A structure with a weak pole that is not field-type

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Deutsche Zusammenfassung (German Summary)

Eine zentrale Eigenschaft von Theorien in der mathematischen Logik ist die Quantorenelimination. Dadurch vereinfacht sich der Formelaufbau wesentlich und es können zum Beispiel definierbare Mengen in Modellen der Theorie besser beschrieben werden. Dies kann genutzt werden um zu zeigen, dass $(\mathbb{R}, <, +, g)$ nicht *field type* ist [HW21].

In dieser Bachelorarbeit arbeiten wir zunächst in Kapitel 5 die Quantoreneliminination für die in [Del97] eingeführte Theorie T aus. Dafür nutzen wir das Theorem 5.24, welches besagt, dass es ausreicht, eine Erweiterung für jede Einbettung zwischen einem Modell $\mathcal{M}$ mit $|\mathcal{M}| \leq \kappa$ und einem Modell $\mathcal{N}$, das κ^+-saturiert ist, zu finden [Hie21, Corollary 4.2.6].

Für die konkrete Konstruktion der erweiterten Einbettung nutzen wir dieselbe Fallunterscheidung und ähnliche Eigenschaften von Modellen von T aus wie in [Del97] genannt werden. Allerdings nutzen wir im Gegensatz zu [Del97] nicht, dass die algebraischen Eigenschaften der Modelle es zulassen, jedes Modell als geordneten Vektorraum über dem Quotientenkörper der endlichen Summen von Elementen aus P zu interpretieren. Stattdessen wenden wir konkret die Axiome der Theorie T an und zeigen damit alle Eigenschaften, die sich aus der Interpretation als Vektorraum ergeben, direkt wenn wir sie für die Konstruktion benötigen. Außerdem werden in dieser Arbeit manche Beweisideen von [Del97] genauer ausgearbeitet und die dafür benötigten Eigenschaften der Presburger Arithmetik zuvor in Kapitel 4.2 gezeigt.

Anschließend diskutieren wir kurz, eine Anwendung der Quantorenelimination. Die Struktur $(\mathbb{R}, <, +, g)$ ist nicht *field type*. Eine Struktur nennen wir *field type*, wenn es in der Struktur definierbare Funktionen gibt, mit denen die Struktur (ggf. eingeschränkt auf ein Intervall) isomorph zum geordneten Körper $(\mathbb{R}, <, +, \cdot)$ ist.

Aus einer kurzen Rechnung in Kapitel 6 folgt, dass $(\mathbb{R}, <, +, g)$ eine *weak pole* zulässt. Dies ist eine in [HW21] eingefürte Notation, die eine Abschwächung der Definition einer *pole* ist und die beschreibt, ob es eine definierbare Menge von Funktionen gibt, die einen bestimmten Definitionsbereich haben und in deren Bild ein kompaktes Intervall liegt.

Dieses Ergebnis, dass die betrachtete Struktur sowohl *field type* ist, als auch eine *weak pole* zulässt, wurde in [HW21] gezeigt. Dafür wird die in dieser Arbeit ausgeführte Quantorenelimination genutzt. Außerdem wird in [HW21] die Bedeutung dieses Ergebnisses herausgestellt. In der Quelle wird bewiesen, dass o-minimale Modelle nur eine *weak pole* zulassen, wenn diese *field type* sind. Dasselbe Ergebnis wird für Expansionen, die eine dichte ω-anordnenbare Menge definieren (*dense ω-orders*), gezeigt. Somit ist das hier analysierte Modell ein interessantes Beispiel, welches belegt, dass sich dieses Ergebnis nicht für alle Expansionen der geordneten, additiven Gruppe der reellen Zahlen verallgemeinern lässt.

Contents

1 Introduction

In this Bachelor thesis we are considering a specific model and its properties. Namely, we follow the ideas from Delon [Del97] to show that the theory T of $\mathcal{R} := (\mathbb{R}, 2^{\mathbb{Z}}, (2^{n\mathbb{Z}})_{n\in\mathbb{N}^*}, <, +, -, g, \lambda, 0, 1)$ admits quantifier elimination. This structure is an expansion by definitions of $(\mathbb{R}, <, +, g)$ which is the real ordered additive group expanded by g, where g denotes the usual multiplication but with the first factor restricted to $2^{\mathbb{Z}}$.

Proving this quantifier elimination will be the main result of this thesis and will take some work. First, we will prove certain results for Presburger Arithmetic in Section 4.2 which we can later translate into results for T. Then we will prove basic results for T in Section 5.2. Applying these results, we will do an embedding test to show that T admits quantifier elimination in Section 5.3.

This quantifier elimination was applied by [HW21] in order to conclude that $(\mathbb{R}, <, +, g)$ is not field type. We will shortly discuss this application in Section 5.5.

With a quick calculation in Chapter 6, we show that $(\mathbb{R}, <, +, g)$ admits a weak pole since g is one. The concept of a weak pole is introduced in [HW21] and is a weakening of the more commonly used notion of a pole.

This concludes the result from [HW21] that $(\mathbb{R}, <, +, g)$ is not field type but admits a weak pole. This result is the main motivation for this thesis and its relevance gets clear from [HW21]. Assuming o-minimality, it is known that structures can be classified into two groups: linear structures and field-type structures [PS98]. Hieronymi and Walsberg used this to show that o-minimal structures could only admit a weak pole if they are field-type. They also showed the same result for expansions which define a dense ω-orderable set: "An ω-orderable set [...] is a definable set that is either finite or admits a definable ordering with order type ω" [HW21]. However, as proven by the example considered in this thesis this result cannot be generalized to all first order expansions of $(\mathbb{R}, <, +)$.

In conclusion, the rather technical and specific constructions in this thesis lay the foundation for this interesting result from [HW21] that there is indeed a structure with a weak pole that is not field-type. This example demonstrates that not all expansions of the additive ordered group of the reals can be classified in the same way as expansions with additional properties like o-minimality.

2 Notations

In the following, we will work in a language that has a binary relation symbol $<$ and some unary relation symbols. In order to shorten the notation, we will use $t_1 > t_2$ to abbreviate $\neg(t_1 < t_2 \vee t_1 = t_2)$, $t_1 \leq t_2$ to abbreviate $t_1 < t_2 \vee t_1 = t_2$, and $t_1 \geq t_2$ to abbreviate $t_1 > t_2 \vee t_1 = t_2$, with t_1 and t_2 being arbitrary $\mathcal{L}$-terms. Note that this agrees with the usual definition of the relation symbols $>$, $\leq$, $\geq$. Another commonly used abbreviation which we use in this thesis is $t_1 < t_2 < t_3$ instead of $t_1 < t_2 \wedge t_2 < t_3$.

For a unary relation symbol R, we will occasionally use the notation $x \in R$ to mean that x is an element such that $R(x)$ holds. For a subset A of the universe of a model of the theory, we use the notation $P(A)$ to refer to the set of all $a \in A$ such that $P(a)$ holds.

For any symbol s in some language $\mathcal{L}$, we will use the notation $s^{\mathcal{M}}$ to denote the interpretation of s in $\mathcal{M}$ for some $\mathcal{L}$-structure $\mathcal{M}$. If it is clear from the context which structure is meant, we will often omit the structure and just write s to mean $s^{\mathcal{M}}$.

We will use $\mathbb{N}^* = \{1, 2, 3, 4, \dots\}$ to denote the smallest inductive set without 0 and $\mathbb{N}_0$ to denote $\mathbb{N}^* \cup \{0\}$.

3 Definitions

In this chapter we will introduce mostly commonly known definitions which will be used in this thesis.

The following seven definitions are taken from [Hie21, Definition 5.1.1, 3.5.1, 2.3.1, 2.3.2, 2.2.4].

Definition 3.1 (universal formula)**.** We say an $\mathcal{L}$-formula ψ is universal if there is a quantifier-free $\mathcal{L}$-formula ψ_1 and an $n \in \mathbb{N}^*$ such that ψ is the formula $\forall x_1 \dots \forall x_n \psi_1$.

Definition 3.2 (κ-saturated)**.** Let κ be an infinite cardinal. We say that an $\mathcal{L}$-structure $\mathcal{M}$ is κ-saturated if every type $p(x_1, \dots, x_n)$ over A is realized in $\mathcal{M}$ for all $A \subset M$ with $|A| < \kappa$ and every $n \in \mathbb{N}^*$.

Definition 3.3 (definable set)**.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure with universe M. A set $X \subset M^n$ is called definable in $\mathcal{M}$ (without parameters) if there is an $\mathcal{L}$-formula $\varphi(x_1, \dots x_n)$ such that $X = \{m : \mathcal{M} \models \varphi(m)\}$.

Definition 3.4 ((first order) expansion)**.** Let $\mathcal{L}', \mathcal{L}$ be languages with $\mathcal{L}' \supset \mathcal{L}$. Let $\mathcal{M}'$ be an $\mathcal{L}'$-structure and $\mathcal{M}$ be an $\mathcal{L}$-structure on the same universe M. $\mathcal{M}'$ is an expansion of $\mathcal{M}$ if

1. $c^{\mathcal{M}'} = c^{\mathcal{M}}$ for each constant symbol c in $\mathcal{L}$,
2. $f^{\mathcal{M}'} = f^{\mathcal{M}}$ for each function symbol f in $\mathcal{L}$,
3. $R^{\mathcal{M}'} = R^{\mathcal{M}}$ for each relation symbol R in $\mathcal{L}$.

Definition 3.5 (expansion by definitions)**.** Let $\mathcal{L}', \mathcal{L}$ be languages with $\mathcal{L}' \supset \mathcal{L}$. Let $\mathcal{M}'$ be an $\mathcal{L}'$-structure and $\mathcal{M}$ be an $\mathcal{L}$-structure on the same universe M. $\mathcal{M}'$ is an expansion by definitions of $\mathcal{M}$ if $\mathcal{M}'$ is an expansion of $\mathcal{M}$ and

1. the set $\{x : \mathcal{M}' \models (x = c)\}$ is definable in $\mathcal{M}$ for each constant symbol $c \in \mathcal{L}'$,
2. the set $\text{graph}(f) = \{(x, y) : \mathcal{M}' \models (y = f(x))\}$ with $x \in M^l$ is definable in $\mathcal{M}$ for each l-ary function symbol $f \in \mathcal{L}'$,

3. the set $\{x : \mathcal{M}' \models (R(x))\}$ is definable in $\mathcal{M}$ for each relation symbol $R \in \mathcal{L}'$.

Definition 3.6 (definable function)**.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure with universe M. A function $F : M^m \to M^n$ is called definable if there exists an $\mathcal{L}$-formula $\varphi(y_1, \ldots, y_{m+n}, z_1, \ldots, z_j)$ and $b \in M^j$, such that:

$$\text{graph}(F) \coloneqq \{(x, F(x)) : x \in M^m\} = \{a \in M^{m+n} : \mathcal{M} \models \varphi(a, b)\}.$$

Definition 3.7 (embedding)**.** Let $\mathcal{M}$,$\mathcal{N}$ be $\mathcal{L}$-structures. A map φ is an embedding if it is an injective map from the universe of $\mathcal{M}$ onto the universe of $\mathcal{N}$ and

1. $\varphi(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for each constant symbol c in $\mathcal{L}$,
2. $\varphi(f^{\mathcal{M}}(m_1, \ldots, m_n)) = f^{\mathcal{N}}(\varphi(m_1), \ldots, \varphi(m_n))$ for each function symbol f in $\mathcal{L}$ and all $m_1, \ldots, m_n$ in the universe of $\mathcal{M}$,
3. $R^{\mathcal{M}}(m_1, \ldots, m_n)$ holds if and only if $R^{\mathcal{N}}(\varphi(m_1), \ldots, \varphi(m_n))$ for each relation symbol R in $\mathcal{L}$ and all $m_1, \ldots, m_n$ in the universe of $\mathcal{M}$.

4 Properties of Presburger Arithmetic

In this chapter, we will introduce Presburger Arithmetic and analyse it. Presburger Arithmetic is one of the most fundamental theories in mathematical logic and has therefore been well studied (see e.g. [PD11, p. 51, pp. 132 ff.] and [Mar02, pp. 81 – 84]). In the theory T, on which we will focus in this thesis, one of the axioms will be that a certain subset models Presburger Arithmetic. Therefore, we want to understand some of the known properties of Presburger Arithmetic here, in order to translate them into results for T later.

Since the Presburger Arithmetic contains the theory of linear ordered abelian groups we will begin by stating that theory, then state Presburger Arithmetic and some of its properties.

4.1 Linear ordered abelian groups

The following theory can be found in [Mar02, p. 17].

Let $\mathcal{L} = \{+, <, 0\}$ be a language with a binary function symbol $+$, a binary relation symbol $<$ and a constant symbol 0. The theory T_{ablo} of linear ordered abelian groups consists of the following axioms:

$T_{ablo}1$. $\forall x\ x + 0 = x$

$T_{ablo}2$. $\forall x\, \forall y\, \forall z\ x + (y + z) = (x + y) + z$

$T_{ablo}3$. $\forall x\, \exists y\ x + y = 0$

$T_{ablo}4$. $\forall x\, \forall y\ x + y = y + x$

$T_{ablo}5$. $\forall x\ \neg(x < x)$

$T_{ablo}6$. $\forall x\, \forall y\, \forall z\ ((x < y \wedge y < z) \to (x < z))$

$T_{ablo}7$. $\forall x\, \forall y\ ((x < y) \vee (x = y) \vee (y < x))$

$T_{ablo}8$. $\forall x\, \forall y\, \forall z\ ((x < y) \to (x + z < y + z))$

Remark 4.1. Let $\mathcal{L}' = \mathcal{L} \cup \{-\}$ be a language with an additional binary function symbol $-$. In that case we call the extension by definitions T'_{ablo} of T_{ablo} such that

$$T'_{ablo} = T_{ablo} \cup \{\forall x\, \forall y\, \forall z\ (x - y = z) \leftrightarrow (x = z + y)\}$$

the theory of linear ordered abelian groups as well.

We call every structure modelling the T_{ablo} or T'_{ablo} a linear ordered abelian group.

4.2 Presburger Arithmetic

For the following theory and Remark 4.2 confer [Mar02, pp. 81 f.].

Let $\mathcal{L} = \{(P_n)_{n\in\mathbb{N}^*}, +, -, <, 0, 1\}$ be a language consisting of unary relation symbols P_n, binary function symbols $+$ and $-$, a binary relation symbol $<$ and two constant symbols $0, 1$. Let T_{Pr} be the theory given by the set of the following sentences:

$T_{Pr}1$. the theory of linear ordered abelian groups T'_{ablo}

$T_{Pr}2$. $0 < 1$

$T_{Pr}3$. $\forall x\ (x \leq 0 \vee x \geq 1)$

$T_{Pr}4.n$. This is a collection of sentences for all $n \in N^*$:

$$\forall x\ (P_n(x) \leftrightarrow \exists y\ x = \underbrace{y + \cdots + y}_{n \text{ times}})$$

$T_{Pr}5.n$. This is a collection of sentences for all $n \in N^*$:

$$\forall x\ \bigvee_{i=0}^{n-1} (P_n(x + \underbrace{1 + \cdots + 1}_{i \text{ times}}) \wedge \bigwedge_{j \neq i} \neg P_n(x + \underbrace{1 + \cdots + 1}_{j \text{ times}}))$$

We call T_{Pr} Presburger Arithmetic.

Remark 4.2. This theory is an expansion by definitions of $Th(\mathbb{Z}, +, <, 0, 1)$, the theory of the ordered group of integers. We call a model of this theory a $\mathbb{Z}$-group.

Notation 4.3. We define the following abbreviations in order to shorten the notation:

$\exists! x\ \psi(x)$ denotes $\exists x\ (\psi(x) \wedge \forall y\ (\psi(y) \rightarrow y = x))$,

$ny = \underbrace{y + \cdots + y}_{n \text{ times}}$ and $i = \underbrace{1 + \cdots + 1}_{i \text{ times}}$.

Remark 4.4. Let $n \in N^*$. By $T_{Pr}4.n$ and $T_{Pr}5.n$, the following sentence has to hold in T_{Pr}:

$$\forall x \ \bigvee_{i=0}^{n-1} (\exists y\ ny = x + i) \wedge \bigwedge_{j \neq i} (\neg \exists z\ nz = x + j).$$

This is equivalent to

$$\forall x\, \exists! y \ \bigvee_{i=0}^{n-1} (ny = x + i).$$

Notation 4.5. Let $n_1, \ldots, n_m \in \mathbb{N}^*$. We write $\text{lcm}(n_1, \ldots, n_m)$ to denote the least common multiple of $n_1, \ldots, n_m$.

Lemma 4.6. $T_{Pr} \models (R_{n_1}(x) \wedge \cdots \wedge R_{n_m}(x)) \leftrightarrow (R_{\text{lcm}(n_1,\ldots,n_m)}(x))$

Proof. The statement follows from $T_{Pr}4.n.$ and the fact that $\bigwedge_{i=1}^{m} n_i \mid z$ if and only if $\text{lcm}(n_1, \ldots, n_m) \mid z$. □

We now want to introduce a universal theory that is similar to the Presburger Arithmetic. Let T^*_{Pr} be the following theory which is taken from [Mar02, p. 82]

$T^*_{Pr}1., 2., 3., 5.n.$ are the same as $T_{Pr}1., T_{Pr}2., T_{Pr}3.$ and $(T_{Pr}5.n.)_{n \in \mathbb{N}^*}$

$T^*_{Pr}6.n.$ This is a collection of sentences for all $n \in N^*$:

$$\forall x\, \forall y\ ((P(x) \wedge P(y)) \rightarrow (P(x + y) \wedge P(x - y)))$$

$T^*_{Pr}7.n.$ This is a collection of sentences for all $n \in N^*$:

$$\forall x\, \forall y\ (\underbrace{y + \cdots + y}_{n \text{ times}} = x) \rightarrow P_n(x)$$

$T^*_{Pr}8.m.n.$ This is a collection of sentences for all $n, m \in N^*$ with $m \mid n$:

$$\forall x\ (P_n(x) \rightarrow P_m(x))$$

$T^*_{Pr}9.n.k.$ This is a collection of sentences for all $n, k \in N^*$:

$$\forall x\ (P_{kn}(\underbrace{x + \cdots + x}_{k \text{ times}}) \rightarrow P_n(x))$$

Remark 4.7. It is easy to check that $T^*_{Pr} \subset T_{Pr}$ (i.e. every sentence from T^*_{Pr} holds true in T_{Pr}). Thus, every model of T_{Pr} is also a model of T^*_{Pr}.

The following is a reformulation of the proof of [Mar02, Lemma 3.1.19] modified slightly at the end to show a similar lemma:

Lemma 4.8. *Let $\mathcal{G} \models T^*_{Pr}$ be a structure with universe G. Then there is a superstructure $\mathcal{H} \supset \mathcal{G}$ with $\mathcal{H} \models T_{Pr}$ and universe H such that for every model $\mathcal{H}' \models T_{Pr}$ with universe H' and every embedding $\zeta : G \to H'$ there is an embedding $\xi : H \to H'$ such that $\xi \restriction_G = \zeta$.*

Proof. Let $H := \{\frac{x}{n} : x \in G, n \in \mathbb{N}^*, P_n(x)\}$. This is a subset of the divisible hull of G which is closed under addition and subtraction:

$\frac{x}{n} \in H$ and $\frac{y}{m} \in H$ imply $P_n(x)$ and $P_m(y)$. Thus $P_{mn}(mx)$ and $P_{mn}(ny)$. Since P_{mn} is closed under subtraction and addition (by $T^*_{Pr}6.n.$), we have $P_{mn}(mx \pm ny)$ and $\frac{x}{n} \pm \frac{y}{m} = \frac{xm \pm yn}{mn} \in H$.

Let $\mathcal{H}$ be the ordered additive subgroup with universe H of the divisible hull of $\mathcal{G}$. Define $P_n^{\mathcal{H}} = nH$.

We can check that $\mathcal{H} \models T_{Pr}$:

1. $\mathcal{H} \models (0 < 1)$ since $\mathcal{G} \models (0 < 1)$.
2. Let $\frac{x}{n} \in H$. $0 < \frac{x}{n} < 1$ would imply $0 < x < n$ in $\mathcal{G}$ and therefore $x \in \{0, 1, \dots, n-1\}$. However, $P_n(n)$ holds by $T^*_{Pr}7.n.$ and thus $\neg P_n(x)$ by $T^*_{Pr}5.n.$ Contradiction to $\frac{x}{n} \in H$.
3. $\mathcal{H} \models T_{Pr}4.n$ by definition of $P_n^{\mathcal{H}}$.
4. Take any $\frac{x}{n} \in H$. Then, $P_n(x)$ holds in $\mathcal{G}$. By $T^*_{Pr}5.n$ there is a unique $i \in \{0, 1, \dots, mn-1\}$ such that $P_{nm}(x+i)$. It follows with $T^*_{Pr}8.n.m$ and $T^*_{Pr}6.n$ that $P_m(x+i)$ and thus $P_m(i)$. $P_m(i)$ implies that there is a unique $l \in \{0, 1, \dots, n-1\}$ with $i = lm$. Then, since $P_{mn}(x + lm)$ holds, we also have that $P_n(\frac{x}{m} + l)$ holds by writing out the definition from the previously proven $T_{Pr}4.n$. The uniqueness follows from the uniqueness of i.

Thus, $\mathcal{H} \supset \mathcal{G}$ with $\mathcal{H} \models T_{Pr}$.

Let $\mathcal{H}' \models T_{Pr}$ with universe H' and an embedding $\zeta : G \to H'$. Let $g \in G$. Since ζ is an embedding, $P_n(g)$ holds if and only if $P_n(\zeta(g))$ holds. $\mathcal{H}' \models P_n(\zeta(g))$ implies that there is a unique $y_{g,n} \in H'$ such that $ny_{g,n} = \zeta(g)$. It is easy to check that $\xi : H \to H', \frac{g}{n} \mapsto y_{g,n}$ is an embedding fixing G. □

Corollary 4.9. *Let $\mathcal{M}, \mathcal{N} \models T_{Pr}$ be structures with universes M, N. Let $\mathcal{G} \models T^*_{Pr}$ with universe G be a substructure of $\mathcal{M}$. Let $\zeta : G \to N$ be an embedding. Then there is some substructure $\mathscr{A} \models T_{Pr}$ of $\mathcal{M}$ such that $G \subset A$ and some embedding $\xi : \mathscr{A} \to \mathcal{N}$ extending ζ. Here A denotes the universe of $\mathscr{A}$.*

Proof. First, apply Lemma 4.8 to G to get some superstructure $\mathcal{H} \models T_{Pr}$ with the properties of Lemma 4.8. Since we have the identity $\zeta_1 : G \to M, g \mapsto g$ as a canonical embedding, we get an embedding $\xi_1 : H \to M$ such that $\xi_1 \restriction_G$ is the identity. Because ξ_1 is an embedding, the image of ξ_1, $im(\xi_1)$, is a substructure of $\mathcal{M}$ and $im(\xi_1) \models T_{Pr}$. To prove this, realize that $\xi_1 : H \to im(\xi_1)$ is an surjective embedding and thus an isomorphism and use [Hie21, Proposition 2.2.3].

Define $\mathscr{A} \models T_{Pr}$ as this substructure of $\mathcal{M}$ with universe $A := im(\xi_1)$. Since $im(\zeta_1) \subset im(\xi_1)$, we have $G \subset A$. By the property of H, there is an embedding $\xi_2 : H \to N$ with $\xi_2 \restriction_G = \zeta$. $\xi_1 : H \to A = im(\xi_1)$ is an isomorphism and has an inverse map $\xi_1^{-1} : A \to H$ which is an embedding as well. Clearly $\xi_1^{-1}(g) = g$ for all $g \in G$. Thus, if we define $\xi := \xi_2(\xi_1^{-1})$, $\xi : A \to N$ is an embedding with $\xi \restriction_G = \xi_2 \restriction_G = \zeta$. □

5 $\mathcal{R}$ admits quantifier elimination

Now we come to the main part of this thesis. In this chapter we introduce T, show some basic properties for it and apply these to show quantifier elimination for T. We will then introduce a model $\mathcal{R} \models T$ with universe $\mathbb{R}$ and conclude that $T = Th(\mathcal{R})$. In the last part of this chapter we shortly consider the application of this quantifier elimination that $(\mathbb{R}, <, +, g)$ is not field type.

5.1 Definition of T

Fix a language $\mathcal{L} = \{P, (R_n)_{n \in \mathbb{N}^*}, <, +, -, f, \lambda, 0, 1\}$ where P and R_n are unary relation symbols, $<$ is a binary relation symbol, $+, -$ and f are binary function symbols, λ is a unary function symbol and $0, 1$ are constant symbols.

We define the theory $T = \{T1, \ldots, T13\}$ in this language that was introduced in [Del97] as the set of the following axioms:

$T1$. $(M, <, +, -, 0)$ is a linear ordered, abelian group for any M being the universe of a structure modelling the theory. (This is axiomized by T'_{ablo} as listed in Section 4.1.)

$T2$. $P(1) \wedge \forall x \ (P(x) \leftrightarrow P(x + x))$

$T3$. $\forall x \, \forall y \ (P(x) \wedge (x < y < x + x)) \to \neg P(y)$

$T4$. $\forall x \ P(x) \to (0 < x)$

$T5$. $\forall x \ (0 < x) \to (\exists p \ (P(p) \wedge (p \leq x) \wedge \neg(\exists q \ P(q) \wedge p < q \leq x)))$

$T6$. $\forall p \, \forall x \ \neg P(p) \to f(p, x) = 0$

$T7.n$. This is a collection of sentences for each $n \in \mathbb{N}^*$

$$\forall y_1, y_2, \dots, y_n \, \forall \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$$
$$((\bigwedge_{j=1}^{n} P(y_j)) \wedge \bigwedge_{i=1}^{n} (\varepsilon_i = 1 \vee \varepsilon_i = -1) \wedge \sum_{i=1}^{n} y_i \varepsilon_i > 0)$$
$$\rightarrow [(\forall y \, \exists x \, \sum \varepsilon_i f(y_i, x) = y)$$
$$\wedge (\forall x_1 \, \forall x_2 \, ((\sum \varepsilon_i f(y_i, x_1) = \sum \varepsilon_i f(y_i, x_2)) \rightarrow (x_1 = x_2)))$$
$$\wedge (\forall x_1 \, \forall x_2 \, (x_1 < x_2 \rightarrow ((\sum \varepsilon_i f(y_i, x_1) < \sum \varepsilon_i f(y_i, x_2))$$
$$\wedge (\sum \varepsilon_i f(y_i, x_1 + x_2) = \sum \varepsilon_i f(y_i, x_1)) + \sum \varepsilon_i f(y_i, x_2)))))]$$

(This means that if $((\bigwedge_{j=1}^{n} P(y_j)) \wedge \bigwedge_{i=1}^{n} (\varepsilon_i = \pm 1) \wedge \sum_{i=1}^{n} y_i \varepsilon_i > 0)$ holds, $\sum \varepsilon_i f(y_i, -)$ is an automorphism of the group defined in $T1$.)

$T8$. $\forall x, y, z \; f(x, f(y, z)) = f(f(x, y), z) \wedge f(1, z) = z$

$T9$. $\forall x \; (x > 0) \rightarrow (\forall y_1 \, \forall y_2 \; (P(y_1) \wedge P(y_2)) \rightarrow ((y_1 < y_2) \rightarrow (f(y_1, x) < f(y_2, x))))$

(This means that the restriction of $f(-, x)$ to P is strictly increasing for every $x > 0$.)

$T10$. $(P, (R_n \restriction_P)_{n \in N^*}, f \restriction P^2, <, 1, 1+1)$ forms a $\mathbb{Z}$-group

(This is axiomized by T_{Pr} (see Section 4.2) with every $\exists x \; \psi(x)$ being replaced by $\exists x \; (P(x) \wedge \psi(x))$ and every $\forall x \; \psi(x)$ being replaced by $\forall x \; (P(x) \rightarrow \psi(x))$ in order to restrict these axioms to P.)

$T11$. $\forall x \; f(1+1, x) = x + x$

$T12$. This is again a collection of sentences for each $n \in \mathbb{N}^*$:

$$\forall x \; R_n(x) \leftrightarrow [P(x) \wedge \exists y \; (P(y) \wedge \underbrace{f(y, f(y, \dots f(y,}_{n \text{ times}} 1 \underbrace{) \dots))}_{n \text{ times}} = x)]$$

$T13$. $\forall x \, \forall y \; \lambda(x) = y \leftrightarrow [((0 < x) \wedge (P(y) \wedge (y \leq x < y + y)))$
$\vee (0 \geq x \wedge y = 0)]$

By $T10$, P forms a $\mathbb{Z}$-group, and thus f applied to P acts on P like the usual multiplication and it makes sense to introduce a similar notation:

Notation 5.1. For any $p \in P$ we will use the notation p^0 to mean 1 and p^n to mean $\underbrace{f(p, f(p, \dots f(p,}_{n \text{ times}} 1 \underbrace{) \dots))}_{n \text{ times}}$, Let p^{-n} be the inverse of p^n in the multiplicative $\mathbb{Z}$-group P. This means p^{-n} is the unique element such that $f(p^{-n}, p^n) = f(p^n, p^{-n}) = 1$ and $P(p^{-n})$ holds.

Remark 5.2. With a case distinction and a simple calculation one can check that in this notation $f(p^{z_1}, p^{z_2}) = p^{z_1 + z_2}$ and $f(p^z, q^z) = (f(p, q))^z$ for $p, q \in P$.

Notation 5.3. In the following let $2 \coloneqq 1+1$. Clearly $P(2)$ holds true. Thus we define 2^n, 2^{-n} with the definition in Notation 5.1.

For the following remarks regarding $T7$ and $T10$ confer [Del97].

Remark 5.4. Note the following direct consequences of the axioms of T:

$T1$: In particular, there are unique additive inverses for all elements in a structure modelling T. We denote the additive inverse of 1 by -1.

$T7$: If $\sum_{i=1}^n y_i\varepsilon_i < 0$, set $\bar{\varepsilon}_i \coloneqq -\varepsilon_i$. Then $\sum_{i=1}^n y_i\bar{\varepsilon}_i > 0$ and by $T7$ the map $x \to \sum \bar{\varepsilon}_i f(y_i, x)$ is an automorphism in $(M, <, +, -, 0)$. Then using $T1$ the map $x \to -(\sum \bar{\varepsilon}_i f(y_i, x)) = \sum(-\bar{\varepsilon}_i)f(y_i, x) = \sum \varepsilon_i f(y_i, x)$ is an automorphism in $(M, +, -, 0)$ as well, that is, however, strictly decreasing.

$T8$: This implies that $f \restriction P^2$ defines a group action on the $\mathbb{Z}$-group introduced in $T10$.

$T10$: By Remark 4.4, there is a unique element y for every $x \in P$ such that

$$P(y) \wedge \bigvee_{j=0}^{n-1} \underbrace{f(y, f(y, \dots f(y,}_{n \text{ times}} 1 \underbrace{) \dots))}_{n \text{ times}} = \underbrace{f(2^{-1}, f(2^{-1}, \dots f(2^{-1},}_{j \text{ times}} x \underbrace{) \dots))}_{j \text{ times}}$$

holds.

$T13$: This implies that for all $x > 0$, $\lambda(x)$ is the unique element p that fulfills the condition $(P(p) \wedge (p \le x) \wedge \neg(\exists q\ P(q) \wedge p < q \le x))$ from $T5$.

Remark 5.5. T is clearly satisfiable (e.g. by the structure $\mathcal{R}$ we define in Section 5.4). In the following let $\mathscr{M}$ be an arbitrary model of T. By $T10$ every model of T must be infinite.

Remark 5.6. Although we do not explicitly use the construction from [Del97] that models of T can be interpreted as ordered vectorspaces over the field of fractions of finite sums of elements of P, it might be useful to keep this idea in mind for the following sections. This might give an intuition for the definitions and the properties we show in the following.

In that setting $\sum_{i=1}^n \varepsilon_i f(y_i, -)$ is the multiplication by $\sum_{i=1}^n \varepsilon_i y_i$, an element of the field.

5.2 Properties of T

The following three definitions, the statement from Lemma 5.13 and the main ideas of the first part of the proof can be found similarly in [Del97].

Definition 5.7 ($f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}$)**.** For $n \in \mathbb{N}^*$, $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ and $y_1, \dots, y_n \in P$ such that $\sum_{i=1}^n \varepsilon_i y_i > 0$ the automorphism in $T7$ is defined and has an inverse map. For $n \in \mathbb{N}^*$, $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ and $y_1, \dots, y_n \in P$ such that $\sum_{i=1}^n \varepsilon_i y_i < 0$ the automorphism from the remark regarding $T7$ in Remark 5.4 is defined and has an inverse map.

Let $n \in \mathbb{N}^*$ and $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$. Define $f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}$ to map $(x, p_1, \dots, p_n)$ to y, such that the automorphism from $T7$ maps y to $\sum \varepsilon_i f(p_i, y) = x$, if $P(p_1), \dots, P(p_n)$ and $\sum \varepsilon_i p_i > 0$. And define $f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}$ to map $(x, p_1, \dots, p_n)$ to y, such that the automorphism from Remark 5.4 maps y to $\sum \varepsilon_i f(p_i, y) = x$, if $P(p_1), \dots, P(p_n)$ and $\sum \varepsilon_i p_i < 0$. Otherwise define $f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}$ as 0.

Remark 5.8. By this definition, $f_1^{-1}(1, p^n) = p^{-n}$ for every $p \in P$ with the notation that is introduced in Notation 5.1.

Remark 5.9. $f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(-, p_1, \dots, p_n)$ is strictly increasing if $\sum \varepsilon_i p_i > 0$, and strictly decreasing if $\sum \varepsilon_i p_i < 0$ since it is the inverse map of $\sum \varepsilon_i f(p_i, -)$.

Definition 5.10 (φ_n)**.** For each $n \in \mathbb{N}$, define the map φ_n: If $x \in P$, let φ_n map x to the unique element y such that the formula

$$P(y) \wedge \bigvee_{j=0}^{n-1} \underbrace{f(y, f(y, \dots f(y,}_{n \text{ times}} 1 \underbrace{) \dots))}_{n \text{ times}} = \underbrace{f(2^{-1}, f(2^{-1}, \dots f(2^{-1},}_{j \text{ times}} x \underbrace{) \dots))}_{j \text{ times}}$$

from Remark 5.4 holds. If $x \notin P$ define $\varphi_n(x) := 0$.

Remark 5.11. Note that $f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}$ and φ_n are definable functions.

Definition 5.12 ($\langle A \rangle$)**.** Let $\mathscr{M}$ be a model of T. Let A be a subset of the universe of $\mathscr{M}$. Define $\langle A \rangle$ to be the closure of the set A regarding the functions λ, $+$, $-$, f, $(\varphi_n)_{n \in \mathbb{N}}$, $(f^{-1}_{\varepsilon_1,\dots\varepsilon_n})_{n \in \mathbb{N}, \varepsilon_1,\dots,\varepsilon_n \in \{-1,1\}}$. This means that $\langle A \rangle$ contains the interpretations in $\mathscr{M}$ of all terms consisting of the function symbols $\lambda, +, -, f, (\varphi_n)_{n \in \mathbb{N}}, (f^{-1}_{\varepsilon_1,\dots\varepsilon_n})_{n \in \mathbb{N}, \varepsilon_1,\dots,\varepsilon_n \in \{-1,1\}}$ and with variables in A. This corresponds to [Hie21, Definition 4.2.1] if we consider $\mathscr{M}$ as a structure in the language $\mathcal{L}' = \mathcal{L} \cup \{(\varphi_n)_{n \in \mathbb{N}}, (f^{-1}_{\varepsilon_1,\dots\varepsilon_n})_{n \in \mathbb{N}, \varepsilon_1,\dots,\varepsilon_n \in \{-1,1\}}\}$.

Lemma 5.13. *Let $\mathscr{M}$ be a model of the theory T. Let A be a non-empty subset of the universe of $\mathscr{M}$. Then $\langle A \rangle \models T$ and $\langle A \rangle$ can be embedded into every model of T whose universe contains A.*

Proof. **$\langle A \rangle$ models T:** The existence of additive inverses and 0 is given because we are closed under $-$: If $a \in \langle A \rangle$, then $a - a = 0$ and $a - a - a = -a$ are in $\langle A \rangle$. That we are closed under $+$ is explicitly claimed in the definition of $\langle A \rangle$. All other properties of $T1$ follow directly because they can be formulated universal and need to be satisfied in $\mathscr{M}$.

The same argument also applies to $T2, T3, T4, T6, T8, T9, T11$ and $T13$.

$T5$ is satisfied because of the condition that $\langle A \rangle$ is closed under λ.

For $T7$, let $a \in \langle A \rangle$ and take any $y_1, \dots, y_n \in \langle A \rangle$ such that $\bigwedge_{j=1}^n P(y_j)$ holds and any $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ such that $\sum_{j=1}^n y_j \varepsilon_j > 0$. Since $\langle A \rangle$ is closed under $+, -$ and f, we get $\sum_{j=1}^n \varepsilon_j f(y_j, a) \in \langle A \rangle$. This shows that the map $m : \langle A \rangle \to \langle A \rangle, a \mapsto \sum_{j=1}^n \varepsilon_j f(y_j, a)$ is well defined. Since this is a restriction of an automorphism, we only need to check that it is surjective in order for it to be an automorphism. m is surjective because we are closed under $f^{-1}_{\varepsilon_1,\dots\varepsilon_n}$. For any

$b \in \langle A \rangle$, we have $b^{-1} := f^{-1}_{\varepsilon_1,\dots\varepsilon_n}(b, y_1, y_2, \dots, y_n) \in \langle A \rangle$. Then $m(b^{-1}) = b$, and since b was chosen arbitrarily, m is surjective. Thus, $T7$ holds for $\langle A \rangle$.

For $T10$ we need to check that $f(x, y), \varphi_n(x), x^{-1} = f_1^{-1}(x, 1) \in P(\langle A \rangle)$ for any $x, z \in P(\langle A \rangle)$ and any $n \in \mathbb{N}$. This is the case since $\langle A \rangle$ is closed under f, φ_n and $f_1^{-1}(-, 1)$. The rest of the properties are again inherited from $\mathscr{M}$.

For $T12$, we need to check for any $x \in \langle A \rangle$ that the existence of an $m \in M$ such that $P(m)$ holds and $x = \underbrace{f(m, f(m, \dots f(m,}_{n \text{ times}} 1 \underbrace{) \dots))}_{n \text{ times}}$ implies $m \in \langle A \rangle$. This is indeed true: If such a m exists, $m = \varphi_n(x)$ and $\varphi_n(x) \in \langle A \rangle$.

$\langle A \rangle$ embeds into every model of T whose universe contains A: For any model containing the set A, the interpretation of all terms containing only elements of A must be in the model again. Thus, it has to be closed under the function symbols λ, $+$,$-$,f. Due to $T7$ and Remark 5.4($T7$,$T10$), any model must be closed regarding $(\varphi_n)_{n \in \mathbb{N}}$, $(f^{-1}_{\varepsilon_1,\dots\varepsilon_n})_{n \in \mathbb{N}, \varepsilon_1,\dots,\varepsilon_n \in \{-1,1\}}$. Thus, any model containing A contains $\langle A \rangle$. □

Later on we will use, that from this lemma follows that $\langle A \rangle$ is a submodel of any model containing A.

Next, we will prove some basic properties of f and $f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}$ which follow from the theory T:

Lemma 5.14. *The following sentences hold in T:*

1. $\forall x, y, z\ f(x, y + z) = f(x, y) + f(x, z) \wedge f(x, y - z) = f(x, y) - f(x, z)$,
2. $\forall x, y \in P\ f(x, y) = f(y, x)$,
3. $\forall p \in P\ f(p, 1) = p$,
4. $\forall p, x\ P(p) \to f(p + p, x) = f(p, x) + f(p, x)$,
5. $\forall x, y\ \neg P(y) \to \neg P(f(x, y))$.

Proof.

1. follows from $T7$ if $x \in P$ (and thus also $x > 0$). If $x \notin P$, all the terms are 0.

2. follows from $T10$ since a $\mathbb{Z}$-group is abelian.

3. Since $P(1)$ and $P(p)$ hold, we have $f(p,1) = f(1,p)$. With $T8$ the claim follows.

4. By $T11$, $f(p+p,x) = f(f(1+1,p),x)$. By $T10$, f is abelian for elements of P, and thus $f(p+p,x) = f(f(p,1+1),x)$. Applying $T8$ gives us $f(p+p,x) = f(p,f(1+1,x))$. Applying $T11$ again we have $f(p+p,x) = f(p,x+x)$. Applying the property from the first claim of this lemma leads to the desired equality.

5. If $\neg P(x)$, then $f(x,y) = 0$ and $\neg P(f(x,y))$. If $P(x)$ holds: Suppose $P(f(x,y))$ holds. Let x^{-1} be the unique inverse of x regarding f. This exists due to $T10$. Then $P(x^{-1})$ holds as well. Since the $\mathbb{Z}$-group must be closed under f, we get $P(f(x^{-1},f(x,y)))$ and by $T10$ $P(f(f(x^{-1},x),y))$. Thus $P(y)$ holds.

□

Notation 5.15. For the next lemma (and also later in this thesis) we will use $\delta_1\varepsilon_1,\dots,\delta_m\varepsilon_n$ to abbreviate $\delta_1\varepsilon_1,\dots,\delta_1\varepsilon_n,\delta_2\varepsilon_1,\dots,\delta_2\varepsilon_n,\dots,\delta_m\varepsilon_1,\dots,\delta_m\varepsilon_n$ and $f(z_1,r_1),\dots,f(z_m,r_n)$ to abbreviate
$f(z_1,r_1),\dots,f(z_1,r_n),f(z_2,r_1),\dots,f(z_2,r_n),\dots,f(z_m,r_1),\dots,f(z_m,r_n)$.

Lemma 5.16. *The following sentences hold in T:*

1. *For all $n \in \mathbb{N}^*$ and $\varepsilon_1,\dots\varepsilon_n \in \{-1,1\}$:*

$$\forall x\ \forall y, r_1,\dots,r_n \in P \quad f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(f(y,x),r_1,\dots,r_n) = f(y, f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(x,r_1,\dots,r_n)).$$

2. *For all $n \in \mathbb{N}^*$ and $\varepsilon_1,\dots\varepsilon_n \in \{-1,1\}$:*

$$\begin{aligned}\forall x,y\ \forall r_1,\dots,r_n \in P \quad & f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(x+y,r_1,\dots,r_n) \\ &= f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(x,r_1,\dots,r_n) + f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(y,r_1,\dots,r_n) \\ \wedge\ & f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(x-y,r_1,\dots,r_n) \\ &= f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(x,r_1,\dots,r_n) - f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(y,r_1,\dots,r_n).\end{aligned}$$

3. *For all $n,m \in \mathbb{N}^*$ and $\varepsilon_1,\dots\varepsilon_n,\delta_1,\dots,\delta_m \in \{-1,1\}$:*

$$\begin{aligned}\forall x\ \forall z_1,\dots,z_m,r_1,\dots,r_n \in P \quad & f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(f^{-1}_{\delta_1,\dots,\delta_m}(x,z_1,\dots,z_m),r_1,\dots,r_n) \\ &= f^{-1}_{\delta_1\varepsilon_1,\dots,\delta_m\varepsilon_n}(x,f(z_1,r_1),\dots,f(z_m,r_n)).\end{aligned}$$

4. *If ι is an embedding from $\mathcal{M}$ to $\mathcal{N}$, $x \in M$, $\varepsilon_1, \dots \varepsilon_n \in \{-1, 1\}$ and $r_1, \dots, r_n \in P(M)$ with $\sum_{i=1}^n \varepsilon_i r_i \neq 0$ we have:*

$$\iota(f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(x, r_1, \dots, r_n)) = f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(\iota(x), \iota(r_1), \dots, \iota(r_n)).$$

Proof.

1. Either $\sum_{i=1}^n \varepsilon_i r_i = 0$ and thus by definition $\forall z \; f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(z, r_1, \dots, r_n = 0)$ (i.e. both terms equal 0 and therefore they are equal) or using the Lemma 5.14 and $T8$ and $T10$, we have:

$$\begin{aligned} & f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(f(y, x), r_1, \dots, r_n) \\ &\quad = f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(f(y, \sum_{j=1}^n \varepsilon_j f(r_j, f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(x, r_1, \dots, r_n)), r_1, \dots, r_n)) \\ &\quad = f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(\sum_{j=1}^n \varepsilon_j f(y, f(r_j, f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(x, r_1, \dots, r_n)), r_1, \dots, r_n)) \\ &\quad = f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(\sum_{j=1}^n \varepsilon_j f(r_j, f(y, f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(x, r_1, \dots, r_n)), r_1, \dots, r_n)) \\ &\quad = f(y, f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(x, r_1, \dots, r_n)). \end{aligned}$$

2. Either $\sum_{i=1}^n \varepsilon_i r_i = 0$ and thus $\forall z \; f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(z, r_1, \dots, r_n) = 0$ (i.e. both terms are 0 and therefore they are equal), or using the Lemma 5.14 and the property just proven before, we have:

$$\begin{aligned} & f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(x + y, r_1, \dots, r_n) \\ &\quad = f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(\sum_{j=1}^n \varepsilon_j f(r_j, f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(x, r_1, \dots, r_n) \\ &\qquad + \sum_{j=1}^n \varepsilon_j f(r_j, f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(y, r_1, \dots, r_n)), r_1, \dots, r_n)) \\ &\quad = f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(\sum_{j=1}^n \varepsilon_j f(r_j, f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(x, r_1, \dots, r_n)) \\ &\qquad + f(r_j, f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(y, r_1, \dots, r_n)), r_1, \dots, r_n) \\ &\quad = f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(\sum_{j=1}^n \varepsilon_j f(r_j, (f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(x, r_1, \dots, r_n) \\ &\qquad + f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(y, r_1, \dots, r_n))), r_1, \dots, r_n) \\ &\quad = f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(x, r_1, \dots, r_n) + f^{-1}_{\varepsilon_1,\dots,\varepsilon_n}(y, r_1, \dots, r_n). \end{aligned}$$

The proof for $-$ follows similarly.

3. $f^{-1}_{\varepsilon_1,\ldots,\varepsilon_n}(f^{-1}_{\delta_1,\ldots,\delta_m}(a, z_1, \ldots, z_m), r_1, \ldots, r_n)$ is either 0 or equal to an element e such that $\sum_{i=1}^n \varepsilon_i f(r_i, \sum_{j=1}^m \delta_j f(z_j, e)) = a$. We have:
$$\begin{aligned}\sum_{i=1}^n \varepsilon_i f(r_i, \sum_{j=1}^m \delta_j f(z_j, e)) &= \sum_{i=1}^n \varepsilon_i \sum_{j=1}^m \delta_j f(r_i, f(z_j, e)) \\ &= \sum_{i=1}^n \sum_{j=1}^m \varepsilon_i \delta_j f(f(r_i, z_j), e).\end{aligned}$$
Thus
$$e = f^{-1}_{\delta_1\varepsilon_1,\ldots,\delta_m\varepsilon_n}(a, f(z_1, r_1), \ldots, f(z_m, r_n)).$$

4. For every y, we have $\iota(\sum_{i=1}^n \varepsilon_i f(r_i, y)) = \sum_{i=1}^n \varepsilon_i f(\iota(r_i), \iota(y))$. Thus
$$\begin{aligned}&\sum_{i=1}^n \varepsilon_i f(\iota(r_i), f^{-1}_{\varepsilon_1,\ldots,\varepsilon_n}(\iota(x), \iota(r_1), \ldots, \iota(r_n))) \\ &= \iota(x) \\ &= \iota(\sum_{i=1}^n \varepsilon_i f(r_i, f^{-1}_{\varepsilon_1,\ldots,\varepsilon_n}(x, r_1, \ldots, r_n))) \\ &= \sum_{i=1}^n \varepsilon_i f(\iota(r_i), \iota(f^{-1}_{\varepsilon_1,\ldots,\varepsilon_n}(x, r_1, \ldots, r_n))).\end{aligned}$$
Since ι is an embedding, $\sum_{i=1}^n \varepsilon_i \iota(r_i) \neq 0$. Thus $\sum_{i=1}^n \varepsilon_i f(\iota(r_i), -)$ is strictly monotone and the previous equation is equivalent to
$$f^{-1}_{\varepsilon_1,\ldots,\varepsilon_n}(\iota(x), \iota(r_1), \ldots, \iota(r_n)) = \iota(f^{-1}_{\varepsilon_1,\ldots,\varepsilon_n}(x, r_1, \ldots, r_n)).$$

□

$<$ defines an order on any model $\mathscr{M}$ of the theory in which $(M, <, +, -, 0)$ is an abelian ordered group by $T1$. Now we want to describe how this order acts on terms containing f and f^{-1}. It is immediate to see that $f(x, y) \leq f(x, z)$ if and only if $y \leq z$ (by $T7$), and $f(x, y) \leq f(z, y)$ if and only if $x \leq z \wedge y > 0$ (by $T9$ and $T7$). In the following lemmata other important terms will be looked at.

Lemma 5.17. $T \models \forall x, y\ ((x < y) \to x < f(2^{-1}, x + y) < y)$

Proof. Let $\mathscr{M} \models T$ and $g_1 < g_2$ be elements of the universe of $\mathscr{M}$. Then

$$\begin{aligned}g_1 &< g_2 && \\ g_1 + g_1 &< g_1 + g_2 && \text{(by } T1) \\ f(2, g_1) &< g_1 + g_2 && \text{(by } T11) \\ f(2^{-1}, f(2, g_1)) &< f(2^{-1}, g_1 + g_2) && \text{(by } T7 \text{ applied to } y_1 = 2^{-1}, \varepsilon_1 = 1) \\ f(f(2^{-1}, 2), g_1) &< f(2^{-1}, g_1 + g_2) && \text{(by } T8) \\ f(1, g_1) &< f(2^{-1}, g_1 + g_2) && \text{(by definition of } 2^{-1}) \\ g_1 &< f(2^{-1}, g_1 + g_2) && \text{(by } T8).\end{aligned}$$

and

$$
\begin{aligned}
g_1 &< g_2 \\
g_1 + g_2 &< g_2 + g_2 && \text{(by } T1) \\
g_1 + g_2 &< f(2, g_2) && \text{(by } T11) \\
f(2^{-1}, g_1 + g_2) &< f(2^{-1}, f(2, g_2)) && \text{(by } T7 \text{ applied to } y_1 = 2^{-1}, \varepsilon_1 = 1) \\
f(2^{-1}, g_1 + g_2) &< f(f(2^{-1}, 2), g_2) && \text{(by } T8) \\
f(2^{-1}, g_1 + g_2) &< f(1, g_2) && \text{(by definition of } 2^{-1}) \\
f(2^{-1}, g_1 + g_2) &< g_2 && \text{(by } T8)
\end{aligned}
$$

□

Lemma 5.18. *For any $n \geq 1$ and any $y_1, \ldots y_n \in P$, $\sum_{i=1}^{n} y_i \varepsilon_i = 0$ implies $\sum_{i=1}^{n} \varepsilon_i f(y_i, x) = 0$.*

Proof. Define $S^+ := \{i : \varepsilon_i = 1\}$, $S^- := \{i : \varepsilon_i = -1\}$ and $S = S^- \cup S^+$. Then, $\sum_{i \in S^+} y_i = \sum_{j \in S^-} y_j > 0$ since $y_i \in P$ and thus $y_i > 0$. We want to show that $\sum_{i \in S^+} f(y_i, x) = \sum_{j \in S^-} f(y_j, x)$. With Lemma 5.14 we can assume that all y_i with $i \in S^-$ are distinct: If $y_{i_1} = y_{i_2}$ with $i_1, i_2 \in S^-$, replace S^- by $(S^- \setminus \{i_1, i_2\}) \cup y'$ with $y' = y_{i_1} + y_{i_2} = y_{i_1} + y_{i_1} \in P$. We have $\sum_{j \in S^-} y_j = y' + \sum_{j \in S^- \setminus \{i_1, i_2\}} y_j$ and

$$
\begin{aligned}
\sum_{j \in S^-} f(y_j, x) &= f(y_{i_1}, x) + f(y_{i_1}, x) + \sum_{j \in S^- \setminus \{i_1, i_2\}} f(y_j, x) \\
&= f(y_{i_1} + y_{i_1}, x) + \sum_{j \in S^- \setminus \{i_1, i_2\}} f(y_j, x) \\
&= f(y', x) + \sum_{j \in S^- \setminus \{i_1, i_2\}} f(y_j, x).
\end{aligned}
$$

Since we always replace two elements of S^- by one and S^- is finite, we will have distinct elements after finitely many steps. Similarly, we can argue that all elements with indicies in S^+ are distinct.

Moreover, we can show that we can assume all elements with indicies in S^- to be distinct from elements with indicies in S^+. If $y_{i^+} = y_{i^-}$ with $i^+ \in S^+$ and $i^- \in S^-$ we have $0 = \sum_{j \in S} \varepsilon_j y_j = \sum_{j \in S \setminus \{i^+, i^-\}} \varepsilon_j y_j$ and

$$
\begin{aligned}
\sum_{j \in S} \varepsilon_j f(y_j, x) &= f(y_{i^+}, x) - f(y_{i^-}, x) + \sum_{j \in S \setminus \{i^+, i^-\}} \varepsilon_j f(y_j, x) \\
&= f(y_{i^+}, x) - f(y_{i^+}, x) + \sum_{j \in S \setminus \{i^+, i^-\}} \varepsilon_j f(y_j, x) \\
&= \sum_{j \in S \setminus \{i^+, i^-\}} \varepsilon_j f(y_j, x)
\end{aligned}
$$

Here we consider the empty sum to equal 0. Thus, we can assume that all y_i with $i \in S$ are distinct.

Since we only have finitely many elements, we can take the maximum. W.l.o.g. assume $y_n > y_{n_1} > \cdots > y_1 > 0$ (since we could permute the indicies). Then clearly $y_1 + y_2 < y_2 + y_2 \leq y_3$ since $y_3 \in P$ and $y_2 + y_2$ is the smallest element of P larger than y_2. By induction we get that $\sum_{i=1}^{n-1} y_i < y_n$.

Thus, $|\sum_{i=1}^{n-1} \varepsilon_i y_i| \leq \sum_{i=1}^{n-1} |y_i| = \sum_{i=1}^{n-1} y_i < y_n$. This is a contradiction to $\sum_{i=1}^{n} y_i \varepsilon_i = 0$. Thus $\{1, \ldots, n\} = S = \emptyset$ and $\sum_{i=1}^{n} \varepsilon_i f(y_i, x) = 0$. □

Lemma 5.19. *Let $z_1, \ldots, z_n \in P$ be distinct. Let $z_n = \max\{z_i : i \in \{1, \ldots, n\}\}$. Let $\delta_n = 1$ and $\delta_1, \ldots, \delta_{n-1} \in \{1, -1\}$. For any $x \in P$ we have*

$$f(2^{-n+1}, f(x, z_n)) \leq \sum_{i=1}^{n} \delta_i f(x, z_i) < f(2, f(x, z_n))$$

and

$$f_1^{-1}(x, f(2^{-n+1}, z_n)) \geq f^{-1}_{\delta_1,\ldots,\delta_n}(x, z_1, \ldots, z_n) \geq f_1^{-1}(x, f(2, z_n)).$$

Proof. Like we showed at the end of the proof of the previous lemma, we have $\sum_{i=1}^{n-1} \delta_i z_i < z_n$. Thus $\sum_{i=1}^{n} \delta_i z_i < z_n + z_n = f(2, z_n)$. Moreover, since the z_i are distinct,

$$\begin{aligned} \sum_{j=1}^{n} \varepsilon_j z_j &= z_n + \sum_{j=1}^{n-1} \varepsilon_j z_j \\ &\geq z_n - \sum_{j=1}^{n-1} f(2^{-j}, z_n) \\ &= f(2^{-(n-1)}, z_n) \end{aligned}$$

For $p \in P$ we have:

$$\begin{aligned} f(2^{-n+1}, z_n) \leq & \sum_{i=1}^{n} \delta_i z_i && \leq f(2, z_n) \\ f(p, f(2^{-n+1}, z_n)) \leq & f(p, \sum_{i=1}^{n} \delta_i z_i) && \leq f(p, f(2, z_n)) \\ f(2^{-n+1}, f(p, z_n)) \leq & \sum_{i=1}^{n} \delta_i f(p, z_i) && \leq f(2, f(p, z_n)) \end{aligned}$$

This shows the first statement.

We will prove the second statement with a contradiction, using the first statement: Suppose

$$f_1^{-1}(x, f(2^{-n+1}, z_n)) < f^{-1}_{\delta_1,\ldots,\delta_n}(x, z_1, \ldots, z_n)$$

or

$$f^{-1}_{\delta_1,\ldots,\delta_n}(x, z_1, \ldots, z_n) < f_1^{-1}(x, f(2, z_n)).$$

Then, since $\delta_n = 1$, we have $\sum_{i=1}^n \delta_i z_i > 0$. Thus, by $T7$, we have for $x < y$: $\sum_{i=1}^n \delta_i f(z_i, x) < \sum_{i=1}^n \delta_i f(z_i, y)$. Hence

$$\sum_{i=1}^n \delta_i f(z_i, f_1^{-1}(x, f(2^{-n+1}, z_n))) < \sum_{i=1}^n \delta_i f(z_i, f_{\delta_1,\ldots,\delta_n}^{-1}(x, z_1, \ldots, z_n)) = x$$

or

$$x = \sum_{i=1}^n \delta_i f(z_i, f_{\delta_1,\ldots,\delta_n}^{-1}(x, z_1, \ldots, z_n)) < \sum_{i=1}^n \delta_i f(z_i, f_1^{-1}(x, f(2, z_n)))$$

and since $x \in P$ implies $f_1^{-1}(x, f(2^{-n+1}, z_n)) \in P$ and $f_1^{-1}(x, f(2, z_n)) \in P$, we have:

$$f(2^{-n+1}, f(f_1^{-1}(x, f(2^{-n+1}, z_n)), z_n)) \leq \sum_{i=1}^n \delta_i f(f_1^{-1}(x, f(2^{-n+1}, z_n)), z_i) < x$$

or

$$x < \sum_{i=1}^n \delta_i f(f_1^{-1}(x, f(2, z_n)), z_i) \leq f(2, f(f_1^{-1}(x, f(2, z_n)), z_n)).$$

This shows $x < x$ or $x < x$. Contradiction. □

Lemma 5.20. *Let $p \in P$. Then for any $\varepsilon_i \in \{-1, 1\}$ and any $y_i \in P$ with $\sum_{i=1}^n \varepsilon_i y_i > 0$, we have*

$$f_1^{-1}(p, \lambda(\sum_{i=1}^n \varepsilon_i y_i)) \geq f_{\varepsilon_1,\ldots,\varepsilon_n}^{-1}(p, y_1, \ldots, y_n).$$

Proof. Since $\sum_{i=1}^n \varepsilon_i y_i > 0$, we have $\sum_{i=1}^n \varepsilon_i y_i > \lambda(\sum_{i=1}^n \varepsilon_i y_i) > 0$. Moreover, $p \in P$ implies $f_1^{-1}(p, \lambda(\sum_{i=1}^n \varepsilon_i y_i)) \in P$. Then we can conclude:

$$\begin{aligned}\sum_{i=1}^n \varepsilon_i f(f_{\varepsilon_1,\ldots,\varepsilon_n}^{-1}(p, y_1, \ldots, y_n), y_i) = p & \\ &= f(f_1^{-1}(p, \lambda(\sum_{i=1}^n \varepsilon_i y_i)), \lambda(\sum_{i=1}^n \varepsilon_i y_i)) \\ &\leq f(f_1^{-1}(p, \lambda(\sum_{i=1}^n \varepsilon_i y_i)), \sum_{i=1}^n \varepsilon_i y_i) \\ &= \sum_{i=1}^n \varepsilon_i f(f_1^{-1}(p, \lambda(\sum_{i=1}^n \varepsilon_i y_i)), y_i).\end{aligned}$$

We have $\sum_{i=1}^n \varepsilon_i y_i > 0$ and thus $\sum_{i=1}^n \varepsilon_i f(-, y_i)$ is strictly increasing. Therefore

$$f_1^{-1}(p, \lambda(\sum_{i=1}^n \varepsilon_i y_i)) \geq f_{\varepsilon_1,\ldots,\varepsilon_n}^{-1}(p, y_1, \ldots, y_n).$$

□

5.3 T admits quantifier elimination

For the following definition confer [Hie21, Definition 4.2.2].

Definition 5.21 ($Sub(\mathcal{M},\mathcal{N})$)**.** Let $\mathcal{L}$ be a language and $\mathcal{M},\mathcal{N}$ be $\mathcal{L}$-structures. $Sub(\mathcal{M},\mathcal{N})$ is the set of all maps ι such that ι is an embedding of some substructure $\mathcal{A}$ of $\mathcal{M}$ into $\mathcal{N}$.

Remark 5.22. For any $\iota \in Sub(\mathcal{M},\mathcal{N})$ we have that the image of the map $im(\iota)$ is a substructure of $\mathcal{N}$.

Proof. Realize that for some embedding $\iota : A \to N$ we have that $\iota' : H \to im(\iota)$, $x \mapsto \iota(x)$ is an surjective embedding. This means ι' is an isomorphism. Applying [Hie21, Proposition 2.2.3] the claim follows. □

The following theorem and a proof of it can be found in [Hie21, Corollary 4.2.6].

Theorem 5.23. *Let T be an $\mathcal{L}$-theory and let κ be a cardinal such that $\kappa \geq |\mathcal{L}|$. Suppose for all models $\mathcal{M},\mathcal{N}$ of T with universes M,N and $|M| \leq \kappa$ and $\mathcal{N}$ being κ^+-saturated and for every $\iota \in Sub(\mathcal{M},\mathcal{N})$ either the domain of ι is M or ι has a proper extension $\iota' \in Sub(\mathcal{M},\mathcal{N})$. Then, T has quantifier-elimination.*

For the following theorem and its proof confer [Del97]. In the proof we distinguish the same cases as in [Del97] and apply ideas regarding how we can describe the elements of $\langle A \cup a\rangle$ from [Del97]. However, as mentioned earlier we elaborate these ideas further, we use a slightly different method for proving the quantifier elimination and we do not focus so much on the underlying algebraic properties. The proof to show that $p(x)$ is finitely satisfiable in Case 1 is a slightly modified version from a similar proof for ordered vectorspaces in [Hie21, Theorem 4.2.7].

Theorem 5.24. *The theory T that is stated above, has quantifier elimination.*

Proof. We will show this using Theorem 5.23.

Take $\mathcal{N},\mathcal{M}$ to be models of T. Let $|\mathcal{M}| \leq \kappa$ and $\mathcal{N}$ κ^+-saturated, let $\mathscr{A}$ be a submodel of $\mathcal{M}$. Let A, N, M be the universes of $\mathscr{A},\mathcal{N},\mathcal{M}$. Take any $\iota : A \to N \in Sub(\mathcal{M},\mathcal{N})$ and any $a \in M$. Let $\mathscr{B} = im(\iota)$ be the image of ι and thus a substructure of $\mathcal{N}$. Let B be the universe of $\mathscr{B}$.

If the domain of ι is M, we are done. If it is not, we can find $a \in M \setminus A$. We will show that in this case we can find an extension $\iota' \in Sub(\mathcal{M},\mathcal{N})$ with a in its domain (i.e. in particular this is a proper extension) using a case distinction.

Case 1 ($P^{\mathcal{M}}(A) = P^{\mathcal{M}}(A \cup \{a\})$): This means for all $x \in \langle A \cup \{a\}\rangle$, $P(x)$ implies $x \in A$. Define the type

$$p(x) := \{\iota(c) < x : c <^{\mathcal{M}} a, c \in A\} \cup \{x < \iota(d) : a <^{\mathcal{M}} d, d \in A\}$$

Note that since $P^{\mathcal{M}}(A) = P^{\mathcal{M}}(A \cup \{a\})$, we have $\lambda(a) < a < \lambda(a) + \lambda(a)$ and thus $\{\iota(c) < x : c <^{\mathcal{M}} a, c \in A\} \neq \emptyset$ and $\{x < \iota(d) : a <^{\mathcal{M}} d, d \in A\} \neq \emptyset$.

Definition of b: Let $c_1, \ldots, c_n, d_1, \ldots d_m$ be in A, such that $c_i < a$ and $a < d_j$ for all $i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}$. Because $\{\iota(c) < x : c <^{\mathcal{M}} a, c \in A\} \neq \emptyset$ and $\{x < \iota(d) : a <^{\mathcal{M}} d, d \in A\} \neq \emptyset$ we can assume that $n \geq 1, m \geq 1$.

Then, $\max\{c_1, \ldots, c_n\} < \min\{d_1, \ldots, d_m\}$ and because ι is a partial isomorphism, we get that $\max\{\iota(c_1), \ldots, \iota(c_n)\} < \min\{\iota(d_1), \ldots, \iota(d_m)\}$. Set

$$b' := f(2^{-1}, \max\{\iota(c_1), \ldots, \iota(c_n)\} + \min\{\iota(d_1), \ldots, \iota(d_m)\}).$$

Then by Lemma 5.17,

$$\max\{\iota(c_1), \ldots, \iota(c_n)\} < b' < \min\{\iota(d_1), \ldots, \iota(d_m)\}.$$

Therefore, $p(x)$ is finitely satisfiable in $\mathcal{N}$. Since $|p(x)| = |A| \leq \kappa$ and $\mathcal{N}$ is κ^+-saturated, $p(x)$ is realized in $\mathcal{N}$. Let b be a realization of $p(x)$ in $\mathcal{N}$.

Show that b $\notin$ P: Since we have that $P^{\mathcal{M}}(A) = P^{\mathcal{M}}(A \cup \{a\})$, it follows that $\lambda(a), \lambda(a) + \lambda(a) \in A$. By $T2, T13$ and since $a \notin P$ which implies $a \neq \lambda(a)$, we have $P^{\mathcal{M}}(\lambda(a)), P^{\mathcal{M}}(\lambda(a) + \lambda(a)), \lambda(a) < a < \lambda(a) + \lambda(a)$. Since ι is an embedding and b satisfies $p(x)$, we have that $P(\iota(\lambda(a)))$ and $\iota(\lambda(a)) < b < \iota(\lambda(a) + \lambda(a)) = \iota(\lambda(a)) + \iota(\lambda(a))$ hold. Because $b, \iota(\lambda(a)), \iota(\lambda(a)) + \iota(\lambda(a)) \in N$ and $T3$ holds for $\mathcal{N}$, we can conclude $b \notin P$.

Elements of $\langle A \cup \{a\} \rangle$: All elements of $\langle A \cup \{a\} \rangle$ are of the form $x_0 + \sum_{i=1}^{n} \varepsilon_i f(y_i, f^{-1}_{\delta_{i_1}, \ldots, \delta_{i_{m_i}}}(a, z_{i_1}, \ldots, z_{i_{m_i}}))$ with $n \in N^*$, $m_1, \ldots, m_n \in \mathbb{N}^*$, $x_0 \in A$ and $y_i, z_i \in P^{\mathcal{M}}(\langle A \cup \{a\} \rangle) = P^{\mathcal{M}}(A)$, $\varepsilon_i, \delta_i \in \{-1, 1\}$.

In order to show this, define S to be the set of all elements of this form. We have $a \in \langle A \cup \{a\} \rangle$ and $A \subset \langle A \cup \{a\} \rangle$. By definition $\langle A \cup \{a\} \rangle$ is closed under $+$,$-$,f, $(f^{-1}_{\delta_{i_1}, \ldots, \delta_{i_{m_i}}})_{i \in \{1, \ldots, n\}}$ and it follows that

$$x_0 + \sum_{i=1}^{n} \varepsilon_i f(y_i, f^{-1}_{\delta_{i_1}, \ldots, \delta_{i_{m_i}}}(a, z_{i_1}, \ldots, z_{i_{m_i}})) \in \langle A \cup \{a\} \rangle.$$

Thus, $S \subset \langle A \cup \{a\} \rangle$.

For the other direction, we have $a = 0 + \sum_{i=1}^{1} f(1, f_1^{-1}(a, 1)) \in S$ and for any $a_1 \in A$ we have $a_1 = a_1 + f(1, f_1^{-1}(a, 1)) - f(1, f_1^{-1}(a, 1)) \in S$. Thus $A \subset S$. Hence, we just have to show that the set S is closed regarding the functions λ,$+$,$-$,f, $(\varphi_n)_{n \in \mathbb{N}}$, $(f^{-1}_{\alpha_1, \ldots, \alpha_n})_{n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in \{-1,1\}}$.

Let $s_0 := x_0 + \sum_{i=1}^{n_0} \varepsilon_i f(y_i, f^{-1}_{\delta_{i_1}, \ldots, \delta_{i_{m_i}}}(a, z_{i_1}, \ldots, z_{i_{m_i}}))$ and $s_1 := x_1 + \sum_{i=1}^{n_1} \mu_i f(p_i, f^{-1}_{\gamma_{i_1}, \ldots, \gamma_{i_{k_i}}}(a, q_{i_1}, \ldots, q_{i_{k_i}}))$ be arbitrary elements of S. Then $s_0 \in \langle A \cup \{a\} \rangle$ and thus $\lambda(s_0) \in \langle A \cup \{a\} \rangle$. By the definition of λ we have $\lambda(s_0) \in P$ or $\lambda(s_0) = 0$. We have $P^{\mathcal{M}}(A) = P^{\mathcal{M}}(A \cup \{a\})$. Thus, $\lambda(s_0) \in A \subset S$.

By $T1$ we get

$$\begin{aligned}
s_0 + s_1 &= (x_0 + \sum_{i=1}^{n_0} \varepsilon_i f(y_i, f^{-1}_{\delta_{i_1},\dots,\delta_{i_{m_i}}}(a, z_{i_1}, \dots, z_{i_{m_i}}))) \\
&\quad + (x_1 + \sum_{i=1}^{n_1} \mu_i f(p_i, f^{-1}_{\gamma_{i_1},\dots,\gamma_{i_{k_i}}}(a, q_{i_1}, \dots, q_{i_{k_i}}))) \\
&= (x_0 + x_1) + (\sum_{i=1}^{n_0} \varepsilon_i f(y_i, f^{-1}_{\delta_{i_1},\dots,\delta_{i_{m_i}}}(a, z_{i_1}, \dots, z_{i_{m_i}})) \\
&\quad + \sum_{i=1}^{n_1} \mu_i f(p_i, f^{-1}_{\gamma_{i_1},\dots,\gamma_{i_{k_i}}}(a, q_{i_1}, \dots, q_{i_{k_i}}))) \in S.
\end{aligned}$$

Define $\bar{\mu_i} = -\mu_i \in \{-1, 1\}$. Then

$$\begin{aligned}
s_0 - s_1 &= (x_0 + \sum_{i=1}^{n_0} \varepsilon_i f(y_i, f^{-1}_{\delta_{i_1},\dots,\delta_{i_{m_i}}}(a, z_{i_1}, \dots, z_{i_{m_i}}))) \\
&\quad - (x_1 + \sum_{i=1}^{n_1} \mu_i f(p_i, f^{-1}_{\gamma_{i_1},\dots,\gamma_{i_{k_i}}}(a, q_{i_1}, \dots, q_{i_{k_i}}))) \\
&= (x_0 - x_1) + (\sum_{i=1}^{n_0} \varepsilon_i f(y_i, f^{-1}_{\delta_{i_1},\dots,\delta_{i_{m_i}}}(a, z_{i_1}, \dots, z_{i_{m_i}})) \\
&\quad + \sum_{i=1}^{n_1} \bar{\mu_i} f(p_i, f^{-1}_{\gamma_{i_1},\dots,\gamma_{i_{k_i}}}(a, q_{i_1}, \dots, q_{i_{k_i}}))) \in S.
\end{aligned}$$

Either $\varphi_n(s_0) = 0 \in S$ or $s_0 \in P$, and since $P^{\mathcal{M}}(A) = P^{\mathcal{M}}(A \cup \{a\})$, we get $\varphi(s_0) \in A \subset S$.

We have $f(s_1, s_0) = 0 \in S$ if $s_1 \notin P$. If $s_1 \in P$, we again have $s_1 \in A$. Then

$$\begin{aligned}
f(s_1, s_0) &= f(s_1, x_0 + \sum_{i=1}^{n_0} \varepsilon_i f(y_i, f^{-1}_{\delta_{i_1},\dots,\delta_{i_{m_i}}}(a, z_{i_1}, \dots, z_{i_{m_i}}))) \\
&= f(s_1, x_0) + f(s_1, \sum_{i=1}^{n_0} \varepsilon_i f(y_i, f^{-1}_{\delta_{i_1},\dots,\delta_{i_{m_i}}}(a, z_{i_1}, \dots, z_{i_{m_i}}))) \\
&= f(s_1, x_0) + \sum_{i=1}^{n_0} \varepsilon_i f(s_1, f(y_i, f^{-1}_{\delta_{i_1},\dots,\delta_{i_{m_i}}}(a, z_{i_1}, \dots, z_{i_{m_i}})))
\end{aligned}$$

by Lemma 5.14. With $T8$, we can conclude that

$$f(s_1, s_0) = f(s_1, x_0) + \sum_{i=1}^{n_0} \varepsilon_i f(f(s_1, y_i), f^{-1}_{\delta_{i_1},\dots,\delta_{i_{m_i}}}(a, z_{i_1}, \dots, z_{i_{m_i}})).$$

This is an element of S since $f(s_1, y_i) \in P$.

Let n be in $\mathbb{N}^*$ and $r_1, \dots, r_n$ be elements of $S \subset \langle A \cup \{a\} \rangle$.

We have $f^{-1}_{\alpha_1,\dots,\alpha_n}(s_0, r_1, \dots, r_n) = 0 \in S$ if any r_j is not in P or if $\sum_{i=1}^n \alpha_i r_i = 0$. Assume $r_1, \dots r_n \in P$, and therefore $r_1, \dots r_n \in A$, and $\sum_{i=1}^n \alpha_i r_i \neq 0$. Then

$$\begin{aligned} &f^{-1}_{\alpha_1,\dots,\alpha_n}(s_0, r_1, \dots, r_n) \\ &\qquad = f^{-1}_{\alpha_1,\dots,\alpha_n}(x_0 + \sum_{i=1}^{n_0} \varepsilon_i f(y_i, f^{-1}_{\delta_{i_1},\dots,\delta_{i_{m_i}}}(a, z_{i_1}, \dots, z_{i_{m_i}})), r_1, \dots, r_n) \\ &\qquad = f^{-1}_{\alpha_1,\dots,\alpha_n}(x_0, r_1, \dots, r_n) \\ &\qquad\quad + \sum_{i=1}^{n_0} \varepsilon_i f^{-1}_{\alpha_1,\dots,\alpha_n}(f(y_i, f^{-1}_{\delta_{i_1},\dots,\delta_{i_{m_i}}}(a, z_{i_1}, \dots, z_{i_{m_i}})), r_1, \dots, r_n) \end{aligned}$$

by Lemma 5.16. Since $\mathscr{A} \models T$ (in particular $\mathscr{A}$ is closed under $f^{-1}_{\alpha_1,\dots,\alpha_n}$) and $x_0, r_1, \dots, r_n \in A$, we have $f^{-1}_{\alpha_1,\dots,\alpha_n}(x_0, r_1, \dots, r_n) \in A$. On the other hand

$$\begin{aligned} &f^{-1}_{\alpha_1,\dots,\alpha_n}(f(y_i, f^{-1}_{\delta_{i_1},\dots,\delta_{i_{m_i}}}(a, z_{i_1}, \dots, z_{i_{m_i}})), r_1, \dots, r_n) \\ &\qquad\qquad = f(y_i, f^{-1}_{\alpha_1,\dots,\alpha_n}(f^{-1}_{\delta_{i_1},\dots,\delta_{i_{m_i}}}(a, z_{i_1}, \dots, z_{i_{m_i}}), r_1, \dots, r_n)) \end{aligned}$$

by Lemma 5.16. With the same lemma, we have
$f^{-1}_{\alpha_1,\dots,\alpha_n}(f^{-1}_{\delta_{i_1},\dots,\delta_{i_{m_i}}}(a, z_{i_1}, \dots, z_{i_{m_i}}), r_1, \dots, r_n) = 0 \in S$ or there are $k_i \in \mathbb{N}$,
$\beta_{i_1}, \dots, \beta_{i_{k_i}} \in \{-1, 1\}$ and $w_{i_1}, \dots, w_{i_k} \in P$ such that
$f^{-1}_{\alpha_1,\dots,\alpha_n}(f^{-1}_{\delta_{i_1},\dots,\delta_{i_{m_i}}}(a, z_{i_1}, \dots, z_{i_{m_i}}), r_1, \dots, r_n) = f^{-1}_{\beta_{i_1},\dots,\beta_{i_{k_i}}}(a, w_{i_1}, \dots, w_{i_{k_i}})$. Thus

$$\begin{aligned} &f^{-1}_{\alpha_1,\dots,\alpha_n}(s_0, r_1, \dots, r_n) \\ &\qquad = f^{-1}_{\alpha_1,\dots,\alpha_n}(x_0, r_1, \dots, r_n) + \sum_{i=1}^{n_0} \varepsilon_i f(y_i, f^{-1}_{\beta_{i_1},\dots,\beta_{i_{k_i}}}(a, w_{i_1}, \dots, w_{i_{k_i}})) \in S. \end{aligned}$$

We can conclude that $S = \langle A \cup \{a\} \rangle$.

Having this, we can actually show an even stronger statement. Every element of $\langle A \cup \{a\} \rangle$ is of the form: $x_0 + \sum_{i=1}^n \varepsilon_i f(y_i, f^{-1}_{\delta_1,\dots,\delta_m}(a, z_1, \dots, z_m))$ with $n \in \mathbb{N}^*$, $m \in \mathbb{N}^*$, $x_0 \in A$, $y_1, \dots, y_n, z_1, \dots, z_m \in P$:

Let $y_1, \dots y_n, w_{i_1}, \dots, w_{i_{m_i}} \in P$ and $\varepsilon_1, \dots, \varepsilon_n, \delta_{i_1}, \dots, \delta_{i_{m_i}} \in \{-1, 1\}$.

Let $I_i = \{i_1, \dots, i_{m_i}\}$ and $I = \bigcup_{j=1}^n I_j$. Then applying the definition of $f^{-1}_{\delta_{i_1},\dots,\delta_{i_{m_i}}}$, Lemma 5.14 and Lemma 5.16, we have for some $h \in \mathbb{N}$, $\gamma_1, \dots, \gamma_h \in \{-1, 1\}$ and $w_1, \dots, w_h \in P$:

$$
\begin{aligned}
&\sum_{i=1}^{n} \varepsilon_i f(y_i, f^{-1}_{\delta_{i_1},\ldots,\delta_{i m_i}}(a, z_{i_1}, \ldots, z_{i m_i})) \\
&\quad = \sum_{i=1}^{n} \varepsilon_i f(y_i, \sum_{j_1 \in I_1} \cdots \sum_{j_{i-1} \in I_{i-1}} \sum_{j_{i+1} \in I_{i+1}} \cdots \sum_{j_n \in I_n} \delta_{j_1} \ldots \delta_{j_{i-1}} \delta_{j_{i-1}} \ldots \delta_{j_n} f(\\
&\qquad f(\ldots f(f(\ldots f(1, z_{j_1}) \ldots, z_{j_{i-1}}), z_{j_{i+1}}), \ldots, z_{j_n}), f^{-1}_{\delta_{1_1},\ldots,\delta_{1 m_1}}(\\
&\qquad\ f^{-1}_{\delta_{2_1},\ldots,\delta_{2 m_2}}(\ldots f^{-1}_{\delta_{n_1},\ldots,\delta_{n m_n}}(a, z_{n_1}, \ldots, z_{n m_n}), \ldots), z_{1_1}, \ldots, z_{1 m_1}))) \\
&\quad = \sum_{i=1}^{n} \sum_{j_1 \in I_1} \cdots \sum_{j_{i-1} \in I_{i-1}} \sum_{j_{i+1} \in I_{i+1}} \cdots \sum_{j \in I_n} \varepsilon_i \delta_{j_1} \ldots \delta_{j_{i-1}} \delta_{j_{i-1}} \ldots \delta_{j_n} f(\\
&\qquad f(\ldots f(f(\ldots f(y_i, z_{j_1}) \ldots, z_{j_{i-1}}), z_{j_{i+1}}), \ldots, z_{j_n}), \\
&\qquad\ f^{-1}_{\gamma_1,\ldots,\gamma_h}(a, w_1, \ldots, w_h)).
\end{aligned} \tag{5.1}
$$

Definition of ι': Define

$$
\begin{aligned}
\iota' : \langle A \cup \{a\} \rangle \to N \\
x_0 + \sum_{i=1}^{n} \varepsilon_i f(y_i, f^{-1}_{\delta_1,\ldots,\delta_m}(a, z_1, \ldots, z_m)) \\
\mapsto \iota(x_0) + \sum_{i=1}^{n} \varepsilon_i f(\iota(y_i), f^{-1}_{\delta_1,\ldots,\delta_m}(b, \iota(z_1), \ldots, \iota(z_m)))
\end{aligned}
$$

In particular: $a \mapsto b$ and $x \mapsto \iota(x)$ if $x \in A$.

ι' is well defined: Take any $s_0 = s_1$ in $\langle A \cup \{a\} \rangle$. Then either $s_0 \in A$ and thus $\iota'(s_0) = \iota(s_0) = \iota(s_1) = \iota'(s_1)$ or we have $\sum_{i=1}^{s} \tau_i p_i \neq 0 \wedge \sum_{i=1}^{t} \alpha_i w_i \neq 0$. Since ι is an embedding, the second case implies $\sum_{i=1}^{s} \tau_i \iota(p_i) \neq 0 \wedge \sum_{i=1}^{t} \alpha_i \iota(w_i) \neq 0$ and it follows

$$
\begin{aligned}
s_0 &= s_1 \\
(x_0 - x_1) &= \sum_{i=1}^{s} \tau_i f(p_i, f^{-1}_{\alpha_1,\ldots,\alpha_t}(a, w_1, \ldots, w_t)) \\
f^{-1}_{\tau_1,\ldots,\tau_s}(x_0 - x_1, p_1, \ldots, p_s) &= f^{-1}_{\alpha_1,\ldots,\alpha_t}(a, w_1, \ldots, w_t) \\
\sum_{i=1}^{t} \alpha_i f(w_i, f^{-1}_{\tau_1,\ldots,\tau_s}(x_0 - x_1, p_1, \ldots, p_s)) &= a.
\end{aligned}
$$

Since b satisfies $p(x)$ and ι is an embedding, this is equivalent to

$$
\begin{aligned}
\iota(\sum_{i=1}^{t} \alpha_i f(w_i, f^{-1}_{\tau_1,\ldots,\tau_s}(x_0 - x_1, p_1, \ldots, p_s))) &= b \\
\sum_{i=1}^{t} \alpha_i f(\iota(w_i), f^{-1}_{\tau_1,\ldots,\tau_s}(\iota(x_0) - \iota(x_1), \iota(p_1), \ldots, \iota(p_s))) &= b
\end{aligned}
$$

$$f^{-1}_{\tau_1,\ldots,\tau_s}(\iota(x_0) - \iota(x_1), \iota(p_1), \ldots, \iota(p_s)) = f^{-1}_{\alpha_1,\ldots,\alpha_t}(b, \iota(w_1), \ldots, \iota(w_t))$$

$$\iota(x_0) - \iota(x_1) = \sum_{i=1}^{s} \tau_i f(\iota(p_i), f^{-1}_{\alpha_1,\ldots,\alpha_t}(b, \iota(w_1), \ldots, \iota(w_t)))$$

$$\iota'(s_0) = \iota'(s_1).$$

We can conclude that $s_0 = s_1$ implies $\iota'(s_0) = \iota'(s_1)$

$s_0 < s_1$ is equivalent to $\iota'(s_0) < \iota'(s_1)$: First, we show that $s_0 < s_1$ implies $\iota'(s_0) < \iota'(s_1)$. Take $s_0 < s_1$ in $\langle A \cup \{a\}\rangle$. Suppose
$s_0 = x_0 + \sum_{i=1}^{n_0} \varepsilon_i f(y_i, f^{-1}_{\delta_1,\ldots,\delta_m}(a, z_1, \ldots, z_m))$,
$s_1 = x_1 + \sum_{i=1}^{n_1}(-\mu_i) f(y_i, f^{-1}_{\gamma_1,\ldots,\gamma_k}(a, q_1, \ldots, q_k))$. Because $T1$ holds for $\mathcal{M}$, the following are equivalent

$$s_0 < s_1$$

$$(x_0 - x_1) < \sum_{i=1}^{n_0} \varepsilon_i f(y_i, f^{-1}_{\delta_1,\ldots,\delta_m}(a, z_1, \ldots, z_m)) + \sum_{i=1}^{n_1} \mu_i f(y_i, f^{-1}_{\gamma_1,\ldots,\gamma_k}(a, q_{i_1}, \ldots, q_{i_{k_i}})).$$

By the previously shown Equation 5.1, we can find $s, t \in \mathbb{N}$, $p_1, \ldots, p_s, w_1, \ldots, w_t \in P$ and $\tau_1, \ldots, \tau_s, \alpha_1, \ldots, \alpha_t$ such that

$$\sum_{i=1}^{n_0} \varepsilon_i f(y_i, f^{-1}_{\delta_1,\ldots,\delta_m}(a, z_1, \ldots, z_m)) + \sum_{i=1}^{n_1} \mu_i f(y_i, f^{-1}_{\gamma_1,\ldots,\gamma_k}(a, q_{i_1}, \ldots, q_{i_{k_i}})) = \sum_{i=1}^{s} \tau_i f(p_i, f^{-1}_{\alpha_1,\ldots,\alpha_t}(a, w_1, \ldots, w_t)).$$

If we do the same calculation for $\sum_{i=1}^{n_0} \varepsilon_i f(\iota(y_i), f^{-1}_{\delta_1,\ldots,\delta_m}(b, \iota(z_1), \ldots, \iota(z_m)))$ and $\sum_{i=1}^{n_1} \mu_i f(\iota(y_i), f^{-1}_{\gamma_1,\ldots,\gamma_k}(b, \iota(q_{i_1}), \ldots, \iota(q_{i_{k_i}})))$ in $\mathcal{N}$, we get

$$\sum_{i=1}^{n_0} \varepsilon_i f(\iota(y_i), f^{-1}_{\delta_1,\ldots,\delta_m}(b, \iota(z_1), \ldots, \iota(z_m))) + \sum_{i=1}^{n_1} \mu_i f(\iota(y_i), f^{-1}_{\gamma_1,\ldots,\gamma_k}(b, \iota(q_{i_1}), \ldots, \iota(q_{i_{k_i}}))) = \sum_{i=1}^{s} \tau_i f(\iota(p_i), f^{-1}_{\alpha_1,\ldots,\alpha_t}(b, \iota(w_1), \ldots, \iota(w_t))).$$

Then:

$$s_0 < s_1$$

$$(x_0 - x_1) < \sum_{i=1}^{s} \tau_i f(p_i, f^{-1}_{\alpha_1,\ldots,\alpha_t}(a, w_1, \ldots, w_t)).$$

Now we will use that by $T7.n$ and the definition of f^{-1}, $\sum_{i=1}^{n} \varepsilon_i f(y_i, -)$ and $f^{-1}_{\varepsilon_1,\ldots,\varepsilon_n}(-, y_1, \ldots, y_n)$ are strictly increasing if $\sum_{i=1}^{n} \varepsilon_i y_i > 0$, and strictly decreasing if $\sum_{i=1}^{n} \varepsilon_i y_i < 0$. Therefore, we do a case distinction.

Case 1.1 ($\sum_{i=1}^{s} \tau_i p_i = 0 \vee \sum_{i=1}^{t} \alpha_i w_i = 0$): We have, either by Lemma 5.18 or the definition of $f^{-1}_{\alpha_1,\ldots,\alpha_t}$, that $\sum_{i=1}^{s} \tau_i f(p_i, f^{-1}_{\alpha_1,\ldots,\alpha_t}(a, w_1, \ldots, w_t)) = 0$. Additionally, since ι is an embedding, $\sum_{i=1}^{s} \tau_i \iota(p_i) = 0 \vee \sum_{i=1}^{t} \alpha_i \iota(w_i) = 0$ and thus $\sum_{i=1}^{s} \tau_i f(\iota(p_i), f^{-1}_{\alpha_1,\ldots,\alpha_t}(b, \iota(w_1), \ldots, \iota(w_t))) = 0$. Then the following inequalities are equivalent since ι is an embedding and $T1$ holds for $\mathcal{N}$:

$$
\begin{aligned}
s_0 &< s_1 \\
(x_0 - x_1) &< 0 \\
\iota(x_0 - x_1) &< \iota(0) \\
\iota(x_0) - \iota(x_1) &< 0 \\
\iota(x_0) - \iota(x_1) &< \sum_{i=1}^{s} \tau_i f(\iota(p_i), f^{-1}_{\alpha_1,\ldots,\alpha_t}(b, \iota(w_1), \ldots, \iota(w_t))) \\
\iota(x_0) - \iota(x_1) &< \sum_{i=1}^{n_0} \varepsilon_i f(\iota(y_i), f^{-1}_{\delta_1,\ldots,\delta_m}(b, \iota(z_1), \ldots, \iota(z_m))) \\
&\qquad + \sum_{i=1}^{n_1} \mu_i f(\iota(y_i), f^{-1}_{\gamma_1,\ldots,\gamma_k}(b, \iota(q_{i_1}), \ldots, \iota(q_{i_{k_i}}))) \\
\iota'(s_0) &< \iota'(s_1).
\end{aligned}
$$

Case 1.2 ($\sum_{i=1}^{s} \tau_i p_i > 0 \wedge \sum_{i=1}^{t} \alpha_i w_i > 0$): Since ι is an embedding, we have $\sum_{i=1}^{s} \tau_i \iota(p_i) > 0 \wedge \sum_{i=1}^{t} \alpha_i \iota(w_i) > 0$. Therefore, the following inequalities are equivalent:

$$
\begin{aligned}
s_0 &< s_1 \\
(x_0 - x_1) &< \sum_{i=1}^{s} \tau_i f(p_i, f^{-1}_{\alpha_1,\ldots,\alpha_t}(a, w_1, \ldots, w_t)) \\
f^{-1}_{\tau_1,\ldots,\tau_s}(x_0 - x_1, p_1, \ldots, p_s) &< f^{-1}_{\alpha_1,\ldots,\alpha_t}(a, w_1, \ldots, w_t) \\
\sum_{i=1}^{t} \alpha_i f(w_i, f^{-1}_{\tau_1,\ldots,\tau_s}(x_0 - x_1, p_1, \ldots, p_s)) &< a.
\end{aligned}
$$

Since b satisfies $p(x)$ and ι is an embedding, this is equivalent to:

$$
\begin{aligned}
\iota(\sum_{i=1}^{t} \alpha_i f(w_i, f^{-1}_{\tau_1,\ldots,\tau_s}(x_0 - x_1, p_1, \ldots, p_s))) &< b \\
\sum_{i=1}^{t} \alpha_i f(\iota(w_i), f^{-1}_{\tau_1,\ldots,\tau_s}(\iota(x_0) - \iota(x_1), \iota(p_1), \ldots, \iota(p_s))) &< b \\
f^{-1}_{\tau_1,\ldots,\tau_s}(\iota(x_0) - \iota(x_1), \iota(p_1), \ldots, \iota(p_s)) &< f^{-1}_{\alpha_1,\ldots,\alpha_t}(b, \iota(w_1), \ldots, \iota(w_t))
\end{aligned}
$$

$$\iota(x_0) - \iota(x_1) < \sum_{i=1}^{s} \tau_i f(\iota(p_i), f^{-1}_{\alpha_1,\ldots,\alpha_t}(b, \iota(w_1), \ldots, \iota(w_t)))$$
$$\iota'(s_0) < \iota'(s_1)$$

Case 1.3 ($\sum_{i=1}^{s} \tau_i p_i > 0 \wedge \sum_{i=1}^{t} \alpha_i w_i < 0$): Since ι is an embedding, we have $\sum_{i=1}^{s} \tau_i \iota(p_i) > 0 \wedge \sum_{i=1}^{t} \alpha_i \iota(w_i) < 0$. Therefore, the following inequalities are equivalent:

$$s_0 < s_1$$
$$(x_0 - x_1) < \sum_{i=1}^{s} \tau_i f(p_i, f^{-1}_{\alpha_1,\ldots,\alpha_t}(a, w_1, \ldots, w_t))$$
$$f^{-1}_{\tau_1,\ldots,\tau_s}(x_0 - x_1, p_1, \ldots, p_s) < f^{-1}_{\alpha_1,\ldots,\alpha_t}(a, w_1, \ldots, w_t)$$
$$\sum_{i=1}^{t} \alpha_i f(w_i, f^{-1}_{\tau_1,\ldots,\tau_s}(x_0 - x_1, p_1, \ldots, p_s)) > a.$$

Since b satisfies $p(x)$ and ι is an embedding, this is equivalent to:

$$\iota(\sum_{i=1}^{t} \alpha_i f(w_i, f^{-1}_{\tau_1,\ldots,\tau_s}(x_0 - x_1, p_1, \ldots, p_s))) > b$$
$$\sum_{i=1}^{t} \alpha_i f(\iota(w_i), f^{-1}_{\tau_1,\ldots,\tau_s}(\iota(x_0) - \iota(x_1), \iota(p_1), \ldots, \iota(p_s))) > b$$
$$f^{-1}_{\tau_1,\ldots,\tau_s}(\iota(x_0) - \iota(x_1), \iota(p_1), \ldots, \iota(p_s)) < f^{-1}_{\alpha_1,\ldots,\alpha_t}(b, \iota(w_1), \ldots, \iota(w_t))$$
$$\iota(x_0) - \iota(x_1) < \sum_{i=1}^{s} \tau_i f(\iota(p_i), f^{-1}_{\alpha_1,\ldots,\alpha_t}(b, \iota(w_1), \ldots, \iota(w_t)))$$
$$\iota'(s_0) < \iota'(s_1).$$

Case 1.4 ($\sum_{i=1}^{s} \tau_i p_i < 0 \wedge \sum_{i=1}^{t} \alpha_i w_i > 0$): Since ι is an embedding, we have $\sum_{i=1}^{s} \tau_i \iota(p_i) < 0 \wedge \sum_{i=1}^{t} \alpha_i \iota(w_i) > 0$. Therefore, the following inequalities are equivalent:

$$s_0 < s_1$$
$$(x_0 - x_1) < \sum_{i=1}^{s} \tau_i f(p_i, f^{-1}_{\alpha_1,\ldots,\alpha_t}(a, w_1, \ldots, w_t))$$
$$f^{-1}_{\tau_1,\ldots,\tau_s}(x_0 - x_1, p_1, \ldots, p_s) > f^{-1}_{\alpha_1,\ldots,\alpha_t}(a, w_1, \ldots, w_t)$$
$$\sum_{i=1}^{t} \alpha_i f(w_i, f^{-1}_{\tau_1,\ldots,\tau_s}(x_0 - x_1, p_1, \ldots, p_s)) > a.$$

Since b satisfies $p(x)$ and ι is an embedding, this is equivalent to:

$$\begin{aligned}\iota(\sum_{i=1}^{t}\alpha_i f(w_i, f^{-1}_{\tau_1,\ldots,\tau_s}(x_0 - x_1, p_1, \ldots, p_s))) &> b \\ \sum_{i=1}^{t}\alpha_i f(\iota(w_i), f^{-1}_{\tau_1,\ldots,\tau_s}(\iota(x_0) - \iota(x_1), \iota(p_1), \ldots, \iota(p_s))) &> b \\ f^{-1}_{\tau_1,\ldots,\tau_s}(\iota(x_0) - \iota(x_1), \iota(p_1), \ldots, \iota(p_s)) &> f^{-1}_{\alpha_1,\ldots,\alpha_t}(b, \iota(w_1), \ldots, \iota(w_t)) \\ \iota(x_0) - \iota(x_1) &< \sum_{i=1}^{s}\tau_i f(\iota(p_i), f^{-1}_{\alpha_1,\ldots,\alpha_t}(b, \iota(w_1), \ldots, \iota(w_t))) \\ \iota'(s_0) &< \iota'(s_1).\end{aligned}$$

Case 1.5 ($\sum_{i=1}^{s}\tau_i p_i < 0 \wedge \sum_{i=1}^{t}\alpha_i w_i < 0$): Since ι is an embedding, we have $\sum_{i=1}^{s}\tau_i\iota(p_i) < 0 \wedge \sum_{i=1}^{t}\alpha_i\iota(w_i) < 0$. Therefore, the following inequalities are equivalent:

$$\begin{aligned}s_0 &< s_1 \\ (x_0 - x_1) &< \sum_{i=1}^{s}\tau_i f(p_i, f^{-1}_{\alpha_1,\ldots,\alpha_t}(a, w_1, \ldots, w_t)) \\ f^{-1}_{\tau_1,\ldots,\tau_s}(x_0 - x_1, p_1, \ldots, p_s) &> f^{-1}_{\alpha_1,\ldots,\alpha_t}(a, w_1, \ldots, w_t) \\ \sum_{i=1}^{t}\alpha_i f(w_i, f^{-1}_{\tau_1,\ldots,\tau_s}(x_0 - x_1, p_1, \ldots, p_s)) &< a.\end{aligned}$$

Since b satisfies $p(x)$ and ι is an embedding, this is equivalent to:

$$\begin{aligned}\iota(\sum_{i=1}^{t}\alpha_i f(w_i, f^{-1}_{\tau_1,\ldots,\tau_s}(x_0 - x_1, p_1, \ldots, p_s))) &< b \\ \sum_{i=1}^{t}\alpha_i f(\iota(w_i), f^{-1}_{\tau_1,\ldots,\tau_s}(\iota(x_0) - \iota(x_1), \iota(p_1), \ldots, \iota(p_s))) &< b \\ f^{-1}_{\tau_1,\ldots,\tau_s}(\iota(x_0) - \iota(x_1), \iota(p_1), \ldots, \iota(p_s)) &> f^{-1}_{\alpha_1,\ldots,\alpha_t}(b, \iota(w_1), \ldots, \iota(w_t)) \\ \iota(x_0) - \iota(x_1) &< \sum_{i=1}^{s}\tau_i f(\iota(p_i), f^{-1}_{\alpha_1,\ldots,\alpha_t}(b, \iota(w_1), \ldots, \iota(w_t))) \\ \iota'(s_0) &< \iota'(s_1).\end{aligned}$$

We can conclude that $s_0 < s_1$ implies $\iota'(s_0) < \iota'(s_1)$.

$\neg(s_0 < s_1)$ implies $s_0 > s_1 \vee s_0 = s_1$. Thus, as just shown, $\iota'(s_0) > \iota'(s_1) \vee \iota'(s_0) = \iota'(s_1)$ and $\neg(\iota'(s_0) < \iota'(s_1))$.

Therefore, $s_0 < s_1$ is indeed equivalent to $\iota'(s_0) < \iota'(s_1)$.

ι′ is injective: By Lemma 5.13, the domain of ι' is a submodel of $\mathcal{M}$. Take $s_0 \neq s_1$ in $\langle A \cup \{a\} \rangle$. W.l.o.g. assume $s_0 < s_1$. As just shown this implies $\iota'(s_0) < \iota'(s_1)$ and thus indeed $\iota'(s_0) \neq \iota'(s_1)$.

ι′ is an embedding: constant symbols: Since $\mathscr{A}$ is a model of the theory, we have $0 \in A$ and $1 \in A$. ι is an embedding, therefore $\iota(0) = 0$ and $\iota(1) = 1$. Thus $\iota'(0) = 0$ and $\iota'(1) = 1$.

function symbols: We know $\lambda(s_0), (\lambda(s_0) + \lambda(s_0)) \in P(A)$ for all $s_0 \in \langle A \cup \{a\} \rangle$. Thus $\iota'(\lambda(s_0)) = \iota(\lambda(s_0)) \in P$ and $\iota'(\lambda(s_0) + \lambda(s_0)) = \iota(\lambda(s_0) + \lambda(s_0))$. We have $\lambda(s_0) \leq s_0 < \lambda(s_0) + \lambda(s_0)$ and thus $\iota(\lambda(s_0)) \leq \iota'(s_0) < \iota'(\lambda(s_0) + \lambda(s_0)) = \iota(\lambda(s_0) + \lambda(s_0)) = \iota(\lambda(s_0)) + \iota(\lambda(s_0))$. Being an element of P and fulfilling this property uniquely characterizes $\lambda(\iota'(s_0))$.

Thus $\lambda(\iota'(s_0)) = \iota(\lambda(s_0)) = \iota'(\lambda(s_0))$.

$$
\begin{aligned}
\iota'(s_1 + s_0) &= \iota'(x_0 + \sum_{i=1}^{n_0} \varepsilon_i f(y_i, f^{-1}_{\delta_1,\ldots,\delta_m}(a, z_1, \ldots, z_m)) \\
&\quad + x_1 + \sum_{i=1}^{n_1} \mu_i f(p_i, f^{-1}_{\gamma_1,\ldots,\gamma_k}(a, q_1, \ldots, q_k))) \\
&= \iota'(x_0 + x_1 + \sum_{i=1}^{n_0} \varepsilon_i f(y_i, f^{-1}_{\delta_1,\ldots,\delta_m}(a, z_1, \ldots, z_m)) \\
&\quad + \sum_{i=1}^{n_1} \mu_i f(p_i, f^{-1}_{\gamma_1,\ldots,\gamma_k}(a, q_1, \ldots, q_k))) \\
&= \iota(x_0 + x_1) + \sum_{i=1}^{n_0} \varepsilon_i f(\iota(y_i), f^{-1}_{\delta_1,\ldots,\delta_m}(b, \iota(z_1), \ldots, \iota(z_m))) \\
&\quad + \sum_{i=1}^{n_1} \mu_i f(\iota(p_i), f^{-1}_{\gamma_1,\ldots,\gamma_k}(b, \iota(q_1), \ldots, \iota(q_k))) \\
&= \iota(x_0) + \iota(x_1) + \sum_{i=1}^{n_0} \varepsilon_i f(\iota(y_i), f^{-1}_{\delta_1,\ldots,\delta_m}(b, \iota(z_1), \ldots, \iota(z_m))) \\
&\quad + \sum_{i=1}^{n_1} \mu_i f(\iota(p_i), f^{-1}_{\gamma_1,\ldots,\gamma_k}(b, \iota(q_1), \ldots, \iota(q_k))) \\
&= \iota(x_0) + \sum_{i=1}^{n_0} \varepsilon_i f(\iota(y_i), f^{-1}_{\delta_1,\ldots,\delta_m}(b, \iota(z_1), \ldots, \iota(z_m))) \\
&\quad + \iota(x_1) + \sum_{i=1}^{n_1} \mu_i f(\iota(p_i), f^{-1}_{\gamma_1,\ldots,\gamma_k}(b, \iota(q_1), \ldots, \iota(q_k))) \\
&= \iota'(s_0) + \iota'(s_1)
\end{aligned}
$$

The proof for $-$ follows similarly.

For f, first show that $P(s_0)$ holds if and only if $P(\iota'(s_0))$ holds. $P(s_0)$ holds if and only if $s_0 \in A$. This implies $\iota'(s_0) = \iota(s_0) \in P$. Suppose $P(\iota'(s_0))$ holds. Then $\iota'(s_0) = \lambda(\iota'(s_0)) = \iota'(\lambda(s_0))$. Since ι' is injective, we have $s_0 = \lambda(s_0)$ and thus $s_0 \in P$.

Now we can apply this statement: $\iota'(f(s_0, s_1)) = 0 = \iota'(0) = f(\iota'(s_0), \iota'(s_1))$ if $s_0 \notin P$ and therefore $\iota'(s_0) \notin P$. If $s_0 \in P$ and thus $s_0 \in A$, we have

$$\begin{aligned}\iota'(f(s_0, s_1)) &= \iota'(f(s_0, x_1 + \sum_{i=1}^{n_1} \mu_i f(p_i, f^{-1}_{\gamma_1,\dots,\gamma_k}(a, q_1, \dots, q_k)))) \\ &= \iota'(f(s_0, x_1) + \sum_{i=1}^{n_1} \mu_i f(s_0, f(p_i, f^{-1}_{\gamma_1,\dots,\gamma_k}(a, q_1, \dots, q_k)))) \\ &= \iota(f(s_0, x_1)) + \sum_{i=1}^{n_1} \mu_i f(\iota(s_0), f(\iota(p_i), f^{-1}_{\gamma_1,\dots,\gamma_k}(b, \iota(q_1), \dots, \iota(q_k)))) \\ &= f(\iota(s_0), \iota(x_1)) + f(\iota(s_0), \sum_{i=1}^{n_1} \mu_i f(\iota(p_i), f^{-1}_{\gamma_1,\dots,\gamma_k}(b, \iota(q_1), \dots, \iota(q_k)))) \\ &= f(\iota(s_0), \iota(x_1) + \sum_{i=1}^{n_1} \mu_i f(\iota(p_i), f^{-1}_{\gamma_1,\dots,\gamma_k}(b, \iota(q_1, \dots, \iota(q_k))))) \\ &= f(\iota'(s_0), \iota'(s_1))\end{aligned}$$

relation symbols: $s_0 < s_1$ is equivalent to $\iota'(s_0) < \iota'(s_1)$ and $P(s_0)$ holds if and only if $P(\iota'(s_0))$ holds (see the previous two paragraphs).

Let s_0 be in $\langle A \cup \{a\}\rangle$. Since $\langle A \cup \{a\}\rangle \models T12$, $R_n(s_0)$ holds if and only if $P(s_0)$, and there is some $s^* \in \langle A \cup \{a\}\rangle$ with $P(s^*)$ and $\underbrace{f(s^*, f(s^*, \dots f(s^*,}_{n \text{ times}} 1 \underbrace{) \dots))}_{n \text{ times}} = s_0$. This implies $s_0, s^* \in A$ and since ι is an embedding, we get

$$\underbrace{f(\iota(s^*), f(\iota(s^*), \dots f(\iota(s^*),}_{n \text{ times}} 1 \underbrace{) \dots))}_{n \text{ times}} = \iota(s_0) \wedge P(\iota(s_0)) \wedge P(\iota(s^*)).$$

Thus,

$$\underbrace{f(\iota'(s^*), f(\iota'(s^*), \dots f(\iota'(s^*),}_{n \text{ times}} 1 \underbrace{) \dots))}_{n \text{ times}} = \iota'(s_0) \wedge P(\iota'(s_0)) \wedge P(\iota'(s^*))$$

and therefore $R_n^{\mathcal{N}}(\iota'(s_0))$ holds. Suppose $R_n(\iota'(s_0))$ holds. Then $P(\iota'(s_0))$ holds, $P(s_0)$ holds and $s_0 \in A$. Thus $\iota'(s_0) = \iota(s_0)$. Therefore $R_n(\iota'(s_0))$ is equivalent to $R_n(\iota(s_0))$ and since ι is an embedding, we have that $R_n(\iota(s_0))$ implies that $R_n(s_0)$ holds.

Case 2 ($a \in P$): This means that $P(a)$ holds in $\mathcal{M}$. Since $a \notin A$, it follows that $P(A) \neq P(A \cup \{a\})$.

Definition of $p(x)$: Define the type

$$p(x) = \{P(x)\} \cup \{\iota(c) < x : c <^{\mathcal{M}} a, c \in A\} \cup \{x < \iota(d) : a <^{\mathcal{M}} d, d \in A\}$$
$$\cup \{R_n(f(2^j, x^{\varepsilon z})) : (\mathcal{M} \models R_n(2^j, a^{\varepsilon z})) \wedge j \in \{0, \dots, n-1\}, \varepsilon \in \{-1, 1\}, z \in \mathbb{N}^*\}$$

Then $|p(x)| \leq \max\{\kappa, \mathbb{N}^*\} = \kappa$ since, by $T_{Pr}5.n$, $\mathcal{M} \models R_n(2^j, a)$ for exactly one $j \in \{0, \ldots, n-1\}$.

Since $P(a) = R_1(a)$ holds, we have $\{R_n(x) : \mathcal{M} \models R_n(a)\} \neq \emptyset$.

$p(x)$ is finitely satisfied in $\mathcal{N}$: Let $p'(x)$ be a finite subset of $p(x)$.

Assume $\varepsilon = z = 1$ for all $R_n(f(2^j, x^{\varepsilon z})) \in p'(x)$.

Take any $m, t, l_0, l_1, \ldots, l_t, k \in \mathbb{N}_0$. Let $c_1, \ldots, c_k, d_1, \ldots d_m$ be in A such that $(c_i < x) \in p'(x)$ and $(x < d_j) \in p'(x)$ for all $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, m\}$. Thus, $c_i < a$ and $a < d_j$ for all $i \in \{1, \ldots, k\}, j \in \{1, \ldots, m\}$.

Let $R_{n_1}(x), \ldots, R_{n_{l_0}}(x) \in p'(x)$, and for $j \in \{1, \ldots, t\}$ let
$R_{n_{1+\sum_{i=0}^{j-1} l_i}}(f(x, 2^j)), \ldots, R_{n_{\sum_{i=0}^{j} l_i}}(f(x, 2^j)) \in p'(x)$.

We can assume that $l_1 \neq 0$ since $\{R_n(x) : \mathcal{M} \models R_n(a)\} \neq \emptyset$ and that $l_j \neq 0$ for $j \in \{1, \ldots, t\}$ since $\mathcal{M} \models R_n(2^j, a)$ for exactly one $j \in \{0, \ldots, n-1\}$.

Assume there are no other elements in $p'(x)$.

Suppose $m, k \neq 0$. Then $\max\{c_1, \ldots, c_k\} < a < \min\{d_1, \ldots, d_m\}$. Define $e \coloneqq \min\{d_1, \ldots, d_m\}$. Since $P(a)$ holds, we have $a > 0$ and thus $0 < e$, $0 < \lambda(e) \leq e$ and $\lambda(e) \in P$ by $T5$. Moreover, since $P(a)$ holds and because $\lambda(x)$ or respectively $f(2, \lambda(x))$ are the closest elements to x that are in $P \cup \{0\}$, it follows that $\max\{c_1, \ldots, c_k\} \leq f(2, \lambda(\max\{c_1, \ldots, c_k\})) \leq a \leq \lambda(e) \leq e$.

Then $d_1, \ldots, d_m \in A$ implies $e = \min\{d_1, \ldots, d_m\} \in A$. Since $A \models T$, we have $\lambda(e) \in A$ and since $c_1, \ldots c_k \in A$ we have $\lambda(\max\{c_1, \ldots, c_k\}) \in A$ and $f(2, \lambda(\max\{c_1, \ldots, c_k\})) \in A$.

Let $n_0^* \coloneqq \operatorname{lcm}(n_1, \ldots, n_l)$ and $n_j^* \coloneqq \operatorname{lcm}(n_{(1+\sum_{i=0}^{j-1} l_i)}, \ldots, n_{(\sum_{i=0}^{j} l_i)})$ for $j \in \{1, \ldots, t\}$. By Lemma 4.6, $R_{n_1^*}(a)$ and $R_{n_j^*}(f(a, 2^j))$ hold for $j \in \{1, \ldots, t\}$.

Define

$$s_0 \coloneqq \varphi_{n_0^*}(\lambda(e)) \in P \text{ and } a_{-1,0} \coloneqq \underbrace{f(s_0, f(s_0, \ldots f(s_0}_{n_0^* \text{ times}}, 1 \underbrace{) \ldots))}_{n_0^* \text{ times}}$$

and for $j \in \{1, \ldots t\}$ define

$$s_j \coloneqq \varphi_{n_j^*}(f(\lambda(e), 2^j)) \in P \text{ and } a_{-1,j} \coloneqq \underbrace{f(s_j, f(s_j, \ldots f(s_j}_{n_j^* \text{ times}}, 1 \underbrace{) \ldots))}_{n_j^* \text{ times}}.$$

Define

$$a_{0,0} \coloneqq \begin{cases} a_{-1,0} & \text{if } a_{-1,0} \neq e, \\ f_1^{-1}(a_{-1,0}, 2^{n_0^*}) & \text{else} \end{cases}$$

and

$$a_{0,j} \coloneqq \begin{cases} f_1^{-1}(a_{-1,j}, 2^j) & \text{if } f_1^{-1}(a_{-1,j}, 2^j) \neq e, \\ f_1^{-1}(f_1^{-1}(a_{-1,j}, 2^j), 2^{n_j^*}) & \text{else.} \end{cases}$$

Define for all $i \in \{1, \ldots, n_j^* \prod_{v \neq j} (n_v^*)^2\}$ and all $j \in \{1, \ldots, t\}$

$$a_{i,j} := f_1^{-1}(a_{i-1,j}, 2^{n_j^*}).$$

Define $m_j^* := n_j^* \prod_{v \neq j} (n_v^*)^2$, $n^* := \prod_v n_v^*$ and $m^* := \prod_v (n_v^*)^2 = (n^*)^2 = n_j^* m_j^*$. Since A is a submodel of $\mathcal{M}$, φ_n is a definable function and $\lambda(e) \in A$, we have $a_{i,j} \in A$ for all $i \in \{-1, 0, 1, \ldots, m_j^*\}$ and all $j \in \{1, \ldots, t\}$. It is easy to see that $R_{n_j^*}(f(a_{i,j}, 2^j))$ holds for $i \in \{-1, 0, 1, \ldots, m_j^*\}$ and $j \in \{0, \ldots, t\}$. Moreover, for any $j \in \{0, \ldots, t\}$ there is no $u_j \in M$ such that $a_{m_j^*,j} \leq u_j < e$, $R_{n_j^*}(f(u_j, 2^j))$ holds and $u_j \neq a_{i,j}$ for all $i \in \{-1, 0, 1, \ldots, m_j^*\}$.

Claim: $X := \bigcap_{j=0}^t \{a_{i,j} : i \in \{0, 1, \ldots, \prod_{v \neq j} n_v^*\}\} \neq \emptyset$.

Proof of claim: Since $R_{n_j^*}(f(a, 2^j))$ holds for all j, we also have that

$$R_{n_j^*}(f_1^{-1}(f(a, 2^j), (\varphi_{m^*}(f(2, a)))^{m^*})).$$

Moreover,

$$R_{n_j^*}(f(f_1^{-1}(f(a, 2^j), (\varphi_{m^*}(f(2, a))^{m^*})), (\varphi_{m^*}(a_{0,j}))^{m^*}))$$

and thus for every j

$$R_{n_j^*}(f(f(f_1^{-1}(a, (\varphi_{m^*}(f(2, a)))^{m^*}), (\varphi_{m^*}(a_{0,j}))^{m^*}), 2^j)).$$

With a short calculation, one can show that

$$2^{(-m^*)} \leq (f_1^{-1}(a, (\varphi_{m^*}(f(2, a)))^{m^*})) \leq 1$$

and

$$\begin{aligned} f(2^{(-m^*)}, (\varphi_{m^*}(a_{0,j}))^{m^*}) & \\ &\leq f(f_1^{-1}(a, (\varphi_{m^*}(a))^{m^*}), (\varphi_{m^*}(a_{0,j}))^{m^*}) \\ &\qquad \leq (\varphi_{m^*}(a_{0,j}))^{m^*}. \end{aligned}$$

(To show this, note that we get $1 \leq f^{-1}(x, (\varphi_{m^*}(x))^{m^*}) \leq 2^{m^*}$ with the definition and the axioms.) Since $f(2^{(-m^*)}, (\varphi_{m^*}(a_{0,j}))^{m^*}) \geq f(2^{-m^*}, f(2^{-m^*}, a_{0,j})) = f(2^{-m^*-m^*}, a_{0,j})$ and $(\varphi_{m^*}(a_{0,j}))^{m^*} \leq a_{0,j}$, we have for every j

$$f(2^{-m^*-m^*}, a_{0,j}) \leq f(f_1^{-1}(a, (\varphi_{m^*}(a))^{m^*}), (\varphi_{m^*}(a_{0,j}))^{m^*}) \leq a_{0,j}.$$

By definition $a_{j,m_j^*} = f(2^{-m^*-m^*}, a_{0,j})$. Thus,

$$f(f_1^{-1}(a, (\varphi_{m^*}(a))^{m^*}), (\varphi_{m^*}(a_{0,j}))^{m^*}) \in X.$$

Choose $a' := \max X$. By the definition of $a_{i,j}$, we have $a' < e$ and $R_{n_j^*}(a', 2^j)$ holds for all $j \in \{0, \ldots, t\}$. By Lemma 4.6, $R_{n_j^*}(a', 2^j)$ implies that

$R_{n_{1+\sum_{i=0}^{j-1} l_i}}(f(a', 2^j)), \ldots, R_{n_{\sum_{i=0}^{j} l_i}}(f(a', 2^j))$ hold for all $j \in \{0, 1, \ldots, t\}$.
By maximality and since the same has to hold for a, we get $a < a'$.

We conclude

$$\max\{c_1, \ldots, c_k\} \leq f(2, \lambda(\max\{c_1, \ldots, c_k\})) \leq a \leq a' < e.$$

Because ι is an embedding, it follows

$$\iota(\max\{c_1, \ldots, c_k\}) \leq \iota(f(2, \lambda(\max\{c_1, \ldots, c_k\}))) \leq \iota(a') < \iota(e).$$

$P(\iota(a'))$ holds since $P(a')$ holds and $R_n(\iota(a'))$ holds if and only if $R_n(a')$.

$\iota(\max\{c_1, \ldots, c_k\}) = \iota(f(2, \lambda(\max\{c_1, \ldots, c_k\})))$ if and only if
$\iota(\max\{c_1, \ldots, c_k\}) = 0$. However, $a' \neq 0$ and thus, $\iota(a') \neq 0$.
Therefore, $\iota(max\{c_1, \ldots, c_k\}) < a'$.

We can conclude $b' := \iota(a')$ is a finite realisation of $p(x)$ in $\mathcal{N}$. This means that $p'(b')$ holds.

If $k = 0$, we can do the same calculation to find a b' with $p'(b')$. We just do not have a lower bound that b' needs to fulfill.

If $k \neq 0$ and $m = 0$, define $e_2 := f(2, \lambda(\max\{1, c_1, \ldots, c_k\}))$.

Then $\max\{1, c_1, \ldots, c_k\} \geq 1 > 0$ implies $e_2 \in P$. Moreover, $e > \max\{c_1, \ldots, c_k\}$.

By $T_{Pr}5.n.$, there is some $i_0^* \in \{0, \ldots, n_0^*\}$ such that $R_{n_0^*}(f(2^{i_0^*}, e_2))$ holds. Define $a_0' := f(2^{i^*}, e_2)$.

Define i_j^* to be an element of $\{0, \ldots, n_j^*\}$ such that $R_{n_j^*}(f(f(a_{j-1}', 2^{i_j^* \prod_{r=0}^{j-1} n_r^*}), 2^j))$ holds.

It is possible to show with a similar calculation like in the setting where $m \neq 0$ that such a i_j^* indeed has to exist if $R_{n_j^*}(f(a, 2^j))$ holds for all j.

Define $a_j' := f(a_{j-1}', 2^{i_j^* \prod_{r=0}^{j-1} n_r^*})$. Then $b' := a_t'$ is a finite realization of $p(x)$ in $\mathcal{N}$.

We can assume that either $k \neq 0$ or $m \neq 0$ since $1 \in A$ and either $a > 1$ or $a < 1$.

Suppose $\neg(\varepsilon_i = z_i = 1)$ for some i. If $f(2, \lambda(\max\{c_1, \ldots, c_k\})) = a$, then clearly $\iota(f(2, \lambda(\max\{c_1, \ldots, c_k\})))$ is a realization of $p'(x)$ in $\mathcal{N}$. If $\lambda(e) = a$, then $\iota(\lambda(e))$ is a realization of $p'(x)$ in $\mathcal{N}$. Thus, assume in the following that
$f(2, \lambda(\max\{c_1, \ldots, c_k\})) \neq a$ and $e \neq a$.

Note that $R_n(f(x, x_1))$ holds if and only if $R_{zn}(f(x^{\varepsilon z}, (x_1)^{\varepsilon z}))$ holds, and for $p, q, x \in P \cup \{0\}$ we have $p < x \leq q$ if and only if $p^{\varepsilon z} < x^{\varepsilon z} \leq q^{\varepsilon z}$. Define $z^* := \prod_i z_i$. For any i^* define $\bar{n} := \frac{nz^*}{z_{i^*}}$, $\bar{j} := \frac{jz^*}{z_{i^*}}$. Then $R_n(f(a^{\varepsilon_{i^*} z_{i^*}}, 2^j))$ holds if and only if $R_{\bar{n}}(f(a^{z^*}, 2^{\bar{j}}))$ holds. $f(2, \lambda(\max\{c_1, \ldots, c_k\})) \leq a \leq \lambda(e)$ if and only if $(f(2, \lambda(\max\{c_1, \ldots, c_k\})))^{z^*} \leq a^{z^*} \leq (\lambda(e))^{z^*}$.

Define a new finite type

$$q_1(y) = \{R_{z^*}(y)\} \cup \{R_{\bar{n}}(f(y, 2^{\bar{j}})) : R_n(f(x^{\varepsilon_{i^*} z_{i^*}}, 2^j)) \in p'(x)\}$$
$$\cup \{(f(2, \lambda(\max\{c_1, \dots, c_k\})))^{z^*} < y < \lambda(e)^{z^*}\}.$$

Then clearly $q(a^{z^*})$ holds. Since this type is of the same form as the one above, we can apply the previous calculation to get $b' \in N$ satisfying $q(y)$. Because $R_{z^*}(b')$ holds, it is easy to see with the previous calculations that $p'(\varphi_{z^*}(b'))$ holds.

Defininition of b: Since $\mathcal{N}$ is κ^+-saturated and $p(x)$ is finitely satisfiable in $\mathcal{N}$, $p(x)$ is satisfiable in $\mathcal{N}$. Let b be a realization of $p(x)$ in $\mathcal{N}$.

Definition of G and ζ: Let G be be the closure under f and f_1^{-1} of $P(A)$ and a. Then $G \subset P(M)$ since $P(A) \subset P(M), a \in P(M)$ and $P(M)$ is a $\mathbb{Z}$-group and thus closed under f and f_1^{-1}.

We have

$$(P(M), (R_n \restriction_{P(M)})_{n\in\mathbb{N}^*}, f \restriction_{P(M)^2}, <\restriction_{P(M)^2}, 1, 1+1) \models T_{Pr}$$

and thus

$$(P(M), (R_n \restriction_{P(M)})_{n\in\mathbb{N}^*}, f \restriction_{P(M)^2}, <\restriction_{P(M)^2}, 1, 1+1) \models T^*_{Pr}.$$

$(G, (R_n \restriction_G)_{n\in\mathbb{N}^*}, f \restriction_{G^2}, <, 1, 1+1)$ is well defined since G is closed under f and $1, 1+1 \in P(A) \subset G$. All the axioms from T^*_{Pr} except for $T_{ablo}3.$ are universal and thus also hold for $(G, (R_n \restriction_G)_{n\in\mathbb{N}^*}, f \restriction_{G^2}, <\restriction_{G^2}, 1, 1+1)$ since they hold for $(P(M), (R_n \restriction_{P(M)})_{n\in\mathbb{N}^*}, f \restriction_{P(M)^2}, <, 1, 1+1)$. $T_{ablo}3.$ (which states the existence of an inverse) is satisfied since we assumed that G is closed under f_1^{-1}.

It is easy to show that all elements of G are of the form $f(a^{\varepsilon n}, a_1)$ with $a_1 \in P(A)$, $n \in N_0$ and $\varepsilon \in \{1, -1\}$. (for the notation see Notation 5.1).

Define $\zeta : G \to N, f(a^{\varepsilon n}, a_1) \mapsto f(b^{\varepsilon n}, \iota(a_1))$. We can show that this is an embedding in the language $\{(R_n)_{n\in\mathbb{N}^*}, f, <, 1, 1+1\}$:

Let $f(a^{z_1}, a_1), f(a^{z_2}, a_2) \in G$ be two arbitrary elements of G. Then the following are equivalent

$$\begin{aligned} f(a^{z_1}, a_1) &< f(a^{z_2}, a_2) \\ f(a_1, a_2^{-1}) &< a^{z_2 - z_1} \\ \varphi_{z_2-z_1}(f(a_1, a_2^{-1})) &< a \\ \iota(\varphi_{z_2-z_1}(f(a_1, a_2^{-1}))) &< b \\ \varphi_{z_2-z_1}(f(\iota(a_1), \iota(a_2)^{-1})) &< b \\ f(\iota(a_1), \iota(a_2)^{-1}) &< b^{z_2-z_1} \\ f(b^{z_1}, \iota(a_1)) &< f(b^{z_2}, \iota(a_2)) \\ \zeta(f(a^{z_1}, a_1)) &< \zeta(f(a^{z_2}, a_2)) \end{aligned} \tag{5.2}$$

since first of all $P(M)$ and $P(N)$ are linear ordered abelian groups, secondly $\varphi_{z_2-z_1}$ is monotone and $x^{z_2-z_1}$ is the smallest element such that $\varphi_{z_2-z_1}(x^{z_2-z_1}) = x$, thirdly b satisfies $p(x)$ and lastly ι is an embedding.

ζ well definded and injective: Let s_1, s_2 be in G. $s_1 = s_2$ if and only if $(\neg s_1 < s_2) \wedge (\neg s_1 > s_2)$. By the previously proven statement this holds if and only if $(\neg\zeta(s_1) < \zeta(s_2)) \wedge (\neg\zeta(s_1) > \zeta(s_2))$ which holds if and only if $\zeta(s_1) = \zeta(s_2)$.

constant symbols: $\zeta(1+1) = \iota(1+1) = \iota(1) + \iota(1) = 1+1$ and $\zeta(1) = \iota(1) = 1$.

function symbols: Let $s_1 := f(a^{z_1}, a_1), s_2 := f(a^{z_2}, a_2)$ be in G.

$$\begin{aligned} f(\zeta(s_1), \zeta(s_2)) &= f(\zeta(f(a^{z_1}, a_1)), \zeta(f(a^{z_2}, a_2))) \\ &= f(f(b^{z_1}, \iota(a_1)), f(b^{z_2}, \iota(a_2))) \\ &= f(f(b^{z_1}, b^{z_2}), f(\iota(a_1), \iota(a_2))) \\ &= f(b^{z_1+z_2}, \iota(f(a_1, a_2))) \\ &= \zeta(f(a^{z_1+z_2}, f(a_1, a_2))) \\ &= \zeta(f(f(a^{z_1}, a_1), f(a^{z_2}, a_2))) \\ &= \zeta(f(s_1, s_2)) \end{aligned}$$

relation symbols: $s_1 < s_2$ if and only if $\zeta(s_1) < \zeta(s_2)$ was shown in Equation 5.2.

Let $s := f(a^z, a_1)$ be in G. $R_n(s)$ holds if and only if $R_n(f(a^z, a_1))$ holds. If $z = 0$, we can apply the fact that ι is an embedding and get that $R_n(a_1)$ holds if and only if $R_n(f(\iota(a_1)))$ holds. Thus, for $z = 0$, $R_n(s)$ holds if and only if $R_n(\zeta(s))$ holds. Suppose $z \neq 0$. By $T_{Pr}5.n.$, there is a unique $j^* \in \{0, \ldots, n\}$ such that $R_n(f(a_1, 2^{-j^*}))$ holds. Thus, by $T^*_{Pr}6.n.$, $R_n(f(a^z, a_1))$ holds if and only if $R_n(f(a^z, 2^{j^*}))$ holds. Since b satisfies $p(x)$, this is equivalent to $R_n(f(b^z, 2^{j^*}))$ and since ι is an embedding $R_n(f(a_1, 2^{-j^*}))$ is equivalent to $R_n(f(\iota(a_1), 2^{-j^*}))$. We can conclude that $R_n(f(a^z, a_1))$ holds if and only if $R_n(f(b^z, \iota(a_1)))$ holds (i.e. $R_n(\zeta(f(a^z, a_1)))$ holds).

Definition of P_1 and ξ: Apply Corollary 4.9 to G. This gives us a $\mathbb{Z}$-group $P_1 \models T_{Pr}$ with $G \subset P_1 \subset P(M)$ and thus $P(A) \subset P_1$ and $a \in P_1$. Moreover, there is some embedding

$$\xi : P_1 \to P(N) \tag{5.3}$$

such that $\xi \restriction_G = \zeta$.

Elements of $\langle A \cup P_1 \rangle$: Since $A \subset M$ and $P_1 \subset P(M) \subset M$, we have $\langle A \cup P_1 \rangle \subset M$.

In the following, we will proof the claim that all elements of $\langle A \cup P_1 \rangle$ are of the form $f^{-1}_{\delta_1,\ldots,\delta_n}(\sum_{j=1}^m \varepsilon_j f(y_j, x_j), z_1, \ldots, z_n)$ with $x_j \in A$, $z_1, \ldots, z_n, y_1, \ldots, y_m \in P_1$, $\delta_j, \varepsilon_j \in \{1, -1\}$ and $\sum_{i=1}^n \delta_i z_i \neq 0$.

Clearly each element of this form must be in $\langle A \cup P_1 \rangle$.

Let S be the set of all elements of the form $f^{-1}_{\delta_1,\ldots,\delta_n}(\sum_{j=1}^m \varepsilon_j f(y_j, x_j), z_1, \ldots, z_n)$ with $n, m \in \mathbb{N}^*$, $x_j \in A$, $z_1, \ldots, z_n, y_1, \ldots, y_m \in P_1$, $\delta_j, \varepsilon_j \in \{1, -1\}$ and $\sum_{i=1}^n \delta_i z_i \neq 0$.

Then we can show that $P(S) = P_1$: Let $f^{-1}_{\delta_1,\ldots,\delta_n}(\sum_{j=1}^m \varepsilon_j f(y_j, x_j), z_1, \ldots, z_n)) \in S$. Either $\lambda(f^{-1}_{\delta_1,\ldots,\delta_n}(\sum_{j=1}^m \varepsilon_j f(y_j, x_j), z_1, \ldots, z_n)) = 0$ or

$f^{-1}_{\delta_1,\dots,\delta_n}(\sum_{j=1}^m \varepsilon_j f(y_j,x_j),z_1,\dots,z_n) > 0$. In the following assume $f^{-1}_{\delta_1,\dots,\delta_n}(\sum_{j=1}^m \varepsilon_j f(y_j,x_j),z_1,\dots,z_n) > 0$. We can assume that $x_1,\dots,x_n \geq 0$ since $\varepsilon_i f(y_i,x_i) = (-\varepsilon_i)f(y_i,(-x_i))$. We can further assume $x_1,\dots,x_n \neq 0$ since $f(y_i,0)=0$. We can choose $z_1,\dots,z_m$ to be distinct and $y_1,\dots,y_m$ to be distinct modulo $P(A)$ (i.e. $f_1^{-1}(y_i,y_j) \notin P(A)$ if $i \neq j$). To show this, we do a similar calculation like in the proof of Lemma 5.18: Suppose $y_i,y_j \in P_1$, $x_i,x_j \in A$, $i \neq j$ and $p \coloneqq f_1^{-1}(y_i,y_j) \in P(A)$. Denote with p^{-1} the unique inverse of p in the $\mathbb{Z}$-group $P(A)$. We have

$$\begin{aligned} f(y_i,x_i)+f(y_j,x_j) &= f(y_i,x_i)+f(f(y_j,1),x_j) \\ &= f(y_i,x_i)+f(f(y_j,f(p,p^{-1})),x_j) \\ &= f(y_i,x_i)+f(f(y_j,p),f(p^{-1},x_j)) \\ &= f(y_i,x_i)+f(y_i,f(p^{-1},x_j)) \\ &= f(y_i,x_i+f(p^{-1},x_j)) \end{aligned}$$

with $y_i \in P_1$ and $(x_i+f(p^{-1},x_j)) \in A$.

For $z_i = z_j$ and $\delta_i = \delta_j$, we have for all x that $\delta_i f(z_i,x)+\delta_j f(z_j,x) = \delta_i f(z_i+z_i,x)$ and thus

$$\begin{aligned} &f^{-1}_{\delta_1,\dots,\delta_i,\dots,\delta_j,\dots,\delta_n}(x,z_1,\dots,z_n) \\ &\qquad = f^{-1}_{\delta_1,\dots,\delta_i,\dots,\delta_{j-1},\delta_{j+1}\dots,\delta_n}(x,z_1,\dots,z_{i-1},z_i+z_i,z_{i+1}\dots,z_{j-1},z_{j+1},\dots,z_n). \end{aligned}$$

If $z_i = z_j$ and $\delta_i \neq \delta_j$, we have $\delta_i f(z_i,x)+\delta_j f(z_j,x) = f(z_i,x)-f(z_i,x) = 0$ and thus

$$\begin{aligned} &f^{-1}_{\delta_1,\dots,\delta_i,\dots,\delta_j,\dots,\delta_n}(x,z_1,\dots,z_n) \\ &\qquad = f^{-1}_{\delta_1,\dots,\delta_{i-1},\delta_{i+1},\dots,\delta_{j-1},\delta_{j+1}\dots,\delta_n}(x,z_1,\dots,z_{i-1},z_{i+1}\dots,z_{j-1},z_{j+1},\dots,z_n). \end{aligned}$$

(Note: If $n=2$, $z_1 = z_2$ and $\delta_1 \neq \delta_2$ we have $\delta_1 z_1 + \delta_2 z_2 = 0$ and thus our assumptions for elements of S are not fulfilled.)

Using this fact, we have since $x_j > 0$ implies $\lambda(x_j), \lambda(x_j)+\lambda(x_j) \in P(A)$ and $\lambda(x_j) \neq \lambda(x_j)+\lambda(x_j)$ for all j, that all elements which are either of the form $f(y_i,\lambda(x_i))$ or of the form $f(y_i,\lambda(x_i)+\lambda(x_i))$ with $i \in \{1,\dots,n\}$ are pairwise distinct. Define

$$\bar{\lambda}(x_i) \coloneqq \begin{cases} \lambda(x_i) & \text{if } \varepsilon_i = 1 \\ \lambda(x_i)+\lambda(x_i) & \text{if } \varepsilon_i = -1. \end{cases}$$

Then all elements of the form $f(y_i,\bar{\lambda}(x_i)) \in P$ are pairwise distinct. Without loss of generality, assume $\max_{i\in\{1,\dots,m\}}\{f(y_i,\bar{\lambda}(x_i))\} = f(y_m,\bar{\lambda}(x_m))$. Assume $\varepsilon_m = 1$. We know that $\lambda(x_j) \leq x_j < \lambda(x_j)+\lambda(x_j)$ since $x_j > 0$. Thus, by the choice of $\bar{\lambda}$, we get $\varepsilon_j f(y_j,x_j) \geq \varepsilon_j f(y_j,\bar{\lambda}(x_j))$ (by $T7$ and Remark 5.4, $\varepsilon_j f(y_j,-)$ is strictly increasing if $\varepsilon_j = 1$ and strictly decreasing if $\varepsilon_j = -1$). Thus $\sum_{j=1}^m \varepsilon_j f(y_j,x_j) \geq \sum_{j=1}^m \varepsilon_j f(y_j,\bar{\lambda}(x_j))$.

Since the $f(y_j,\bar{\lambda}(x_j)) \in P$ are distinct, we can apply Lemma 5.19 and get

$$\sum_{j=1}^{m} \varepsilon_j f(y_j, \bar{\lambda}(x_j)) \geq f(2^{-(m-1)}, f(y_m, \bar{\lambda}(x_m))).$$

We can conclude

$$\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j) \geq \sum_{j=1}^{m} \varepsilon_j f(y_j, \bar{\lambda}(x_j)) \geq f(2^{-(m-1)}, f(y_m, \bar{\lambda}(x_m))).$$

Define

$$\lambda'(x_i) := \begin{cases} \lambda(x_i) & \text{if } \varepsilon_i = -1 \\ \lambda(x_i) + \lambda(x_i) & \text{if} \varepsilon_i = 1. \end{cases}$$

Since $f(y_m, \bar{\lambda}(x_m)) = f(y_m, \lambda(x_m)) \geq f(y_j, \lambda(x_j))$, we also have

$$\begin{aligned} f(y_m, \lambda'(x_m)) &= f(y_m, \lambda(x_m)) + f(y_m, \lambda(x_m)) \\ &\geq f(y_j, \lambda(x_j)) + f(y_j, \lambda(x_j)) \\ &= f(y_j, \lambda(x_j) + \lambda(x_j)) \\ &\geq f(y_j, \lambda'(x_j)). \end{aligned}$$

By Lemma 5.19, $\sum_{j=1}^{m} \varepsilon_j f(y_j, \lambda'(x_j)) < f(2, f(y_m, \lambda'(x_m)))$. By the choice of λ', we have $\varepsilon_j f(y_j, x_j) \leq \varepsilon_j f(y_j, \lambda'(x_j))$ and thus
$\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j) \leq \sum_{j=1}^{m} \varepsilon_j f(y_j, \lambda'(x_j)) < f(2, f(y_m, \lambda'(x_m)))$.

We can conclude

$$f(2^{-(m-1)}, f(y_m, \bar{\lambda}(x_m))) \leq \sum_{j=1}^{m} \varepsilon_j f(y_j, x_j) \leq f(2, f(y_m, \lambda'(x_m))).$$

W.l.o.g. assume $z_n = \max\{z_i : i \in \{1, \ldots, n\}\}$. Since $f(2^{-(m-1)}, f(y_m, \bar{\lambda}(x_m)))$ and $f(2, f(y_m, \lambda'(x_m)))$ are elements of P, we further can apply Lemma 5.19 and get

$$\begin{aligned} &f_1^{-1}(f(2, f(y_m, \lambda'(x_m))), f(2^{-n+1}, z_n)) \\ &\qquad \geq f_{\delta_1, \ldots, \delta_n}^{-1}(f(2, f(y_m, \lambda'(x_m))), z_1, \ldots, z_n) \end{aligned}$$

and

$$\begin{aligned} &f_{\delta_1, \ldots, \delta_n}^{-1}(f(2^{-(m-1)}, f(y_m, \bar{\lambda}(x_m))), z_1, \ldots, z_n) \\ &\qquad \geq f_1^{-1}(f(2^{-(m-1)}, f(y_m, \bar{\lambda}(x_m))), f(2, z_n)). \end{aligned}$$

Additionally, since $\sum_{j=1}^{n} \delta_i z_i > 0$, we have

$$\begin{aligned} &f_{\delta_1, \ldots, \delta_n}^{-1}(f(2^{-(m-1)}, f(y_m, \bar{\lambda}(x_m))), z_1, \ldots, z_n) \\ &\qquad \leq f_{\delta_1, \ldots, \delta_n}^{-1}(\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j), z_1, \ldots, z_n) \\ &\qquad\qquad \leq f_{\delta_1, \ldots, \delta_n}^{-1}(f(2, f(y_m, \lambda'(x_m))), z_1, \ldots, z_n). \end{aligned}$$

We conclude

$$\begin{aligned}
&f_1^{-1}(f(2, f(y_m, \lambda'(x_m))), f(2^{-n+1}, z_n)) \\
&\qquad\qquad\qquad \geq f^{-1}_{\delta_1,\ldots,\delta_n}(\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j), z_1, \ldots, z_n) \\
&\qquad\qquad\qquad\qquad\qquad \geq f_1^{-1}(f(2^{-(m-1)}, f(y_m, \bar{\lambda}(x_m))), f(2, z_n)).
\end{aligned}$$

We know that all elements of P between

$$\begin{aligned}
&f_1^{-1}(f(2, f(y_m, \lambda'(x_m))), f(2^{-n+1}, z_n)) \\
&\qquad = f_1^{-1}(f(2, f(y_m, \lambda(x_m) + \lambda(x_m))), f(2^{-n+1}, z_n)) \\
&\qquad\qquad = f_1^{-1}(f(f(2,2), f(y_m, \lambda(x_m))), f(2^{-n+1}, z_n))
\end{aligned}$$

and

$$\begin{aligned}
&f_1^{-1}(f(2^{-(m-1)}, f(y_m, \bar{\lambda}(x_m))), f(2, z_n)) \\
&\qquad\qquad = f_1^{-1}(f(2^{-(m-1)}, f(y_m, \lambda(x_m))), f(2, z_n))
\end{aligned}$$

are elements of the form

$$f_1^{-1}(f(2^i, f(y_m, \lambda(x_m))), f(2^j, z_n))$$

with $i \in \{2, 1, 0, \ldots, -(m-1)\}$ and $j \in \{1, 0, -1, \ldots, -n+1\}$. Thus these elements are in P_1 because $y_m \in P_1$, $\lambda(x_m) \in P(A) \subset P_1$ and P_1 is closed regarding f and f_1^{-1}. But

$$\lambda(f^{-1}_{\delta_1,\ldots,\delta_n}(\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j), z_1, \ldots, z_n))$$

is the largest element of P smaller than

$$f^{-1}_{\delta_1,\ldots,\delta_n}(\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j), z_1, \ldots, z_n)$$

and thus clearly between

$f_1^{-1}(f(2^2, f(y_m, \lambda(x_m))), f(2^{-n+1}, z_n))$ and $f_1^{-1}(f(2^{-(m-1)}, f(y_m, \lambda(x_m))), f(2, z_n))$.

It follows that

$$\lambda(f^{-1}_{\delta_1,\ldots,\delta_n}(\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j), z_1, \ldots, z_n)) \in P_1.$$

Suppose $\delta_n = -1$. Then $\sum_{i=1}^{n} \delta_i z_i < 0$ and $f^{-1}_{\delta_1,\ldots,\delta_n}(-, z_1, \ldots, z_n)$ is strictly decreasing. Remember that we have

$$\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j) \geq f(2^{-(m-1)}, f(y_m, \bar{\lambda}(x_m))) > 0.$$

Thus,

$$f^{-1}_{\delta_1,\dots,\delta_n}(\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j), z_1, \dots, z_n) < f^{-1}_{\delta_1,\dots,\delta_n}(0, z_1, \dots, z_n) = 0.$$

This is a contradiction to our assumptions.

Suppose $\varepsilon_m = -1$. Define

$$\bar{\bar{\lambda}}(x_i) := \begin{cases} \lambda(x_i) & \text{if } \varepsilon_i = -1 \\ \lambda(x_i) + \lambda(x_i) & \text{if } \varepsilon_i = 1. \end{cases}$$

Then

$$\max_{i\in\{1,\dots,m\}}\{f(y_i, \bar{\lambda}(x_i))\} = f(y_m, \bar{\lambda}(x_m)) \quad \text{and} \quad \max_{i\in\{1,\dots,m\}}\{f(y_i, \bar{\bar{\lambda}}(x_i))\} = f(y_{m_1}, \bar{\bar{\lambda}}(x_{m_1}))$$

imply

$$f(y_m, \bar{\lambda}(x_m)) = f(y_m, f(2, \lambda(x_m))) \geq f(y_{m_1}, \bar{\lambda}(x_{m_1})) \quad \text{and} \quad f(y_m, \bar{\bar{\lambda}}(x_m)) = f(y_m, \lambda(x_m)) \leq f(y_{m_1}, \bar{\bar{\lambda}}(x_{m_1})).$$

Suppose $\varepsilon_{m_1} = 1$. Then we have $f(y_m, f(2, \lambda(x_m))) \geq f(y_{m_1}, \lambda(x_{m_1}))$ and $f(y_m, \lambda(x_m)) \leq f(y_{m_1}, f(2, \lambda(x_{m_1})))$. The second inequality is equivalent to $f(2^{-1}, f(y_m, \lambda(x_m))) \leq f(y_{m_1}, \lambda(x_{m_1}))$. Thus,

$$f(y_m, f(2, \lambda(x_m))) \geq f(y_{m_1}, \lambda(x_{m_1})) \geq f(2^{-1}, f(y_m, \lambda(x_m)))$$

Since the only elements of P between $f(y_m, f(2, \lambda(x_m)))$ and $f(y_m, f(2^{-1}, \lambda(x_m)))$ are $f(y_m, f(2, \lambda(x_m)))$, $f(y_m, \lambda(x_m))$ and $f(y_m, f(2^{-1}, \lambda(x_m)))$, this is a contradiction to $f(y_m, f(2, \lambda(x_m)))$, $f(y_m, f(2, \lambda(x_m)))$, $f(y_{m_1}, \lambda(x_{m_1}))$ and $f(y_{m_1}, f(2, \lambda(x_{m_1})))$ being pairwise distinct. This proves that $\varepsilon_{m_1} = 1$.

Thus, we can apply the previous calculations to $\bar{\varepsilon}_1 := -\varepsilon_1, \dots, \bar{\varepsilon}_m := -\varepsilon_m$ and $\bar{\delta}_1 := -\delta_1, \dots, \bar{\delta}_n := -\delta_n$ and $\bar{\varepsilon}^*_j = 1$ for $\max_{i\in\{1,\dots,m\}}\{f(y_i, \bar{\lambda}(x_i)) = f(y_{j^*}, \bar{\lambda}(x_{j^*}))$. We have

$$\begin{aligned} f^{-1}_{\delta_1,\dots,\delta_n}(\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j), z_1, \dots, z_n) &= -f^{-1}_{\delta_1,\dots,\delta_n}(-\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j), z_1, \dots, z_n) \\ &= f^{-1}_{-\delta_1,\dots,-\delta_n}(\sum_{j=1}^{m} \bar{\varepsilon}_j f(y_j, x_j), z_1, \dots, z_n) \\ &= f^{-1}_{\bar{\delta}_1,\dots,\bar{\delta}_n}(\sum_{j=1}^{m} \bar{\varepsilon}_j f(y_j, x_j), z_1, \dots, z_n) \end{aligned}$$

Thus, $\lambda(f^{-1}_{\delta_1,\dots,\delta_n}(\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j), z_1, \dots, z_n)) \in P_1$.

In particular $\lambda(P(S)) = P(S) \subset P_1$.

$\underline{\langle A \cup P_1 \rangle \subset S}$: We have to show that S is closed regarding the functions λ, $+$, $-$, f, $(\varphi_n)_{n\in\mathbb{N}}$, $(f^{-1}_{\alpha_1,\ldots,\alpha_n})_{n\in\mathbb{N},\alpha_1,\ldots,\alpha_n\in\{-1,1\}}$ since clearly $A = f_1^{-1}(f(1, A), 1) \subset S$ and $P_1 = f_1^{-1}(f(P_1, 1), 1) \subset S$.

$$\lambda(f^{-1}_{\delta_1,\ldots,\delta_n}(\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j), z_1, \ldots, z_n)) \in P_1 \subset S$$

was shown in the proof of $P_1 = P(S)$ before.

By Lemma 5.16, we have

$$f^{-1}_{\delta_1,\ldots,\delta_{n_0}}(\sum_{j=1}^{m_0} \varepsilon_j f(y_j, x_j), z_1, \ldots, z_{n_0}) + f^{-1}_{\gamma_1,\ldots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1} \varepsilon_j f(y_j, x_j), p_1, \ldots, p_{n_1})$$

$$\begin{aligned}
&= f^{-1}_{\delta_1,\ldots,\delta_{n_0}}(f^{-1}_{\gamma_1,\ldots,\gamma_{n_1}}(\sum_{i=1}^{n_1} \gamma_i f(p_i, \sum_{j=1}^{m_0} \varepsilon_j f(y_j, x_j)), p_1, \ldots, p_{n_1}), z_1, \ldots, z_{n_0}) \\
&\quad + f^{-1}_{\gamma_1,\ldots,\gamma_{n_1}}(f^{-1}_{\delta_1,\ldots,\delta_{n_0}}(\sum_{i=1}^{n_0} \delta_i f(z_i, \sum_{j=m_0+1}^{m_1} \varepsilon_j f(y_j, x_j)), z_1, \ldots, z_{n_0}), p_1, \ldots, p_{n_1})
\end{aligned}$$

$$\begin{aligned}
&= f^{-1}_{\delta_1\gamma_1,\ldots,\delta_{n_0}\gamma_{n_1}}(\sum_{i=1}^{n_1} \gamma_i f(p_i, \sum_{j=1}^{m_0} \varepsilon_j f(y_j, x_j)), f(z_1, p_1), \ldots, f(z_{n_0}, p_{n_1})) \\
&\quad + f^{-1}_{\delta_1\gamma_1,\ldots,\delta_{n_0}\gamma_{n_1}}(\sum_{i=1}^{n_0} \delta_i f(z_i, \sum_{j=m_0+1}^{m_1} \varepsilon_j f(y_j, x_j)), f(z_1, p_1), \ldots, f(z_{n_0}, p_{n_1}))
\end{aligned}$$

$$\begin{aligned}
&= f^{-1}_{\delta_1\gamma_1,\ldots,\delta_{n_0}\gamma_{n_1}}(\sum_{i=1}^{n_1}\sum_{j=1}^{m_0} \varepsilon_j \gamma_i f(f(p_i, y_j), x_j) + \sum_{i=1}^{n_0}\sum_{j=m_0+1}^{m_1} \varepsilon_j \delta_i f(f(y_j, z_i), x_j), \\
&\qquad\qquad f(z_1, p_1), \ldots, f(z_{n_0}, p_{n_1})) \in S.
\end{aligned}$$

That S is closed regarding $-$ follows directly if we define $\bar{\varepsilon}_j = -\varepsilon_j$ for all $j \in \{m_0, \ldots, m_1\}$ and apply the previous equation to $\bar{\varepsilon}_j$.

$f(s_0, f^{-1}_{\gamma_1,\ldots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1} \varepsilon_j f(y_j, x_j), p_1, \ldots, p_{n_1})) = 0 \in A \subset S$ if $s_0 \notin P_1$. By Lemma 5.16 and Lemma 5.14, $s_0 \in P_1$ implies

$$\begin{aligned}
&f(s_0, f^{-1}_{\gamma_1,\ldots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1} \varepsilon_j f(y_j, x_j), p_1, \ldots, p_{n_1})) \\
&\qquad = f^{-1}_{\gamma_1,\ldots,\gamma_{n_1}}(f(s_0, \sum_{j=m+1}^{m_1} \varepsilon_j f(y_j, x_j)), p_1, \ldots, p_{n_1}) \\
&\qquad\qquad = f^{-1}_{\gamma_1,\ldots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1} \varepsilon_j f(f(s_0, y_j), x_j), p_1, \ldots, p_{n_1}) \in S.
\end{aligned}$$

Let n be in $\mathbb{N}^*$. Then $\varphi_n(s_0) = 0 \in S$ if $s_0 \notin P(S)$. Suppose $s_0 \in P(S) = P_1$. Since P_1 is a $\mathbb{Z}$-group, we have that $\varphi_n(s_0) \in P_1 \subset S$.

Let $n_1 \in N^*$, $\gamma_1, \ldots, \gamma_{n_1} \in \{1, -1\}$, $p_1, \ldots, p_{n_1} \in P(S) = P_1$. Then by Lemma 5.16, we have

$$f^{-1}_{\gamma_1,\ldots,\gamma_{n_1}}(f^{-1}_{\delta_1,\ldots,\delta_n}(\sum_{i=1}^{m} \varepsilon_i f(y_i, x_i), z_1, \ldots, z_n), p_1, \ldots, p_{n_1})$$
$$= f^{-1}_{\delta_1\gamma_1,\ldots,\delta_n\gamma_{n_1}}(\sum_{i=1}^{m} \varepsilon_i f(y_i, x_i), f(z_1, p_1), \ldots, f(z_n, p_{n_1})) \in S$$

since $f(z_i, p_j) \in P_1$ and $\varepsilon_i\delta_j \in \{1, -1\}$.

We can conclude that $S = \langle A \cup P_1 \rangle$.

Definition of ι': We define

$$\iota' \colon \langle A \cup P_1 \rangle \to \mathcal{N}$$
$$f^{-1}_{\delta_1,\ldots,\delta_n}(\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j), z_1, \ldots, z_n) \mapsto f^{-1}_{\delta_1,\ldots,\delta_n}(\sum_{j=1}^{m} \varepsilon_j f(\xi(y_j), \iota(x_j)), \xi(z_1), \ldots, \xi(z_n)).$$

In particular $x \mapsto \iota(x)$ if $x \in A$ and $x \mapsto \xi(x)$ if $x \in P_1$, with ξ as defined in 5.3)

ι' is well defined: Let

$$f^{-1}_{\delta_1,\ldots,\delta_{n_0}}(\sum_{j=1}^{m_0} \varepsilon_j f(y_j, x_j), z_1, \ldots, z_{n_0}) = f^{-1}_{\gamma_1,\ldots,\gamma_{n_1}}(\sum_{j=1}^{m_1} \tau_j f(r_j, x_j), p_1, \ldots, p_{n_1})$$

be in $\langle A \cup P_1 \rangle$. This is equivalent to

$$\sum_{i=1}^{n_1} \gamma_i f(p_i, (\sum_{j=1}^{m_0} \varepsilon_j f(y_j, x_j))) = \sum_{i=1}^{n_0} \delta_i f(z_i, (\sum_{j=1}^{m_1} \tau_j f(r_j, x_j)))$$

and

$$0 = \sum_{i=1}^{n_1} \sum_{j=1}^{m_0} -\gamma_i\varepsilon_j f(f(p_i, y_j), x_j) + \sum_{i=1}^{n_0} \sum_{j=1}^{m_1} \delta_i\tau_j f(f(z_i, r_j), x_j).$$

Since we can do a similar calculation in $\mathcal{N}$, we only need to show that for any $n \in \mathbb{N}^*$ $x_i \in A$, any $y_i \in P_1$ and any $\varepsilon_i \in \{1, -1\}$: $\sum_{i=1}^{n} \varepsilon_i f(y_i, x_i) = 0$ implies $\sum_{i=1}^{n} \varepsilon_i f(\xi(y_i), \iota(x_i)) = 0$. This is easy to show: $0 \in A$ implies $\sum_{i=1}^{n} \varepsilon_i f(y_i, x_i) \in A$ and thus

$$\sum_{i=1}^{n} \varepsilon_i f(\xi(y_i), \iota(x_i)) = \iota'(\sum_{i=1}^{n} \varepsilon_i f(y_i, x_i)) = \iota(\sum_{i=1}^{n} \varepsilon_i f(y_i, x_i)) = \iota(0) = 0.$$

$s_1 < s_2$ is equivalent to $\iota'(s_1) < \iota'(s_2)$: First, we want to show that $\iota'(s_2 - s_1) = \iota'(s_2) - \iota'(s_1)$.

Applying Lemma 5.16(3), we have

$$\iota'(f^{-1}_{\delta_1,\dots,\delta_{n_0}}(\sum_{j=1}^{m_0}\varepsilon_j f(y_j,x_j),z_1,\dots,z_{n_0}) - f^{-1}_{\gamma_1,\dots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1}\varepsilon_j f(y_j,x_j),p_1,\dots,p_{n_1}))$$

$$= \iota'(f^{-1}_{\delta_1\gamma_1,\dots,\delta_{n_0}\gamma_{n_1}}(\sum_{i=1}^{n_1}\sum_{j=1}^{m_0}\varepsilon_j\gamma_i f(f(p_i,y_j),x_j)$$
$$-\sum_{i=1}^{n_0}\sum_{j=m_0+1}^{m_1}\varepsilon_j\delta_i f(f(y_j,z_i),x_j), f(z_1,p_1),\dots,f(z_{n_0},p_{n_1})))$$

$$= f^{-1}_{\delta_1\gamma_1,\dots,\delta_{n_0}\gamma_{n_1}}(\sum_{i=1}^{n_1}\sum_{j=1}^{m_0}\varepsilon_j\gamma_i f(\xi(f(p_i,y_j)),\iota(x_j))$$
$$\sum_{i=1}^{n_0}\sum_{j=m_0+1}^{m_1} -\varepsilon_j\delta_i f(\xi(f(y_j,z_i)),\iota(x_j)),\xi(f(z_1,p_1)),\dots,\xi(f(z_{n_0},p_{n_1})))$$

$$= f^{-1}_{\delta_1\gamma_1,\dots,\delta_{n_0}\gamma_{n_1}}(\sum_{i=1}^{n_1}\sum_{j=1}^{m_0}\varepsilon_j\gamma_i f(f(\xi(p_i),\xi(y_j)),\iota(x_j))$$
$$-\sum_{i=1}^{n_0}\sum_{j=m_0+1}^{m_1}\varepsilon_j\delta_i f(f(\xi(y_j),\xi(z_i)),\iota(x_j)), f(\xi(z_1),\xi(p_1)),\dots,f(\xi(z_{n_0}),\xi(p_{n_1}))).$$

A similar calculation for

$$f^{-1}_{\delta_1,\dots,\delta_{n_0}}(\sum_{j=1}^{m_0}\varepsilon_j f(\xi(y_j),\iota(x_j)),\xi(z_1),\dots,\xi(z_{n_0}))$$
$$- f^{-1}_{\gamma_1,\dots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1}\varepsilon_j f(\xi(y_j),\iota(x_j)),\xi(p_1),\dots,\xi(p_{n_1}))$$

in $\mathcal{N}$ like the one we did for

$$f^{-1}_{\delta_1,\dots,\delta_{n_0}}(\sum_{j=1}^{m_0}\varepsilon_j f(y_j,x_j),z_1,\dots,z_{n_0}) - f^{-1}_{\gamma_1,\dots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1}\varepsilon_j f(y_j,x_j),p_1,\dots,p_{n_1})$$

in $\mathcal{M}$ leads to

$$\iota'(f^{-1}_{\delta_1,\dots,\delta_{n_0}}(\sum_{j=1}^{m_0}\varepsilon_j f(y_j,x_j),z_1,\dots,z_{n_0}))$$
$$- \iota'(f^{-1}_{\gamma_1,\dots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1}\varepsilon_j f(y_j,x_j),p_1,\dots,p_{n_1}))$$

$$= f^{-1}_{\delta_1,\ldots,\delta_{n_0}}(\sum_{j=1}^{m_0} \varepsilon_j f(\xi(y_j), \iota(x_j)), \xi(z_1), \ldots, \xi(z_{n_0}))$$

$$- f^{-1}_{\gamma_1,\ldots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1} \varepsilon_j f(\xi(y_j), \iota(x_j)), \xi(p_1), \ldots, \xi(p_{n_1}))$$

$$= f^{-1}_{\delta_1\gamma_1,\ldots,\delta_{n_0}\gamma_{n_1}}(\sum_{i=1}^{n_1}\sum_{j=1}^{m_0} \varepsilon_j \gamma_i f(f(\xi(p_i), \xi(y_j)), \iota(x_j))$$
$$- \sum_{i=1}^{n_0}\sum_{j=m_0+1}^{m_1} \varepsilon_j \delta_i f(f(\xi(y_j), \xi(z_i))), \iota(x_j)),$$
$$f(\xi(z_1), \xi(p_1)), \ldots, f(\xi(z_{n_0}), \xi(p_{n_1})).$$

We can conclude that for $s_1, s_2 \in \langle A \cup P_1 \rangle$ we have $\iota'(s_1) - \iota'(s_2) = \iota'(s_1 - s_2)$.

Note that we can do the same calculation for $+$ to get $\iota'(s_1) + \iota'(s_2) = \iota'(s_1 + s_2)$.

Now, we show that $s_1 < s_2$ implies $\iota'(s_1) < \iota'(s_2)$.

We only need to consider $s_0 > 0$ since for any s_1, s_2, we have that $s_2 > s_1$ implies $s_2 - s_1 > 0$, $s_1 - s_2$ is again an element of $\langle A \cup P_1 \rangle$ and $\iota'(s_2 - s_1) = \iota'(s_2) - \iota'(s_1)$ (thus $\iota'(s_2 - s_1) > 0$ implies $\iota'(s_2) > \iota'(s_1)$). Suppose $s_1 = f^{-1}_{\delta_1,\ldots,\delta_n}(\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j), z_1, \ldots, z_n)$. Like in the proof of $P(S) = P_1$, either, we have $s_0 = 0$ and thus $\iota'(s_0) = \iota(s_0) = 0$, or, we can assume that the y_i are distinct modulo $P(A)$ and that the z_i are distinct. Because ξ is an embedding, the $\xi(y_i)$ are again distinct modulo $\xi(P(A)) = \zeta(P(A)) = \iota(P(A))$ and the $\xi(z_i)$ are distinct as well.

We can do a case distinction:

Case 2.1 ($\sum_{i=1}^{n} \delta_i z_i = 0$): This is a contradiction to the assumption that the z_i are distinct.

Case 2.2 ($\sum_{i=1}^{n} \delta_i z_i > 0$): Let i^* be the index such that $z_{i^*} = \max\{z_i\}$. Since the z_i are distinct elements of P, we have that $z_{i^*} > \sum_{i \neq i^*} z_i$ and thus, $\sum_{i=1}^{n} \delta_i z_i > 0$ implies $\delta_{i^*} = +1$. Since ξ is an embedding, we have $\xi(\max\{z_i\}) = \max\{\xi(z_i)\}$. Because all of the $\xi(z_i)$ are distinct, it follows that $\sum_{i=1}^{n} \delta_i \xi(z_i) > 0$. Thus, the following inequalities are equivalent:

$$s_0 > 0$$

$$f^{-1}_{\delta_1,\ldots,\delta_n}(\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j), z_1, \ldots, z_n) > 0$$

$$\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j) > \sum_{i=1}^{n} \delta_i f(z_i, 0) = 0$$

and the following inequalities are again equivalent:

$$\iota'(s_0) > 0$$

$$f^{-1}_{\delta_1,\ldots,\delta_n}(\sum_{j=1}^{m} \varepsilon_j f(\xi(y_j), \iota(x_j)), \xi(z_1), \ldots, \xi(z_n)) > 0$$

$$\sum_{j=1}^{m} \varepsilon_j f(\xi(y_j), \iota(x_j)) > \sum_{i=1}^{n} \delta_i f(\xi(z_i), 0) = 0$$

If $m = 1$, we know that $\varepsilon_1 f(\xi(y_1), \iota(x_1)) > 0$ if and only if $\varepsilon_1 = 1$, and $\varepsilon_1 f(y_1, x_1) > 0$ if and only if $\varepsilon_1 = 1$. Thus, we can conclude $s_0 > 0$ if and only if $\iota'(s_0) > 0$.

If $m > 1$: Let i^* be the index such that $f(y_{i^*}, x_{i^*}) = \max\{f(y_i, x_i)\}$. This means that $f(y_{i^*}, x_{i^*}) \geq f(y_i, x_i)$ and therefore $f(y_{i^*}, \lambda(x_{i^*})) \geq f(y_i, \lambda(x_i))$. Since all the y_i are distinct modulo $P(A)$, $f(y_{i^*}, \lambda(x_{i^*})) > f(y_i, \lambda(x_i))$ for all $i \neq i^*$. This is equivalent to $f(y_{i^*}, \lambda(x_{i^*})) \geq f(y_i, f(\lambda(x_i), 2))$. Again, the fact that the y_i are distinct modulo $P(A)$ implies $f(y_{i^*}, \lambda(x_{i^*})) > f(y_i, f(\lambda(x_i), 2))$. Since ξ is an embedding,

$$\xi(f(y_{i^*}, \lambda(x_{i^*}))) > \xi(f(y_i, f(\lambda(x_i), 2)))$$
$$f(\xi(y_{i^*}), \iota(\lambda(x_{i^*}))) > f(\xi(y_i), \iota(f(\lambda(x_i), 2))).$$

Since the y_i are distinct modulo $P(A)$, the $f(y_i, f(\lambda(x_i), 2))$ are distinct and we have

$$f(y_{i^*}, x_{i^*}) \geq f(\xi(y_{i^*}), \lambda(x_{i^*})) \geq \sum_{i \neq i^*} f(y_i, f(\lambda(x_i), 2)) > \sum_{i \neq i^*} f(y_i, x_i).$$

Because the $\xi(y_i)$ are distinct modulo $P(A)$, the $f(\xi(y_i), f(\lambda(\iota(x_i)), 2))$ are distinct and we get

$$\begin{aligned} f(\xi(y_{i^*}), \iota(x_{i^*})) &\geq f(\xi(y_{i^*}), \lambda(\iota(x_{i^*}))) \\ &\geq \sum_{i \neq i^*} f(\xi(y_i), f(\lambda(\iota(x_i)), 2)) > \sum_{i \neq i^*} f(\xi(y_i), \iota(x_i)) \end{aligned}$$

It follows that $\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j) > 0$ if and only if $\varepsilon_{i^*} = +1$, and $\varepsilon_{i^*} = +1$ if and only if $\sum_{j=1}^{m} \varepsilon_j f(\xi(y_j), \iota(x_j)) > 0$. Thus, we can conclude that $s_0 > 0$ is equivalent to $\iota'(s_0) > 0$.

Case 2.3 ($\sum_{i=1}^{n} \delta_i z_i < 0$): With a consideration similar to the one in Case 2.2, we get that the following are equivalent

$$s_0 > 0$$

$$f^{-1}_{\delta_1,\ldots,\delta_n}(\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j), z_1, \ldots, z_n) > 0$$

$$\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j) < \sum_{i=1}^{n} \delta_i f(z_i, 0) = 0$$

and the following inequalities are again equivalent

$$\iota'(s_0) > 0$$

$$f^{-1}_{\delta_1,\dots,\delta_n}(\sum_{j=1}^{m} \varepsilon_j f(\xi(y_j), \iota(x_j)), \xi(z_1), \dots, \xi(z_n)) > 0$$

$$\sum_{j=1}^{m} \varepsilon_j f(\xi(y_j), \iota(x_j)) < \sum_{i=1}^{n} \delta_i f(\xi(z_i), 0) = 0.$$

Like in Case 2.2, we can show that $\sum_{j=1}^{m} \varepsilon_j f(y_j, x_j) < 0$ if and only if $\varepsilon_{i^*} = -1$, and $\varepsilon_{i^*} = -1$ if and only if $\sum_{j=1}^{m} \varepsilon_j f(\xi(y_j), \iota(x_j)) < 0$. Thus, we can again conclude that $s_0 > 0$ is equivalent to $\iota'(s_0) > 0$.

In conclusion $s_0 < s_1$ implies $\iota'(s_0) < \iota'(s_1)$.

We can also show the other direction: $\neg(s_0 < s_1)$ implies $s_0 > s_1 \lor s_0 = s_1$. Thus, as just shown, $\iota'(s_0) > \iota'(s_1) \lor \iota'(s_0) = \iota'(s_1)$ and $\neg(\iota'(s_0) < \iota'(s_1))$.

Therefore, we have that indeed $s_0 < s_1$ is equivalent to $\iota'(s_0) < \iota'(s_1)$.

ι' is injective: Let s_1, s_2 be in $\langle A \cup P_1 \rangle$. $s_1 = s_2$ if and only if $(\neg(s_1 < s_2)) \land \neg(s_1 > s_2)$. By the previously proven statement, this holds if and only if $(\neg(\iota'(s_1) < \iota'(s_2))) \land \neg(\iota'(s_1) > \iota'(s_2))$ which holds if and only if $\iota'(s_1) = \iota'(s_2)$.

ι' is an embedding: constant symbols: $\iota'(0) = \iota(0) = 0$, $\iota'(1) = \iota(1) = 1$.

function symbols: Let $s_0, s_1 \in \langle A \cup P_1 \rangle$.

We have shown before in the paragraph "$s_1 < s_2$ is equivalent to $\iota'(s_1) < \iota'(s_2)$" that $\iota'(s_1 + s_2) = \iota'(s_1) + \iota'(s_2)$ and $\iota'(s_1 - s_2) = \iota'(s_1) - \iota'(s_2)$.

If $s_0 \le 0$, we have $\iota'(s_0) \le 0$ and $\iota'(\lambda(s_0)) = \iota'(0) = 0 = \lambda(\iota'(s_0))$. If $s_0 > 0$, we have that $\lambda(s_0)$ is the unique element in P fulfilling $\lambda(s_0) \le s_0 < f(2, \lambda(s_0))$. We have shown before that ι' is injective and strictly increasing. Thus, $\lambda(s_0) \le s_0 < f(2, \lambda(s_0))$ implies $\iota'(\lambda(s_0)) \le \iota'(s_0) < \iota'(f(2, \lambda(s_0))) = \iota'(\lambda(s_0) + \lambda(s_0)) = \iota'(\lambda(s_0)) + \iota'(\lambda(s_0)) = f(2, \iota'(\lambda(s_0)))$. Since this is the characterizing inequality for $\lambda(\iota'(s_0))$, we have $\lambda(\iota'(s_0)) = \iota'(\lambda(s_0))$.

For f, first, we show that $s_0 \in P$ if and only if $\iota'(s_0) \in P$. $\iota'(s_0) \in P$ if and only if $\iota'(s_0) = \lambda(\iota'(s_0)) = \iota'(\lambda(s_0))$. Since ι' is injective, this is equivalent to $s_0 = \lambda(s_0)$. However, this is true if and only if $s_0 \in P$.

Now, we can apply this property: Suppose $s_0 \notin P$ and thus $\iota'(s_0) \notin P$, then

$$\iota'(f(s_0, f^{-1}_{\gamma_1,\dots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1} \varepsilon_j f(y_j, x_j), p_1, \dots, p_{n_1}))) = \iota'(0) = 0$$

$$= f(\iota'(s_0), f^{-1}_{\gamma_1,\dots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1} \varepsilon_j f(y_j, x_j), p_1, \dots, p_{n_1})).$$

If $s_0 \in P_1$, we have shown before that

$$f(s_0, f^{-1}_{\gamma_1,\dots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1} \varepsilon_j f(y_j, x_j), p_1, \dots, p_{n_1}))$$
$$= f^{-1}_{\gamma_1,\dots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1} \varepsilon_j f(f(s_0, y_j), x_j), p_1, \dots, p_{n_1}).$$

Again, a similar calculation leads to

$$f(\xi(s_0), f^{-1}_{\gamma_1,\dots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1} \varepsilon_j f(\xi(y_j), \iota(x_j)), \xi(p_1), \dots, \xi(p_{n_1})))$$
$$= f^{-1}_{\gamma_1,\dots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1} \varepsilon_j f(f(\xi(s_0), \xi(y_j)), \iota(x_j)), \xi(p_1), \dots, \xi(p_{n_1})).$$

for $s_0 \in P$ also implying $\xi(s_0) \in P$. Thus,

$$\begin{aligned}
&\iota'(f(s_0, f^{-1}_{\gamma_1,\dots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1} \varepsilon_j f(y_j, x_j), p_1, \dots, p_{n_1})))\\
&= \iota(f^{-1}_{\gamma_1,\dots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1} \varepsilon_j f(f(s_0, y_j), x_j), p_1, \dots, p_{n_1}))\\
&= f^{-1}_{\gamma_1,\dots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1} \varepsilon_j f(f(\xi(s_0), \xi(y_j)), \iota(x_j)), \xi(p_1), \dots, \xi(p_{n_1}))\\
&= f(\xi(s_0), f^{-1}_{\gamma_1,\dots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1} \varepsilon_j f(\xi(y_j), \iota(x_j)), \xi(p_1), \dots, \xi(p_{n_1})))\\
&= f(\iota'(s_0), \iota'(f^{-1}_{\gamma_1,\dots,\gamma_{n_1}}(\sum_{j=m+1}^{m_1} \varepsilon_j f(y_j, x_j), p_1, \dots, p_{n_1}))).
\end{aligned}$$

relation symbols: $s_1 < s_2$ is equivalent to $\iota'(s_1) < \iota'(s_2)$ and $P(\iota(s_1))$ if and only if $P(s_1)$ was shown before.

Suppose $R_n(s_0)$ holds. Thus, $P(s_0)$ holds, and hence, $s_0 \in P_1$. This implies $\iota'(s_0) = \xi(s_0)$ since ξ is an embedding, $R_n(\iota(s_0)) = R_n(\xi(s_0))$ holds. Suppose $R_n(\iota'(s_0))$ holds. Thus, $P(\iota'(s_0))$ and $P(s_0)$ hold. Again, this implies $\iota'(s_0) = \xi(s_0)$ and since ξ is an embedding $R_n(\iota(s_0)) = R_n(\xi(s_0))$ holds.

Case 3: If neither of the previous two cases hold (i.e. $a \notin P$ but $P(\langle A \cup \{a\}\rangle) \neq P(A)$), then define

$$\tilde{\lambda}(a) := \begin{cases} \lambda(a) & \text{if } a > 0 \\ \lambda(-a) & \text{if } a < 0. \end{cases}$$

Remember that $a \notin A$ and thus $a \neq 0$. In particular $\tilde{\lambda}(a) \in P$. By Case 2, there is an embedding $\iota'_1 : \langle A \cup \{\tilde{\lambda}(a)\}\rangle \to \mathcal{N}$ extending ι. Clearly, we have $A \subset \langle A \cup \{\tilde{\lambda}(a)\}\rangle \subset \langle A \cup \{a\}\rangle$.

We can show that: $P(\langle A \cup \{\tilde{\lambda}(a)\}\rangle) = P(\langle A \cup \{a\}\rangle)$.

Suppose $q \in P(\langle A \cup \{a\}\rangle)$. This means $q = \lambda(q)$. Then by the construction of $\langle\ \rangle$, there are $a_1, \ldots a_n \in A$ and a term $t(x, x_1, \ldots, x_n)$ consisting of λ, $+$, $-$, f , $(\varphi_n)_{n\in\mathbb{N}}$, $(f^{-1}_{\varepsilon_1,\ldots,\varepsilon_n})_{n\in\mathbb{N},\varepsilon_1,\ldots,\varepsilon_n\in\{1,-1\}}$ such that $q = t(a, a_1, \ldots, a_n)$.

Let $t_0(x, x_1, \ldots, x_n), t_1(x, x_1, \ldots, x_n), \ldots, t_m(x, x_1, \ldots, x_n)$ be arbitrary terms consisting of λ, $+$, $-$, f , $(\varphi_n)_{n\in\mathbb{N}}$, $(f^{-1}_{\varepsilon_1,\ldots,\varepsilon_n})_{n\in\mathbb{N},\varepsilon_1,\ldots,\varepsilon_n\in\{1,-1\}}$. In order to shorten the notation, we will write t_i to mean $t_i(a, a_1, \ldots, a_n)$

We collect some properties of terms: $t_1 - t_2 = t_1 + (-t_2)$, $\lambda(\lambda(t_1)) = \lambda(t_1)$, $\lambda(-\lambda(t_1)) = 0$

$$\lambda(t_1 + t_2) = \begin{cases} 0 & \text{if } t_1 + t_2 \leq 0 \\ f(2, \lambda(t_1)) & \text{if } t_1 + t_2 \geq 0 \wedge t_1 \geq t_2 \geq f(2, \lambda(t_1)) - t_1 \\ \lambda(t_1) & \text{if } t_1 + t_2 \geq 0 \wedge t_1 > t_2 \wedge t_2 < f(2, \lambda(t_1)) - t_1 \\ f(2, \lambda(t_2)) & \text{if } t_1 + t_2 \geq 0 \wedge t_2 > t_1 \geq f(2, \lambda(t_2)) - t_2 \\ \lambda(t_2) & \text{if } t_1 + t_2 \geq 0 \wedge t_2 > t_1 \wedge t_1 < f(2, \lambda(t_2)) - t_2 \end{cases}$$

(Note that for $t_1 = t_2$ we always have $t_2 \geq f(2, \lambda(t_1)) - t_1$.) Similarly:

$$\lambda(-(t_1+t_2)) = \begin{cases} 0 & \text{if } t_1 + t_2 \geq 0 \\ f(2, \lambda(-t_1)) & \text{if } t_1 + t_2 \leq 0 \wedge t_1 \leq t_2 \leq -f(2, \lambda(-t_1)) + t_1 \\ \lambda(-t_1) & \text{if } t_1 + t_2 \leq 0 \wedge t_1 < t_2 \wedge t_2 > -f(2, \lambda(-t_1)) + t_1 \\ f(2, \lambda(-t_2)) & \text{if } t_1 + t_2 \leq 0 \wedge t_2 < t_1 \leq -f(2, \lambda(-t_2)) + t_2 \\ \lambda(-t_2) & \text{if } t_1 + t_2 \leq 0 \wedge t_2 < t_1 \wedge t_1 > -f(2, \lambda(-t_2)) + t_2. \end{cases}$$

$$\lambda(f(t_1, t_2)) = \begin{cases} 0 & \text{if } t_1 \notin P \text{ or } t_2 \leq 0 \\ f(\lambda(t_1), \lambda(t_2)) & \text{else} \end{cases}$$

$$\lambda(-f(t_1, t_2)) = \begin{cases} 0 & \text{if } t_1 \notin P \text{ or } t_2 \geq 0 \\ f(\lambda(t_1), \lambda(-t_2)) & \text{else} \end{cases}$$

$$\lambda(\varphi_n(t_1)) = \begin{cases} 0 & \text{if } t_1 \notin P \\ \varphi_n(\lambda(t_1)) & \text{else} \end{cases}$$

$\lambda(-\varphi_n(t_1)) = 0$

Lastly, we want to consider $\lambda(f^{-1}_{\varepsilon_1,\ldots\varepsilon_m}(t_0, t_1, \ldots, t_m))$ and $\lambda(-f^{-1}_{\varepsilon_1,\ldots\varepsilon_m}(t_0, t_1, \ldots, t_m))$. It is immediate to see that $\lambda(f^{-1}_{\varepsilon_1,\ldots\varepsilon_m}(t_0, t_1, \ldots, t_m)) = 0$ if $(\bigwedge_{i=1}^m t_i \notin P) \vee (\sum_{i=1}^m \varepsilon_i t_i = 0) \vee (\sum_{i=1}^m \varepsilon_i t_i > 0 \wedge t_0 \leq 0) \vee (\sum_{i=1}^m \varepsilon_i t_i < 0 \wedge t_0 \geq 0)$. If $\sum_{i=1}^m \varepsilon_i t_i > 0$ and $t_0 > 0$ we have $t_0 \geq \lambda(t_0) > 0$. We can apply Lemma 5.20 and use the monotonicity of $f^{-1}_{\varepsilon_1,\ldots,\varepsilon_m}(-, t_1, \ldots, t_m)$ to get

$$\begin{aligned} f_1^{-1}(f(\lambda(t_0), 2), \lambda(\sum_{i=1}^m \varepsilon_i t_i)) \\ &\geq f^{-1}_{\varepsilon_1,\ldots,\varepsilon_m}(f(\lambda(t_0), 2), t_1, \ldots, t_m) \\ &> f^{-1}_{\varepsilon_1,\ldots,\varepsilon_m}(t_0, t_1, \ldots, t_m) \end{aligned}$$

and

$$f_1^{-1}(\lambda(t_0), f(2, \lambda(\sum_{i=1}^{m} \varepsilon_i t_i)))$$
$$\leq f^{-1}_{\varepsilon_1,\ldots,\varepsilon_m}(\lambda(t_0), t_1, \ldots, t_m)$$
$$\leq f^{-1}_{\varepsilon_1,\ldots,\varepsilon_m}(t_0, t_1, \ldots, t_m).$$

Thus, $\lambda(f^{-1}_{\varepsilon_1,\ldots,\varepsilon_m}(t_0, t_1, \ldots, t_m)) = f(2^{-1}, f_1^{-1}(\lambda(t_0), \lambda(\sum_{i=1}^m \varepsilon_i t_i)))$ or $\lambda(f^{-1}_{\varepsilon_1,\ldots,\varepsilon_m}(t_0, t_1, \ldots, t_m)) = f_1^{-1}(\lambda(t_0), \lambda(\sum_{i=1}^m \varepsilon_i t_i))$.

If $\sum_{i=1}^m \varepsilon_i t_i < 0$ and $t_0 < 0$, notice that

$$f^{-1}_{\varepsilon_1,\ldots,\varepsilon_m}(t_0, t_1, \ldots, t_m) = f^{-1}_{\varepsilon_1,\ldots,\varepsilon_m}(-t_0, -t_1, \ldots, -t_m)$$

and by the previous calculation

$$\begin{aligned}
&f^{-1}_{\varepsilon_1,\ldots,\varepsilon_m}(t_0, t_1, \ldots, t_m) = 0 \\
&\vee f^{-1}_{\varepsilon_1,\ldots,\varepsilon_m}(t_0, t_1, \ldots, t_m) = f(2^{-1}, f_1^{-1}(\lambda(-t_0), \lambda(\sum_{i=1}^{m} \varepsilon_i - t_i))) \\
&\vee f^{-1}_{\varepsilon_1,\ldots,\varepsilon_m}(t_0, t_1, \ldots, t_m) = f_1^{-1}(\lambda(-t_0), \lambda(\sum_{i=1}^{m} \varepsilon_i - t_i)).
\end{aligned}$$

Moreover, $\lambda(-f^{-1}_{\varepsilon_1,\ldots\varepsilon_m}(t_0, t_1, \ldots, t_m)) = \lambda(f^{-1}_{\varepsilon_1,\ldots\varepsilon_m}(-t_0, t_1, \ldots, t_m))$. With the previous calculation we get that

$$\begin{aligned}
&(\lambda(-f^{-1}_{\varepsilon_1,\ldots\varepsilon_m}(t_0, t_1, \ldots, t_m)) = 0) \\
\vee&(\lambda(-f^{-1}_{\varepsilon_1,\ldots\varepsilon_m}(t_0, t_1, \ldots, t_m)) = f(2^{-1}, f_1^{-1}(\lambda(-t_0), \lambda(\sum_{i=1}^{m} \varepsilon_i t_i)))) \\
\vee&(\lambda(-f^{-1}_{\varepsilon_1,\ldots\varepsilon_m}(t_0, t_1, \ldots, t_m)) = f_1^{-1}(\lambda(-t_0), \lambda(\sum_{i=1}^{m} \varepsilon_i t_i))) \\
\vee&(\lambda(-f^{-1}_{\varepsilon_1,\ldots\varepsilon_m}(t_0, t_1, \ldots, t_m)) = f(2^{-1}, f_1^{-1}(\lambda(t_0), \lambda(\sum_{i=1}^{m} \varepsilon_i t_i)))) \\
\vee&(\lambda(-f^{-1}_{\varepsilon_1,\ldots\varepsilon_m}(t_0, t_1, \ldots, t_m)) = f_1^{-1}(\lambda(t_0), \lambda(\sum_{i=1}^{m} \varepsilon_i t_i))).
\end{aligned}$$

Using all these properties, one can prove inductively that there is a term $\bar{t}(x, y, x_1, \ldots, x_n, y_1, \ldots y_n)$ in $\mathscr{L} = \{0, 1, +, f, (f^{-1}_{\varepsilon_1,\ldots,\varepsilon_n})_{n\in\mathbb{N},\varepsilon_1,\ldots,\varepsilon_n\in\{1,-1\}}, (\varphi_n)_{n\in\mathbb{N}}\}$ such that $\bar{t}(\lambda(a), \lambda(-a), \lambda(a_1), \ldots \lambda(a_n), \lambda(-a_1), \ldots, \lambda(-a_n)) = t(a, a_1, \ldots, a_n)$. Clearly $\bar{t}(\lambda(a), \lambda(-a), \lambda(a_1), \ldots \lambda(a_n), \lambda(-a_1), \ldots, \lambda(-a_n)) \in \langle A \cup \tilde{\lambda}(a)\rangle$.

Thus, we have $p \in \langle A \cup \tilde{\lambda}(a)\rangle$.

Since $P(\langle A \cup \{\tilde{\lambda}(a)\}\rangle) = P(\langle A \cup \{a\}\rangle)$, we can apply Case 1 to get an embedding $\iota_2' : \langle A \cup \{a\}\rangle \to \mathcal{N}$ extending ι_1' and therefore also extending ι.

□

Remark 5.25. The image of ι' as it is defined in the proof is $\langle B \cup \{b\} \rangle$.

For the following theorem and a proof of it confer [Del97].

Theorem 5.26. *The theory T is complete.*

5.4 Definition of $\mathcal{R}$

Now, we want to introduce a structure that models the theory T.

Definition 5.27 ($\mathcal{R}$)**.** Let

$$\mathcal{R} := (\mathbb{R}, P^{\mathcal{R}}, (R_n^{\mathcal{R}})_{n \in \mathbb{N}^*}, <, +, -, g, \lambda^{\mathcal{R}}, 0, 1)$$

be an expansion of the real ordered field with

$$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (t, t') \mapsto \begin{cases} tt' & \text{if } t \in 2^{\mathbb{Z}} \\ 0 & \text{else.} \end{cases}$$

$P^{\mathcal{R}}(x)$ holds if and only if $x \in 2^{\mathbb{Z}}$,$R_n^{\mathcal{R}}(x)$ holds if and only if $x \in 2^{n \cdot \mathbb{Z}}$ and

$$\lambda^{\mathcal{R}}(x) = \begin{cases} \max\{y \in 2^{\mathbb{Z}} : y \leq x\} & \text{if } x > 0 \\ 0 & \text{else.} \end{cases}$$

It is easy to check that $\mathcal{R} \models T$.

Corollary 5.28. *T is the theory of $\mathcal{R}$, i.e. $T = Th(\mathcal{R})$*

Proof. Follows directly since $T \models \mathcal{R}$ and T is complete. □

Lemma 5.29. *$\mathcal{R}$ is an expansion by definitions of* $(\mathbb{R}, <, +, g)$.

Proof. We can explicitly state the definition of the other symbols:

$$2^{\mathbb{Z}} = \{x \colon f(x, 1) \neq 0\},$$

$$R_n^{\mathcal{R}}(\mathbb{R}) = \{x \colon P(x) \wedge \exists y \ (P(y) \wedge y^n = x)\},$$

$$\text{graph}(-) = \{(x, y, z) \colon (x = z + y)\},$$

$$\text{graph}(\lambda) = \{(x, y) \colon (x \leq 0 \wedge y = 0) \vee (P(y) \wedge (y \leq x \wedge x < y + y))\},$$

$$0 = \{x \colon x + x = x\},$$

$$1 = \{x \colon \forall y f(x, y) = y\}.$$

□

Remark 5.30. By Lemma 5.29, $(\mathbb{R}, <, +, g)$ and $\mathcal{R}$ have the same definable sets.

5.5 $\mathcal{R}$ is not field type

In this chapter we will shortly discuss the application that mainly motivated us to show the quantifier elimination for $\mathcal{R}$. The quantifier elimination for $\mathcal{R}$ allows us to define all definable sets by quantifier free formulas. Thus, the definable sets are boolean combinations of sets defined by atomic formulas. As [HW21] suggested one can use this quantifier elimination result to show that $\mathcal{R}$ is field type. Though we will not prove this result, we want to take a short look on it.

First, let us define the property to be field type. The following definition is taken from [HW21].

Definition 5.31 (field-type)**.** Let $\mathcal{R} = (\mathbb{R}, <, +, \dots)$ be a first order expansion of the ordered additive group of the real numbers. $\mathcal{R}$ is field-type if there is a definable, bounded, open, non-empty subinterval $I \subset \mathbb{R}$ and there are definable functions $\oplus, \otimes : I^2 \to I$ such that $(I, <, \oplus, \otimes)$ is an ordered field isomorphic to $(\mathbb{R}, <, +, \cdot)$.

Remark 5.32. By Remark 5.30, $\mathcal{R}$ and $(\mathbb{R}, <, +, g)$ agree on the definable sets. Therefore, the structure $(\mathbb{R}, <, +, g)$ is field-type if and only if $\mathcal{R}$ is field-type.

Theorem 5.33. $(\mathbb{R}, <, +, g)$ *is not field-type.*

This theorem is taken from [HW21]. Main ideas, how to conclude this result from the quantifier elimination we showed, can be found there.

6 $(\mathbb{R}, <, +, g)$ admits a weak pole

In this chapter we define a weak pole for $(\mathbb{R}, <, +, g)$. Together with the property, that $(\mathbb{R}, <, +, g)$ is field type, which was discussed in the previous chapter, this concludes the main outcome that motivates this thesis. $(\mathbb{R}, <, +, g)$ is an expansion of the ordered additive group of real numbers that is not field type but admits a weak pole.

First, we define what a weak pole is. The following definition is taken from [HW21].

Definition 6.1 (weak pole)**.** Let $\mathcal{R} = (\mathbb{R}, <, +, \dots)$ be a first order expansion of the ordered additive group of the real numbers. A weak pole is a definable family $\{h_d : d \in E\}$ of continuous maps $h_d : [0, d] \to \mathbb{R}$ such that

1. $E \subset \mathbb{R}_{>0}$ is closed in $\mathbb{R}_{>0}$ and $(0, \varepsilon) \cap E \neq \emptyset$ for all $\varepsilon > 0$,
2. there is $\delta > 0$ such that $[0, \delta] \subset h_d([0, d])$ for all $d \in E$.

For the following lemma we will prove the statement that the set $\{g_t : t \in 2^{\mathbb{Z}}\}$ is a weak pole from [HW21].

Theorem 6.2. $(\mathbb{R}, <, +, g)$ *admits a weak pole.*

Proof. Set $E = 2^{\mathbb{Z}} \subset \mathbb{R}_{>0}$ and $\delta = 1$. We can show that E is closed in $\mathbb{R}_{>0}$: Let x be in the closure of E in $\mathbb{R}_{>0}$. Then $x > 0$. Clearly $x \in (\frac{x}{2}, 2x)$. However, it is easy to check that for any $x > 0$, $(\frac{x}{2}, 2x) \cap 2^{\mathbb{Z}}$ is finite. Since x is in the closure of a finite set and the closure of a finite set is the set itself, we have $x \in 2^{\mathbb{Z}}$. For all $\varepsilon > 0$ we have $f(\lambda(\varepsilon), 2^{-1}) \in E \cap (0, \varepsilon)$ and thus $(0, \varepsilon) \cap E \neq \emptyset$.

Take an arbitrary $2^z \in E$. We have to show that there is a continuous definable function h_z with $[0, \delta] = [0, 1] \subset h_z([0, 2^z])$.

Define $\varphi(z_1, z_2)$ as

$$(0 \leq z_1 \leq 2^z \wedge \exists m((P(m) \wedge f(m, 2^z) \geq 1) \wedge (\neg \exists n(n < m \wedge P(n) \wedge f(n, d) \geq 1)) \\ \wedge f(m, z_1) = z_2)).$$

Define $\mu_z := 2^{-z} > 0$. Then $g(\mu_z, 2^z) = 1$ and by $T9$ there is no $p \in P$ such that $p < \mu_z$ and $g(p, 2^z) \geq 1$. Define $h_z(x) := g(\mu_z, x)$. Then clearly

$$\{(x, y) \in [0, 2^z] \times \mathbb{R} : y = h_z(x)\} = \{(a, b) \in \mathbb{R}^2 : (\mathbb{R}, <, +, g) \models \varphi(a, b)\}$$

and therefore h_z is a definable function.

Let $\varepsilon_1 > 0$. Define $\varepsilon_2 := \frac{\varepsilon_1}{\mu_z} > 0$. Then, $|x_1 - x_2| < \varepsilon_2$ implies

$$|h_z(x_1) - h_z(x_2)| = |g(\mu_z, x_1) - g(\mu_z, x_2)| = g(\mu_z, |x_1 - x_2|) < g(\mu_z, \frac{\varepsilon_1}{\mu_z}) = \varepsilon_1.$$

Thus h_z is continuous.

$x \in [0, 1]$ implies $0 \leq x \leq g(\mu_z, 2^z)$ and by monotonicity of g (by $T7$) $0 \leq g(\mu_z^{-1}, x) \leq 2^z$. Thus $g(\mu_z^{-1}, x) \in [0, 2^z]$ and $h_z(g(\mu_z^{-1}, x)) = x$. Since x was chosen arbitrarily, $[0, \delta] = [0, 1] \subset h_d([0, d])$.

□

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