a limited number of types. Specifically, in such model $\mathcal{M}$ for any $\kappa \geq \aleph_0$ and $A \subseteq M$, the number of types realized in $S_n^{\mathcal{M}}(A)$ is at most $|A| + \aleph_0$. Additionally, he applied the Erdős–Rado Theorem to build large models with indiscernibles that omit certain types. Morley also expanded Vaught's notion of prime models by allowing dependence on a parameter set A. Essentially, prime models over A are prime models for the enriched $L(A)$-theory. Morley demonstrated that in ω-stable theories such prime models always exist, and Shelah later proved their uniqueness [She72].

Morley's contributions to model theory were widely recognized, earning him the Leroy P. Steele Prize for Seminal Contribution to Research from the American Mathematical Society in 2003. The prize committee praised his work, stating

[6] The term "totally transcendental" originates from transcendental extensions in field theory.

[7] Using the monster model $\mathfrak{C}$ removes dependence on a specific model. It is also sufficient to verify this condition for $n = 1$, though it must hold for all sets A of size κ.

that he “set in motion an extensive development of pure model theory by proving the first deep theorem in this subject and introducing entirely new tools to analyze theories and their models”. Baldwin and Lachlan similarly highlighted the significance of Categoricity in Power, asserting that it “marked the beginning of modern model theory and laid the foundation for decades of future developments” [BM21]. Even more concisely, Poizat referred to Morley’s work as the “second birth” of model theory [Poi00].

3.5 On Strongly Minimal Sets

After Morley unified the understanding of uncountably categorical theories, model theorists turned to a deeper exploration of their internal structure. Zilber highlighted that ”the main logical problem, after resolving Łoś’s question, was to determine what properties of T ensure κ-categoricity for uncountable κ” [Zil10]. In 1971, Baldwin and Lachlan addressed this issue by refining Morley’s results, providing a precise characterization of $\aleph_1$-categorical theories [BL71].

Their work built on Vaught’s study on *two-cardinal models*. Vaught sought to understand the possible cardinalities of definable subsets of a model relative to the cardinality of the model itself. To formalize this, he introduced the concept of a (κ, λ)-model.

Definition 3.5.1. Let $\kappa > \lambda \geq \aleph_0$. An $\mathcal{L}$-theory T is said to have a (κ, λ)-model if there exists $\mathcal{M} \models T$ and an $\mathcal{L}$-formula $\varphi(v)$ such that $|\mathcal{M}| = \kappa$ and $|\varphi(\mathcal{M})| = \lambda$.

The first two-cardinal theorem was established by Vaught himself, who showed that an $\aleph_1$-categorical theory T has no two-cardinal model [Vau61]. This result was later generalized to state that for any $\kappa > \lambda$ a two cardinal model for (κ, λ) implies that for (ω_1, ω) [MV62]. Seeking a Löwenheim-Skolem-like behavior, Vaught then posed the question of which quadruples of cardinals $\kappa > \lambda$, $\kappa' \geq \lambda'$ satisfy the implication $(\kappa, \lambda) \Rightarrow (\kappa', \lambda')$ [Vau65], making this a prominent line of inquiry in set theory.

Ultimately, a solution was provided for stable theories by Shelah (see Definition 4.1.1) and Lachlan [She69; Lac72], and for o-minimal theories by Bays see Definition 5.3.1) [Bay98]. In these cases, it was shown that if T admits a (κ, λ)-model for $\kappa > \lambda$ and $\kappa' \geq \lambda'$, then T also admits a (κ', λ')-model. Baldwin noted that this answer recast Vaught’s original set-theoretic question, which asked “for which cardinals. . . ?”, to a model-theoretic one, reframing it as “for which theories. . . ?”—highlighting a crucial aspect of the mid-century shift [Bal18]. In words of Baldwin, “for most practitioners of late 20th century model theory and especially for applications in traditional mathematics the effect of this shift was to lessen the links with set theory that had seemed evident in the 1960’s.”

The Baldwin-Lachlan theorem is often described using the notion of a Vaughtian pair, which Tent and Ziegler introduced as a more semantic and robust analogue of two-cardinal models [TZ12].

Definition 3.5.2. Two models $\mathcal{M}$ and $\mathcal{N}$ of T form a *Vaughtian pair*, denoted $(\mathcal{N}, \mathcal{M})$, if $\mathcal{M} \neq \mathcal{N}$, $\mathcal{M} \prec \mathcal{N}$, and there exists an $L_{\mathcal{M}}$-formula $\varphi(\bar{v})$ such that $\varphi(\mathcal{M})$ is infinite and $\varphi(\mathcal{M}) = \varphi(\mathcal{N})$.

By compactness, the existence of a Vaughtian pair serves as an obstacle to categoricity, and, in fact, it is the only such obstacle aside from a theory not being ω-stable. The theorem oof Baldwin and Lachlan is as follows:

Theorem 3.5.3 (Baldwin-Lachlan)**.** *Let T be a complete theory in a countable language with infinite models, and let κ be an uncountable cardinal. Then T is κ-categorical if and only if T is ω-stable and has no Vaughtian pairs.*

This result directly leads to Morley's Categoricity Theorem, as the characterization does not depend on a specific uncountable cardinal κ. Diving deeper, it also reveals the structure of uncountable categorical theories through the notion of a strongly minimal set, introduced by Marsh [Mar66].

Definition 3.5.4. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $D \subseteq M^n$ be an infinite definable set. D is said to be *minimal* in $\mathcal{M}$ if for any definable $Y \subseteq D$, either Y is finite or $D \setminus Y$ is finite. If $\varphi(x, a)$ is the formula that defines D, then $\varphi(x, a)$ is also called *minimal.* Moreover, D and φ are said to be *strongly minimal* if φ is minimal in any elementary extension $\mathcal{N}$ of $\mathcal{M}$.

A theory T is called *strongly minimal* if the formula $x = x$ is strongly minimal, i.e., if for all models $\mathcal{M} \models T$, every definable subset of $\mathcal{M}$ is either finite or cofinite.

One important remark is that if a set D defined in $\mathcal{M}$ by $\psi(x, \bar{a})$ with parameters A_0, the property of being strongly minimal depends only on the type of A_0, rather than the particular model. Using this observation, Baldwin and Lachlan introduced a model-theoretic abstract dependence relation by restricting the algebraic closure relation to the strongly minimal set D, denoted as $\mathrm{acl}_D(A)$, where the subscript D is often ommitted.

Definition 3.5.5. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. An element $b \in \mathcal{M}$ is *algebraic over* A if there exists a formula $\varphi(x, \bar{a})$ with $\bar{a} \in A$ such that $\mathcal{M} \models \varphi(b, \bar{a})$ and $\varphi(\mathcal{M}, \bar{a}) := \{y \in \mathcal{M} : \mathcal{M} \models \varphi(y, \bar{a})\}$ is finite. The *algebraic closure* of A, $\mathrm{acl}(A)$, is defined as the set of all elements in $\mathcal{M}$ that are algebraic over A.

Given a set $D \subseteq \mathcal{M}$ definable over $A_0 \subseteq \mathcal{M}$, define: $\mathrm{acl}_D(A) := \mathrm{acl}(A \cup A_0) \cap D$

A strongly minimal set can be viewed as a pregeometry, as it satisfies all the axioms of a pregeometry, providing a natural associated notion of independence that generalizes linear independence in vector spaces and algebraic independence in algebraically closed fields.

Definition 3.5.6. Let X be a set and let $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$. The pair (X, cl) is called a *pregeometry* (or: *combinatorial pregeometry*, if for all $A, B \subseteq X$ and for all $a, b \in X$:

1. $A \subseteq \mathrm{cl}(A)$ and $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$,
2. if $A \subseteq B$, then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$,

3. if $a \in \mathrm{cl}(A)$, then there exists a finite set $F \subseteq A$ such that $a \in \mathrm{cl}(F)$,
4. if $b \in \mathrm{cl}(A \cup \{a\}) \setminus \mathrm{cl}(A)$, then $a \in \mathrm{cl}(A \cup \{b\})$.

Thus, a notion of independence and basis can be defined analogously. Given an $\mathcal{L}$-theory T and a model $\mathcal{M} \models T$, let $D \subseteq \mathcal{M}$ be a strongly minimal set defined over A_0. A set $A \subseteq D$ is called *independent over* A_0 if $a \notin \mathrm{acl}_D(A \setminus \{a\})$ for every $a \in A$. This notion is localized to depend on the set of parameters A_0 defining the strongly minimal set. Given $C \subseteq D$, A is said to be *independent of* C *over* A_0 if $a \notin \mathrm{acl}_D((C \cup A) \setminus \{a\})$ for every $a \in A$.

A subset $B \subseteq Y$ is called a *acl-basis* for Y if B is independent over A_0 and $\mathrm{acl}_D(B) = \mathrm{acl}_D(Y)$. As for any pregeometry, this yields a well-defined notion of dimension. Thus, for any $Y \subseteq D$, the *dimension* of Y, denoted by $\dim(Y)$, is precisely the cardinality of any basis of Y.

In strongly minimal theories, this notion of dimension aligns with Morley rank. More importantly, it is sufficient to show that every model of a strongly minimal theory is uniquely determined up to isomorphism by its dimension.

Theorem 3.5.7. *Suppose T is a strongly minimal theory and $\mathcal{M}, \mathcal{N} \models T$, then $\mathcal{M} \cong \mathcal{N}$ if and only if $dim(\mathcal{M}) = dim(\mathcal{N})$. Therefore, if T is a strongly minimal theory, then T is κ-categorical for $\kappa \geq \aleph_1$.*

More precisely, given a strongly minimal formula ϕ, two models are isomorphic if and only if $\phi(\mathcal{M})$ and $\phi(\mathcal{N})$ have the same dimension. This isomorphism is obtained by extending the bijection between two acl-bases of $\phi(\mathcal{M})$ and $\phi(\mathcal{N})$ [8]. This strategy was precisely the core of the Baldwin and Lachlan theorem.

Baldwin and Lachlan showed that being ω-stable and having no Vaughtian pairs implied the existence of a strongly minimal formula ϕ. They used this formula, together with a dimension argument, to construct a partial isomorphism from $\phi(\mathcal{M})$ to $\phi(\mathcal{N})$ which extends to an isomorphism from $\mathcal{M}$ to $\mathcal{N}$. This extension was justified by relying on the properties of prime models over sets to extend the domain and on the non-existence of Vaughtian pairs to establish surjectivity.

It is worth noting that the proof by Baldwin and Lachlan also showed that any two models with same dimension are isomorphic. In particular, this implies an effective bound on the number of countable models of uncountably categorical theories:

Theorem 3.5.8. *If T is uncountably categorical, then $I(T, \aleph_0) \leq \aleph_0$.*

This result had been previously established by Morley [Mor67]. However, using their detailed understanding of the spectrum function, Baldwin and Lachlan extended it to the following result, which resolved affirmatively another conjecture posed by Vaught [Vau61]:

Theorem 3.5.9. *If T is $\aleph_1$- categorical but not $\aleph_0$-categorical, $I(T, \aleph_0) = \aleph_0$.*

[8] The crucial detail in constructing such a map is to define the strongly minimal formula ϕ over a set of parameters in the prime model [Iba23].

Across the previous two sections, several implications have been established, but two are especially significant for the current aim. On one hand, Morley demonstrated (Theorem 3.4.6) that $\aleph_1$-categoricity implies ω-stability. On the other hand, Baldwin and Lachlan showed that strong minimality implies $\aleph_1$-categoricity. Therefore, strong minimality implies ω-stability, which is one of the implications in Conant's *Map of the Universe*:

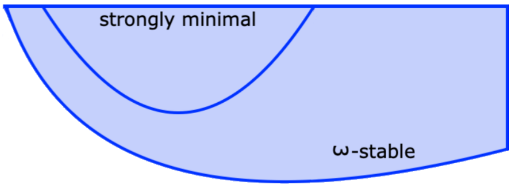

Figure 2: First Building Block of the *Map of the Universe*.

3.6 Geometric Model Theory

Morley's seminal paper concluded with seven provocative questions that had a crucial impact on the subsequent development of model theory. Among these, the second question asked whether there exists an $\aleph_1$-categorical but not $\aleph_0$-categorical theory that is finitely axiomatizable. Fifteen years later, Peretyatkin [Per80] provided an affirmative answer by constructing such a theory. More importantly, Zilber and, independently, Cherlin, Harrington, and Lachlan proved that no theory countable in *all* infinite powers is finitely axiomatizable [CHL85; Zil84]. Although this problem had been phrased as a logical one, its solution revealed deep structural properties of such theories, leading to the emergence of geometric model theory, one of the most fruitful branches of model theory.

Geometric model theory, sometimes known as geometric stability theory, focuses on the study of combinatorial geometries that emerge from sets with well-behaved independence relations, such as strongly minimal sets. For example, Zilber's early result demonstrated that every model of an $\aleph_1$-categorical theory can be decomposed into finite "ladders" of strongly minimal sets [Zil93]. This field also relies on the study of classical algebraic structures, such as groups and fields, that are interpretable within these sets. Notably, the proof by Cherlin, Harrington, and Lachlan drew on the classification of finite simple groups.

Initially motivated by analogies with algebraic geometry, Zilber delved into the study of $\aleph_1$-categorical theories. His work soon expanded into an ambitious program aimed at classifying uncountably categorical theories—a problem that, after Baldwin and Lachlan's theorem, ultimately reduces to the classification of strongly minimal sets. Zilber succeeded in classifying the pregeometries of strongly minimal sets into three distinct classes: trivial or degenerate, non-trivial locally modular, and non-modular [Zil81].

The first class, the trivial or degenerate class, consists of structures where every set is closed under the algebraic closure operator, meaning $\mathrm{acl}(X) = X$ for all subsets X. An example is the dependence relation where an element is dependent only on sets containing it. The second class, non-trivial locally modular, includes structures that are not trivial but in which the lattice of closed subsets of the geometry is a modular lattice, meaning $\dim(A \cup B) = \dim(A) + \dim(B) - \dim(A \cap B)$ holds for all closed sets $A, B \subset X$, provided that $\dim(A \cap B) > 0$. A classical example is a vector space. The third class, non-modular pregeometries, encompasses structures that do not fit into the previous categories. An example are algebraically closed fields, where the algebraic closure of a set A is the smallest subfield containing A such that every non-zero polynomial with coefficients in the field has its roots within the field.

Building upon this classification, at the 1984 International Congress Zilber conjectured the following:

Conjecture 3.6.1 (Zilber's Trichotomy Conjecture)**.** If $\mathcal{M}$ is strongly minimal, then one of the following must hold: the geometry of $\mathcal{M}$ is either trivial, or it is an affine or projective geometry over a division ring, or $\mathcal{M}$ interprets an algebraically closed field.

However, in 1993, Ehud Hrushovski refuted the conjecture in his paper *A new strongly minimal set* [Hru93]. He introduced special forms of Fraïssé limits, now known as the Hrushovski construction, to produce a non-locally modular strongly minimal set that does not interpret an algebraically closed field. Despite the refutation, his contruction remains *artificial*, and Zilber's conjecture has indeed been proved in several restricted scenarios.

An important restricted version of the trichotomy conjecture was established by Hrushovski and Zilber [HZ96]. They strengthened the hypothesis from a strongly minimal set to a so-called Zariski geometry, demonstrating that certain of these geometries are mutually interpretable with an algebraically closed field. These results were utilized by Hrushovski to prove, for the first time, the Mordell–Lang Conjecture in all characteristics [Hru96]. This is a central result in Diophantine geometry which, roughly speaking, describes the properties of the intersection of a subvariety X of a semi-abelian variety A, both defined by polynomial equations over a function field K, with specific subgroups Γ of A. The application of model theory to prove the Mordell–Lang Conjecture is thus historical for bridging seemingly distinct areas of mathematics.

Hrushovski later applied a similar approach to prove the Manin–Mumford Conjecture in algebraic geometry [Hru01]. This conjecture for number fields, is a deep and important finiteness question regarding the intersection of a curve with the torsion subgroup of its Jacobian. Given this remarkable unification, in his *Short Model Theory*, Hodges references Hrushovski's work as part of the motivation for viewing model theory as *algebraic geometry - fields* [Hod97].

Research in the field remains highly active and productive. A significant advancement was made in 2022 when Benjamin Castle resolved the following restricted trichotomy, which had been long overdue: If M is a non-locally modular strongly minimal structure interpretable in an algebraically closed field K of characteristic 0, then M interprets K [Cas22]. In 2024, Castle further extended this work by proving the higher-dimensional case of the o-minimal variant of the conjecture [Cas24], by building on earlier contributions by Peterzil and Starchenko in o-minimal theories [PS98].

4 The Classification Program

4.1 The Stability Hierarchy

Morley's third question among his seven sought to extend his Categoricity Theorem to uncountable languages. In 1971, Saharon Shelah addressed this question with a proof that introduced the notion of stability [She74a]. This concept, along with Shelah's pioneering work, became foundational to what would soon be known as stability theory, which, as Väänänen notes, "is now the accepted state-of-the-art and the focus of research for all those working in model theory" [Vää20].

The intuition behind stability, as for ω-stability, is that it limits the complexity of type spaces by bounding their cardinalities. Since types describe the potential behaviors of elements in models (or in elementary extensions), constraining the number of possible types over parameter sets implies that the theory exhibits a controlled and well-behaved structure. In his first paper in 1969, Shelah formally introduced the concepts of a stable theory and its stronger form, superstability, extending Rowbottom's notion of κ-stability.

Definition 4.1.1. Recall that given an infinite cardinal κ, a theory is κ-stable if $|S_n(A)| \leq \kappa$ for every set $A \subseteq M$ of size at most κ.

- A theory T is *stable* if it is κ-stable for some cardinal $\kappa \geq \aleph_0$. Otherwise, it is called *unstable*.
- A theory T is *superstable* if there exists some cardinal λ such that T is κ-stable for all $\kappa \geq \lambda$.

While studying these properties, Shelah discovered that crucial syntactic properties of ω-stable theories could be generalized by restricting some of Morley's notions from types involving all formulas to types containing only instances of a single formula $\phi(x, y)$. He introduced the notion of complete ϕ-types, which are not complete types in the usual sense but are "complete for ϕ" as follows:

Definition 4.1.2. Let $\phi(x, y)$ be a formula. A *complete ϕ-type* over a set of parameters $A \subseteq M$ is a maximal consistent collection of formulas of the form $\phi(x, b)$ or $\neg\phi(x, b)$, where b ranges over A. The space of all complete ϕ-types over A is denoted by $S_\phi(A)$.

Shelah defined a formula ϕ as *stable* if there exists a cardinal λ such that for all sets B with $|B| \leq \lambda$, the size of $S_\phi(B)$ is at most λ. The name is due to the following key equivalence: a theory T is stable in Rowbottom's original sense if and only if all its formulas are stable in terms of ϕ-types. Therefore, if a theory T is unstable, meaning it is not κ-stable for any κ, this instability can be detected by a single formula ϕ for which there are too many ϕ-types.

By extending Moley's implication that, in ω-stable theories, $|A| \leq \aleph_0$ leads to $|S_n(M, A)| \leq \aleph_0$ for ϕ-types, Shelah was able to generalize Morley's result that ω-stability implies stability in all ordinals. The outcome was a mere implication such as "κ-stable implies κ'-stable", but an overarching result. Shelah demonstrated that there are only a limited number of possibilities for the set of cardinals κ in which a theory can be κ-stable, resulting in a classification of *all* theories into four disjoint classes [She69].

Theorem 4.1.3 (The Stability Hierarchy)**.** *Every countable complete first-order theory T falls into exactly one of the following classes:*

1. *Unstable: There are no cardinals κ such that T is κ-stable.*
2. *Strictly Stable: T is stable in exactly those cardinals κ such that $\kappa^\omega = \kappa$.*
3. *Strictly Superstable: T is stable in exactly those cardinals $\kappa \geq 2^{\aleph_0}$.*
4. *ω-Stable: T is stable in all infinite cardinals κ.*

This result was formulated in terms of the stability spectrum function $g_T(\lambda)$, defined as $g_T^m(\lambda) = \sup\{|S_m(M)| : M \models T, |M| \leq \lambda\}$, where, if m is omitted, $m = 1$. Shelah proved that there are only four possible stability spectrum functions for any theory. To further analyze this, Shelah introduced various notions of topological rank for the space of ϕ-types, based on Morley's work and variations of the Cantor-Bendixson rank. Using these ranks, Shelah translated the four possible stability functions into specific syntactic conditions, thus providing a purely syntactic description of the stability hierarchy.

However, the most notable syntactic characterization remains that of unstable theories. By 1971 [She71b], Shelah demonstrated that instability can be characterized not only by counting types but also by a local combinatorial property involving a single formula and a countable set of elements. This property, known as the order property:.

Definition 4.1.4. A formula $\varphi(x, y)$ has the *order property* (OP) in a model $\mathcal{M}$ if there exist sequences $(a_i)_{i<\omega}$ and $(b_j)_{j<\omega}$ in M such that $\mathcal{M} \models \varphi(a_i, b_j) \iff i < j$. A complete first-order theory T has the *order property* if there is a formula $\varphi(x, y)$ that has the order property in some model of T.

Theorem 4.1.5. *Let T be a complete theory. Then, T is stable if and only if T does not have the order property.*

Morley had used linear orderings to construct Ehrenfeucht–Mostowski models, and Shelah recognized their potential as a powerful tool for characterizing unstable theories, particularly in relation to the number of models, as seen in Ehrenfeucht's work (Theorem 3.2.5). For any unstable theory, Shelah constructed 2^κ linear orderings and demonstrated that the corresponding Ehrenfeucht–Mostowski models remain non-isomorphic [She71b], proving that unstable theories have the maximum number of models. A few years later, he extended this argument to unsuperstable theories by replacing the linear order with trees of width κ and height $\omega + 1$ [She74b].

Theorem 4.1.6. *Let T be a complete theory. If T is either unstable or unsu-*

perstable, then for any uncountable κ, T has 2^κ models of size κ.

Before introducing Shelah's notion of forking, it's worthwhile to visualize the implications discussed in this section. With the introduction of the stability spectrum, Shelah moved beyond analysing individual classes of theories, such as ω-stable or $\aleph_1$-categorical theories, and, for the first time, established an overarching hierarchy that encompasses all theories. In doing so, Shelah had a comprehensive view of the *Map of the Universe*.

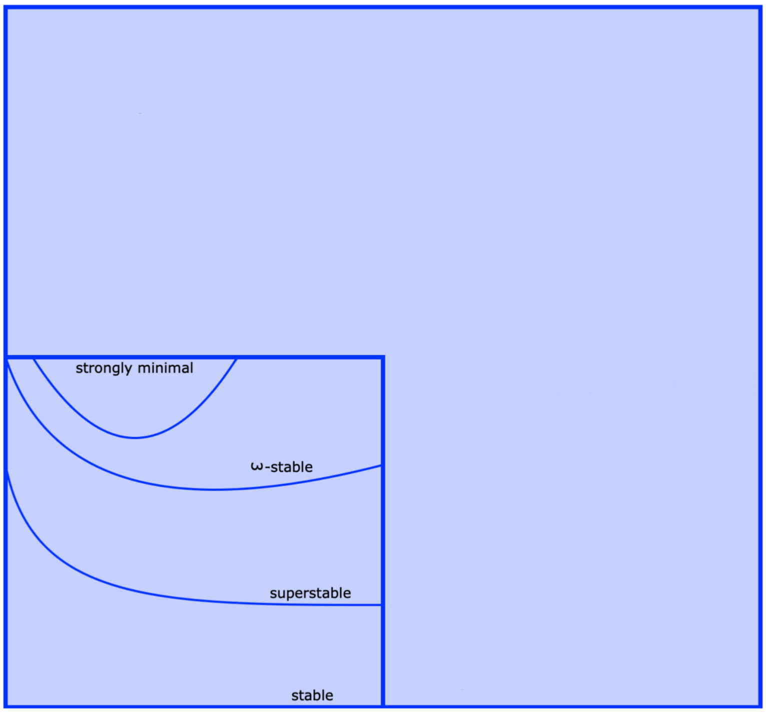

Figure 3: Second Building Block of the *Map of the Universe*.

4.2 Forking Independence

The central concept underlying the analysis of stability is forking, which provides a combinatorial framework of dimension rather than a topological one. As Kim describes:

> "The merit of forking theory is that it supplies such an independence notion to any stable structure. Two stable structures may not have any common phenomena without observing those via model theory. Furthermore, 'forking' is a basic tool of stability theory for proving classification theorems. Forking can be said to be a core notion of stability theory."[Kim98]

Shelah was initially motivated by the desire to understand the possible "bad behavior" of a type when extending it over a larger set of parameters, or conversely, the "good behavior" of an extension of a type that does not introduce new constraints to its set of realizations. Given a type $p \in S_n(A)$ with $A \subseteq B$, Shelah sought to identify a type $q \in S_n(B)$ extending p such that q remains as "free" as possible. He termed such extensions non-forking extensions.[1] To formalize this characterization of types, Shelah defined forking for a single formula, which itself relies on the weaker notion of dividing [She78].

Definition 4.2.1. Let $A \cup \{a\} \cup \{a_j : j < n\}$ be a subset of a model of T.

1. A set of formulas $X = \{\varphi(x, a_i) : i < \omega\}$ is *k-inconsistent* if every k-element subset of X is inconsistent.
2. The formula $\varphi(x, b)$ *k-divides* over A if there exists a set $I = \{b_i : i < \omega\}$ such that $\{\varphi(x, b_i) : i < \omega\}$ is k-inconsistent and all the b_i realize $\mathrm{tp}(b/A)$. The formula φ divides over A if it k-divides over A for some k. A type p divides over A if it implies some $\varphi(x, b)$ which divides over A.
3. A formula $p(x, b)$ *forks* over A if there exists a finite set of formulas $\{\psi_j(x, b_j) : j < n\}$ such that $p(x, b)$ implies $\bigvee_{j<n} \psi_j(x, b_j)$ and each $\psi_j(x, b_j)$ divides over A. A type p forks over A if it implies some $\varphi(x, b)$ which forks over A.

Intuitively, if a formula $\varphi(x, b)$ divides over A, any element c satisfying $\varphi(c, b)$ is more constrained by $A \cup \{b\}$ than by A alone. To illustrate these definitions, consider the theory of dense linear orders (DLO). In this theory, the formula $b_1 < x < b_2$ 2-divides over the empty set, whereas the type $p = \{x > a \mid a \in \mathbb{Q}\}$ does not divide over the empty set for any natural number k. Moreover, while dividing implies forking, there are formulas that fork but do not divide, as seen in the following example. Let T be the theory of the circle S, with a ternary relation $R(x, y, z)$ which describes that y lies on the arc between x and z, ordered clockwise. Since this theory has quantifier elimination, there is a unique 2-type $p(x, y)$ that is consistent with the formula $x \neq y$. As in DLO, the formula $R(a, y, b)$ divides over the empty set for any elements a, b. However, the formula $x = x$ forks over the empty set but does not divide.

Building upon the notion of forking for types, Shelah introduced the non-forking independence relation, denoted $\mathop{\smile\!\!\!\!\!\!|}\,$, which provides a notion of independence between subsets of the monster model.

Definition 4.2.2. Given A and B and C (usually with $C \subseteq B$) subsets of the monster model, A is said to be independent from B over C, denoted by $A \mathop{\smile\!\!\!\!\!\!|}_C B$, if $\mathrm{tp}(a/B \cup C)$ does not fork over C, for any finite tuple a from A. This is sometimes written as $\mathrm{tp}(A/B \cup C)$ does not fork over C .

Since $C \subseteq B$, stating that $\mathrm{tp}(a/B \cup C)$ does not fork over C is equivalent to saying that $\mathrm{tp}(a/B \cup C)$ is a non-forking extension of $\mathrm{tp}(a/C)$ to the larger set $B \cup C$, often abbreviated as BC.

In parallel to the non-forking independence relation, Shelah introduced the

[1] Shelah originally called these *non-splitting extensions* (in Hebrew) and asked Chang for a similar English term, who suggested *forking* [Bal88b].

non-dividing independence relation $A \underset{C}{\perp}^{d} B$, which holds when $\mathrm{tp}(A/B \cup C)$ does not divide over C. In stable theories, these two concepts are equivalent, but this distinction becomes crucial when forking is extended to unstable theories.

Despite being named an "independence relation", forking, with its associated closure operator, given by $a \in \mathrm{cl}(B)$ over C if $\mathrm{tp}(a/B)$ forks over C, does not satisfy all the properties of a pregeometry (see Definition 3.5.6). For stable theories, it fails to satisfy transitivity. Shelah addressed this issue by restricting the definition to *regular types.* Initially, he considered global types, which are types over the model. By working in the monster model $\mathfrak{C} \models T$, a global type can be written as $p \in S(\mathfrak{C})$ [2] Shelah proved that in stable theories, global types have a unique non-forking extension to any superset, a property called *stationarity.* This is a crucial characteristic, but it is not sufficient to achieve transitivity, as the unique non-forking extension must also be independent, which is precisely satisfied by regular types. Shelah's result can then be stated as follows [She78].

Theorem 4.2.3. *Let $p \in S(C)$ be a regular type and X the set of realizations of p. Then for $a \in X$ and $B \subset X$, the relation $\mathrm{tp}(a/B)$ forks over C defines a pregeometry on X.*

Similar to how strongly minimal sets and algebraic closure lead to a notion of independence, this theorem introduces a forking dimension for subsets of the model. Specifically, for any regular type p, it becomes possible to define the dimension of the set of realizations of p within the monster model $\mathfrak{C}$. In fact, Shelah proved a striking unification. In ω-stable theories, $\mathrm{tp}(a/B \cup C)$ is a non-forking extension of $\mathrm{tp}(a/C)$ if and only if $RM(a/B \cup C) = RM(a/C)$, showing that regular types generalize Morley rank 1 types, which are precisely strongly minimal sets.

Shelah proved that in stable theories, non-forking independence, with its underlying pregeometry, exhibits many notable features. Some of these are more intuitive when viewed through the lens of independence, while others are better understood through non-forking extensions. The complexity of forking often stems from the interplay between these two. In terms of extension, Shelah proved the following result [She90].

Theorem 4.2.4. *Let T is a stable theory. Then the following hold:*

1. *Local Character: For any type $p(\bar{x}) \in S(A)$ (where $\bar{x}$ is a finite tuple), there exists a subset $A_0 \subseteq A$ of cardinality at most $|T|$ such that p does not fork over A_0.*
2. *Extension: For any $p(\bar{x}) \in S(A)$ and $B \supseteq A$, there is a type $q(\bar{x}) \in S(B)$ such that $p \subseteq q$ and q does not fork over A (Such a q is called a non-forking extension of p over B).*
3. *Symmetry: For any $A \subseteq B$ and tuple $\bar{a}$, $\mathrm{tp}(\bar{a}/B)$ does not fork over A if*

[2] While $S(\mathfrak{C})$ is more precise, it is standard to use the more sober font, especially as the monster model effectively becomes the new standard model in the context of these results. The same convention will be followed from here onwards.

and only if for every tuple $\bar{b}$ *from* B, $\operatorname{tp}(\bar{b}/A \cup \{\bar{a}\})$ *does not fork over* A.

4. *Transitivity: For any* $A \subseteq B \subseteq C$ *and tuple* $\bar{a}$, $\operatorname{tp}(\bar{a}/C)$ *does not fork over* A *if and only if* $\operatorname{tp}(\bar{a}/C)$ *does not fork over* B *and* $\operatorname{tp}(\bar{a}/B)$ *does not fork over* A.

5. *Stationarity over models: If* $\mathcal{M}$ *is a model and* $p(\bar{x}) \in S(M)$, *then for any* $B \supseteq M$, p *has a unique non-forking extension over* B.

By counting types, these properties also establish the converse, showing that any theory satisfying them is indeed stable. The fifth property is especially significant, as it guarantees that any global type is stationary. Interestingly, the axiomatic characterization of non-forking independence in stable theories may have been first recognized not by Shelah, but by Harnik and Harrington in 1984 [HH84]. This result is central and will play a key role in extending the concept of forking beyond the study of stability alone.

However, the dissemination of forking and stability was not solely Shelah's effort. Shelah's presentation of the basics of stability theory was notably complex, written in a unique expository style that, combined with the inherent intricacy of the subject, made the material challenging to digest even for experienced mathematicians. In the 1970s, Daniel Lascar and Bruno Poizat introduced a clearer, more accessible approach to forking theory. Because they were based in Paris, this approach became known as the French school.

In their 1979 joint publication, *Introduction to Forking*, Lascar and Poizat reformulated forking theory by replacing Shelah's original combinatorial definition of forking with one more closely tied to the notion of definability. In 1971 [She71b], Shelah had established the definability of types as a characterizing property of stability, so Lascar and Poizat used this result as the foundation for deriving other properties.

Definition 4.2.5. Let $\varphi(x;y)$ be a partitioned formula. A type $p(x) \in S_\varphi(B)$ is said to be *A-definable* if there exists a formula $\psi(y) \in \mathcal{L}_A$ such that, for all $b \in B$, $\varphi(x;b) \in p \iff \models \psi(b)$.

Lascar and Poizat based their characterisation on the notions of *heir* and *coheir*. This terminology stemmed from the concept of viewing an extension as a *fils* (son), with a non-forking extension being a *fils aîné* (eldest son), a term which Harnik and Harrington later translated to *heir* [HH84]. These notions were first introduced by Lascar [Las73; Las75], and subsequently characterized in greater generality by Poizat [Poi77].

Definition 4.2.6. Let p be a type over a model $\mathcal{M} \models T$, and let $q \in S(B)$ be an extension of p to some $B \supset M$.

1. The type q is called a *heir* of p if, for every $\mathcal{L}_M$-formula $\varphi(x,y)$ such that $\varphi(x,b) \in q$ for some $b \in B$, there exists $m \in M$ with $\varphi(x,m) \in p$.

2. The type q is called a *coheir* of p if q is finitely satisfiable in $\mathcal{M}$.

Heirs, coheirs, and definable type extensions all extend types to larger sets while maintaining the essential properties of the original type without adding significant new information. Lascar and Poizat showed that, in stable theories, these extensions are equivalent and correspond to non-forking extensions [LP79].

Theorem 4.2.7. *Let T be a stable theory, p a type over a model $\mathcal{M}$, and A an extension of M. Then p has a unique extension $q \in S(A)$ that satisfies the following equivalent properties: q does not fork over M, q is definable over M, q is an heir of p, and q is a coheir of p.*

The approach of the French school significantly enhanced the accessibility of forking, fostering broader engagement with stability theory. A notable example is Pillay's *An Introduction to Stability Theory* [Pil83], published in 1983, which was framed in these terms. This broader understanding, in turn, propelled advancements in specific areas of geometric model theory, such as the model theory of modules [Zie84] and the model theory of groups of finite Morley rank [Che79].

4.3 Uncountable Spectra

Shelah's work on the stability hierarchy and his use of forking to assign dimensions to regular types were the stepping stones of his greater aim: the Classification Program. This ambitious program was originally motivated by what is perhaps the most influential of Morley's questions.[3] In the 1960s, Robert Vaught highlighted the importance of the countable spectrum function $I(T, \aleph_0)$, which counts the number of non-isomorphic countable models of a theory T. Morley expanded this by looking at the uncountable spectrum function $I(T, \kappa)$, which counts the non-isomorphic models of T for a given uncountable cardinality κ. Morley conjectured that it is non-decreasing.

Conjecture 4.3.1 (Morley's Conjecture)**.** Let T be a complete first-order theory, and let κ and κ^+ be uncountable cardinals with $\kappa^+ \geq \kappa$. Then $I(T, \kappa^+) \geq I(T, \kappa)$.

Shelah's first step toward this problem was proving that for any unstable theory T, the number of models $I(T, \kappa) = 2^\kappa$ for every uncountable κ (Theorem 4.1.6). For Shelah, this distinction represented a crucial division between *good* theories, which have relatively few models of the same cardinality, and *bad* theories, which have many. This division led him to reformulate Morley's question into a broader classification program centered on the number of (non-isomorphic) models a theory can have. In 1978, Shelah published his landmark book *Classification Theory and the Number of Nonisomorphic Models* [She78], which was later expanded in a second edition in 1990 [She90]. Most results related to the classification program are found in these works. Shelah outlined the program's goals as follows:

> "The basic thesis of the Classification Program is that reasonable families of classes of mathematical structures should have natural dividing lines. Here a dividing line means a partition into low, an-

[3] The conjecture is listed as Problem 19 in Friedman's 1975 work [Fri75], where it is credited to Morley; however, it appears that Morley never formally published it.

alyzable, *tame* classes on the one hand, and high, complicated, *wild* classes on the other. These partitions will generate a tameness hierarchy. For each such partition, if the class is on the tame side one should have useful structural analyses applying to all structures in the class, while if the class is on the wild side one should have strong evidence of chaotic behavior (set theoretic complexity). These results should be complementary, proving that the dividing lines are not merely sufficient conditions for being low complexity, or sufficient conditions for being high complexity." [She12]

Shelah's classification program had two intertwined goals, first, to classify first-order theories, and second, to classify the mathematical structures that these theories describe.

The first goal is to classify theories by distinguishing between *wild* theories, also referred to as "non-structures", and *tame* theories, sometimes called "structures". This requires identifying dividing lines—specific properties of a theory that, when present or absent, lead to significant structural outcomes. If a theory has certain undesirable features, it is categorized as non-structural, meaning its models are chaotic, numerous, and hard to differentiate from one another. On the other hand, when these problematic features are not present, the theory is considered to have a *structure theory*, enabling a more systematic and constructive analysis. In such cases, theories have relatively few non-isomorphic models.[4]

The second goal is to classify the models of *tame* theories through a system of invariants. These invariants are based on the forking dimensions of specific systems of regular types. Typically, they are arranged in a tree-like structure, representing the interactions between various parts of the model, with each part having its own dimension. Strongly minimal theories are the simplest example, where a single dimension provides complete control over the models.

In 1982, Shelah published a paper with the title *Why Am I So Happy?* [She82] celebrating the success of the classification program with the formulation of the Main Gap Theorem, which established his long-sought division between between *tame* and *wild* theories. As Hodges noted

> "He had just brought to a successful conclusion a line of research which had cost him fourteen years of intensive work and nearly a hundred published books and papers. During this work, he established a new range of questions about mathematics with implications extending far beyond mathematical logic." [Hod87]

The Main Gap Theorem essentially states that a theory either has the maximum possible number of models in every uncountable cardinality or has a structure theory, revealing a surprising regularity in the landscape of countable theo-

[4] Shelah consistently highlighted that having many non-isomorphic models indicates non-structure. However, scholars like Button and Walsh argue that he provides limited reasoning for linking structure theory with classifiability in a straightforward sense. They suggest stronger criteria, like using topological measures of complexity, to assess classifiability [BW18].

ries. Moreover, the two scenarios described by the theorem can be distinguished purely through model-theoretic properties of T, without needing to count the models themselves. Following the exposition by Button and Walsh [BW18], the theorem, as stated in 1985 [She85], is as follows:

Theorem 4.3.2 (Main Gap Theorem)**.** *Let T be a complete theory in a countable language. Then exactly one of the following holds:*

1. *$I(T, \kappa) = 2^\kappa$ for all uncountable cardinals κ.*
2. *$I(T, \aleph_\gamma) \leq \beth_\omega(\max(|\gamma|, \omega))$, and T has a structure theory with countable depth.*[5]

The name "Main Gap" refers to the gap between $\beth_\omega(\max(|\gamma|, \omega))$ and $2^{\aleph_\gamma}$. The theorem asserts that no theory T has a number of models of size κ between these two quantities. Depending on γ, there may be no gap, but generally, $\beth_\omega(\max(|\gamma|, \omega))$ grows much more slowly than $2^{\aleph_\gamma}$.

The "structure theory with countable depth" characterising the *tame* case, where $I(T, \aleph_\gamma) \leq \beth_\omega(\max(|\gamma|, \omega))$, is described using cardinal invariants Inv_α. These invariants reflect the fundamental building blocks of models and the methods for constructing models from them. Shelah provided a recursive definition:

Definition 4.3.3. For an infinite cardinal κ and an ordinal α:

1. $\mathrm{Inv}_0(\kappa) = \{\lambda \mid \lambda \leq \kappa\}$.
2. $\mathrm{Inv}_{\alpha+1}(\kappa)$ consists of sequences of length less than or equal to the continuum, where each element is a function $f : \mathrm{Inv}_\alpha(\kappa) \to \{\lambda \mid \lambda \leq \kappa\}$.
3. For limit ordinals α, $\mathrm{Inv}_\alpha(\kappa) = \bigcup_{\beta<\alpha} \mathrm{Inv}_\beta(\kappa)$.

Finally, Inv_α is the union of $\mathrm{Inv}_\alpha(\lambda)$ as λ ranges over all infinite cardinals.

Intuitively, α represents the depth of the construction process for these invariants, while κ corresponds to the cardinality of the models. Using these invariants, Shelah defined what it means for a theory to have a structure theory:

Definition 4.3.4. A theory T has a *structure theory of depth α* if there exists a function ι from the set of models of T to Inv_α such that:

1. For any model M of size κ, $\iota(M) \in \mathrm{Inv}_\alpha(\kappa)$.
2. For any models M, N of T, $M \cong N$ if and only if $\iota(M) = \iota(N)$.

Finally, T has a *structure theory* if there is an α such that T has a structure theory of depth α. In this case, the theory is called *shallow*; otherwise, if there is no such α, the theory is called *deep*.

The idea behind Shelah's structure theory is that the regular types realized in a model provide the skeleton of that model. The dimensions of these regular types give rise to cardinal invariants that classify models up to isomorphism. Thus, the Main Gap Theorem essentially shows that regular types are sufficient to control the structure of a model.

[5] In this case, $\kappa = \aleph_\gamma$, where $\aleph_\gamma$ represents the γ-th infinite cardinal, since the bound depends on the γ defining κ.

The notion of *depth* arises as a dividing line because its absence (shallowness) leads to significant structural consequences and a well-behaved structure theory. Meanwhile, its presence leads to the maximum number of models. While Shelah's program began by observing that unstable theories have the maximum number of models, stability itself is not *strictly* a dividing line[6] as some stable theories also have the maximum number of models. The dividing line within the stability hierarchy is superstability.[7]

In the classical versions of the Main Gap Theorem, two other dividing lines come into play, the dimensional order property and the omitting types order property.[8] When combined with shallowness and superstability, these lead to an even stronger structure theory.

Theorem 4.3.5 (Main Gap Theorem)**.** *Let T be a countable, complete first-order theory. If T is superstable and lacks the omitting types order property or the dimensional order property, and is shallow, then each model of cardinality λ can be decomposed into countable models indexed by a tree of countable height and width λ. Thus, for any ordinal $\alpha > 0$, $I(T, \aleph_\gamma) < \beth_\delta(|\alpha|)$ (for a countable ordinal δ depending on T); otherwise, $I(T, \aleph_\gamma) = 2^{\kappa_\alpha}$.*

Bradd Hart, Ehud Hrushovski, and Michael C. Laskowski summarised this result:

> "Shelah identified several dividing lines among complete theories, which, although defined without reference to uncountable objects, play a crucial role in distinguishing between different classes of uncountable models. On one side of these dividing lines, for theories in the non-structure category, Shelah demonstrated that their models embed a certain amount of set theory. As a result, the spectrum of such theories is maximal, meaning $I(T, \kappa) = 2^\kappa$ for all uncountable κ. This is seen as a negative feature, as it prevents the existence of a reasonable structure theorem for the class of models of the theory.
>
> Conversely, for theories on the structure side of these dividing lines, their models can be associated with a system of combinatorial geometries. The isomorphism type of a model of such a theory is determined by local information, specifically the behavior of countable substructures, along with numerical invariants such as dimensions for the corresponding geometries. This implies that the uncountable spectrum of such a theory cannot be maximal. Therefore, the uncountable spectrum of a complete theory in a countable language is not maximal if and only if every model of the theory can be described,

[6] Different authors use this term in varying ways. For Baldwin [Bal88b], a property is a "dividing line" if both it and its complement are virtuous, i.e., they impact the understanding of the models of T. This *impact* is, however, somewhat relative, and indeed many authors now employ this notion more broadly.

[7] One might consider ω-stability as a potential dividing line. While ω-stability enables strong structural theorems through the construction of prime models over sets, it is not an effective dividing line because many non-ω-stable theories have relatively few models.

[8] For a detailed definition of these properties, see [Bal88b].

up to isomorphism, by a well-founded, independent tree of countable substructures." [HHL00]

Returning to Morley's conjecture, Shelah had bounded the uncountable spectra, but the conjecture itself remained partly unresolved. In 2000, Hart, Hrushovski, and Laskowski provided a complete solution. They introduced three new dividing lines, the first two measuring how far a theory is from being ω-stable (or totally transcendental), while the third refined the distinction between different types of spectra. When combined with Shelah's divisions, these lines fully characterized the uncountable spectra of all complete first-order theories, effectively determining $I(T, \kappa)$ for any uncountable cardinal $\kappa = \aleph_\gamma$.

Their classification showed that spectrum functions are either maximal or fall into one of twelve well-understood forms. At one extreme, the constant function 1 represents uncountably categorical theories; at the other, $2^{\aleph_\gamma}$ represents maximal complexity. Between these extremes lie intermediate forms, such as $\min(2^{\aleph_\gamma}, \beth_{\alpha+1}(|\gamma + \omega|))$, where α represents the depth of the decomposing tree and $|\gamma + \omega|$ the number of cardinals below $\aleph_\gamma$, indicating the possible dimensions of a component. For example, a theory with this spectrum function might include equivalence relations E_β for each β, where every E_θ class is a union of infinitely many E_β classes, and each E_0 class is infinite.

By proving that each of these spectrum functions is non-decreasing, Hart, Hrushovski, and Laskowski fully answered Morley's conjecture regarding uncountable cardinalities. However, the problem for countable cardinalities remains unsolved. In particular, Vaught's conjecture asking whether a countable first-order theory can have exactly $\aleph_1$ countable models (see Conjecture 3.3.5), is still the only unanswered problem regarding the possible values of $I(T, \kappa)$ for any infinite cardinal κ.

5 Classification Beyond Stability

When Shelah began developing classification theory, the test question was counting the possible number of models. However, the most profound progress emerged from identifying dividing lines that provided deeper insights into *well-behaved* theories, with stable theories being central. There was a widespread belief that the techniques and machinery applied only within this narrow scope, and for nearly two decades, pure model theory largely remained confined to this stable framework. Yet, many other notions of *well-behaved* theories, some of them arising from Shelah's earliest works, have recently come to light.

In 1971 [She71b], while establishing that every unstable theory exhibits the order property (OP), Shelah introduced two weaker properties: the strict order property (sOP)[1] and the independence property (IP) . These properties, like OP, are syntactic conditions for a single formula that lead to the maximal number of models in each cardinality.

Definition 5.0.1. 1. A formula $\varphi(x, y)$ has the strict order property (sOP) if there are $(a_i)_{i<\omega}$ such that $\vdash \exists x\,(\varphi(x, a_j) \wedge \neg\varphi(x, a_i)) \iff i < j$. A theory is NsOP if no formula has the strict order property.

2. A formula $\varphi(x, y)$ has the independence property (IP) in a model M if for every m, there are $\{a_i : i < m\}$ and $\{b_X : X \subseteq m\}$ such that $\varphi(a_i, b_X)$ if and only if $i \in X$. A theory is NIP (dependent) if no formula has the independence property.

Shelah demonstrated that every unstable theory must exhibit at least one of these two syntactic conditions [She71b]. Therefore, if a theory possesses neither, it is stable. Consequently, NIP and NsOP theories emerge as two natural, orthogonal generalizations of stability within the realm of unstable structures.

Theorem 5.0.2. *A theory T is stable if and only if it is NsOP and NIP.*

Interest in these classes of theories grew substantially throughout the 1980s and 1990s. Major progress came from studying simple theories, which belong to the NsOP class, and o-minimal theories, which fall under the NIP class. These classes have proven valuable not only for their robust structural properties but also for their remarkable applications across various fields of mathematics. In 1998, two years after their work on simple theories, Kim and Pillay reflected on this growing momentum:

> "The question arises, in this post-classification theory period, what, if anything, is the aim or final purpose of 'pure' model theory? [...] Many of the notions that emerged from classification theory have found meaning and implications in various mathematical contexts,

[1] The strict order property is denoted as sOP to distinguish it from the strong order property, denoted SOP. This notation follows Conant [Con13].

> such as number theory and differential equations. The generalization of these notions outside the context of stable theories is of great importance.
>
> Let us mention some general themes that could inform future directions in model theory. One is *dimension theory*, and another is the search for general notions of *independence*. [...] One more theme is the *classification of first-order theories*. Note that arithmetic (the theory of the ring of integers, or the field of rationals) is, in all possible senses, 'wild', although the search for an understanding of it drives much of mathematics. It is interesting that the auxiliary structures which mathematicians have used, or even invented, to help understand arithmetic—such as the local fields $\mathbb{R}$, $\mathbb{Q}_p$, and the finite fields $\mathbb{F}_q$—are key unstable structures that support dimension theories, which model theorists are currently generalizing." [KP98]

The themes Kim and Pillay highlighted have become central to the past 30 years of model theory. By extending many of the tools developed for stable theories, numerous dividing lines have been established within the NIP and NsOP classes, consolidating a field now known as "Neostability Theory".[2]

Thus, Neostability Theory arose from the following *Map of the Universe*, focusing, though not restricted, on the top-left and bottom-right quadrants.

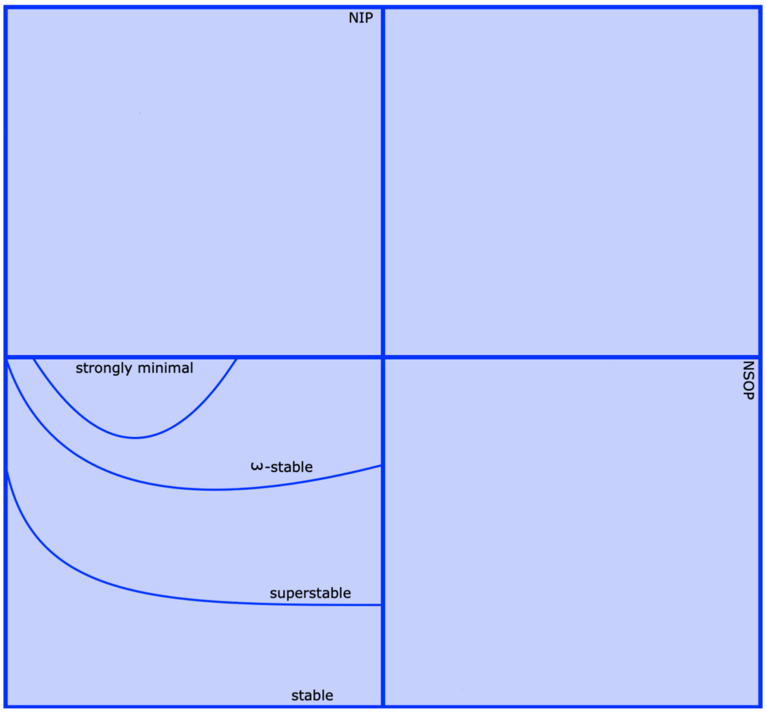

Figure 4: Third Building Block of the *Map of the Universe*.

[2] Several BIRS meetings under the title "Neostability Theory" took place in 2009, 2012, 2015, 2018, and 2023.

It is worth noting that arithmetic exhibits both the strict order property and the independence property, placing it in the top-right *totally unstable* quadrant. As Baldwin suggests, this may help explain why much of modern algebraic number theory doesn't take place directly in first-order Peano arithmetic. Instead, more tame auxiliary structures, such as algebraically closed or valued fields, serve as the framework for proving number-theoretic results [Bal18].

5.1 Simple Theories

Simple theories stem from Shelah's 1978 work, *Classification Theory*, where, while investigating stable theories, he identified local character (see Theorem 4.2.4) as a key property of non-forking independence. In 1980, Shelah formalized the definition of simple theories in *Simple Unstable Theories* [She80], defining them as those where local character holds:

Definition 5.1.1. A theory T is simple if, for all sets B and complete types p in $S_n(B)$, there exists $A \subseteq B$ such that $|A| \leq |T|$ and p does not fork over A.[3]

From this definition, the simplicity of the random graph, which is the prototypical example of a simple theory, can be derived. The theory of the random graph is axiomatized as follows: for all distinct $x_1, \ldots, x_n, y_1, \ldots, y_n$, there exists z such that $R(z, x_i)$ for $i = 1, \ldots, n$ and $\neg R(z, y_i)$ for all $i = 1, \ldots, n$. In this theory, forking is characterized by the condition $A \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{C} B \iff A \cap B \subseteq C$. Using this characterization, it can be shown that any complete 1-type does not fork over a finite set, thereby confirming that the random graph is simple.

In the same paper, Shelah provided several equivalent definitions of simplicity, the most notable being based on a combinatorial property of formulas known as the tree property.

Definition 5.1.2. The formula $\varphi(x; y)$ has the *tree property (TP)* if there is $k < \omega$ and a tree of tuples $(a_\eta)_{\eta \in \omega^{<\omega}}$ in M such that

1. for all $\eta \in \omega^\omega$, $\{\varphi(x; a_{\eta|\alpha}) : \alpha < \omega\}$ is consistent,
2. for all $\eta \in \omega^{<\omega}$, $\{\varphi(x; a_{\eta\frown\langle i \rangle}) : i < \omega\}$ is k-inconsistent.

A complete theory T is *simple (or NTP)* if no formula has the tree property.

Shelah proved that stable theories do not have the tree property.[4] Therefore, all stable theories are simple. Furthermore, since the strict order property implies the tree property, simple unstable theories do not have the strict order property and thus belong to the NsOP class.

Shelah observed that the breakdown of local character occurs in one of two ways, depending on whether the tree property manifests as TP_1 (the tree property of the first kind) or TP_2 (the tree property of the second kind). These represent opposite extremes, based on the behavior of tuples a_η and a_ν for in-

[3] As is often the case with such properties, it suffices to consider complete 1-types.
[4] Shelah showed that stability is equivalent to a stronger concept called NBTP, which means no formula exhibits the *binary order property* [She71b].

comparable pairs η and ν that are not siblings. TP_1 requires that all such pairs are inconsistent, while TP_2 requires that all such pairs are consistent. This distinction gives rise to two classes of theories: NTP_1 (no formula has TP_1) and NTP_2 (no formula has TP_2).

Definition 5.1.3. 1. The formula $\varphi(x; y)$ has the *tree property of the first kind (TP_1)* if there is a tree of tuples $(a_\eta)_{\eta\in\omega^{<\omega}}$ in M such that

- for all $\eta \in \omega^\omega$, $\{\varphi(x; a_{\eta|\alpha}) : \alpha < \omega\}$ is consistent,
- for all $\eta \perp \nu$ in $\omega^{<\omega}$, $\{\varphi(x; a_\eta), \varphi(x; a_\nu)\}$ is inconsistent.

2. The formula $\varphi(x; y)$ has the *tree property of the second kind (TP_2)* if there is a $k < \omega$ and an array $(a_{\alpha,i})_{\alpha<\omega,i<\omega}$ in M such that

- for all functions $f : \omega \to \omega$, $\{\varphi(x; a_{\alpha,f(\alpha)}) : \alpha < \omega\}$ is consistent,
- for all α, $\{\varphi(x; a_{\alpha,i}) : i < \omega\}$ is k-inconsistent.

T has one of the above properties if some formula does.

Shelah established that a theory has the tree property if and only if it exhibits either TP_1 or TP_2. This can be restated as:

Theorem 5.1.4. *A theory T is simple (NTP) if and only if it has NTP_1 and NTP_2.*

Building on the notion of local character, Shelah sought to extend key properties from stable to simple theories. Although he succeeded in proving the extension axiom, symmetry and transitivity eluded him (see Theorem 4.2.4). Additionally, Lascar and Poizat's notions of heir and coheir were unsuccessful, as types in simple theories are generally not definable. As a result, interest in simple theories waned in the 1990s. However, simple-like behavior began to emerge in certain unstable structures, driven by algebraic questions in geometric model theory [CH99]. Several authors demonstrated that pseudo-finite fields and ACFA are simple, which later played an important role in Hrushovski's proof of the Manin-Mumford conjecture [CDM92; Hru93; Hru01].

Inspired by these discoveries, Byunghan Kim revisited simple theories in 1996. Together with Pillay, they quickly developed a comprehensive understanding of simple theories, closely tied to the concept of forking. To properly explain their results, it is necessary to introduce the notion of a Morley sequence–an indiscernible sequence tied to a specific type.

Definition 5.1.5. Let $(I, <)$ be an ordered set, $(a_i : i \in I)$ be a sequence of distinct elements of M, and A a set of parameters. A sequence $(a_i : i \in I)$ is indiscernibles over A, or A-indiscernibles if if for every $n \in \omega$, and for every set of indices $i_0 < \cdots < i_n$ and $j_0 < \cdots < j_n$, the following holds: $\mathrm{tp}(a_{i_0}, a_{i_1}, \ldots, a_{i_n}/A) = \mathrm{tp}(a_{j_0}, a_{j_1}, \ldots, a_{j_n}/A)$.[5]

Definition 5.1.6. Let p be a complete type over A. A *Morley sequence* in p of length α, is an A-indiscernible sequence $(b_i : i < \alpha)$ of realizations of p such that $\mathrm{tp}(b_i/A \cup \{b_j : j < i\})$ does not fork over A for each $i < \alpha$.

[5] The original definition (see Definition 3.2.2) can be recovered as $\emptyset$-indiscernible.

The central result of Kim's 1996 paper is an equivalence between dividing and Morley sequences. However, it built upon Shelah's well-known characterization of dividing in terms of indiscernibles [She78], which states that a formula $\varphi(x, b)$ divides over A if there exists an A-indiscernible sequence $(b_i : i < \omega)$ starting with b, such that $(\varphi(x, b_i) : i < \omega)$ is inconsistent. Kim's contribution was to transform the existential quantifier in this definition "there exists an A-indiscernible sequence such that..." into a universal one "for every Morley sequence over A..." [Kim98]. This result is now known as Kim's Lemma:

Theorem 5.1.7 (Kim's Lemma). *Let T be simple, let $\varphi(x, b)$ be a formula, and A a set of parameters. Then the following are equivalent:*

1. *The formula $\varphi(x, b)$ divides over A,*
2. *There is Morley sequence $(b_i : i < \omega)$ in $\mathrm{tp}(b/A)$, such that $\{\varphi(x, b_i) : i < \omega\}$ is inconsistent,*
3. *For any Morley sequence $(b_i : i < \omega)$ in $\mathrm{tp}(b/A)$, $\{\varphi(x, b_i) : i < \omega\}$ is inconsistent,*
4. *The formula $\varphi(x, b)$ forks over A.*

From this lemma, Kim concluded that in simple theories, as in stable theories, forking and dividing are equivalent. Additionally, he derived the long-sought result that forking in simple theories satisfies both transitivity and symmetry. To illustrate the latter, consider the case where $\mathrm{tp}(b/A \cup \{a\})$ does not fork over A. Using local character and standard techniques to obtain indiscernibles, one can construct a sequence $(b_i : i < \omega)$ that is indiscernible over $A \cup \{a\}$, with $b_0 = b$, and such that $\mathrm{tp}(b_i/A \cup \{a\})$ does not fork over A for all i. This sequence is a Morley sequence in $\mathrm{tp}(b/A)$. Additionally, for any formula $\varphi(x, y)$ over A that holds for (a, b), the element a satisfies $\varphi(x, b_i)$ for all i. By the equivalence of the third and fourth properties in Kim's Lemma, $\mathrm{tp}(a/A \cup \{b\})$ does not fork over A. Later, Kim demonstrated the equivalence between simplicity and the symmetry of forking, establishing that Kim's Lemma characterizes simple theories [Kim01].

With this result, the only significant aspect of forking in stable theories that had not been extended to simple theories was stationarity over models. However, stationarity could not hold in simple theories, as it would lead to their equivalence with stable theories (see Theorem 4.2.4). Instead, Kim and Pillay sought an appropriate analogue of stationarity for simple theories. On his work on pseudo-finite fields, Hrushovski had identified a crucial property of independence [HP94], and building on this foundation, in 1997, Kim and Pillay formulated the Independence Theorem for simple theories [KP97].

Theorem 5.1.8 (The Independence Theorem). *Let T be a simple, then T satisfies the independence theorem over a models, which is as follows. Let $\mathcal{M}$ be a model and p a complete type over $\mathcal{M}$. Let $A \supset M$ and $B \supset M$ be such that A is independent from B over M (that is, $\mathrm{tp}(A/B)$ does not fork over M). Let $p_1 \in S(A)$ and $p_2 \in S(B)$ be non-forking extensions of $p(x)$. Then p_1 and p_2 have a common non-forking extension $q(x)$ over $A \cup B$.*

For stable theories, this property is a direct consequence of stationarity over models, as the unique non-forking extension of p over $A \cup B$ must extend both p_1 and p_2. In fact, stationarity over models, along with local character, extension, symmetry, and transitivity, is sufficient to characterize stable theories. Kim and Pillay established a parallel result for simple theories [KP97], stating "the Independence Theorem is to simple theories what the stationarity of types over models is to stable theories" [KP97].

Theorem 5.1.9 (Kim-Pillay)**.** *Suppose that T supports some notion of independence $\mathop{\smile\!\!\!\!\!\!|}'$ satisfying properties of local character, extension, symmetry, transitivity and the independence theorem over a model. Then T must be simple, and this notion of independence must coincide with non-forking.*

Recognising the close relationship between simple and stable theories, Kim and Pillay wrote in the introduction to their 1997 paper:

> "It should be clear from the material in this paper that the study of simple theories is essentially just the study of forking in full generality, and that stable theories should be viewed just as a particularly nice class of simple theories." [KP97]

As the characterisation of simple theories, the Kim-Pillay theorem is a pivotal result in the characterization of simple theories, offering significant practical applications. To determine whether a theory is simple, instead of verifying that no formula $\varphi(x, y)$ has the tree property, which requires a deep understanding of definable sets, one can instead identify a ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ and show that it satisfies the axioms of the Kim-Pillay theorem. While establishing the independence theorem can be challenging, it is generally more feasible than proving the absence of the tree property. Furthermore, applying the Kim-Pillay theorem provides valuable insights into the independence relation, ensuring it coincides with forking and can be expressed through formulas.

In addition to their work on simple theories, Kim and Pillay also defined supersimple theories, to name the theories which are both stable and superstable. Their are defined analogously to simple theories, by strengthening local character.

Definition 5.1.10. A theory T is supersimple if for all sets B and complete types p in $S_n(B)$, there exists a finite $A \subseteq B$ such that p does not fork over A. A stable, supersimple theory is called *superstable*.

Kim proved a result concerning the countable spectrum for supersimple theories [Kim99], generalizing Lachlan's result for superstable theories, which in turn was a generalization of the corresponding result for $\aleph_1$-categorical theories by Baldwin and Lachlan (see Theorem 3.5.9).

Theorem 5.1.11. *If T is a supersimple, not ω-categorical theory, $I(T, \aleph_0) = \aleph_0$.*

Theorem 5.1.12. *If T is a superstable, not ω-categorical theory, $I(T, \aleph_0) = \aleph_0$.*

A deeper exploration of supersimplicity involves various ranks that offer stronger, equivalent characterizations of local character. One such rank is the *Lascar rank*.

The Lascar rank of a type $\mathrm{tp}(a/B)$ measures the length of the longest forking chain starting with $\mathrm{tp}(a/B)$. In superstable theories, the Lascar rank satisfies $U(a/B) < \infty$ for every n. Indeed this is also a sufficient condition. For example, in the theory of the random graph, the formula $x = x$ has a Lascar rank of 1, and hence the random graph is supersimple.

By collecting the results from this section, a more refined classification appears on the *Map of the Universe*. Most notably, simplicity emerges as a generalization of stability. Due to the orthogonality between NsOP and NIP, stable theories are often viewed as the intersection of NIP with simplicity or NTP_1. Although certain equivalences, such as NIP implying NTP_2 or NTP_1 implying NsOP, were not explicitly stated, they follow directly from the definitions and are found in several works such as Shelah's [She78; She80]. It is also interesting to note that a suitable definition of ω-*simplicity*, or *total simplicity*, has not yet been proposed [Wag02]. The next properties to be investigated will offer a finer subdivision within the categories of simple and NsOP theories.

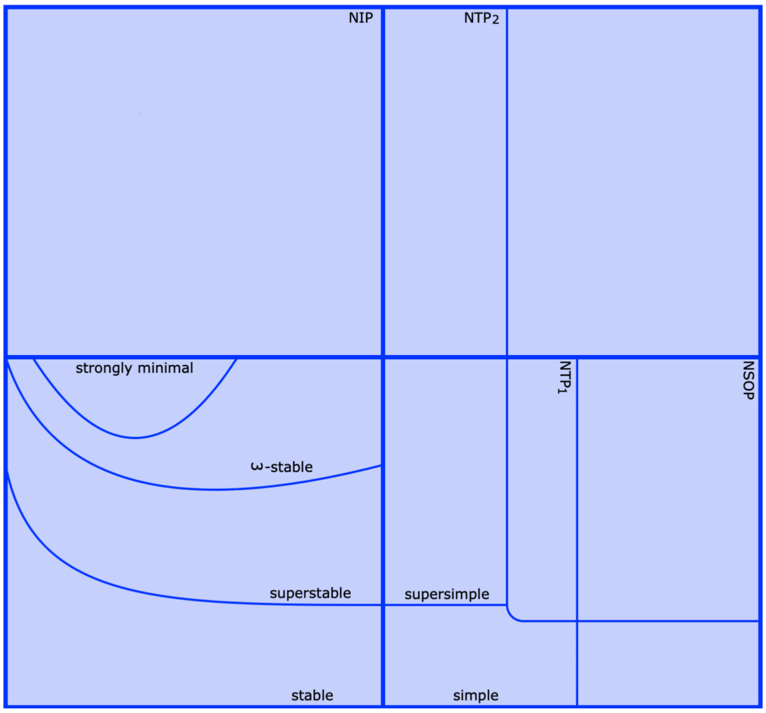

Figure 5: Fourth Building Block of the *Map of the Universe*.

5.2 NsOP Theories

In 1996's *Toward Classifying Unstable Theories* Shelah aimed to explore new instances of instability while focusing on theories that do not have the strict order property (NsOP). When considering the strict order property in terms of saturation, several weaker properties naturally emerge. A theory T is said to have the strict order property (SOP) if there exists a formula $\varphi(x, y)$ such that, in every $\aleph_0$-saturated model of T, $\varphi(x, y)$ defines a partial order on M^n containing an infinite chain. This concept led Shelah to introduce a sequence of weaker properties, known as the n-strong order property (SOP$_n$), which he defined as follows [She96]:

Definition 5.2.1. Fix $n \geq 3$ and a formula $\varphi(x, y)$ with $|x| = |y|$. Then:

- Then $\varphi(x, y)$ has the n-strong order property (SOP$_n$) if the set $\{\varphi(x_1, x_2), \ldots, \varphi(x_{n-1}, x_n), \varphi(x_n, x_1)\}$ is inconsistent, and there is a sequence $(a_i)_{i<\omega}$ such that $\varphi(a_i, a_j)$ holds for all $i < j$. A theory is NSOP$_n$ if no formula has SOP$_n$.
- Then $\varphi(x, y)$ has the strong order property (SOP) if it has the n-strong order property for all n. A theory is NSOP if no formula has SOP.
- A theory had the no finitary strong order property (NFSOP) if no formula has the strong order property with $|x|$ finite.

Intuitively, a theory T has the n-strong order property, SOP$_n$, for any ($n \geq 3$) if there is a formula $\varphi(x, y)$ that defines a directed graph with an infinite chain but no cycle of length $\leq n$. By extending this to all n, a theory T has the strong order property, SOP, if a type $p(x, y)$ defines a directed graph that has an infinite chain but no cycle. Shelah established a natural hierarchy among these properties:

Theorem 5.2.2. *The following implications hold: OP $\Rightarrow$ sOP $\Rightarrow$ FSOP $\Rightarrow$ SOP $\Rightarrow \cdots \Rightarrow$ SOP$_{n+1} \Rightarrow$ SOP$_4 \Rightarrow$ SOP$_3$*

None of these implications are reversible, except that T has SOP if and only if it has SOP$_n$ for all $n \geq 3$.

Shelah originally restricted his definition of the n-strong order property to $n \geq 3$ because extending it to $n = 1$ or $n = 2$ would result in trivial properties. Specifically, SOP$_1$, using the formula $x \neq y$, would merely indicate the existence of an infinite model, while SOP$_2$ would reduce to the order property (OP). To address this, Mirna Džamonja and Shelah proposed alternative definitions for SOP$_1$ and SOP$_2$ in 2004, using tree structures to frame these concepts in a non-trivial way.

Definition 5.2.3.

1. The formula $\varphi(x; y)$ has SOP$_2$ if there is a collection of tuples $(a_\eta)_{\eta \in 2^{<\omega}}$ so that.
   - For all $\eta \in 2^\omega$, $\{\varphi(x; a_{\eta\restriction\alpha}) : \alpha < \omega\}$ is consistent.
   - If $\eta, \nu \in 2^{<\omega}$ and $\eta \perp \nu$, then $\{\varphi(x; a_\eta), \varphi(x; a_\nu)\}$ is inconsistent.
2. A formula $\varphi(x; y)$ exemplifies SOP$_1$ if and only if there are $(a_\eta)_{\eta \in 2^{<\omega}}$ so that
   - For all $\eta \in 2^\omega$, $\{\varphi(x; a_{\eta\restriction n}) : n < \omega\}$ is consistent,
   - If $\eta^\frown 0 \trianglelefteq \nu \in 2^{<\omega}$, then $\{\varphi(x; a_{\eta^\frown 1}), \varphi(x; a_\nu)\}$ is inconsistent.

T has one of the above properties if some formula does.

Džamonja and Shelah demonstrated that these properties fall into a specific hierarchy. The implication from the first strong order property, SOP_1, to the tree property tp was proven to be strict, showing that not every theory with SOP_1 is simple. However, whether the implications between SOP_1, SOP_2, and SOP_3 are strict remained an open question.

Theorem 5.2.4. *The implication relations among the notions for T are as follows: $SOP_3 \Rightarrow SOP_2 (\iff \mathrm{TP}_1) \Rightarrow SOP_1 \Rightarrow \mathrm{TP}$ (simple).*

To deepen the understanding of these properties, Byunghan Kim and Hyeung-Joon Kim introduced two infinite families of properties in 2011: k-TP_1 and weak k-TP_1 for $k \geq 2$ [KK11]. Initially, they established the following chain of implications: $\mathrm{TP}_1 \iff k\text{-}\mathrm{TP}_1 \iff$ weak 2-$\mathrm{TP}_1 \iff$ weak 3-$\mathrm{TP}_1 \Rightarrow \cdots \Rightarrow \mathrm{SOP}_1$. However, in their later work *On Model-Theoretic Tree Properties*, Artem Chernikov and Nicholas Ramsey demonstrated that this hierarchy collapses and that all these properties are, in fact, equivalent to TP_1 [CR16].

In that same work, Chernikov and Ramsey characterized NSOP_1 theories through a weak independence theorem for invariant types, extending one direction of the Kim-Pillay theorem. Through this characterization, many algebraic theories naturally emerged as NSOP_1. Building on earlier work by Chatzidakis on ω-free pseudo-algebraically closed fields and by Granger on infinite-dimensional vector spaces with a generic bilinear form, Chernikov and Ramsey demonstrated that both of these theories are NSOP_1 [Gra99; Cha02].

These advancements marked significant progress in understanding NSOP_1 theories. However, the next challenge was to prove the counter part of the the Kim-Pillay theorem, creating a full theory of independence for NSOP_1 theories. In 2017, Itay Kaplan and Nicholas Ramsey addressed this with their work *On Kim-Independence* [KR17]. They introduced a new notion, Kim-dividing,[6] which states that a formula $\varphi(x; b)$ Kim-divides over a model M if it divides along a Morley sequence for a global M-invariant type extending $\mathrm{tp}(b/M)$. Kaplan and Ramsey demonstrated that Kim-dividing serves as a natural extension of dividing to NSOP_1 theories.

Much like Kim's 1996 paper, the centerpiece of Kaplan and Ramsey's work was a generalization of Kim's lemma to NSOP_1 theories. They proved that in NSOP_1 theories, forking is not witnessed by generic sequences, unless it is simple. Consequently, forking and dividing do not always coincide in NSOP_1 theories. Nevertheless, their work, along with earlier results by Chernikov and Ramsey, paved the way for a Kim-Pillay style characterization of independence in NSOP_1 theories. This characterization is distinct from the one in simple theories, particularly as it does not involve the local character property, which would otherwise imply simplicity. Nevertheless, their Kim's Lemma, along with earlier results by Chernikov and Ramsey, paved the way for a Kim-Pillay style characterization of independence in NSOP_1 theories. This characterization is framed in terms of independence.

[6] Kim-dividing was named after Kim, who had suggested defining an independence relation based on dividing witnessed by every relevant Morley sequence [Kim09].

Theorem 5.2.5. *Let $\perp'$ be an independence relation on small subsets of the monster model $\mathcal{M} \models T$, invariant under $Aut(M)$, and satisfying the following properties for any arbitrary $\mathcal{M} \models T$ and tuples from M:*

1. *Strong finite character: If $a \not\perp'_M b$, then there exists a formula $\varphi(x, b, m) \in \mathrm{tp}(a/bM)$ such that for any $a' \models \varphi(x, b, m)$, $a' \not\perp'_M b$.*
2. *Existence over models: For any $\mathcal{M} \models T$, $a \perp'_M M$ holds for any a.*
3. *Monotonicity: $a \cup a' \perp'_M b \cup b' \Rightarrow a \perp'_M b$.*
4. *Symmetry: $a \perp'_M b \iff b \perp'_M a$.*
5. *Independence theorem for invariant types: If $c_0 \perp'_M c_1$, $b_0 \perp'_M c_0$, $b_1 \perp'_M c_1$, and $\mathrm{tp}(b_0/c_0 \cup M) = \mathrm{tp}(b_1/c_0 \cup M)$, then there exists b such that $\mathrm{tp}(b/c_1 \cup M) = \mathrm{tp}(b_1/c_1 \cup M)$ and $\mathrm{tp}(b/c_0 \cup M) = \mathrm{tp}(b_0/c_0 \cup M)$.*

Then T is NSOP$_1$ and $\perp'$ strengthens $\perp^K$ *. If, moreover, $\perp'$ satisfies*

6. *Witnessing: if $a \not\perp'_M b$ is witnessed by $\varphi(x; b)$ and $(b_i)_{i<\omega}$ is a Morley sequence over M in a global M-invariant type extending $\mathrm{tp}(b/M)$, then $\{\varphi(x; b_i) : i < \omega\}$ is inconsistent.*

then $\perp' = \perp^K$ over models, i.e., if $\mathcal{M} \models T$, $a \perp_M b$ if and only if $a \perp^K_M b$.

The result above underscored the central role of Kim's Lemma, but in showcasing this, Kaplan and Ramsey built on the earlier work on a Kim's Lemma for NTP$_2$ by Chernikov and Kaplan[CK11]. Chernikov and Kaplan demonstrated that in an NTP$_2$ theory, a formula $\varphi(x; b)$ divides over a model $\mathcal{M}$ if and only if it divides along Morley sequences for every M-invariant type extending $\mathrm{tp}(b/M)$ [CK11]. From this, they derived the equivalence of forking and dividing over models in NTP$_2$ theories. This generalized form of Kim's Lemma was shown to characterize NTP$_2$ theories in [Che14].

In the last years, the most remarkable result come by Scott Mutchnik who, answering a question that had been open for many years, established the equivalence of NSOP$_1$ and NSOP$_2$ [Mut23], thereby eliminating the dividing line between the two.

These results can once again be depicted on the *Map of the Universe*. For the infinitely many n-strong order properties, only the lower ones are shown as representatives of their entire class. Additionally, the strong order property (NSOP) is included in the hierarchy, though not explicitly illustrated (the outermost class represents the strict order property, here denoted as NsOP). The classes of k-TP$_1$ and weak k-TP$_1$ are omitted, as Chernikov and Ramsey demonstrated their equivalence to TP$_1$. After Mutchnik's result, NSOP$_1$ and NSOP$_2$ collapse into the same class, along with TP$_1$, based on earlier work by Džamonja and Shelah. However, whether NSOP$_3$ and NSOP$_2$ are equivalent remains an open question, and their dividing line is shown as discontinuous.

Possible new dividing lines include NATP and NBTP. The Antichain Tree Property (ATP) was recently introduced by Ahn and Kim [AK24], who proved that ATP properly implies both SOP1 and TP$_2$, meaning the class of NATP theories (those without ATP) contains both NSOP_1 and NTP_2 theories. On the other hand, the Bizarre Tree Property (BTP) was introduced more recently by Kruckman and Ramsey using a variant of Kim's Lemma [KR24]. They showed that the class of NBTP theories (those without BTP) similarly includes both NTP_2 and NSOP_1. They established that NBTP implies NATP, but the reverse implication remains an open question.

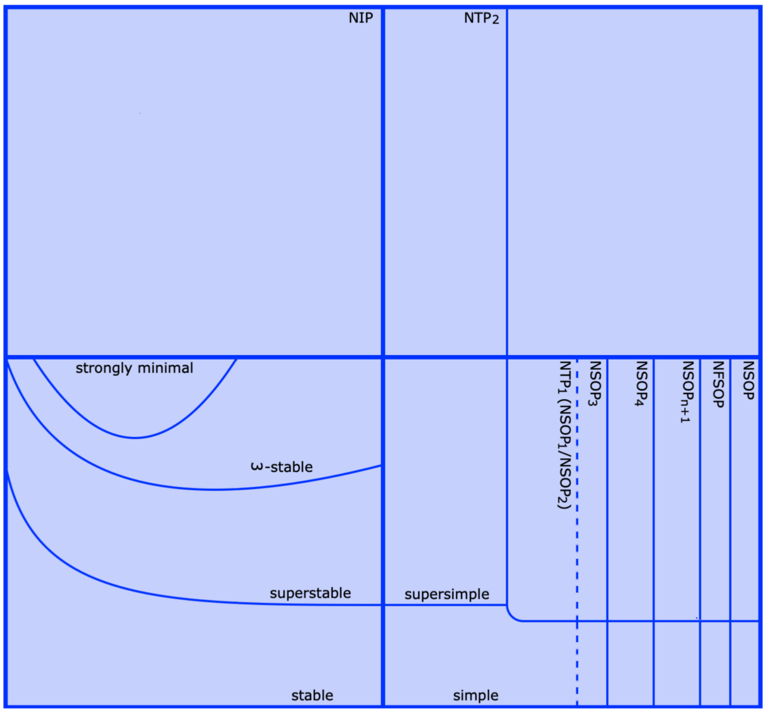

Figure 6: Fifth Building Block of the *Map of the Universe*.

5.3 O-Minimality

Among the various dividing lines in model theory, o-minimality stands out as one of the oldest, tracing its origins back to Alfred Tarski's work on the real field [Tar31]. In his 1931 monograph, Tarski explored whether his results could extend to $\mathbb{R}_{\exp}$, where exp denotes the real exponential function. In that same work, he observed that the definable subsets of the real numbers are precisely finite unions of intervals an points or, in other words, of semialgebraic sets (see Theorem 2.2.7). This significant insight was largely overlooked until Lou van den Dries revived it in the 1980s.

In 1982, while van den Dries did not resolve Tarski's question regarding $\mathbb{R}_{\exp}$, he recognized that adopting the properties of semialgebraic sets as axioms enabled him to derive many known characteristics of definable sets in higher dimensions [Dri84]. Building upon this idea, in 1986, Anand Pillay and Charles Steinhorn introduced the concept of o-minimality [PS86]. In their paper *Definable Sets in Ordered Structures I*, Pillay and Steinhorn provided a concise set of axioms that both characterized and generalized semialgebraic sets.

Definition 5.3.1. Let $\mathcal{L}$ be a language containing a binary relation $<$. An $\mathcal{L}$-structure $\mathcal{M}$ is said to be *o-minimal* if any definable subset of $\mathcal{M}$ is a a finite union of points in M and intervals (a, b) where $a \in M$ or $a = -\infty$ and $b \in M$ or $b = +\infty$. Moreover, $\mathcal{M}$ is sad to be *strongly o-minimal* if any $\mathcal{N}$ elementary equivalent of $\mathcal{M}$ is o-minimal.

An $\mathcal{L}$-theory T is said to be (strongly) o-minimal if every model of T is o-minimal.

Initially, the term strongly o-minimal was used, paralleling the terminology of strongly minimal theories. However, in a subsequent paper with Julia Knight, Pillay and Steinhorn proved that any structure elementary equivalent to an o-minimal structure is also o-minimal, implying that any o-minimal structure is strongly o-minimal [KPS86]. Since then, the term has been shortened to o-minimal both for structures and for theories. This finding was also crucial in demonstrating that not only the real field but all real closed fields are o-minimal.

Pillay and Steinhorn developed the concept of o-minimality by extending ideas from stability theory, particularly drawing from the notion of strongly minimal structures (see Definition 3.5.4). They observed that, in strongly minimal $\mathcal{L}$-structures, every definable subset is either finite or cofinite, making the definable sets as simple as possible given that $\mathcal{L}$ includes the symbol $=$, as it is always assumed. Likewise, in $\mathcal{L}'$-structures, where $\mathcal{L}'$ also includes $<$, any Boolean combination of intervals can be reduced to a finite union of intervals. Thus, they defined a structure as o-minimal if its definable subsets are no more complex than those necessarily arising from its linear ordering. Hence the term o-minimal, short for order-minimal.

Knight, Pillay, and Steinhorn further showed that the similarities with strongly minimal theories go beyond definitions. In both cases, definable sets can be interpreted geometrically. In o-minimal structures, this leads to a generalization of semialgebraic geometry, dealing with polynomial equations and inequalities over $\mathbb{R}$. In the strongly minimal case, it parallels classical algebraic geometry, involving polynomial equations over $\mathbb{C}$. Among the results that transfer from the semialgebraic geometry to the setting of o-minimal structures, are the Monotonicity Theorem and the Cell Decomposition Theorem for definable sets, which generalizes the Cylindrical Algebraic Decomposition Theorem of semialgebraic sets. These results were initially established by van den Dries for semialgebraic sets [Dri84] and later extended to o-minimal structures by Pillay, Steinhorn, and Knight [PS86; KPS86].

Theorem 5.3.2 (Monotonicity-Continuity Theorem)**.** *Let $\mathcal{R}$ be an o-minimal expansion of the dense linear order $(R, <)$. In particular, it is not assumed that $\mathcal{R}$ carries a group or ring structure. Let $f : R \to R$ be definable, meaning that its graph is a definable subset. Then, there exist $a_1, \ldots, a_n \in R$ with $-\infty =: a_0 < a_1 < \cdots < a_n < a_{n+1} := \infty$ such that for every $i \in \{0, \ldots, n\}$, the restriction $f|_{(a_i, a_{i+1})}$ is continuous and either constant or strictly monotone.*

Building on this, Pillay, Steinhorn and Knight considered a closure operator analogous to the algebraic closure in strongly minimal theories (see Definition 3.5.5). The definable closure of a set $A \subseteq R$, denoted $\mathrm{dcl}(A)$, consists of all elements $b \in R$ such that $\{b\}$ is definable over A [KPS86]. Using the Monotonicity-Continuity Theorem, they demonstrated that (R, dcl) forms a pregeometry, establishing a well-defined notion of dimension on particularly "well-behaved" definable subsets of R^n, known as cells.

Definition 5.3.3. Let $\mathcal{R}$ be an o-minimal expansion of the dense linear order $(R, <)$. A cylindrical definable cell decomposition (cdcd) C of R^n is a finite partition of R^n into definable sets $(c_i)_{i \in I}$ such that:

- If $n = 1$, then there are $a_1 < \cdots < a_\ell \in R$ with $C = \{\{a_i\} : i \in \{1, \ldots, \ell\}\} \cup \{(a_i, a_{i+1}) : i \in \{0, \ldots, \ell\}\}$
- If $n > 1$, there is a cdcd D of R^{n-1} such that for each $D \in D$, there are continuous definable functions $g_{D,1}, \ldots, g_{D,\ell_D} : D \to R$ such that $g_{D,1} < \cdots < g_{D,\ell_D}$ and $C = \bigcup_{D \in D} (\{\Gamma(g_i) \mid i \in [\ell_D]\} \cup \{(g_i, g_{i+1}) \mid i \in \{0, \ldots, \ell_D\}\})$

  With the above notation, a set of the form (g_i, g_{i+1}) is called a *band*.

An element of C is called a *cell*. A cell in R is called a *1-cell* if it is an interval, and a *0-cell* if it is a point. If C is a cdcd of R^n arising from a cdcd D in R^{n-1} as above, then a cell $C \in C$ is called an $(i_1, \ldots, i_{n-1}, 1)$-cell if it is a band over an $(i_1, \ldots, i_{n-1})$-cell in D, or a $(i_1, \ldots, i_{n-1}, 0)$-cell if it is a graph over an $(i_1, \ldots, i_{n-1})$-cell in D. The dimension of an $(i_1, \ldots, i_n)$-cell is $i_1 + \cdots + i_n$. If $X \subseteq R^n$ is a finite union $C_1 \cup \cdots \cup C_n$ of cells, then the dimension of X is $\max_{i=1}^n \dim(C_i)$.

Pillay, Steinhorn, and Knight recognized that in an o-minimal structure, the mere definability of cells does not always ensure desirable geometric properties. To address this, they demonstrated that every cell $C \subseteq R^n$ satisfies a stronger property called definable connectedness. This result was key in establishing the Cylindrical Definable Cell Decomposition for o-minimal structures. While o-minimality provides insights into definable sets in one variable, this theorem extends this understanding to definable sets in multiple variables. The theorem comprises three fundamental results, all of which are closely interrelated in their proofs.

Theorem 5.3.4. *Let $\mathcal{R}$ be an o-minimal expansion of the dense linear order $(R, <)$.*

1. *Cell Decomposition (CDCDn): Let $A_1, \ldots, A_k$ be definable subsets of R^n. There is a cdcd of R^n such that each A_i is a union of cells. A cdcd of R^n satisfying the property of the theorem will be called* adapted to $A_1, \ldots, A_k$.
2. *Uniform Finiteness (UFn): Let $A \subseteq R^n$ be definable such that $A_x = \{y \in R\ ;\ (x, y) \in A\}$ is finite for every $x \in R^{n-1}$. Then there exists $k \in \mathbb{N}$ such that $|A_x| \leq k$ for every $x \in R^{n-1}$.*
3. *Piecewise Continuity (PCn): Let A be a definable subset of R^n and $f : A \to R$ a definable function. There is a cdcd of R^n adapted to A such that,*

for every cell C contained in A, the restriction $f|_C$ is continuous.

With this theorem, Pillay, Steinhorn, and Knight extended the notion of dimension from cells to any definable set in an o-minimal structure. The dimension of a cell is defined inductively in a natural way and is then extended to all definable sets $A \subseteq R^n$ by defining $\dim A := \max\{\dim(C) \mid C \subseteq A \text{ is a cell}\}$ for nonempty sets, and $\dim(\emptyset) = -\infty$. Although this definition of dimension relies on the choice of a cylindrical definable cell decomposition (cdcd) adapted to A, they proved that the resulting dimension is intrinsic to A and does not depend on the particular cdcd used.

The geometric properties of o-minimal structures resonate with Alexander Grothendieck's concept of a "tame topology". In his influential 1984 proposal, the *Esquisse d'un Programme* [Gro97], Grothendieck proposed a new foundation for topology and geometry that would avoid the problematic, counter-intuitive examples often found in classical geometry. He aimed for a framework more in line with geometric intuition, incorporating ideas like stratification, which breaks sets into progressively simpler, lower-dimensional parts, as seen in the moduli spaces. In the view of van den Dries, o-minimality can be regarded as a realization of Grothendieck's idea of tame topology through an axiomatic, model-theoretic approach. Van den Dries, quoting Hrushovski, captures this connection:

> "Another significant influence is Grothendieck's 1984 work *Esquisse d'un Programme*, which presents an eloquent argument for developing tame topology (topologie modérée). Many suggestions from sections 5 and 6 of his program bear a strong resemblance to current o-minimal results. This is logical, as much of model theory focuses on discovering and mapping out the 'tame' regions of mathematics, those areas where wild phenomena, like space-filling curves and Gödel incompleteness, are either absent or under control. As Hrushovski recently described: Model Theory = Geography of Tame Mathematics." [Dri99]

Beyond its foundational significance, o-minimality has, in recent decades, become one of the most productive tools for applications of model theory [Hod00]. Perhaps the most renowned example is Alex Wilkie's work in 1996 [Wil96]. Building on a geometric result by Khovanskii [Kho80], Wilkie proved that $\mathbb{R}_{\exp}$, the real field with exponentiation, is model-complete and therefore o-minimal. However, Tarski's original question regarding the decidability of the real field with exponentiation remains unsolved. This problem likely depends on achieving quantifier elimination, which is closely tied to a deep conjecture in number theory known as Schanuel's Conjecture. Nevertheless, Macintyre and Wilkie showed that, assuming the real version of Schanuel's Conjecture holds, $T_{\exp}$ would be decidable [MW96].

These results sparked a continuing study of o-minimal structures, which have many connections with real algebraic geometry. The main theme of current research is to include increasingly richer classes of functions and sets in o-minimal structures on $\mathbb{R}$. In 1986, building on the work of Łojasiewicz [Łoj65] and Gabrielov [Gab68], van den Dries established the o-minimality of globally subanalytic sets, often denoted $\mathbb{R}_{\text{an}}$ [Dri86]. This culminated in the proof by Lou van den Dries and Chris Miller that the structure $\mathbb{R}_{\text{an,exp}}$ is also o-minimal [DM94]. Along similar lines, Wilkie proved that the structure of the real field with Pfaffian functions—which include, for instance, the exponential function—denoted $\mathbb{R}_{\text{Pfaff}}$, is o-minimal as well [Wil99].

More recently, and in a slightly different direction, a celebrated theorem from 2006 by Jonathan Pila and Alex Wilkie demonstrated that, excluding subsets defined using only polynomial inequalities, definable subsets in an o-minimal expansion of the real field have few rational points [PW06]. Following a strategy first employed by Pila and Umberto Zannier to reprove the Manin-Mumford conjecture [PZ08], various authors have used this o-minimal counting theorem to solve significant open problems in Diophantine geometry. Notably, in 2021, Pila, Ananth Shankar, and Jacob Tsimerman settled the André-Oort conjecture regarding Shimura varieties [Pil+22].

These applications have gone hand-in-hand with significant advancements beyond o-minimality. In 2006, Miller introduced *d-minimality*, a generalization of o-minimality where "points" are replaced by "discrete sets" [Mil05]. Another important notion was introduced by Toffalori and Vozoris in 2009, known as *locally o-minimal* structures. In these structures, for every $a \in R$ and every definable set $X \subseteq R$, there exists an interval I containing a such that $X \cap I$ is a finite union of points and intervals [TV09]. Other notable classes of structures include *weakly o-minimal* structures [Dic87], *quasi o-minimal* structures [BPW00], *strongly locally o-minimal* structures [TV09], and structures with *o-minimal open cores* [DMS09]. These classes represent variations of o-minimality that maintain a certain level of tameness.[7]

To position o-minimality within the *Map of the Universe* it is first necessary to examine how it related to the other dividing lines among NIP theories. Hence, this visualization will be postponed until these relationships are clarified, which will be addressed in the next section.

5.4 NIP Theories

Theories without the independence property (NIP), which Shelah refers to as *dependent* theories, generalize stable theories while allowing for an ordering. Initially, this class received limited attention. However, over time, various structures were found to be unstable yet NIP, such as the field of p-adic numbers $\mathbb{Q}_p$ [Bél99] and ordered abelian groups [GS84]. Despite these discoveries, NIP theories were not extensively studied as a distinct class until the development of o-minimality, which brought forth numerous examples and techniques highlighting their significance. Adler suggests that the importance of NIP theories lies

[7] In the case of d-minimality, it is not a model-theoretic tameness notion, as there is a locally o-minimal structure $\mathcal{R}$ that interprets first-order arithmetic [FM05; Hie22].

in their ability to unify stability and o-minimality, two of the most influential classes of theories:

> "One could say that the main point about NIP is that it is a natural common generalization of stability and o-minimality. It appears that stability theory takes much of its strength from the coincidence of a nice combinatorial machinery and nice geometric notions. O-minimal theories behave nicely from both points of view, but the combinatorial and geometric notions do not coincide. NIP generalizes the nice combinatorial aspects that are common to stable and o-minimal theories." [Adl08]

Interestingly, the same year that Shelah introduced the independence property in model theory, Vladimir Vapnik and Alexey Chervonenkis developed the concept of the *VC dimension* in statistical learning theory [VC71]. The VC dimension measures the complexity of a class of data sets and plays a crucial role in understanding the capacity of statistical models [Vap99].

Definition 5.4.1. Let X be a set, and let $S \subseteq \mathcal{P}(X)$ be a family of subsets of X. A subset $A \subseteq X$ is said to be *shattered* by S if every subset of A can be obtained by intersecting A with some member of S; that is, $\mathcal{P}(A) = \{A \cap S \mid S \in S\}$.

The *VC-dimension* of S is the maximum cardinality $n \in \omega$ such that there exists a subset of X of size n that is shattered by S, or ∞ if S shatters finite sets of arbitrarily large cardinality.

In model theory, the VC dimension can be interpreted by viewing formulas as uniformly definable families of subsets of a model. Consider a formula $\varphi(x; y)$, where the free variables are partitioned into *object variables* x, ranging over elements of the model, and *parameter variables* y, ranging over parameters from the model. This allows us to associate φ with a family of definable sets $\Phi = \{\varphi(x; b) : b \in M\}$, where the subsets of the model are defined by the formula as y varies across the model $\mathcal{M}$.

This connection between model theory and VC dimension was first recognized by Laskowski, who in 1992 demonstrated that a complete first-order theory is NIP, as defined by Shelah, if and only if, in each model, every definable family of sets has finite VC dimension [Las92].

Theorem 5.4.2. *A formula $\varphi(x; y)$ has IP if and only if $\{\varphi(x; b) : b \in C\} \subseteq P(C)$ has infinite VC-dimension.*

Over time, this line of research has expanded. Efforts include establishing explicit bounds on the VC dimension for definable families in o-minimal structures [KM97], and exploring related concepts such as *VC density* and *VC minimality* [Asc+16; GL13]. This research connects model theory to fields like learning theory, computational geometry, and probability theory.

In *pure* model theory, the characterization of NIP theories often relies on indiscernible sequences, as studied by Shelah and Poizat in the 1980s [She80; Poi81]. Poizat, in particular, showed that determining whether a theory is NIP can be done by examining formulas where the tuple of object variables consists of a single variable [Poi00].

Theorem 5.4.3. *If all formulas in T of the form $\varphi(x; y)$ with $|x| = 1$ are NIP, then the theory T is NIP.*

This criterion is particularly useful when one has a good understanding of one-dimensional definable sets. For example, in an o-minimal theory, a formula $\varphi(x; y)$ with $|x| = 1$ defines a finite union of intervals and points. By compactness, there is an integer n such that every instance of $\varphi(x; b)$, for $b \in M$, can be expressed as a union of at most n intervals and points. Thus, the VC dimension of $\varphi(x; y)$ satisfies $\mathrm{VC}(\varphi(x; y)) \leq 2n$, implying finiteness. Consequently, o-minimal theories belong to the class of NIP theories.

A related area of research involves counting types in NIP theories. For any theory T, it holds that $\kappa \leq g_T(\kappa) \leq 2^\kappa$ for all cardinals $\kappa \geq |T|$. Building on Shelah's work on the stability hierarchy, Keisler's Classification Theorem showed that the stability spectrum function for any countable theory T takes one of six forms: κ, $\kappa + 2^{\aleph_0}$, $\kappa^{\aleph_0}$, $\mathrm{ded}(\kappa)$, $(\mathrm{ded}(\kappa))^{\aleph_0}$, 2^κ, where $\mathrm{ded}(\kappa)$ is the supremum of the cardinalities of all linear orders containing a dense subset of cardinality κ [Kei78]. Since $\kappa < \mathrm{ded}(\kappa) \leq (\mathrm{ded}(\kappa))^{\aleph_0} \leq 2^\kappa$ for all infinite cardinals κ, the Generalized Continuum Hypothesis implies a collapse of the last three functions. As NIP theories include unstable ones for which $g_T(\kappa) = 2^\kappa$, separating NIP from IP theories by counting types, as it is done for stable and unstable theories, is not feasible without additional set-theoretic assumptions. Under the assumption $\mathrm{ded}(\kappa) < 2^\kappa$, in [She78] Shelah proved the following:

Theorem 5.4.4. *If T is NIP, then $g_T(\kappa) \leq (ded(\kappa))^{\aleph_0}$. Conversely, if T has the independence property (IP), then $g_T(\kappa) = 2^\kappa$.*

The significance of this result lies in the fact that, in 1972, Mitchell had demonstrated that for an appropriate κ, there exists an extension of the set-theoretic universe in which $\mathrm{ded}(\kappa) < 2^\kappa$ [Mit72]. In such an extension, a formula $\varphi(x; y)$ has the independence property if and only if $g_T(\kappa) = 2^\kappa$. Therefore, the presence of the independence property can always be detected by counting types, although not necessarily in the standard model of ZFC.

Inspired by this insight, in 2012, Chernikov, Kaplan, and Shelah introduced the *non-forking spectrum* as a generalization of the stability function, aiming to identify finer dividing lines in unstable theories by also counting types. It is a function of two cardinals κ and λ, representing the supremum of the possible number of types over a model of size λ that do not fork over a submodel of size κ. They succeeded in showing that the possible values a non-forking spectrum can take are limited [CKS12].

One of the most remarkable subclasses of NIP theories emerged from Shelah's work in 2009 [She09]. Shelah introduced the concept of *strongly dependent* theories, which were meant to play a role in NIP theories similar to how superstability strengthens stability. It is defined in terms of the the dp-rank:

Definition 5.4.5. Let $(I_t : t \in X)$ be a family of sequences and A a set of parameters. The sequences $(I_t : t \in X)$ is said to be *mutually indiscernible over*

A if for each $t \in X$, the sequence I_t is indiscernible over $A \cup I_{\neq t}$.

Let p be a partial type over a set A, and let κ be a cardinal. The dp-rk$(p, A) < \kappa$ if for every family $(I_t : t < \kappa)$ of mutually indiscernible sequences over A and $b \models p$, there is $t < \kappa$ such that I_t is indiscernible over $A \cup b$.

The dp-rank serves as a measure of the complexity of types within a theory. In any NIP theory, given a type p over a set A, there exists a cardinal κ such that dp-rk$(p, A) < \kappa$. Notably, the converse holds as well, providing a characterization of NIP theories in terms of dp-rank. In 2009, Shelah defined strong dependence by setting a stricter upper bound on dp-rank [She09]. To years later, building on Shelah's strong dependence, Alf Onshuus and Alexander Usvyatsoc introduced dp-minimality as a notion of minimality for dependent theories which generalises many of the usual notions of minimality [OU11].

Definition 5.4.6. The NIP theory T is *strongly dependent* if for any finite tuple of variables x, dp-rk$(x = x, \emptyset) < \aleph_0$. The theory T is *dp-minimal* if dp-rk$(x = x, \emptyset) = 1$, for x a singleton.

In particular, strongly minimal theories and o-minimal theories are classic examples of dp-minimal theories [OU11]. Recently, there have been several important developments regarding dp-rank and dp-minimal theories. Key results include the additivity of dp-rank [KOU13], and Guingona's proof of the uniform definability of types over finite sets in dp-minimal theories [Gui12]. This result was later generalized to all NIP theories by Chernikov and Simon [CS15], implying that types over finite sets behave in a stable-like manner, which relates to long-standing problems in learning theory.

One of the most recent dividing line within NIP theories is that of distality, introduced by Pierre Simon in 2012 [Sim13]. Distal theories aim to characterize NIP structures that are *completely* unstable. As Simon expressed:

> "One basic intuition we have about NIP structures is that they are somehow built out of stable components and linear orders. In this view, the theory ACVF of algebraically closed valued fields is archetypical: it has a stable part embodied in the residue field, an order part which is the value group, and the whole structure is in some sense dominated by those two components. Stable theories appear as NIP theories which are degenerate in a certain way. The idea of distality is to characterize the other extreme: NIP theories which are as far away from stable as possible." [Sim13]

A key theorem in stability states that T is stable if and only if every indiscernible sequence is totally indiscernible. For distal theories, no infinite non-constant indiscernible sequence is totally indiscernible. Moreover, if the theory is also dp-minimal, the converse holds. This demonstrates the intuition that distal theories represent the opposite extreme of NIP. In terms of indiscernibles, distal theories are defined as follows:

Definition 5.4.7. A theory T is *distal* if, for any parameter set A, any A-indiscernible sequence I, and any tuple b, if $I = I_1 + I_2$ for some sequences I_1

and I_2 without endpoints, and $I_1 + b + I_2$ is indiscernible, then $I_1 + b + I_2$ is A-indiscernible.

Although this definition is often the simplest way to verify a theory's distality, there is another valuable characterization based on honest definitions. Honest definitions, introduced by Chernikov and Simon, offer a deeper understanding of definable sets in NIP theories [CS10]. They demonstrate that a weak form of definability of types exists in NIP theories, with the *weakest* case being that of distal theories.

Using the descriptions from honest definitions, one can prove that any dp-minimal linearly ordered theory, such as o-minimal or weakly o-minimal theories, is distal. These are implications among classes of theories that therefore can be depicted in the *Map of the Universe*.

As Simon envisioned, distal theories occupy the opposite end of NIP theories relative to stable theories. Moreover, distal theories include o-minimal theories, which, both theoretically and pictorially, share multiple features with strongly minimal theories. Both classes also fall within the broader framework of dp-minimality. Here, the dividing line parallels that of NIP, since dp-minimality directly strengthens NIP in terms of dp-rank. Looking ahead, several new dividing lines are emerging, including variations of o-minimality like weak o-minimality and local o-minimality, as well as further refinements of NIP, such as theories with finite dp-rank and strongly dependent theories.

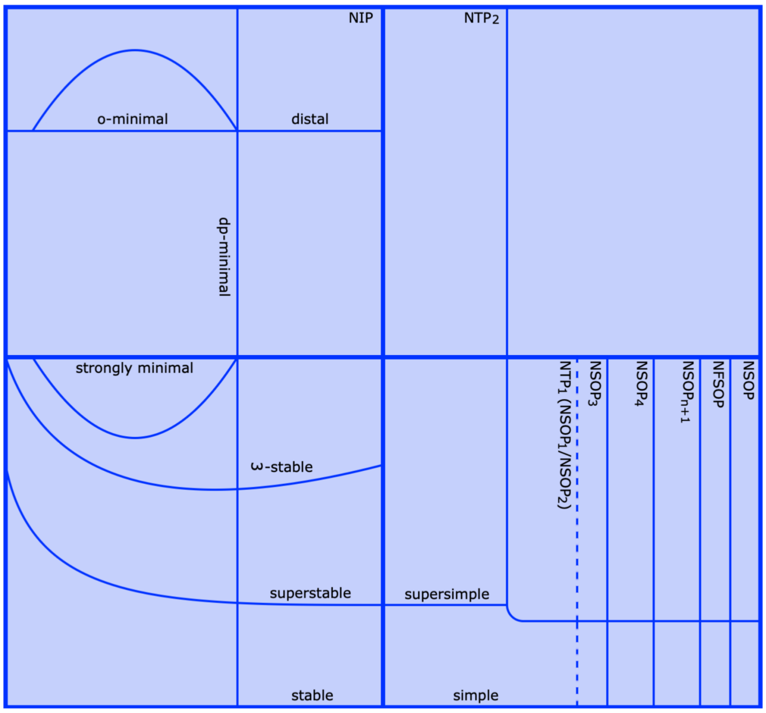

Figure 7: Sixth Building Block of the *Map of the Universe*.

6 Conclusion

In 1988, Stephen Hawking told *Der Spiegel*:

> "We are just an advanced breed of monkeys on a minor planet of a very average star. But we can understand the Universe. That makes us something very special."

The modern understanding of the Universe might be said to have been triggered by Albert Einstein's general relativity, introduced in *The Field Equations of Gravitation*, which he published in 1915. That same year, Leopold Löwenheim published *On Possibilities in the Calculus of Relatives*, quietly triggering, in parallel, the understanding of the Model Theoretic Universe.

Over the next half-century, the work of Skolem, Tarski, Henkin, and Robinson, along with the development of key concepts like completeness, decidability, and quantifier elimination, led to the emergence of first-order logic. This, in turn, enabled the description of first-order theories, which became central objects in the model-theoretic universe. Contributions from Łoś, Vaught, Ryll-Nardzewski, and Morley further advanced the field by introducing syntactic characterizations of semantic properties. This shift created a new perspective, where models were no longer viewed in isolation but rather as part of the broader family of models associated with their corresponding theories. However, there was still no clear framework for organizing families of theories.

A major breakthrough came in the 1970s with Shelah's stability hierarchy and the subsequent Classification Program. These developments showed that theories in the universe are either *chaotic* or governed by structure theorems that provide detailed descriptions of their models. Shelah, together with Hart, Hrushovski, and Laskowski, proved that, rather than there existing 2^ω distinct spectrum functions corresponding to the 2^ω complete theories, only a finite number of such functions exist. This discovery highlighted an underlying regularity in the landscape of theories, bringing order to what was previously seen as a chaotic expanse.

Since then, model theorists such as Zilber, Hrushovski, Lascar, Baldwin, van den Dries, and more recently Kaplan, Ramsey, Simon, Pila, Wilkie, and of course Shelah, have continued to develop a systematic classification of complete first-order theories. Beyond stability, new ideas have been introduced—simplicity, NSOP_3, distality, strong dependence, NTP_2, and o-minimality—to further explore and understand the more complex unstable regions of the model-theoretic universe.

As of 2024, the field, now known as Neostability theory, remains a thriving and interconnected area of study, influencing domains such as combinatorics, Diophantine geometry, and differential equations. While long-standing questions like Vaught's conjecture remain unsolved, new lines of research are constantly emerging. In this dynamic era, Gabriel Conant's *Map of the Universe* visually captures the progress and reveals both established insights and areas ripe for exploration, making it an invaluable resource for navigating the field.

In the end, despite the many unanswered questions, after a century of exploration, this advanced breed of monkeys can proudly claim to understand the Model Theoretic Universe. And perhaps, by invoking Tarski's theory of truth, viewing model theory as a branch of mathematics, and, in turn, seeing mathematics as the appropriate metalanguage in which to define the truth of the world, it can be said that a deeper understanding of the Universe itself has also been gained.

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