Current Developments in Local O-Minimality

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1 Introduction

Throughout the last decades o-minimality, as introduced in [36], has been a thriving field of study with many groundbreaking results exploiting geometric tameness properties.[1]

This raises the question what other classes of geometrically tame structures[2] exist. How do these classes correlate and are all other structures wild? Local o-minimality, the topic of this thesis, is one such class of geometrically tame structures.

In addition to the geometric tameness of o-minimal structures, there is model theoretic tameness of the theories of o-minimal structures, so called o-minimal theories. For other classes of geometrically tame structures, like local o-minimality, this is not necessarily the case. So, what kind of geometrically tame properties are we able to show for these classes even without the tools provided by the model theoretic tameness of o-minimality?

And can we also classify the universe of structures into categories of nice geometric properties?

These questions have led to many interesting insights regarding reasonable classifications of structures, and in restricted cases there are substantial results:

First and foremost, in the very restricted case of expansions of the real ordered field most notably Miller presented several tameness notions (like o-minimality, d-minimality and noiselessness) in [33] all imposing restrictions on the definable subsets of the universe to either be "large" or "small" in some sense for tame structures. Depending on the kind of "largeness" and "smallness" different tame properties are implied. Moreover, Miller conjectured that all expansions of the real ordered field can be divided into tame classes of structures, the most general being structures with a noiseless open core, and a wild class of structures which define $\mathbb{Z}$. While this conjecture is not yet proven there has been progress in this direction. It is shown for all D_Σ sets that these are noiseless in expansions of the real ordered field not defining $\mathbb{Z}$. A comprehensive elaboration of the historical developments and current progress on this topic can be found in [23].

Secondly, for the more general setting of expansions of the real ordered additive group, [24] gives a tetrachotomy into wild structures and different kinds of tame structures. The results are also included in [23].

In the most general case of arbitrary dense linear orders without endpoints[3], the ques-

[1] In particular, there are relevant applications to algebraic geometry e. g. the Pila Wilkie Theorem [34] counting rational points and the proof of the Manin-Mumford Conjecture about finiteness of torsion points on algebraic subvarieties of abelian varieties [35]. These results are also nicely revised in [39].

[2] What is still considered as geometrically tame is inconsistent throughout the literature and quite subjective to the author. Here, we also consider structures as tame that might have some wild phenomena.

[3] We only consider dense linear orders without endpoints, since these naturally allow us to consider

tion, how to classify structures which are geometrically tame, seems to be substantially harder to answer. O-minimality is the main example where it is well established that many tame geometric properties hold (confer e. g. [7]). Millers results for the less restrictive notions than o-minimality, introduced for his consideration of expansions of the real ordered field, do not generalize in any way: Neither do generic dense linear orders necessarily behave wild if they define $\mathbb{Z}$ nor is it clear to what extent generalizations of the tameness notions from Miller imply any geometric tameness in the general case. Thus, other notions, more general than o-minimality, are required to further investigate tameness in this setting.

Although there is by far not as good of an understanding for tameness in the general case as for expansions of the real ordered field or real ordered additive group, there is also some progress beyond o-minimality in the general case. For this, we consider a slight generalization of o-minimality, which coincides with o-minimality in the case of expansions of the real ordered field: local o-minimality. Local o-minimality and variations of it have been the theme of many recent papers which prove several different versions of geometric tameness for various settings.

The goal of this thesis is to introduce the reader to this current field of study. In general, the different notions of local o-minimality and familiar notions require unary definable sets to be the union of "large" and "small" sets, similar to Millers tameness notions. Local o-minimality itself was first introduced in [40] as a quite natural generalization of o-minimality. While o-minimality requires every definable set to be a finite union of intervals and points, local o-minimality requires every definable set to be a finite union of intervals and points locally (i. e. in an interval around every point). Other notions require the intervals to be chosen in some uniform way, require the same condition to hold also for definable gaps or relax the criterion to being a finite union of convex sets.

In the first part of this thesis, we introduce the common notions of local o-minimality and familiar notions, relate them to each other and summarize the results in a visualization.

Afterwards, we present which of these notions are fulfilled for several examples from the literature.

In the next part, we summarize the most foundational and general tameness results that have been shown for these notions, in order to evaluate to which degree these can be considered geometrically tame. For all fundamental results, we also present a proof only assuming some generally known technical lemmata from [7].

While the case of locally o-minimal ordered groups and fields is an interesting topic for itself where many interesting results have been shown, it is not the main focus of our studies. Therefore, we only discuss this topic as a short outlook. We summarize how our tameness results can be strengthened in these settings and state some other

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geometric properties like continuity of functions and dimensions of sets. Theoretically, it is also possible to consider geometric tameness for other structures, but this is beyond the scope of this thesis.

interesting results for these structures, both without proofs.

Note that [17] is a prior summary of tameness results of locally o-minimal structures. However, many of the results presented there are outdated now and stronger results have been shown since it was published in 2021. In this thesis, we present an updated and more conclusive summary of this field of study, also including proofs of central results.

Throughout this thesis, we add citations to all definitions, theorems and lemmata where either the statements themselves, trivially equivalent statements or stronger statements implying the presented statements can be found in the literature. The proofs are also mainly taken from these cited sources (or referenced statements in there) with minor modifications and adaptions for our purposes. Notable exceptions are Propositions 6.27 and 6.36 and Theorem 6.31.

In conclusion, this thesis should give the reader a conclusive overview over the currently known tameness results for arbitrary expansions of locally o-minimal dense linear orders without endpoints and refer the interested reader to suitable literature where more detailed presentations of the results and topics can be found.

2 Preliminaries

In this section, we introduce our notations which are coherent with most of the literature on model theory and tame geometry and recall fundamental definitions from tame geometry and basic point set topology. A basic familiarity with first order logic and model theory as in the first chapter of [29] will be assumed.[4]

Throughout this thesis, let $\mathcal{M} = (M, <, \dots)$ be a model theoretic structure expanding a dense linear order without endpoints $(M, <)$ (i. e. a structure modeling the theory DLO e. g. stated in [29, Example 1.2.2]).[5] All other structures introduced in this thesis are also always assumed to be dense linear orders and by dense linear orders, we always mean dense linear orders without endpoints, unless explicitly stated otherwise. By definable, we always mean definable with parameters, unless explicitly stated otherwise.

Let $\mathcal{L}_\mathcal{M}$ be the language of $\mathcal{M}$. Note that the order on M naturally induces a topology. We say a set is *open* if it is an open set in this topology. This means $X \subseteq M$ is open if for every $x \in X$ there exist $y, z \in M$ with $y < x < z$ such that for every $x' \in M$ with $y < x' < z$, $x' \in X$. A set $X \subseteq M^n$ is called open if for every $x \in X$ there is some open box $B \subseteq M^n$ with $x \in B \subseteq X$. A set $Y \subseteq M^n$ is called *closed* if $M^n \setminus Y$ is open. A set $Z \subseteq M^n$ is called *discrete* if for every $z \in Z$ there is some open set $X \subseteq M^n$ with $X \cap Z = \{z\}$. A definable function $f : A \to B$ with $A \subseteq M^m, B \subseteq M^n$ is called *continuous* if for every definable open set $V \subseteq B$ the definable set $U = \{a \in A : f(a) \in V\}$[6] is also open.

For this thesis, regarding intervals, we follow the notation of [7, p. 17] and an *interval* is always meant to be an open interval, i. e. a set of the form $]a, b[:= \{x \in M : a < x < b\}$ for some $-\infty \leq a < b \leq +\infty$ where $\pm\infty$ is are a positive and negative endpoint added to M. Similarly, an (open) box is a set of the form $B := \{(x_1, \dots, x_n) \in M^n : \bigwedge_{i=1}^n (a_i < x_i < b_i)\}$ for some $a_1, \dots, a_n, b_1, \dots, b_n$ with $-\infty \leq a_i < b_i \leq +\infty$.

Moreover, we use the following common notations with the usual meaning:

$$]a, b] = \{x \in M : a < x \leq b\}, [a, b[= \{x \in M : a \leq x < b\}, [a, b] = \{x \in M : a \leq x \leq b\}.$$

A set $X \subseteq M$ is called *convex* if for any $a, b \in X$ and $c \in M$ with $a < c < b$, $c \in X$.

[4] For a reader unfamiliar with model theory, refer to [7, (2.1) and (3.2)] for an alternative (and for our purposes – we only consider definable sets – equivalent) definition of a structure, that might be easier to understand. If the reader chooses to work with that definition of structures they can also find a coherent definition of o-minimality there. In a similar way (definable sets correspond to sets in the structure) one can easily translate the following definitions regarding local o-minimality to that setting. For this, one may also confer [17].

[5] Note that a priori the definition of (local) o-minimality would not require $\mathcal{M}$ to be a dense linear order without endpoints, but for this thesis we only consider dense linear orders. This is still a quite broad setting and the broadest setting in which local o-minimality has been discussed so far.

[6] Throughout this thesis, we use commonly known abbreviations for formulas and the definitions of definable sets, like $\{a \in A : \phi(a)\}$ as an abbreviation for $\{a : a \in A \wedge \phi(a)\}$ and $(\forall\, a \in A \quad \phi(a))$ as an abbreviation for $(\forall\, a\, (a \in A \to \phi(a)))$.

For some set $A \subseteq M^n$ and some $a \in M^m$ with $m < n$, the notation A_a refers to the *fiber of a in A*, defined by $A_a := \{b \in M^{n-m} : (a, b) \in A\}$. Recall that the fiber of a definable subset is also definable.

Throughout this thesis, π denotes projections:

$$\begin{aligned}
\pi_i :& M^n \to M, \\
& (x_1, \ldots, x_n) \mapsto x_i && \text{is the projection onto the } i\text{-th coordinate,} \\
\pi_{\leq j} :& M^n \to M, \\
& (x_1, \ldots, x_n) \mapsto (x_1, \ldots x_j) && \text{is the projection on the first } j \text{ coordinates,} \\
\pi_{\geq j} :& M^n \to M, \\
& (x_1, \ldots, x_n) \mapsto (x_j, \ldots, x_n) && \text{is the projection on the last } n - j + 1 \text{ coordinates.}
\end{aligned}$$

Now, we formally recall the most important fundamental definitions.

Definition 2.1 (O-Minimality, [7, Definition 5.7]). We call the structure $\mathcal{M}$ *o-minimal* if every definable subset $X \subseteq M$ is a finite union of points and intervals.

Another important fundamental property, that is often required for structures in order to potentially have tame properties is definable completeness, first introduced in [32, Corollary 1.5] as the intermediate value property:

Definition 2.2 (Definable Completeness (DC)). We call the structure $\mathcal{M}$ *definably complete* if every definable, bounded subset of M has an infimum and a supremum in M.

Every o-minimal structure is already definably complete.[7] For local o-minimality this is not necessarily the case as many examples in Section 5 illustrate.

Moreover, it is easy to check the following equivalence, which gives an alternative definition of o-minimality, that is sometimes used in the literature.

Remark 2.3. $\mathcal{M}$ is o-minimal if and only if $\mathcal{M}$ is definably complete and every definable set either has interior or is finite.

By [32, Corollary 1.5], in the case of expansions of dense linear orders without endpoints (which is in particular any structure we consider in this thesis)$\mathcal{M}$ being DC is equivalent to M being definably connected:

Definition 2.4 (Definably Connected, [7, Definition 3.5]). A set $X \subseteq M^n$ is called *definably connected* if X is definable and X is not the union of two disjoint nonempty definable open subsets of X.

An important topological property is compactness:

Definition 2.5 (Compactness, [1, Definition 7.2]). A set $X \subseteq M$ is *compact* if for every open covering $\bigcup_{i \in I} U_i \supseteq X$ of X with $U_i \subseteq M$, there exists a finite subcovering, i. e. a finite $J \subseteq I$ such that $\bigcup_{i \in J} U_i \supseteq X$.

[7]For a proof, confer Proposition 3.6.

Other commonly known topological definitions used in this thesis are:

Definition 2.6 (Homeomorphism, [1, Definition 2.7])**.** A function $f : A \to B$ is called *homeomorphism* if it is continuous, bijective (i. e. $f^{-1} : B \to A$ exists) and f^{-1} is also continuous.

Definition 2.7 (Closure, [1, Proposition 3.4])**.** Let $X \subseteq M^n$. Then, $\overline{X}$ denotes the *closure of* X which is the smallest closed set $C \subseteq M^n$ such that $X \subseteq C$.

Definition 2.8 (Interior, [1, Proposition 3.3])**.** Let $X \subseteq M^n$. Then, $\mathring{X}$ denotes the *interior of* X which is the largest open set $U \subseteq M^n$ such that $U \subseteq X$.

Definition 2.9 (Boundary, [23])**.** Let $X \subseteq M^n$. Then, $\mathrm{bd}(X)$ denotes the *boundary of* X defined as $\mathrm{bd}(X) = \overline{X} \setminus \mathring{X}$.

Definition 2.10 (Frontier, [23])**.** Let $X \subseteq M^n$. Then, $\mathrm{fr}(X)$ denotes the *frontier of* X defined as $\mathrm{fr}(X) = \overline{X} \setminus X$.

Remark 2.11. It is well known, that the closure, interior, boundary and frontier of a definable set are definable.

3 Notions of Local O-Minimality

In this section, we introduce the various different notions of local o-minimality which are used in publications and discussed throughout this thesis. Moreover, we evaluate how these notions relate to each other. These relations are also visualized in Figure 1. For our considerations, we view the notions being some condition of a (possibly uniform) way to choose bounded intervals on which a definable set is a finite union of intervals and points as the notions of local o-minimality. In the general case of arbitrary dense linear orders, these restrictions on definable sets have been one of the least restrictive conditions for which significant results can be shown. Therefore, these are the notions of main interest to us in this thesis.

Historically, the concept of local o-minimality was first introduced by Toffalori and Vozoris in [40] with the notion of local o-minimality and strong local o-minimality:

Definition 3.1 (Local O-Minimality, [40, Definition 2.1])**.** We call the structure $\mathcal{M}$ *locally o-minimal* if for every $m \in M$ and every definable $X \subseteq M$, there is an interval I around m such that $X \cap I$ is a finite union of intervals and points.

Definition 3.2 (Strong Local O-Minimality, [40, Definition 3.1])**.** We call the structure $\mathcal{M}$ *strongly locally o-minimal* if for every $m \in M$, there is an interval I around m such that for every definable $X \subseteq M$, $X \cap I$ is a finite union of intervals and points.

Moreover, in [26] uniform local o-minimality of the first kind is first introduced as uniform local o-minimality.

Definition 3.3 (Uniform Local O-Minimality of the First Kind, [26, Definition 4])**.** $\mathcal{M}$ is *uniformly locally o-minimal of the first kind* if for every definable $X \subseteq M^{n+1}$ and $a \in M$, there is an open interval $I \ni a$ such that for every $b \in M^n$, the set $I \cap X_b$ is a finite union of intervals and points.

Fujita then added several different other notions of local o-minimality throughout various papers. Here, we present all of the notions for which substantial results have been shown. The notions matching our restricted understanding of notions of local o-minimality are the following two:

Definition 3.4 (Uniform Local O-Minimality of the Second Kind, [16, Definition 2.1])**.** $\mathcal{M}$ is *uniformly locally o-minimal of the second kind* if for every definable $X \subseteq M^{n+1}$, $a \in M$ and $b \in M^n$, there is an open interval $I \ni a$ and an open box $B \ni b$ such that for every $b' \in B$, the set $I \cap X_{b'}$ is a finite union of intervals and points.

Definition 3.5 (Almost O-Minimal, [10, Definition 1.2])**.** $\mathcal{M}$ is *almost o-minimal* if every bounded definable subset $X \subseteq M$ is a finite union of points and open intervals.

In Definition 2.2 we defined a structure to have DC, if every bounded definable set has an infimum and a supremum. Notice that this property is guaranteed in every almost o-minimal structure:

Proposition 3.6 ([10, Lemma 4.6])**.** *Every almost o-minimal structure is definably* *complete.*

Proof. Let $X \subseteq M$ be a bounded, definable subset. By almost o-minimality, X is a finite union of intervals and points. Thus, $\text{bd}(X)$ is finite and $\sup(X) = \max(\text{bd}(X)) \in M$ and $\inf(X) = \min(\text{bd}(X)) \in M$.

□

For all weaker notions of local o-minimality, this is not necessarily the case, as presented in Section 5. This crucially complicates efforts to prove tameness results for locally o-minimal structures which are not almost o-minimal since many of the proofs for o-minimal structures use definable completeness. It turns out to be meaningful to consider these properties in the case where DC holds and in the case where DC does not hold separately. Moreover, we later introduce univariate $*$-continuity as a slight generalization of DC, preserving most of the tameness results.[8]

For locally o-minimal dense linear orders, the following easy, but important, observation holds:

Lemma 3.7. *Let $\mathcal{M}$ be locally o-minimal. Then, the finite union of definable sets $X_i \subseteq M$ without interior has no interior.*

Proof. Towards a contradiction, suppose $X = \bigcup_{i=1}^{n} X_i$ has interior and all X_i do not have interior. Let $x \in X$ be a point in the interior. Since all the X_i do not have interior, by local o-minimality, these are locally finite. Thus, we can find a (non-empty, open) interval I around x in the interior of X, such that for all X_i: $(I \cap X_i) \setminus \{x\} = \emptyset$. This contradicts $\bigcup_{i=1}^{n} X_i \cap I = X \cap I = I$. □

We use this basic fact numerous times throughout this thesis, without explicitly mentioning it every time.

Remark 3.8. For subsets of M^n, the statement, that $X = \bigcup_{i=1}^{n} X_i$ does not have interior if all X_i do not have interior, is a lot harder to show and not generally true. In this thesis, we only present a proof for locally o-minimal structures admitting local cell decomposition and $*$-locally weakly o-minimal structures enjoying the univariate $*$-continuity property.[9] In [15, Theorem 3.9], this statement is additionally proven for all uniformly locally weakly o-minimal structures of the second kind. For other locally o-minimal structures it is not known if the statement holds.

Proposition 3.9. *O-minimality implies almost o-minimality, almost o-minimality implies strong local o-minimality, strong local o-minimality implies uniform local o-minimality of the first kind, uniform local o-minimality of the first kind implies uniform local o-minimality of the second kind and uniform local o-minimality of the second kind implies local o-minimality.*

[8] see Definition 4.20 and Section 6

[9] see Lemma 6.29 and Corollary 6.23

Proof. All implications follow from the definitions. Here, we mention the least obvious ones:

ALMOST LOCAL O-MINIMALITY $\Rightarrow$ STRONG LOCAL MINIMALITY For some x and X, pick any bounded interval I around x. Then, $X \cap I$ is bounded and, thus, a finite union of intervals and points.

STRONG LOCAL O-MINIMALITY $\Rightarrow$ UNIFORM LOCAL MINIMALITY OF THE FIRST KIND Note that the definition of strong local o-minimality can be reformulated as: For $a \in \mathcal{M}$, there is an open interval $I \ni a$ such that for every definable $X \subseteq M^{n+1}$ and every $b \in M^n$, the set $I \cap X_b$ is a finite union of intervals and points.

UNIFORM LOCAL O-MINIMALITY OF THE SECOND KIND $\Rightarrow$ LOCAL MINIMALITY Note that the definition of local o-minimality can be reformulated as: For every definable $X \subseteq M^{n+1}$, every $a \in M$ and $b \in M^n$, there is an open interval $I \ni a$ such that $I \cap X_b$ is a finite union of intervals and points.

□

Remark 3.10. None of the implications from the previous remark are equivalences as shown later in Section 5.

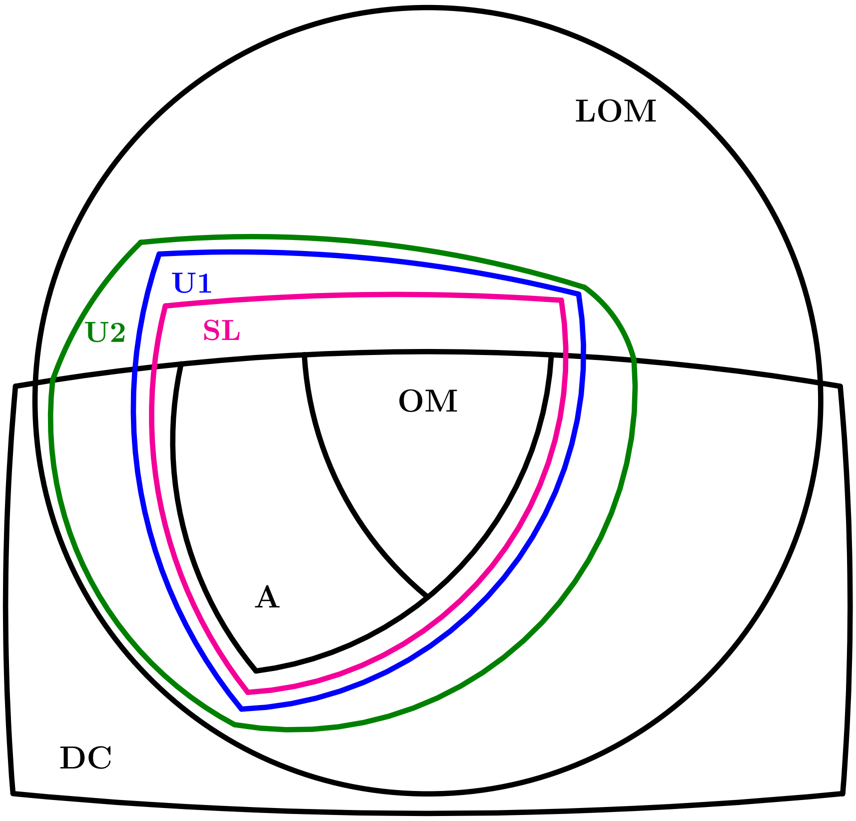
If we do not consider arbitrary expansions of dense linear orders without endpoints but special restricted cases instead, then equivalences for some of these notions can occur. In the following, we present some examples of such cases.

Proposition 3.11 ([16, Proposition 2.1]). *If an ordered field $\mathcal{M} = (M, <, +, \cdot, \dots)$ is uniformly locally o-minimal of the second kind, then $\mathcal{M}$ is o-minimal.*

Proof. Let $\mathcal{M} = (M, <, +, \cdot, \dots)$ be a uniformly locally o-minimal ordered field of the second kind. Let X be a definable subset of M. We show that X is a finite union of intervals and points. We first consider the case in which X is bounded. Consider the set $Y = \{(x, r) \in M^2 : r > 0, \frac{x}{r} \in X\}$. There is some interval $I \ni 0$ and some $r_1 > 0$ such that for all $r_1 > r > 0$ and all fibers $Y_r = \{x \in M : \frac{x}{r} \in X\}$, the set $I \cap Y_r$ is a finite union of intervals and points. Since X is bounded, there is some r' with $r_1 > r' > 0$ such that $I \cap Y_{r'} = Y_{r'}$. This implies that $Y_{r'}$ is a finite union of intervals and points. Thus, the set X is also a finite union of intervals and points.

Let X be unbounded, let $X_1 := X \cap [-1, 1]$ and $X_2 = X \setminus X_1$. By the previous reasoning, the set X_1 is a finite union of intervals and points because X_1 is bounded. Consider the set $Z = \{x \in M : \frac{1}{x} \in X_2\}$. It is bounded and, hence, a finite union of intervals and points. Therefore, X_2 is also a finite union of intervals and points. □

While in the following reference the statement only considered strong o-minimal structures, the proof presented in [40, Corollary 3.4] already implies equivalence to almost o-minimality:



A	almost o-minimality	**SL**	strong local o-minimality
DC	definable completeness	**U1**	uniform local o-minimality of the 1. kind
LOM	local o-minimality	**U2**	uniform local o-minimality of the 2. kind
OM	o-minimality		

Figure 1: Schematic visualization of the relations between the notions discussed in Section 3. All the labels refer to the smallest bounded convex set they are contained in. A few of the labels and associated polygons are additionally colored for clarity. Note that the sizes of the areas have no meaning but only differ due to technical reasons. All of the inclusions have been shown in this thesis. However, not for every cut an example is presented that the cut is indeed not empty. It might be that some of the cuts are empty, but we suspect them all to indeed be non-empty.

Proposition 3.12 ([40, Corollary 3.4])**.** *If $\mathcal{M} = (\mathbb{R}, <)$ expands the real line and is locally o-minimal, it is almost o-minimal.*

Proof. Take any $a \in \mathbb{R}$ and $[b, c] \in \mathbb{R}$ with $a \in [b, c]$. Note that any $[b, c]$ is compact for $b, c \in \mathbb{R}$. Let $X \subseteq M$ be definable. By local o-minimality, for every $x \in [b, c]$, there is an interval I_x around x such that $X \cap I_x$ is a finite union of intervals and points. By compactness, there is a finite subcover $\bigcap_{i \in J} I_i \supseteq [b, c]$ with $J \subset [b, c]$ being finite.

Then, $X \cap \bigcap_{i \in J} I_i = \bigcap_{i \in J} X \cap I_i$ is a finite union of intervals and points. It is easy to check, that the same holds for the subset $X \cap [b, c]$. □

Remark 3.13. The necessary condition to prove the equivalence of strong local o-minimality and local o-minimality is actually just that for any $a \in M$, there is a $b, c \in M$ such that $b < a < c$ and $[b, c]$ is compact.

Now, we combine the previous two statements:

Corollary 3.14 ([6, 2.13 (3)])**.** *If* $\mathcal{M} = (\mathbb{R}, <, +, \cdot, \dots)$ *expands the real field and is locally o-minimal, then it is o-minimal.*

Proof. Let $\mathcal{M} = (\mathbb{R}, <, +, \cdot, \dots)$ expand the real field and be locally o-minimal. By Proposition 3.12, $\mathcal{M}$ is almost o-minimal and, thus, uniformly locally o-minimal of the second kind. By Proposition 3.11, o-minimality follows. □

4 Familiar Tameness Notions

There are some other tameness notions closely related to o-minimality and local o-minimality. Here, we introduce some of these notions and how they relate to our notions of local o-minimality. First, we recall well known notions which are of general interest due to their properties. Secondly, some of our tameness results are shown in the most general setting they are known for.[10] This is not always one of our local o-minimality notions but instead one of the notions introduced here.

Most of the notions introduced here imply local o-minimality. Only the univariate $*$-continuity property, d-minimality and local l-viscerality do not. The last two of these notions are indeed generalizations of local o-minimality. D-minimality generalizes the notion of local o-minimality slightly in the DC setting, while local l-viscerality generalizes local o-minimality slightly including more structures which are not DC. The univariate $*$-continuity property is a slight generalization of the DC property, essentially specifying the actual necessary condition for the tameness results implied by DC. Thus, it is a rather unrelated property to local o-minimality which plays a crucial role as an additional property for structures to have in order to ensure tameness.

Note that some of these notions are quite closely related to local o-minimality and could even be considered as a definition of some kind of local o-minimality.

At the end of the subsections, Figures 2 to 5 visualize the relations between the different tameness notions, discussed in the corresponding subsection.

4.1 TC and DCTC

In this subsection, we discuss TC which is a slightly stronger version of local o-minimality also requiring a similar condition for intervals with $\pm\infty$ as a boundary. TC was extensively studied in [37] by Schoutens, who actually considered it to be the canonical definition of local o-minimality: "(TC) is stronger than local o-minimality, since we also have this condition at $\pm\infty$, which seems to [Schoutens] an omission in the original definition" [37, p. 8].[11]

Definition 4.1 (Type Complete (TC))**.** $\mathcal{M}$ is called *type complete (TC)* if for any definable $Y \subseteq M$, there exist $y_1, y_2 \in M$ such that $]-\infty, y_1[\, \subseteq Y$ or $]-\infty, y_1[\, \cap Y = \emptyset$ and $]y_2, \infty[\, \subseteq Y$ or $]y_2, \infty[\, \cap Y = \emptyset$. Additionally, for every $x \in M$, there exist z_1, z_2 with $z_1 < x < z_2$ such that $]x, z_2[\, \subseteq Y$ or $]x, z_2[\, \cap Y = \emptyset$ and $]z_1, x[\, \subseteq Y$ or $]z_1, x[\, \cap Y = \emptyset$.

Schoutens showed many properties of structures which are type complete and definably complete, and thus, also dedicated an extra name to these structures, called DCTC structures:

[10] There are exceptions where we do not present the most general setting, but a reasonable one. This is always remarked in these cases.

[11] However, the results presented in this thesis should convince the reader, that it is also reasonable to consider local o-minimality without the TC condition.

Definition 4.2 (DCTC)**.** A structure is called *DCTC* if it is both, DC and TC.

Note that for a particular definable set the properties DC and TC can be expressed as a $\mathcal{L}_{\mathcal{M}}$-sentence. Thus, the following structures are DCTC:

Definition 4.3 (O-Minimalistic, [37])**.** A structure is called *o-minimalistic* if it models all sentences of T_{o-min}, the theory of o-minimal structures, which contains all sentences that hold in all o-minimal structures.

Schoutens suggested that there probably are DCTC structures which are not o-minimalistic. However, he did not provide an example of such a structure and, thus, this is still an unsolved question. In the following, we only consider DCTC as a tameness criterion, but the reader interested in this possibly tamer subclass of o-minimalistic structures may refer to [37] for an intensive evaluation of these structures.

As the definitions of almost o-minimality and TC restrict complementary sets to be o-minimal, we would naturally assume the following, indeed true, theorem:

Proposition 4.4 ([10, Proposition 4.8])**.** *$\mathcal{M}$ is o-minimal if and only if $\mathcal{M}$ is almost o-minimal and TC.*

Proof. By definition, every o-minimal structure is almost o-minimal and TC. For the other implication, let $\mathcal{M}$ be almost o-minimal and TC. Let X be some definable subset. By TC, there are $a, b \in M$ such that $]-\infty, a[\cap X$ and $]b, \infty[\cap X$ are finite unions of intervals and points. By almost o-minimality, $[a, b] \cap X$ is a finite union of intervals and points. Hence, $X = \big(]-\infty, a[\cap X\big) \cup \big([a, b] \cap X\big) \cup \big(]b, \infty[\cap X\big)$ is a finite union of intervals and points.

□

4.2 Weak Notions

In the setting of structures which are not DC, convex sets are not necessarily points and intervals anymore. Therefore, it is only reasonable to wonder what happens if we not only consider finite unions of intervals and points but finite unions of convex sets instead. Replacing this in the definition of o-minimality results in the well known and extensively studied class of weakly o-minimal structures which turns out to be a subset of locally o-minimal structures and hence, of particular interest to us. Moreover, we could also apply the same changes to the different notions of local o-minimality. This results in a particularly interesting new definition in the case of uniform local weak o-minimality of the second kind.

Weak o-minimality was first introduced in [3] and its properties were further studied in several publications, e. g. [28].

Definition 4.5 (Weak O-Minimality)**.** We call a structure $\mathcal{M}$ *weakly o-minimal* if every definable subset $X \subseteq M$ is a finite union of convex definable subsets.

Recall, a set $Y \subseteq M$ is convex if for any $a, b \in Y$ and $c \in M$ with $a < c < b$, $c \in Y$.

Proposition 4.6. *$\mathcal{M}$ is weakly o-minimal and has DC if and only if $\mathcal{M}$ is o-minimal.*

Proof. Note that convex sets with definable infimum and supremum are equivalent to the union of the interval between the infimum and supremum (if existent) and possibly the infimum or the supremum, i. e. one or two points. □

Proposition 4.7 ([40, Proposition 2.2])**.** *Every weakly o-minimal structure $\mathcal{M}$ is locally o-minimal.*

Proof. Let $\mathcal{M}$ be weakly o-minimal, $X \subseteq M$ definable and $x \in M$. Then, X is a finite union of convex subsets. First, assume $x \in X$:

We consider two cases:

1. There is some $\varepsilon < x$ in M such that $]\varepsilon, x] \cap X = \{x\}$.
2. For every $\varepsilon < x$ in M, there is some $c \in M$ such that $\varepsilon < c < x$ and $c \in X$. Then, by weak o-minimality, there is some $\varepsilon < x$ in M such that $]\varepsilon, x] \subseteq X$.

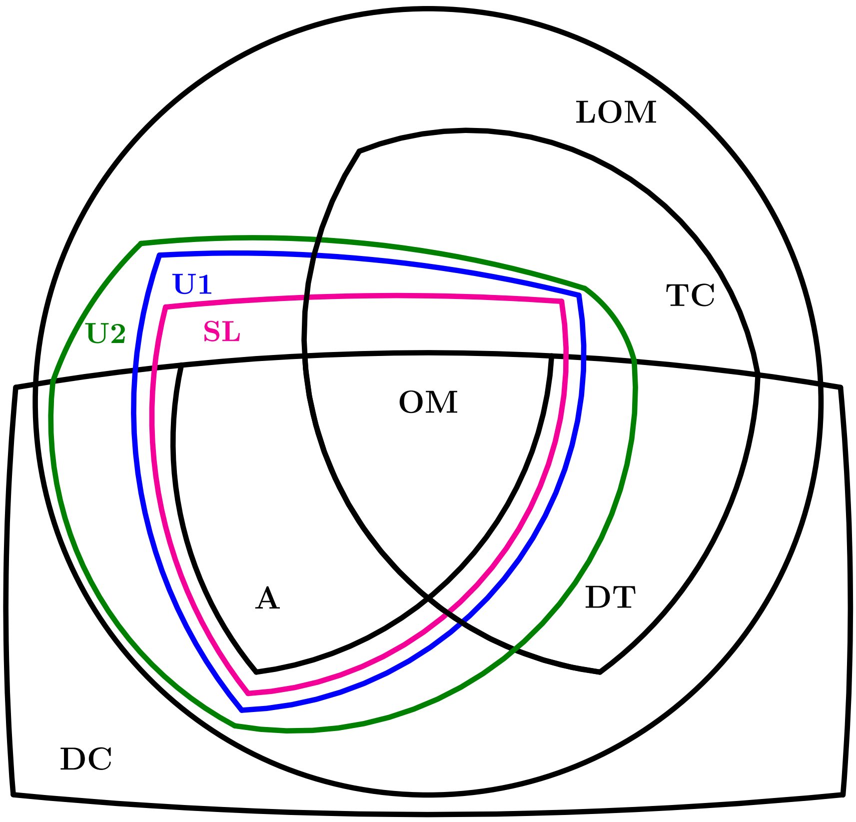
Similarly, for each of these two cases we can again do a case distinction:

(a) There is some $\delta > x$ in X, such that $[x, \delta[\cap X = \{x\}$.

(b) There is some $\delta > x$ in M, such that $[x, \delta[\subseteq X$.

Define $I :=]\varepsilon, \delta[$. By the case distinction, $X \cap I$ equals one of these four sets: $\{x\}$, $]\varepsilon, x]$, $[x, \delta[$, I, each of which are obviously a finite union of intervals and points. For $x \notin X$, a similar construction for $M \setminus X$ results in $X \cap I$ equaling one of the four sets $\emptyset$, $]\varepsilon, x[$, $]x, \delta[$, $I \setminus \{x\}$. □

Proposition 4.8. *Every weakly o-minimal structure is TC.*



A	almost o-minimality	**SL**	strong local o-minimality
DC	definable completeness	**TC**	type completeness
DT	DCTC	**U1**	uniform local o-minimality of the 1. kind
LOM	local o-minimality	**U2**	uniform local o-minimality of the 2. kind
OM	o-minimality		

Figure 2: Expanded version of Figure 1, also including the notions discussed in subsection 4.1. Again, all the labels refer to the smallest bounded convex set they are contained in. A few of the labels and associated polygons are additionally colored for clarity. All of the inclusions have been shown in this thesis. However, not for every cut an example is presented that the cut is indeed not empty. It might be that some of the cuts are empty, but we suspect them all to indeed be non-empty.

Proof. Let $\mathcal{M}$ be a weakly o-minimal structure and $Y \subseteq M$ be definable. Since $\mathcal{M}$ is locally o-minimal, it remains to show, that there exist $y_1, y_2 \in M$ such that $]-\infty, y_1[\subseteq Y$ or $]-\infty, y_1[\cap Y = \emptyset$ and $]y_2, \infty[\subseteq Y$ or $]y_2, \infty[\cap Y = \emptyset$. By weak o-minimality, Y is a finite union of convex sets $\{Y_i\}_{i=1}^n$. Without loss of generality, we can assume the convex sets Y_i to be pairwise disjoint. By finiteness, we can choose the convex set $Y_{i_{max}}$ containing elements which are larger than all elements contained in Y_i for all $i \neq i_{max}$. Similarly, we define $Y_{i_{min}}$ containing the smallest elements of Y. Pick $z_1 \in Y_{i_{min}}$ and $z_2 \in Y_{i_{max}}$. Then, $M \setminus (Y_{i_{min}})$ either contains $y_1 < z_1$ implying $]-\infty, y_1[\cap Y = \emptyset$ or there is no such element implying $]-\infty, z_1[\subseteq Y$. Moreover, $M \setminus (Y_{i_{max}})$ either contains $y_2 > z_2$ implying $]-\infty, y_2[\cap Y = \emptyset$ or there is no such element implying $]-\infty, z_2[\subseteq Y$.

□

Moreover, we can introduce weak versions of our previously discussed notions of local o-minimality, requiring a finite union of convex sets instead of intervals and points. One example would be local weak o-minimality defined as follows:

Definition 4.9 (Local Weak O-Minimality, [15, Definition 2.3])**.** We call a structure $\mathcal{M}$ *locally weakly o-minimal* if for every $m \in M$ and every definable $X \subseteq M$, there is an interval I around m such that $X \cap I$ is a finite union of convex definable subsets.

However, this definition is rather uninteresting as the following proposition shows:

Proposition 4.10 ([15, Proposition 2.4])**.** *Every locally weakly o-minimal structure is locally o-minimal.*

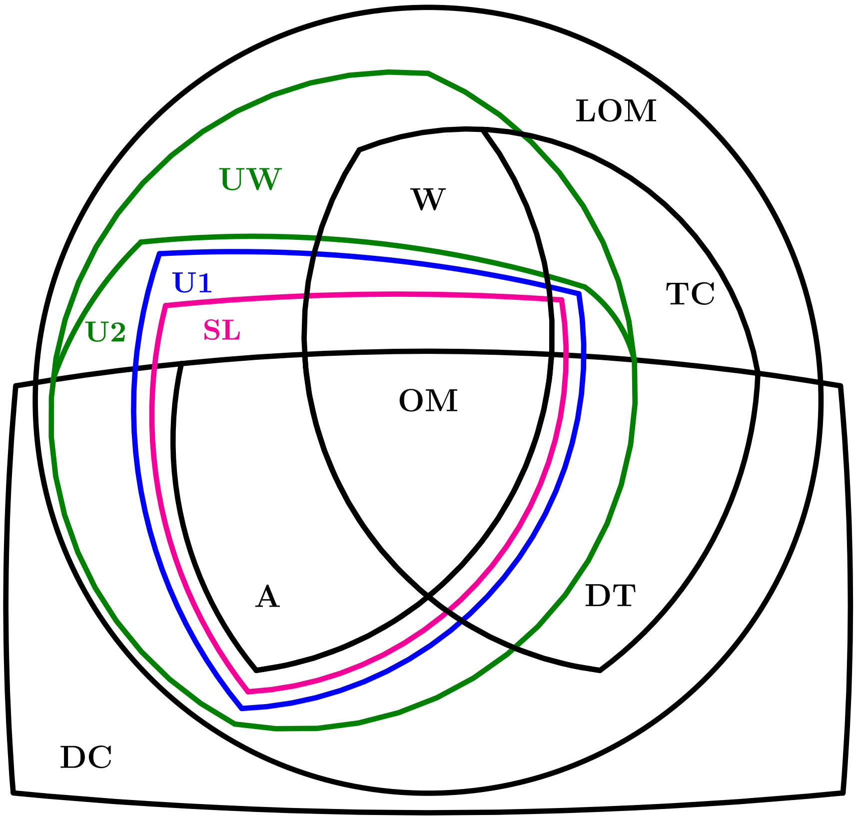
Proof. Let $\mathcal{M}$ be locally weakly o-minimal, $X \subseteq M$ definable and $x \in M$. By local weak o-minimality, there is an interval $I \ni x$ such that $I \cap X$ is a finite union of convex sets. We can apply the proof of Proposition 4.7 to x and $I \cap X$. Thus, there is some interval $J \ni x$ such that $J \cap I \cap X$ is a finite union of intervals and points. Hence, $J \cap I$ is an interval satisfying the condition for local o-minimality. □

A more interesting example is uniform local weak o-minimality:

Definition 4.11 (Uniform Local Weak O-Minimality, [15, Definition 2.3])**.** $\mathcal{M}$ is called *uniformly locally weakly o-minimal of the second kind* if for every definable set $X \subseteq M^{n+1}$, every $a \in M$ and every $b \in M^n$, there exist an open interval $I \ni a$ and an open box $B \ni b$ such that $X_x \cap I$ is a finite union of convex definable subsets for all $x \in B$.

If the statement holds for $B = M^n$, the structure is called a *uniformly locally weakly o-minimal structure of the first kind.*

Remark 4.12. By the definitions, weak o-minimality implies uniform local weak o-minimality of the first kind. Uniform local weak o-minimality of the first kind implies uniform local weak o-minimality of the second kind and uniform local weak o-minimality of the second kind implies local weak o-minimality, which is equivalent to local o-minimality. Moreover, uniform local o-minimality of the second kind clearly implies uniform local weak o-minimality of the second kind.



A	almost o-minimality	**TC**	type completeness
DC	definable completeness	**U1**	uniform local o-minimality of the 1. kind
DT	DCTC	**U2**	uniform local o-minimality of the 2. kind
LOM	local o-minimality	**UW**	uniform local weak o-minimality of the 2. kind
OM	o-minimality	**W**	weak o-minimality
SL	strong local o-minimality		

Figure 3: Expanded version of Figure 1, also including the notions discussed in subsections 4.1 and 4.2. Again, all the labels refer to the smallest bounded convex set they are contained in. A few of the labels and associated polygons are additionally colored for clarity. All of the inclusions have been shown in this thesis. However, not for every cut an example is presented that the cut is indeed not empty. It might be that some of the cuts are empty, but we suspect them all to indeed be non-empty.

4.3 $*$-Local Notions

As already hinted previously, the absence of DC imposes significant challenges in proving tameness results, since not all convex sets are intervals in this case. Thus, it is a intuitive idea to consider the definable Dedekind completion of our structure as a familiar structure which has DC, and imposing conditions also onto the Dedekind completion instead of just on our structure.[12] This actually gives rise to a classification for structures which inherit some tameness from their Dedekind completion. Here, we only introduce notions for which significant tameness results have been proven, but obviously one could introduce other familiar notions.[13]

Definition 4.13 (Definable Dedekind Completion, [15, Definition 2.9, Definition 2.10])**.** Let $\mathcal{M} = (M, <, \dots)$ be an expansion of a dense linear order without endpoints. If there is a non-empty, open and closed, convex, definable set $A \subsetneq M$ with $\inf(A) = -\infty$, we call the pair $(A, M \setminus A)$ a *definable gap.*

Set $\overline{\overline{M}} = M \cup \{\text{definable gaps in } M\}$. We can naturally extend the order $<$ on M to an order on $\overline{\overline{M}}$, which is denoted by the same symbol $<$. The linearly ordered set $(\overline{\overline{M}}, <)$ is called the *definable Dedekind completion* of $(M, <)$. For any arbitrary open interval $I = (a_1, b_1)$ in M, where $a_1, b_1 \in M \cup \{\pm\infty\}$, we set $\overline{\overline{I}} = \{x \in \overline{\overline{M}} : a_1 < x < b_1\}$.

Moreover, $f : X \to \overline{\overline{M}}$ with $X \subseteq M^n$ is called a *definable function* if there exists a definable set $Y \subseteq M^{n+1}$ such that $\pi_{\leq n}(Y) = X$ and $\sup(Y_x) = f(x)$ for all $x \in X$.

Remark 4.14. Note, that we can easily check that this definition of definable gaps actually coincides with [15, Definition 2.9].

If we have a definable gap $(A, M \setminus A)$ as defined here, then clearly $A \cup M \setminus A = M$. Moreover, $A \neq M$ implies that $M \setminus A$ is non-empty. For every $a \in A, b \in M \setminus A$, we have $a < b$ since $\inf(A) = -\infty$ and A is convex. Additionally, A is open and it has no largest element. Because A is closed and convex, $M \setminus A$ is open and has no smallest element.

For the converse: If (A, B) is a definable gap according to [15, Definition 2.9], then $B = M \setminus A$, since $A \cup B = M$. Moreover, B non-empty implies $A \neq M$. The sets A, $M \setminus A$ are convex and $\inf(A) = -\infty, \sup(M \setminus A) = \infty$ since $a < b$ for all $a \in A, b \in B = M \setminus A$. With this and A not having a largest element and $M \setminus A$ not having a smallest element, A and $M \setminus A$ are open sets.

Fact 4.15. *Let $\mathcal{M}$ be definably complete. Then, $\overline{\overline{M}} = M$.*

Proof. We need to show that a definably complete structure has no definable gaps. Suppose there was a definable gap $(A, M \setminus A)$. By definable completeness, there is some $\sup(A) \in M \cup \{\pm\infty\}$. The sets A, $M \setminus A$ are non-empty, $\inf(A) = -\infty$ and A is

[12]Note that the Dedekind completion, as defined here, is also a common definition in the literature used for tameness results for weak o-minimal structures.

[13]Interested readers should consider [15].

convex. Hence, $\sup(A) \in M$ and since A is open, we have $\sup(A) \in M \setminus A$. It is easy to check, that $\sup(A) = \inf(M \setminus A)$, but $\inf(M \setminus A) \in M \setminus A$ contradicts that $M \setminus A$ is open. □

Naturally, a first idea for a reasonable new tameness notion would be to impose the local o-minimality condition for all points of $\overline{\overline{M}}$, as follows:

Definition 4.16 (∗-Local O-Minimality, [15, Definition 2.11])**.** $\mathcal{M}$ is *∗-locally o-minimal* if for every definable subset $X \subseteq M$ and for every point $\overline{\overline{a}} \in \overline{\overline{M}}$, there exists an open interval I such that $\overline{\overline{a}} \in \overline{\overline{I}}$ and $X \cap I$ is a finite union of intervals and points.

However, this definition describes actually just the set of definably complete locally o-minimal structures, as implied by Fact 4.15 and the next proposition:

Proposition 4.17 ([15, Proposition 2.13])**.** *A ∗-locally o-minimal structure is definably complete.*

Proof. Let $\mathcal{M}$ be ∗-locally o-minimal and let $X \subseteq M$ be definable and bounded. We need to show that $\sup(X) \in M$ and $\inf(X) \in M$. Suppose not. The set $X_l := \{m \in M : \exists x \in X \quad m \leq x\}$ is definable and $(X_l, M \setminus X_l)$ is a definable gap. Therefore, $\overline{\overline{x_l}} := (X_l, M \setminus X_l) \in \overline{\overline{M}}$ and there is some interval I_l with $\overline{\overline{x_l}} \in \overline{\overline{I_l}}$ such that $I_l \cap X$ is a finite union of intervals and points. In particular, $\sup(I_l \cap X) \in M$ since it is equal to the maximal point or boundary point of an interval and X is bounded. By the definition of X_l, there is some $x \in X \cap I_l$. We conclude that $\sup(X) = \sup(I_l \cap X) \in M$ by construction. Similarly, $X_r := \{m \in M : \exists x \in X \quad m \geq x\}$ is definable and $(X_r, M \setminus X_r)$ is a definable gap. Again, we can find some interval I_r around the definable gap such that $X \cap I_r$ is a finite union of intervals and points and non-empty. This results in $\inf(X) = \inf(X \cap I) \in M$ which is a contradiction.

□

However, there is also a different definition, which has several significant results:

Definition 4.18 (∗-Local Weak O-Minimality, [15, Definition 2.11])**.** $\mathcal{M}$ is *∗-locally o-minimal* if for every definable subset $X \subseteq M$ and for every point $\overline{\overline{a}} \in \overline{\overline{M}}$, there exists an open interval I such that $\overline{\overline{a}} \in \overline{\overline{I}}$ and $X \cap I$ is a finite union of convex definable subsets.

Remark 4.19. By definition, any weak o-minimal structure is also ∗-locally weakly o-minimal. Moreover, any ∗-locally o-minimal structure (i. e. any definably complete, locally o-minimal structure by Proposition 4.17) is ∗-locally weakly o-minimal.

Moreover, the ∗-notation allows to denote a slight generalisation of DC[14] that already suffices as a condition instead of DC for most of the tameness results that are known for definably complete, locally o-minimal structures.

[14]It is a generalization in the case of locally o-minimal structures, as we prove in Proposition 4.21. In the general setting of arbitrary structures, these notions are unrelated.

Definition 4.20 (Univariate $*$-Continuity Property, [15, Definition 3.11])**.** Let $\mathcal{M}$ be an expansion of a dense linear order without endpoints. We say that $\mathcal{M}$ has the *univariate $*$-continuity property* if, for every definable function $f : I \to \overline{\overline{M}}$ from a nonempty, open interval I, there exists a nonempty open subinterval J of I such that the restriction of f to J is continuous.

Proposition 4.21 ([21, Lemma 2.1, Theorem 2.3])**.** *Let $\mathcal{M}$ be a definably complete, locally o-minimal structure. Then, $\mathcal{M}$ has the univariate $*$-continuity property.*

Proof. Let I be a nonempty, open interval and let $f : I \to \overline{\overline{M}}$ be definable. Since $\mathcal{M}$ is definably complete, we have $\overline{\overline{M}} = M$ and thus, $f : I \to M$. It remains to show, that there is some $J \subseteq I$ such that f is continuous on J.

Step 1. There is some interval $I_1 \subseteq I$ such that f is strictly monotone.

Consider the following formulas:

$$\begin{aligned}
\phi_{co} &= \exists\, x_1 \quad (x_1 > x) \wedge (\forall\, t(x < t < x_1) \to (f(x) = f(t))) \\
\phi_{+} &= \exists\, x_1 \quad (x_1 > x) \wedge (\forall\, t(x < t < x_1) \to (f(x) < f(t))) \\
\phi_{-} &= \exists\, x_1 \quad (x_1 > x) \wedge (\forall\, t(x < t < x_1) \to (f(x) > f(t))) \\
\psi_{co} &= \exists\, x_0 \quad (x_0 < x) \wedge (\forall\, t(x_0 < t < x) \to (f(x) = f(t))) \\
\psi_{-} &= \exists\, x_0 \quad (x_0 < x) \wedge (\forall\, t(x_0 < t < x) \to (f(x) < f(t))) \\
\psi_{+} &= \exists\, x_0 \quad (x_0 < x) \wedge (\forall\, t(x_0 < t < x) \to (f(x) > f(t)))
\end{aligned}$$

And the definable sets:

$$\begin{aligned}
A_{\phi_i} &:= \{x \in I : \mathcal{M} \models \phi_i(x)\} \\
A_{\psi_j} &:= \{x \in I : \mathcal{M} \models \psi_j(x)\} \\
A_{\phi_i\psi_j} &:= A_{\phi_i} \cap A_{\psi_i} = \{x \in I : \mathcal{M} \models (\phi_i \wedge \psi_j)(x)\}
\end{aligned}$$

with $i, j \in \{+, -, co\}$.

Note that, for any x and t one of the formulas $(f(x) = f(t))$, $(f(x) > f(t))$ and $(f(x) < f(t))$ has to hold. The sets of t's such that each of these formulas hold are definable. By local o-minimality, there is an interval around x where each of these sets are a finite union of intervals and points. Thus, at least one of these sets has some interior of the form (x_0, x) and at least one has interior of the form (x, x_1) with $x_0 < x < x_1$. Hence, for every x, at least one ϕ_i and at least one ψ_j holds true. Again, for any x, due to local o-minimality, one A_{ϕ_i} has interior of the form (y_1, x) and at least one of the A_{ψ_i} has interior of the form (y_2, x) with $y_1 < x$ and $y_2 < x$. Therefore, there is some $y_3 < x$ and some i, j such that $]y_3, x[\subseteq A_{\phi_i\psi_j}$.

Obviously, the sets $A_{\phi_+\psi_{co}}, A_{\phi_-\psi_{co}}, A_{\phi_{co}\psi_+}$ and $A_{\phi_{co}\psi_-}$ cannot have interior.

Towards a contradiction, suppose $A_{\phi_+\psi_-}$ has interior. Let I_A be an interval in the interior. Let $x_1 < x_2$ be in I_A such that $f(x_1) < f(x_2)$. It is easy to check, that such points have to exist. Define the set

$$L_{x_2} = \{y \in M : (y < x_2) \wedge (\forall z \in]y, x_2[\quad f(z) > f(x_2))\},$$

which has interior since $x_2 \in A_{\phi_+\psi_-}$. By definable completeness, the infimum $x_L \in I_A$ of the set exists, and clearly $x_L \geq x_1$. This leads to a contradiction, since $x_L \in A_{\phi_+\psi_-}$ but for all $z \in]x_L, x_2[$, we have $f(z) > f(x_2) \geq f(x_L)$ by the definition of L_{x_2}. Similarly, we can argue that $A_{\phi_-\psi_+}$ has empty interior.

Thus, either $A_{\phi_{co}\psi_{co}}$, $A_{\phi_+\psi_+}$ or $A_{\phi_-\psi_-}$ has to contain interior.

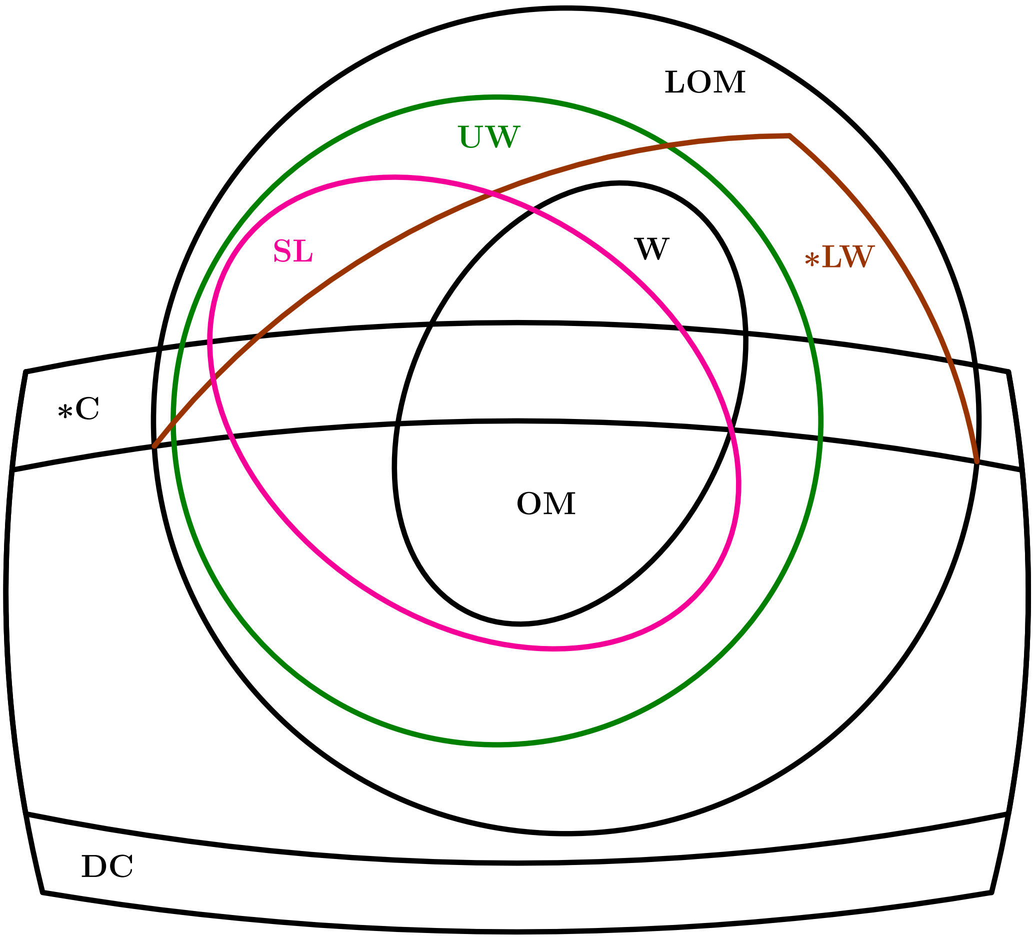
Step 2. There is some interval $J \subseteq I_1$ such that f is continuous.

If f is constant on I_1, we are done. Suppose f is strictly increasing. First, if $f(I_1)$ contains interior, then let $I_2 \subseteq f(I_1)$ be an interval in this interior. It is easy to check that $f^{-1}(I_2) \subseteq I_1$ is an interval on which f is continuous. Therefore, it remains to consider the case of $f(I_1)$ not having interior. By local o-minimality, $f(I_1)$ is discrete and closed in this case.[15]. Let $a, b \in I_1$ with $a < b$. Then, $f(]a, b[) \subseteq]f(a), f(b)[$. Thus, $f(]a, b[)$ is discrete, closed and bounded. Since the set is bounded and the structure is DC, there is a supremum $s = \sup(f(]a, b[)) \in M$. Because the set is discrete and closed, this supremum is attained and $s \in f(]a, b[)$. Let $x \in (]a, b[$ such that $s = f(x)$. Then, for $y \in]x, b[$, we have $f(x) < f(y)$ contradicting the maximality of s. Thus, $f(I_1)$ cannot have no interior.

The proof is similar if f is strictly decreasing.

□

[15]For a proof, confer Proposition 4.28.



DC	definable completeness	**UW**	uniform local weak o-minimality of the 2. kind
LOM	local o-minimality	**W**	weak o-minimality
OM	o-minimality	**∗C**	∗-continuity property
SL	strong local o-minimality	**∗LW**	∗-local weak o-minimality

Figure 4: New schematic visualization of the relation between tameness notions. Here, we include only a selection of the most relevant notions with regard to their implications presented in the next sections. Note that while the colors and the labels still refer to the same properties as in the previous figures, the shapes of the associated areas have changed for some properties, in order to allow for a reasonable visualization including the new definitions. Again, all the labels refer to the smallest bounded convex set they are contained in. A few of the labels and associated polygons are additionally colored for clarity.

4.4 Viscerality

Viscerality is a tameness criterion which is not only studied in the context of dense linear orders without endpoints. For an extensive discussion of the general viscerality property, one may consider [5] and [25]. In the case of dense linear orders without endpoints viscerality has the following equivalent definition which was introduced in [15]:

Definition 4.22 (Viscerality)**.** The expansion of a dense linear order without endpoints $\mathcal{M} = (M, <, \dots)$ is *visceral* if for every structure $\mathcal{N} = (N, <, \dots)$ elementary equivalent to $\mathcal{M}$, every definable subset $X \subseteq N$ is the union of an open set and a finite set.

Moreover, [15] also introduced a weakened version of viscerality:

Definition 4.23 (L-Viscerality, [15, Definition 2.2])**.** A structure $\mathcal{M} = (M, <, \dots)$ is called *l-visceral* (or *lesser-visceral*) if every definable subset of M is the union of an open set and a finite set.

Moreover, t-minimality as discussed in [30] is another familiar tameness notion that coincides with l-viscerality in the case of dense linear orders.

We can again localize the definition:

Definition 4.24 (Local L-Viscerality, [15])**.** A structure $\mathcal{M} = (M, <, \dots)$ is called *locally l-visceral* if for every definable subset of M and every $x \in M$, there is an interval I around x such that $I \cap X$ is a union of a finite set and an open set.

This has the following equivalent definition:

Proposition 4.25 ([15, Proposition 2.7])**.** *An expansion of a dense linear order without endpoints is locally l-visceral if and only if any definable $X \subseteq M$ is a union of an open set and a discrete closed set.*

By [13, Lemma 2.3], in the DC setting, local o-minimality and local l-viscerality coincide:

Lemma 4.26. *Consider a definably complete structure $\mathcal{M} = (M, <, \dots)$. Then, $\mathcal{M}$ is a locally o-minimal structure if and only if it is locally l-visceral.*

Thus, local l-viscerality is a more general notion than local o-minimality in the general setting, that coincides with local o-minimality in the DC setting. For local l-viscerality, however, only a few tameness results have been shown. Note that even for the general locally o-minimal setting there are not many significant tameness results and thus, the results for local l-viscerality are even more sparse.

4.5 D-Minimality

In the case of expansions of the real ordered field, as considered by Miller in his program, the slightest generalization of o-minimality is d-minimality. Therefore, it might be an intuitive thought to also try to consider some general definition of d-minimality as a more general tameness condition. However, such a notation only makes sense if any tameness properties can be shown for it. For d-minimality in a general setting no such results have been shown so far. The most general setting, for which d-minimality has been defined, is the following already requiring the dense linear order to be definably complete:

Definition 4.27 (D-Minimality, [31])**.** A structure $\mathcal{M} = (M, <, \dots)$ is called *d-minimal* if every elementary equivalent structure $\mathcal{M}'$ is DC and every definable set $X \subseteq M'$ in every $\mathcal{M}'$ either has interior or is a finite union of discrete sets.

By the next two propositions, we have that if we assume definable completeness, every locally o-minimal structure is d-minimal:

Proposition 4.28 ([15, Lemma 3.1])**.** *Let $\mathcal{M}$ be a locally o-minimal structure. A definable subset of M is discrete and closed if it has empty interior.*

Proof. Let X have empty interior. Let $x \in M$ be arbitrary. By local o-minimality, there is an interval I around x such that $X \cap I$ is finite. Thus, X is discrete and closed. □

Proposition 4.29 ([40, Corollary 2.5])**.** *Local o-minimality is preserved under elementary equivalence.*

Proof. Let $\mathcal{M}$ be locally o-minimal, $x \in M$ and $X \subseteq M$ definable. Then, there is some $I_x \ni x$ such that $X \cap I_x$ is a finite union of intervals and points. This implies that $\mathrm{bd}(I_x \cap X)$ is finite. Choosing $a = \max\{y \in \mathrm{bd}(I_x \cap X) \cup \mathrm{bd}(I_x) : y < x\}$ and $b = \min\{y \in \mathrm{bd}(I_x \cap X) \cup \mathrm{bd}(I_x) : y > x\}$ implies $x \in]a, b[$ and $]a, b[\cap X$ equals one of the following sets: $\emptyset$, $\{x\}$, $]a, x[$, $]a, x]$, $]x, b[$, $[x, b[$, $]a, b[\setminus \{x\}$ or $]a, b[$.

For every formula $\varphi(x, z_1, \dots, z_n)$ and every $c \in M^n$, we can write down a sentence saying that for all x there is some a, b with $a < x < b$ and such that $]a, b[\cap \varphi(x, c)$ is equal to one of the sets $\emptyset$, $\{x\}$, $]a, x[$, $]a, x]$, $]x, b[$, $[x, b[$, $]a, b[\setminus \{x\}$ or $]a, b[$.

Thus, every elementary equivalent structure $\mathcal{N}$ models these sentences and has the property that for every unary definable[16] set $X = \varphi(x, c)$ and every $n \in N$ there are $a, b \in N$ such that $]a, b[\cap \varphi(x, c)$ is equal to one of the sets $\emptyset$, $\{x\}$, $]a, x[$, $]a, x]$, $]x, b[$, $[x, b[$, $]a, b[\setminus \{x\}$ or $]a, b[$. Thus, $\mathcal{N}$ is locally o-minimal. □

The converse is not true and there are several examples of structures which are d-minimal but not locally o-minimal. In particular, by Corollary 3.14, every expansion of the real ordered field which is d-minimal but not o-minimal is an example. There are several such examples presented in [33], e. g. $(\mathbb{R}, <, +, \cdot, 2^{\mathbb{Z}})$.

Thus, in the DC setting, d-minimality is a generalization where we could hope for geometric tameness in an even more general setting than local o-minimality. However, no geometric tameness results have been shown for this setting so far, leaving local

[16] Recall, that we always consider definable sets with parameters.

o-minimality as the most general setting where significant geometric tameness results for arbitrary definably complete dense linear orders have been shown.

Remark 4.30. Note, that we could also choose to work with a generalization of the stronger definition for d-minimality, which is discussed in [23] and first introduced in [33]. In this definition it is additionally required that the number of discrete sets is bounded in a uniform way. More precisely, for every definable $A \subseteq M^{n+1}$, there is some $N \in \mathbb{N}$ such that for every $x \in M^n$, the fiber A_x has interior or is the union of N discrete sets. With the same reasoning as before local o-minimality still implies d-minimality for this definition. However, the additional value of using this stricter definition for d-minimality is questionable since it is unknown, whether it is equivalent to the previously given definition or not. Moreover, there are also no interesting results shown for this definition in the case of arbitrary dense linear orders without endpoints.

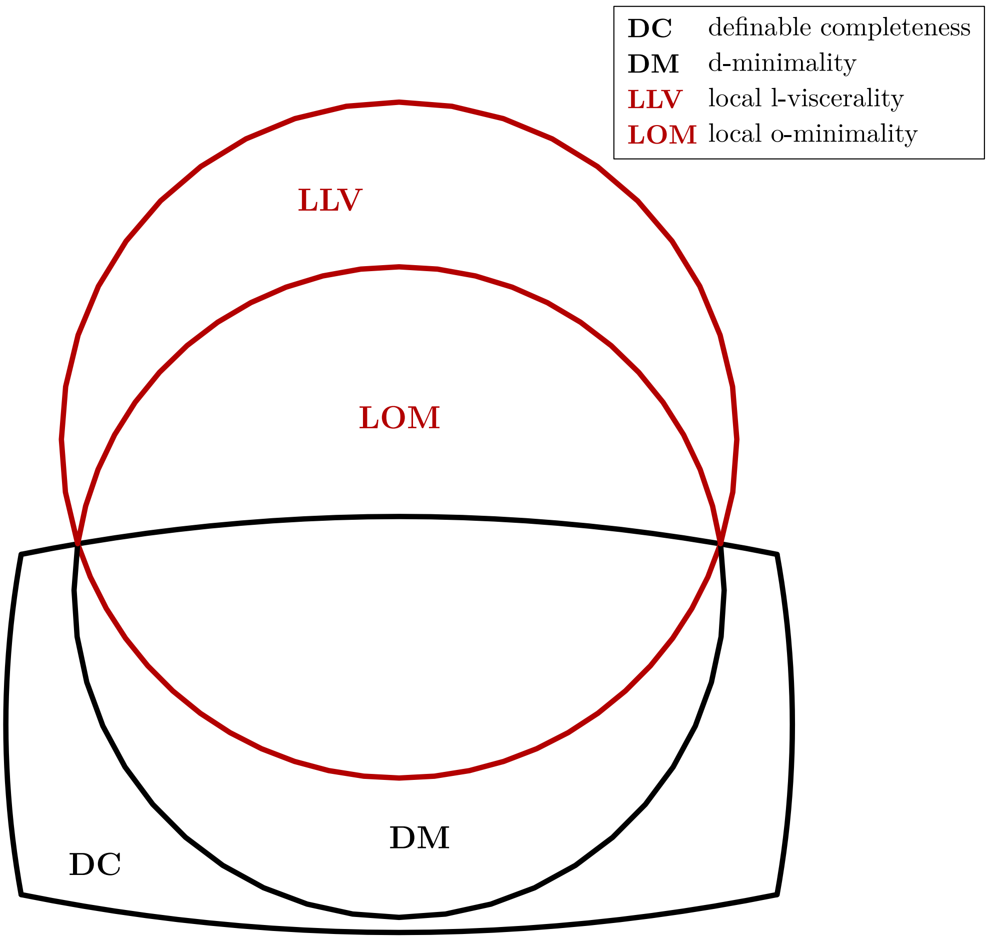


Figure 5: New schematic visualization of the relations between local o-minimality, d-minimality, definable completeness, and local l-viscerality. Again, all the labels refer to the smallest bounded convex set they are contained in. A few of the labels and associated polygons are additionally colored for clarity. Note that in this figure the area associated with local o-minimality is not a circle unlike in the other figures due to technical reasons in the construction of the image.

5 Examples

In this section, several different concrete examples of locally o-minimal structures possibly fulfilling some of the previously discussed notions are summarized. In particular, the examples showcase which of the definitions are not implied or equivalent to some others. For all examples, we discuss the following properties: univariate $*$-continuity, DC, weak o-minimality, $*$-local weak o-minimality, uniform local weak o-minimality of the second kind, uniform local o-minimality of the second kind, uniform local o-minimality of the first kind, strong local o-minimality, almost o-minimality, o-minimality, DCTC, TC and local o-minimality. To this end, we proceed as follows: First, we present the examples stating key properties which do or do not hold for the structure. For these properties, we either provide a reference or a short proof.[17] These key properties then imply which of the other properties do or do not hold, simply because of the relations between the notions shown in the previous sections. Afterwards, we summarize which properties hold for each example in Table 1.[18]

Throughout this section, note that every definable set in a reduct is also definable in the expansion. Thus, all the tameness properties shown for some example here, also hold for every reduct (which is still a dense linear order). Moreover, if a tameness property does not hold for a reduct, it certainly does not hold in every expansion as well.[19]

Example 1 ([8])**.** The expansion of the real field $(\mathbb{R}, <, +, \cdot, \exp)$ by the exponential function $\exp : \mathbb{R} \to \mathbb{R}, x \mapsto \exp(x)$, is o-minimal.

In particular, as mentioned before, all reducts of $(\mathbb{R}, <, +, \cdot, \exp)$ are also o-minimal. In particular, the real ordered field, the real additive group and the real line are o-minimal.

Example 2 ([37, Example 2.3])**.** The structure $(\mathbb{Q}, <)$ is o-minimal.

Example 3 (Marker-Steinhorn, [40, Theorem 2.7])**.** The ordered additive real group expanded by the usual sin-function $(\mathbb{R}, <, +, \sin)$ is almost o-minimal but not o-minimal.

Proof. $(\mathbb{R}, <, +, \sin)$ is locally o-minimal by [40, Theorem 2.7]. By Proposition 3.12, almost o-minimality follows. However, $(\mathbb{R}, <, +, \sin)$ is not o-minimal since $\{x : \sin x = 0\}$ is an infinite set without interior. □

Remark 5.1. Since $\mathbb{Z} = \{x : \sin(\pi x) = 0\}$ is definable in $(\mathbb{R}, <, +, \sin)$, we have that $(\mathbb{R}, <, +, \mathbb{Z})$ is also almost o-minimal. Clearly, $(\mathbb{R}, <, +, \mathbb{Z})$ is also not o-minimal.

[17] As we are not so much interested in the proofs but only in the results, we choose to only present proofs for the properties which are not proven in a reference.

[18] There are some properties for a few examples, where we are uncertain if these hold. We suspect them all to be true, so far none of them are proven. These are marked with a question mark in the table.

[19] In both cases, the converse is clearly false.

Example 4 ([16, Example 2.4])**.** Let $\mathcal{U}$ to be a ultrafilter on $\mathbb{N}$ containing the sets $\{n \in \mathbb{N} : n > k\}$ for all $k \in \mathbb{N}$, $\mathcal{F}$ to be the family of all maps from $\mathbb{N}$ to $\mathbb{Q}^+$, and $\mathcal{S}_n$ to be the structure $\mathcal{S}_n = (\mathbb{R}, <, (P_f^{\mathcal{S}_n})_{f\in\mathcal{F}})$ for all $n \in \mathbb{N}$ with $P_f^{\mathcal{S}_n} = \{a \in \mathbb{R} : f(n)\cdot a \text{ in } \mathbb{Z}\}$.

The ultraproduct $\mathcal{M} = \Pi_{n\in\omega}\mathcal{S}_n/\mathcal{U} = (M, <, (P_f)_{f\in\mathcal{F}})$ is a definably complete uniformly locally o-minimal structure of the first kind which is not strongly locally o-minimal and not TC.

Proof. In the reference, the reader can find a proof that the structure is a definably complete uniformly locally o-minimal structure of the first kind which is not strongly locally o-minimal. It remains to show, that the structure is not TC. To show this, consider the identity map $\mathrm{Id} : \mathbb{N} \to \mathbb{Q}^+, x \mapsto x$ and the corresponding predicate P_{Id}. Suppose there is $b = [(b(i))_{i\in\mathbb{N}}] \in M$ such that either $P_{\mathrm{Id}} \cap]b, \infty[=]b, \infty[$ or $P_{\mathrm{Id}} \cap]b, \infty[= \emptyset$. In the first case, note that for every $i \in \mathbb{N}$, we can find $b_1(i) \in \mathbb{R} \setminus \mathbb{Q}$ with $b_1(i) > b(i)$. Thus, $b_1(i) \notin P_{\mathrm{Id}}^{\mathcal{S}_i}$ for all $i \in \mathbb{N}$, implying $b_1 = [(b_1(i))_{i\in\mathbb{N}}] \notin P_{\mathrm{Id}}$. If $b_1(i) > b(i)$ holds for all $i \in \mathbb{N}$, this implies $b < b_1$. In the second case, let $b_2(i) \in \mathbb{Z}$ with $b_2(i) > b(i)$ in $\mathbb{R}$. Then, $b_2(i) \notin P_{\mathrm{Id}}^{\mathcal{S}_i}$ for all $i \in \mathbb{N}$, implying $b_2 = [(b_2(i))_{i\in\mathbb{N}}] \notin P_{\mathrm{Id}}$. But $b_2(i) > b(i)$ for all $i \in \mathbb{N}$ implies $b < b_2$. So neither of the cases hold. This is a contradiction. □

Example 5 ([16, Example 2.2], [26, Example 6])**.** The structure $\mathcal{M} = (\mathbb{Q}, <, (S_q)_{q\in\mathbb{Q}^+})$ with $S_q = \{(a, b) \in \mathbb{Q}^2 : a + q \cdot \sqrt{2} \leq b \text{ in } \mathbb{R}\}$ is not uniformly locally o-minimal of the first kind, but uniformly locally o-minimal of the second kind. Moreover, the structure is weakly o-minimal and has the univariate $*$-property but is not definably complete.

Proof. [16, Example 2.2] shows that $\mathcal{M}$ is uniformly locally o-minimal of the second kind. By [26, Example 6], $\mathcal{M}$ is not uniformly locally o-minimal of the first kind and $Th(\mathcal{M})$ admits quantifier elimination. It is easy to check, that by the quantifier elimination, $\mathcal{M}$ is weakly o-minimal and has the univariate $*$-continuity property. The set $\left\{a \in \mathbb{Q} : 0 \leq a < \sqrt{2}\right\}$ is definable and bounded but has no supremum. Thus, $\mathcal{M}$ is not definably complete. □

Example 6 ([16, Example 2.3])**.** The structure $\mathcal{M} = (\mathbb{Q}, <, P_{\pi 2})$ with $P_{\pi 2} = \{(a, b) \in \mathbb{Q}^2 : a < \pi \cdot b \text{ in } \mathbb{R}\}$ is not uniformly locally o-minimal of the second kind, but weakly o-minimal. Moreover, $\mathcal{M}$ has the $*$-continuity property.

Proof. By [16, Example 2.3], $\mathcal{M}$ is not uniformly locally o-minimal of the second kind, but weakly o-minimal and all definable sets are a finite union of finite intersections of definable sets each of which is equal to one of the following for some $k \in \mathbb{N}$, $q \in \mathbb{Q}$ and

$i, j \in \{1, \ldots, n\}$:

$$\{(x_1, \ldots, x_n) \in \mathbb{Q}^n : x_i = x_j\}$$
$$\{(x_1, \ldots, x_n) \in \mathbb{Q}^n : x_i = q\}$$
$$\{(x_1, \ldots, x_n) \in \mathbb{Q}^n : x_i > \pi^k \cdot x_j\}$$
$$\{(x_1, \ldots, x_n) \in \mathbb{Q}^n : x_i > \pi^k \cdot q\}$$
$$\{(x_1, \ldots, x_n) \in \mathbb{Q}^n : x_i < \pi^k \cdot x_j\}$$
$$\{(x_1, \ldots, x_n) \in \mathbb{Q}^n : x_i < \pi^k \cdot q\}$$

Thus, for every definable unary function, the domain can be divided into finitely many convex sets on each of which the function is either the identity or continuous. In particular, $\mathcal{M}$ has the univariate $*$-continuity property. □

The set $P_\pi = \{a \in \mathbb{Q} : a < \pi\} = \{a \in \mathbb{Q} : (a, b) \in P_{\pi 2}\}$ is definable in $(\mathbb{Q}, <, P_{\pi 2})$ and therefore, the structure $\mathcal{M} = (\mathbb{Q}, <, P_\pi)$ discussed in the next example is a reduct of $\mathcal{M} = (\mathbb{Q}, <, P_{\pi 2})$.

Example 7 ([4, Proposition 2.5])**.** The structure $\mathcal{M} = (\mathbb{Q}, <, P_\pi)$ with $P_\pi = \{a \in \mathbb{Q} : a < \pi\}$ is weakly o-minimal. Moreover, $\mathcal{M}$ is strongly locally o-minimal and has the $*$-continuity property but is not definably complete.

Proof. By [4, Proposition 2.5], $(\mathbb{Q}, <, P_\pi)$ is weakly o-minimal and $Th((\mathbb{Q}, <, P_\pi))$ has quantifier elimination. This implies that all definable sets are a finite union of finite intersections of sets defined by atomic formulas. Thus, for every definable unary function, the domain can be divided into finitely many convex sets on each of which the function is either the identity or continuous. In particular, $\mathcal{M}$ has the univariate $*$-continuity property.

For every $x \in \mathbb{Q}$, there is some interval $I_x \ni x$ such that $\pi \notin \overline{\overline{I_x}}$. By quantifier elimination, every definable subset of I_x is a finite union of intervals and points. Thus, $(\mathbb{Q}, <, P_\pi)$ is strongly o-minimal.

The set $\{a \in \mathbb{Q} : 0 \leq a < \pi\}$ is definable and bounded but has no supremum. Therefore, $(\mathbb{Q}, <, P_\pi)$ is not definably complete. □

Example 8 ([38, Example 3.2])**.** The structure $\mathcal{M} = (\mathbb{Q}^2, <_{lex}, f)$ with $f : \mathbb{Q}^2 \to \mathbb{Q}^2, (a, b) \mapsto (b, 0)$ is strongly locally o-minimal. Moreover, $(\mathbb{Q}^2, <_{lex}, f)$ is $*$-locally weakly o-minimal but is not TC and does not have the univariate $*$-continuity property.

Proof. By [38, Example 3.2], the expansion $(\mathbb{Q}^2, <_{lex}, f, E)$ by the binary relation symbol $E = \{((a_1, b_1), (a_2, b_2)) \in (\mathbb{Q}^2)^2 : a_1 = a_2 \text{ in } \mathbb{Q}\}$ is strongly locally o-minimal and $Th((\mathbb{Q}^2, <_{lex}, f, E))$ has quantifier elimination. It is easy to check that the intersection of some set defined by atomic formulas in $(\mathbb{Q}^2, <_{lex}, f, E)$ and $\{q\} \times \mathbb{Q}$ is a finite union of convex sets for any $q \in \mathbb{Q}$. Moreover, for every $\overline{\overline{x}}$ in the Dedekind closure of $(\mathbb{Q}^2, <_{lex}, f, E)$, we can find an interval around $\overline{\overline{x}}$ which is contained in the union of at most two sets of the form $\{q\} \times \mathbb{Q}$. Thus, $(\mathbb{Q}^2, <_{lex}, f, E)$ is $*$-locally weakly o-minimal. Therefore, the reduct $(\mathbb{Q}^2, <_{lex}, f)$ is also strongly locally o-minimal and $*$-locally weakly o-minimal.

Since the set $f(\mathbb{Q}^2) = \mathbb{Q} \times \{0\}$ is definable, $(\mathbb{Q}^2, <_{lex}, f)$ is not type complete.

Since f is nowhere continuous, $(\mathbb{Q}^2, <_{lex}, f)$ does not have the univariate $*$-continuity property. □

Remark 5.2. A more general version of examples similar to Example 8 is discussed in [26, Example 12] and [16, Example 5.2]. For any o-minimal structure $\mathcal{M}$ and $a \in M$, the structure $\mathcal{N} = (M^2, <_{lex}, f)$ with $f : \{a\} \times M \to M^2, (a, b) \mapsto (b, a)$ is strongly locally o-minimal but does not have the univariate $*$-continuity property.

Example 9 ([26, Example 6])**.** The structure $\mathcal{M} = (\mathbb{Q}, <, (Sb_n)_{n \in \mathbb{N}})$ with $Sb_n = \{a \in \mathbb{Q} : a < 2^{-n}\sqrt{2} \text{ in } \mathbb{R}\}$ is not strongly locally o-minimal, but uniformly locally o-minimal of the first kind. Moreover, it is not DC since $Sb_0 = \{a \in \mathbb{Q} : a < \sqrt{2}\}$ has no supremum.

Example 10 ([40, Proposition 2.13, Proposition 3.6])**.** The structure $\mathcal{N} = (N, <_{lex}, f)$ with the universe $N := \{(a, 0) : a \in \mathbb{Q}^+\} \cup \{(a, b) : a \in \mathbb{Q}^- \setminus \mathbb{Z}^-, b \in \mathbb{Q}\} \cup \{(a, 0) : a \in \mathbb{Z}_0^-\}$ and the unary function $f : N^- \to N, (a, b) \mapsto (-a, 0)$[20] is locally o-minimal but not uniformly locally o-minimal of the second kind and not definably complete.

Proof. By [40, Proposition 2.13], $\mathcal{N}$ is locally o-minimal.

With the same reasoning as in [40, Proposition 3.6], we show that $\mathcal{N}$ is not uniformly locally o-minimal of the second kind: $X = \{(a, b) \in N^2 : f(a) = f(b)\}$ is a definable set. Let I, J be intervals around 0. Let $x \in I$ and $x < 0$. Pick some $y \in J \cap \,]x, 0[$. Then, $X_y \cap I$ is a non-empty convex set, but neither an interval nor a point. Thus, $\mathcal{N}$ is not uniformly locally o-minimal of the second kind.

The set $f^{-1}((1, 0)) = \{(a, b) \in N : a = -1\}$ is definable and bounded but has no supremum. Thus, $\mathcal{N}$ is not definably complete.

□

Example 11 ([40, Example 2.3])**.** Every expansion of $(\mathbb{R}, <, \mathbb{Q})$ is not locally o-minimal, as $\mathbb{Q}$ intersected with any interval is a dense and co-dense subset of the interval.

Remark 5.3. Obviously $(\mathbb{R}, <, \mathbb{Q})$ is DC. But note, that $\mathbb{Q}$ is not the finite union of finitely many discrete sets and thus, $(\mathbb{R}, <, \mathbb{Q})$ is not even d-minimal.

[20] Here, N^- refers to the interval consisting of all elements of N less than $(0, 0)$ with respect to $<_{lex}$.

Examples	univariate ∗-continuity	DC	weak o-minimality	∗-local weak o-minimality	uniform local weak o-minimality of the second kind	uniform local o-minimality of the second kind	uniform local o-minimality of the first kind	strong local o-minimality	almost o-minimality	o-minimality	DCTC	TC	local o-minimality
$(\mathbb{R},<,+,\cdot,\exp)$, Example 1	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$(\mathbb{Q},<)$, Example 2	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$(\mathbb{R},<,+,\sin)$, Example 3	✓	✓	✗	✓	✓	✓	✓	✓	✓	✗	✗	✗	✓
$\Pi_{n\in\omega}(\mathbb{R},<,(P_f^{S_n})_{f\in\mathcal{F}})/\mathcal{U}$, Example 4	✓	✓	✗	✓	✓	✓	✓	✗	✗	✗	✗	✗	✓
$(\mathbb{Q},<,(S_q)_{q\in\mathbb{Q}^+})$, Example 5	✓	✗	✓	✓	✓	✓	✗	✗	✗	✗	✗	✓	✓
$(\mathbb{Q},<,P_{\pi 2})$, Example 6	✓	✗	✓	✓	✓	✗	✗	✗	✗	✗	✗	✓	✓
$(\mathbb{Q},<,P_\pi)$, Example 7	✓	✗	✓	✓	✓	✓	✓	✓	✗	✗	✗	✓	✓
$(\mathbb{Q}^2,<_{lex},f)$, Example 8	✗	✗	✗	✓	✓	✓	✓	✓	✗	✗	✗	✗	✓
$(\mathbb{Q},<,(Sb_n)_{n\in\mathbb{N}})$, Example 9	?	✗	?	?	✓	✓	✓	✗	✗	✗	✗	?	✓
$(N,<_{lex},f)$, Example 10	?	✗	?	?	?	✗	✗	✗	✗	✗	✗	?	✓
$(\mathbb{R},<,\mathbb{Q})$, Example 11	?	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗

Tameness properties

Table 1: Summary of the tameness properties which hold for the examples presented in this section. The blue highlighted cells are the key properties that are proven in this section. All other properties are deduced by the relations proven in previous sections.

6 Properties of Locally O-Minimal Structures

In this section, we present the fundamental tameness results for arbitrary dense linear orders that fulfill one of the different notions of local o-minimality. Note that the notions of local o-minimality imply no model theoretic tameness (besides elementary equivalence) but only geometric tameness. The geometric tameness properties are usually some localized version of tame properties that o-minimal structures have, like monotonicity, definable cell decomposition and dimensional tameness.

For the fundamental geometrical results in this section, we present the proofs given in the literature in versions adapted to our purposes. In this thesis, we present comprehensive versions of the proofs keeping the use of auxiliary lemmata and references to results of the literature to a minimum. Moreover, in [15] several results are presented in the most general setting possible. These settings tend to be quite technical. Here, we only consider the most general setting with regards to the commonly considered notions of local o-minimality for which the statements are true. This sometimes allows for simplified versions of the proofs. Additionally, a few proofs where changed due to personal preference of the author.

6.1 Model-Theoretic Properties

One very desirable characteristic of tameness notions for the study of model theoretic tameness is preservation under elementary equivalence.[21] If a property is preserved under elementary equivalence, the theory of any structure having the property has only models with the property. This warrants the consideration of the theories and their properties.

Recall, local o-minimality is preserved under elementary equivalence as proven in Proposition 4.29. The same holds for o-minimality (confer e. g. [27, Theorem 0.2]). Moreover, $*$-local weak o-minimality is preserved under elementary equivalence by [15, Proposition 2.16] and the $*$-continuity property is preserved under elementary equivalence by [15, Proposition 3.12].[22]

However, not all the notions of local o-minimality preserve under elementary equivalence. Strong local o-minimality is not preserved under elementary equivalence by [40, Corollary 3.9]. Also, almost o-minimality does not preserve under elementary equivalence by [10, Proposition 4.14]. Moreover, every ω-saturated elementary extension of some almost o-minimal but not o-minimal structure is not almost o-minimal.

[21] Note, that elementary equivalence is not a condition which is artificially added to the definitions of notions of local o-minimality, as is the case for d-minimality. Adding this criterion to the definitions of the notions of local o-minimality, while technically also possible, would possibly restrict the set of structures which fulfill these definitions and further complicate the proof that structures fulfill these definitions. Thus, this has not been considered so far.

[22] The proofs for $*$-local weak o-minimality and $*$-continuity property are also straightforward: Note that there is a first order formula expressing that a pair of sets is a definable gap and a first order formula expressing that a function is continuous. The rest of the proofs is analogous to the proof of Proposition 4.29

Besides elementary equivalence, there are no notable results showing model theoretic tameness for locally o-minimal structures. O-minimal theories are dp-minimal, NIP, distal and NTP2 (see e. g. [2]). For locally o-minimal theories (i. e. theories which only have locally o-minimal models), however, this is not the case: In [14], the reader can find a proof that on the one hand there are distal, NIP and NTP2 locally o-minimal theories, but on the other hand there are also locally o-minimal theories which are not distal, not NIP and not NTP2. Another example of a locally o-minimal structure which has the Independence Property (i. e. is not NIP) can be found in [9, Example 5.17]. Therefore, local o-minimality is rather uninteresting from a model theoretic point of view and there are no model theoretic tools available for our work with locally o-minimal structures.

6.2 Geometric Properties

Now, we turn our attention to the main motivation for the discussion of local o-minimality, namely the geometric properties that are implied by these notions.

6.2.1 Monotonicity

One of the foundational and groundbreaking geometric tameness results regarding o-minimality is the fact that every definable unary function is strictly monotone or constant everywhere except for a finite set:

Proposition 6.1 (Monotonicity Theorem, [7, p. 3])**.** *Let $\mathcal{M}$ be o-minimal and $f : M \to M$ a definable function. Then, M can be divided into a finite union of points and open intervals such that f is either constant, strictly increasing or strictly decreasing on each of these intervals. Moreover, there is a partition of M into a finite union of points and open intervals such that f is additionally continuous on each of the intervals.*

For the purpose of any geometric tameness study – in our case the study of locally o-minimal structures – it is advisable to seek some similar kind of monotonicity and continuity result for unary functions, as it is the basis for many other tameness properties.

Therefore, we define a weakened version of monotonicity, we call local monotonicity, and what we call the local continuity property.

Definition 6.2 (Local Monotonicity)**.** A locally o-minimal structure $\mathcal{M}$ has *local monotonicity* if for every definable unary function $f : M \to M$ and every point $a \in M$, there exists an interval I around a such that $I \cap \mathrm{dom}(f)$ can be partitioned into a finite union of points and open intervals, on each of which f is locally constant, locally strictly increasing or locally strictly decreasing.

Definition 6.3 (Local Continuity Property)**.** We say $\mathcal{M}$ has the *local continuity property* if for every definable unary function $f : M \to M$ and every point $a \in M$, there exists an interval I around a such that $I \cap \mathrm{dom}(f)$ can be partitioned into a finite union of points and open intervals such that f is continuous on the intervals.

Remark 6.4. Note, that continuity is a local property and it is enough for a function to be everywhere locally continuous to be continuous. In particular, there is not necessarily a continuous function $f : \overline{\overline{I}} \to \overline{\overline{M}}$ with $f|_M = g$ for every continuous $g : I \to \overline{\overline{M}}$.

Recall the univariate $*$-continuity property as defined in Definition 4.20. One of the most general monotonicity results for locally o-minimal structures is that every locally o-minimal structure with the univariate $*$-continuity property has local monotonicity and the local continuity property.

Theorem 6.5 ([15, Corollary 3.18])**.** *Let $\mathcal{M} = (M, <, \dots)$ be a locally o-minimal structure with the univariate $*$-continuity property and $f : I \to \overline{\overline{M}}$ be a definable map defined on an open interval I. Then, the interval $I \subseteq M$ can be decomposed into definable sets $X_+, X_-, X_{const}, X_{dis}$ satisfying the following conditions:*

- *X_{dis} is discrete and closed,*
- *X_{const} is open and the restriction of f to X_{const} is locally constant,*
- *X_- is open and the restriction of f to X_- is locally strictly decreasing and continuous,*
- *X_+ is open and the restriction of f to X_+ is locally strictly increasing and continuous.*

Proof. Similar as in Step 1 of the proof of Proposition 4.21, consider the following formulas:

$$\begin{aligned}
\phi_{co} &= \exists\, x_1\ (x_1 > x) \wedge (\forall\, t\ ((x < t < x_1) \to (f(x) = f(t))))\\
\phi_{+} &= \exists\, x_1\ (x_1 > x) \wedge (\forall\, t\ ((x < t < x_1) \to (f(x) < f(t))))\\
\phi_{-} &= \exists\, x_1\ (x_1 > x) \wedge (\forall\, t\ ((x < t < x_1) \to (f(x) > f(t))))\\
\psi_{co} &= \exists\, x_0\ (x_0 < x) \wedge (\forall\, t\ ((x_0 < t < x) \to (f(x) = f(t))))\\
\psi_{-} &= \exists\, x_0\ (x_0 < x) \wedge (\forall\, t\ ((x_0 < t < x) \to (f(x) < f(t))))\\
\psi_{+} &= \exists\, x_0\ (x_0 < x) \wedge (\forall\, t\ ((x_0 < t < x) \to (f(x) > f(t))))
\end{aligned}$$

And the definable sets:

$$\begin{aligned}
A_{\phi_i} &\coloneqq \{x \in I : \mathcal{M} \models \phi_i(x)\}\\
A_{\psi_j} &\coloneqq \{x \in I : \mathcal{M} \models \psi_j(x)\}\\
A_{\phi_i\psi_j} &\coloneqq A_{\phi_i} \cap A_{\psi_i} = \{x \in I : \mathcal{M} \models (\phi_i \wedge \psi_j)(x)\}
\end{aligned}$$

with $i, j \in \{+, -, co\}$.

Note that, for any x and t one of the formulas $(f(x) = f(t))$, $(f(x) > f(t))$ and $(f(x) < f(t))$ has to hold. The sets of t's such that each of these formulas hold are definable. By local o-minimality, there is an interval around x where each of these sets are a finite union of intervals and points. By Lemma 3.7, at least one of these sets has some interior of the form $]x_0, x[$ and at least one has interior of the form $]x, x_1[$ with $x_0 < x < x_1$. Thus, for every x at least one ϕ_i and at least one ψ_j holds true.

Define $X_{const} := A_{\phi_{co}\psi_{co}}$. This is an open set and $f|_{X_{const}}$ is constant, as desired.

It is obvious that the sets $A_{\phi_+\psi_{co}}, A_{\phi_-\psi_{co}}, A_{\phi_{co}\psi_+}$ and $A_{\phi_{co}\psi_-}$ cannot have interior.

Next, we show that $A_{\phi_+\psi_-}$ has empty interior: Towards a contradiction, assume that

$$A_{\phi_+\psi_-} = \{x \in I : \exists\, x_0, x_1\ (x_0 < x < x_1) \\ \wedge \forall\, t, s\ ((x_0 < s < x < t < x_1) \to (f(s) < f(x) \wedge f(t) < f(x)))\}$$

has interior and let I_1 be a bounded interval in this interior. Then, the following function is definable

$$g : I_1 \to \overline{\overline{I_1}},\ x \mapsto \inf\{r \in I_1 : \forall\, s \in M\ ((r < s < x) \to (f(s) < f(x)))\}$$

Let $a \in I_1$ be arbitrary. Then, by the definition of $A_{\phi_+\psi_-}$, there exist $x_0 < a < x_1$ such that for all s, t with $(x_0 < s < x < t < x_1)$, we have $(f(s) < f(x) \wedge f(t) < f(x))$. Thus, $g(a) \leq x_0 < a$. But for all t with $a < t < x_1$, we have $g(t) \geq a$. Thus, g is discontinuous at a and since a was arbitrarily chosen, it is discontinuous everywhere. This contradicts the univariate $*$-continuity property. Thus, $A_{\phi_+\psi_-}$ has empty interior.

Similarly, one can show that $A_{\phi_-\psi_+}$ has empty interior. Moreover, note that this proves that any definable function $h : J \to \overline{\overline{M}}$ cannot have local minima or local maxima throughout the interval J. This is applied in the next step of the proof.

Now, we want to consider $A_{\phi_+\psi_+}$ and show that f is locally strictly increasing everywhere except for a set without interior. Let $Y_{++} \subset A_{\phi_+\psi_+}$ be the definable subset of points where f is not locally strictly increasing. Consider the two formulas

$$\chi_0(x) := \forall\, x_0\ ((x_0 < x) \to (\exists\, s, t\ ((x_0 < s < t < x) \wedge (f(s) \geq f(t)))))$$
$$\chi_1(x) := \forall\, x_1 \in ((x < x_1) \to (\exists\, s, t\ ((x < s < t < x_1) \wedge (f(s) \geq f(t))))).$$

Note that for every $x \in Y_{++}$ at least one of these has to hold. By Lemma 3.7, Y_{++} has empty interior if $\chi_0(I)$ and $\chi_1(I)$ have empty interior. The proofs for both of these facts are similar, so we only present the proof for $\chi_0(I)$ here.

Suppose, towards a contradiction, that $\chi_0(I)$ has interior. Let J be a bounded interval in this interior. Define h to be the definable function

$$h : J \to \overline{\overline{M}}, x \mapsto \inf(\{x\} \cup \{y \in J : \forall\, t \in [y, x[\quad (f(t) < f(x))\}).$$

Since $J \subseteq A_{\phi_+\psi_+}$, we have $h(x) < x$ for all $x \in J$. Let $a \in J$ be arbitrary. By local o-minimality, we can find a_0 such that either $h(a) < h(t)$ for all $t \in [a_0, a[$ or $h(a) \geq h(t)$ for all $t \in [a_0, a[$. Without loss of generality, we can assume $h(a) \leq a_0$.

Suppose $h(a) \geq h(t)$ for all $t \in [a_0, a[$. Since $\chi_0(a)$ holds true, we can find $s, t \in]a_0, a[$ with $s < t$ and $f(s) \geq f(t)$. By definition, $h(t) \geq s$, which implies $h(t) \geq s >$

$a_0 \geq h(a)$. Thus, $h(a) < h(t)$ for all $t \in [a_0, a[$. Similarly, we can find a_1 such that $h(a) < h(t)$ for all $t \in]a, a_1]$ or $h(a) \geq h(t)$ for all $t \in]a, a_1]$. Again, since $a_1 \in A_{\phi_+\psi_+}$, we can find $s, t \in]a, a_1[$ such that $h(t) \geq s > a \geq h(a)$. Hence, $h(a) < h(t)$ for all $t \in]a, a_1]$. Therefore, a is a local minimum. But since a was chosen arbitrarily, h has local minima throughout J, which is a contradiction to our previous conclusion. Thus, $\chi_0(x)$ has empty interior and we can show the same for $\chi_1(x)$. Therefore, Y_{++} has empty interior. Similarly, one can proof that the set $Y_{--} \subseteq A_{\phi_-\psi_-}$ of points of $A_{\phi_-\psi_-}$ where f is not locally strictly decreasing, has empty interior as well.

By the univariate $*$-continuity property, the set Y_{dis} of points where f is discontinuous cannot have interior. If it had, there would be a subinterval in the interior where f is continuous, which is a contradiction.

In conclusion, the definable set $X_{dis} = Y_{dis} \cup Y_{\phi_+\psi_+} \cup Y_{\phi_-\psi_-} \cup A_{\phi_-\psi_+} \cup A_{\phi_+\psi_-} \cup A_{\phi_+\psi_{co}} \cup A_{\phi_-\psi_{co}} \cup A_{\phi_{co}\psi_+} \cup A_{\phi_{co}\psi_-}$ is a finite union of definable sets without interior and by Lemma 3.7, also has no interior. Thus, by local o-minimality, it is discrete and closed.

It is easy to check that the sets of points $X_+ = A_{\phi_+\psi_+} \setminus (Y_{\phi_+\psi_+} \cup Y_{dis})$, where f is locally strictly increasing and locally continuous and $X_- = A_{\phi_-\psi_-} \setminus (Y_{\phi_-\psi_-} \cup Y_{dis})$, where f is locally strictly decreasing and locally continuous, are definable and open. Moreover, by construction, $I = X_{dis} \cup X_{const} \cup X_+ \cup X_-$.

□

Remark 6.6. By local o-minimality, there is some interval $I \ni a$ such that the definable sets $X_{const} \cap I, X_+ \cap I$ and $X_- \cap I$ are finite unions of intervals and $X_{dis} \cap I$ is a finite set. Thus, the shown statement implies local monotonicity and the local continuity property.

Theorem 6.5 has the following immediate corollary since every definably complete locally o-minimal structure already has the univariate $*$-continuity property by Proposition 4.21.

Corollary 6.7 ([21, Theorem 2.3]). *Let $\mathcal{M} = (M, <, \dots)$ be a definably complete locally o-minimal structure. Let I be an interval and $f : I \to M$ be a definable function. Then, the interval $I \subseteq M$ can be decomposed into definable sets $X_+, X_-, X_{const}, X_{dis}$ satisfying the following conditions:*

- *X_{dis} is discrete and closed,*
- *X_{const} is open and f is locally constant on X_{const},*
- *X_+ is open and f is locally strictly increasing and continuous on X_+,*
- *X_- is open and f is locally strictly decreasing and continuous on X_-.*

Moreover, in this definable complete setting, there is even a slightly stronger version of local monotonicity that holds.

Definition 6.8 (Strong Local Monotonicity[23], [40, Definition 2.12])**.** A locally o-minimal structure $\mathcal{M}$ has *strong local monotonicity* if, for every unary function f definable in $\mathcal{M}$ and every point $a \in M$, there exists an interval I around a such that $I \cap \operatorname{dom}(f)$ can be partitioned into a finite union of points and open intervals, on each of which f is constant, strictly increasing, or strictly decreasing.

Proposition 6.9. *Every definably complete locally o-minimal structure has strong local monotonicity and local continuity.*

Proof. It is sufficient to show that the definitions of strong local monotonicity and local monotonicity coincide for definably complete locally o-minimal structures.

Let $\mathcal{M}$ be a definably complete locally o-minimal structure and $x \in M$. Let $I =]a, b[$ be some interval around x on which f is locally strictly increasing. Then, the set

$$L_x \coloneqq \{y \in I : (y < x) \wedge \forall\, z_1, z_2 \in [y, x] \quad ((z_1 < z_2) \to f(z_1) < f(z_2))\}$$

is definable and non-empty, since f is locally strictly increasing. By definable completeness, L_x has an infimum $x_{inf} \in [a, x[$.

Suppose $x_{inf} > a$. This implies that f is locally strictly increasing at x_{inf}.

If $x_{inf} \notin L_x$, there is some $z_0 \in L_x$ such that $f(x_{inf}) \geq f(z_0)$. But for all $z \in]x_{inf}, z_0[$, we have $f(z) < f(z_0) \leq f(x_{inf})$, contradicting that f is locally strictly increasing at x_{inf}.

If not, we have $x_{inf} \in L_x$. Recall, f is locally strictly increasing at x_{inf}. Thus, there is some $x'_{inf} < x_{inf}$ such that for all $z_1, z_2 \in \left]x'_{inf}, x_{inf}\right]$, $(z_1 < z_2)$ implies $f(z_1) < f(z_2)$. Using this and transitivity, $z_1 \in L_x$ follows. However, this is a contradiction to x_{inf} being the infimum of L_x.

Thus, we have a contradiction in both cases, which implies that $x_{inf} = a$.

In the same manner, we define

$$R_x \coloneqq \{y \in I : (x < y) \wedge \forall\, z_1, z_2 \in [x, y] \quad ((z_1 < z_2) \to f(z_1) < f(z_2))\}$$

and one can prove that the supremum of this set is equal to b. Thus, f is strictly increasing on I. The proof is similar for intervals on which f is locally constant or locally strictly decreasing. □

For almost o-minimal structures, we can show an even stronger result:

Proposition 6.10. *Let $\mathcal{M}$ be almost o-minimal, $f : M \to M$ a definable function and $a, b \in M$. Then, $]a, b[$ can be divided into a finite union of points and open intervals such that f is either constant, strictly increasing or strictly decreasing on each of these intervals.*

[23] In [40, Definition 2.12] this property is called local monotonicity, but we adapted the name to distinguish it from the previous definition and to empathize its connection to strong local monotonicity.

Proof. Every almost o-minimal structure is definably complete and locally o-minimal. Thus, $]a, b[= X_{dis} \cup X_{const} \cup X_+ \cup X_-$ as in Corollary 6.7. By almost o-minimality, X_{dis} is finite and X_{const}, X_+ and X_- are finite unions of intervals. By the proof of Proposition 6.9, f is constant on an interval if it is locally constant on that interval, decreasing on an interval if it is locally decreasing on that interval and increasing on an interval if it is locally increasing on that interval. □

The second very general result in this section states that every uniformly weakly locally o-minimal structure of the second kind has local monotonicity.

Theorem 6.11 ([15, Lemma 3.2, Theorem 3.17])**.** *Let $\mathcal{M} = (M, <, \dots)$ be a uniformly weakly locally o-minimal structure of the second kind and $f : I \to \overline{\overline{M}}$ be a definable map defined on an open interval I. Then, the interval $I \subseteq M$ can be decomposed into definable sets $X_+, X_-, X_{const}, X_{dis}$ satisfying the following conditions:*

- *X_{dis} is discrete and closed,*
- *X_{const} is open and the restriction of f to X_{const} is locally constant,*
- *X_- is open and the restriction of f to X_- is locally strictly decreasing,*
- *X_+ is open and the restriction of f to X_+ is locally strictly increasing.*

Proof. Note that in the first part of the proof of Theorem 6.5, the univariate $*$-continuity property is only applied to show that any definable function $h : J \to \overline{\overline{M}}$ cannot have local minima or maxima throughout the interval J. Thus, with the same proof, we only need to show the following claim, to prove the statement.

Claim 1. Any definable function $g : I_g \to \overline{\overline{M}}$ on some interval I_g cannot have a local minimum at every $x \in I_g$ or a local maximum at every $x \in I_g$.

Proof of Claim 1. Towards a contradiction, suppose we have a function g which has a local minimum at every $x \in I_g$. For all $a \in I_g$, define

$$\begin{aligned} U_a^+ &= \{x \in I_g : x > a \land \forall y \ \big((a < y \le x) \to (g(y) > g(a))\big)\} \\ U_a^- &= \{x \in I_g : x < a \land \forall y \ \big((x \le y < a) \to (g(y) > g(a))\big)\} \\ U_a &= U_a^- \cup \{a\} \cup U_a^+. \end{aligned}$$

Throughout this proof, we define several different sets and successively show properties for these sets to eventually get some contradiction.

First, we show that $U_b \subsetneq U_a$ is equivalent to $a \neq b$ and $b \in U_a$. Note that for any $a \in I_g$, we have $a \in U_a$. Thus, $U_b \subseteq U_a$ implies $b \in U_a$. Moreover, $U_b \neq U_a$ implies $a \neq b$.

For every $a, b \in I_g$ with $a \neq b$, $b \in U_a$ implies $g(b) > g(a)$ which then implies $a \notin U_b$. Therefore, $a \neq b$ implies $U_a \neq U_b$. If we have $a, b \in I_g$ with $a > b$ and $b \in U_a$, pick any $c \in U_b$. Since $a \notin U_b$ and $b < a$, we have $c < a$. By the definition of U_a, we have

$c \in U_a$ if $a > c \geq b$. If $c < b$, we have for any $b > d \geq c$ that $g(d) > g(b) > g(a)$ and thus $c \in U_a$. We conclude that $U_b \supsetneq U_a$. For $a < b$ and $b \in U_a$, we can show $U_b \subsetneq U_a$ with the same reasoning.

Secondly, if we have $U_a \cap U_b \neq \emptyset$ and $g(a) \neq g(b)$, then $U_b \subsetneq U_a$ or $U_a \subsetneq U_b$. By the previous statement, it is enough to show $a \in U_b$ or $b \in U_a$. Let $d \in U_a \cap U_b$ be an element in the cut. Without loss of generality, assume $a < b$. Clearly, $d \leq a$ implies $a \in U_b$ and $b \leq d$ implies $b \in U_a$. Thus, it only remains to consider the case $a < d < b$. For any $d' \in \,]a, d]$, we have $g(a) < g(d')$ and for any $d'' \in [d, b[$, we have $g(b) < g(d'')$. Hence, for any $d''' \in \,]a, b[$, we have $\min\{g(a), g(b)\} < g(d''')$. In conclusion, $g(a) < g(b)$ implies $b \in U_a$ and $g(b) < g(a)$ implies $a \in U_b$.

Define the definable sets $C_a = \{x \in I_g : a \in U_x\}$ and $C = \{(a, x) \in I_g \times I_g : x \in C_a\}$. Shrinking I_g if necessary, C_a is a finite set for all $a \in I_g$. To prove this, assume towards a contradiction that C_a has interior for some $a \in I_g$. Let $c < b < d$ be contained in an interval J of this interior, such that $c, d \in U_b$. If $g(c) = g(d)$, as g attains a local minimum at c, there is a c_1 such that for all $c_2 \in \,]c, c_1[$, we have $g(c) < g(c_2)$. We can use the denseness of $\mathcal{M}$ to pick $c' \in \,]c, c_1[\, \cap \,]c, b[$. Then, $c' < b < d$, $c' \in J$ and $c' \in U_b$. Thus, we can assume $g(c) \neq g(d)$. Note that $c, d \in C_a$ implies $a \in U_c \cap U_d$ and thus $U_d \subsetneq U_c$ or $U_c \subsetneq U_d$. Since $c, d \in U_b$, we have $b \notin U_c$ and $b \notin U_d$. By convexity of U_c and U_d, $c \notin U_d$ and $d \notin U_c$ follows. However, this is a contradiction to $U_d \subsetneq U_c$ or $U_c \subsetneq U_d$. Therefore, C_a cannot have interior for any $a \in I_g$ and by uniform local weak o-minimality of the second kind, we can find a subinterval $I'_g \subseteq I_g$ such that C_a is finite for all $a \in I'_g$. By possibly shrinking I_g, we can assume that C_a is finite for all $a \in I_g$.

Define the definable sets $D_a = \{x \in I_g : U_x \subsetneq U_a \wedge (\neg \exists\, y \in I_g \; (U_x \subsetneq U_y \subsetneq U_a))\}$ for all $a \in I_g$ and $K = \{x \in I_g : \neg \exists\, y \in I_g \setminus \{x\} \; (x \in U_y)\}$. Shrinking I_g if necessary, these sets are finite: By uniform local weak o-minimality of the second kind, we can shrink I_g such that these sets are all a finite union of points and open convex sets. Suppose D_a has interior for some $a \in I_g$. Then, let J_2 be an open interval in the interior and let $b \in J_2$. Clearly, $J_2 \cap U_b \neq \emptyset$ but for every $d \in J_2 \cap U_b$, we have $U_d \subsetneq U_b \subsetneq U_a$ contradicting $d \in J_2$. Suppose K has interior. Then, choose J_2 and b again in the same way. For every $d \in J_2 \cap U_b$, we have $U_d \subsetneq U_b$ contradicting $d \in K$.

By definition, the sets D_a are pairwise disjoint for all $a \in I_g$. Moreover, since $K = \{x \in I_g : \neg \exists\, y \in I_g \; (U_x \subsetneq U_y)\}$, D_a and K are disjoint for all $a \in I_g$.

Recall, that $C_a = \{x \in I_g : U_a \subsetneq U_x\}$ is finite for all $a \in I_g$. Thus, for every $x \in I_g$, if there is some $y \in I_g$ such that $U_x \subsetneq U_y$, then there is some $y_{min} \in I_g$ such that $U_x \subsetneq U_{y_{min}}$ and there is no $y \in I_g$ such that $U_x \subsetneq U_y \subsetneq U_{y_{min}}$. With this fact, we can conclude that

$$I_g = K \cup \bigcup_{a \in I_g} D_a.$$

Since I_g clearly has interior but K and D_a are finite, there must be infinitely many non-empty D_a. Recall, the sets D_a are pairwise disjoint for all $a \in I_g$. Thus, we can define a definable infinite set Y as follows:

$$Y \coloneqq \{x \in I_g : \exists\, a \in I_g\ ((x \in D_a) \wedge (\neg\exists\, y \in D_a\ (y > x)))\}$$

Possibly shrinking I_g, we can assume that Y is a finite union of points and open convex sets. Since Y is infinite, it must contain interior. Define $Z \subseteq Y$ to be a inclusion-wise maximal convex set such that for $z \in Z$, $Y \setminus \big(Z \cup]-\infty, z[\big) = \emptyset$ (i. e. Z is the rightmost convex set or point of Y). Clearly, Z is definable.

Let $a \in Z$. Note that $a \in I_g$ implies that g attains a local minimum at a and thus, $a \in \overset{\circ}{U}_a$. Pick $b_1, b_2 \in U_a$ such that $b_1 < a < b_2$. The sets C_{b_1} and C_{b_2} are finite and hence, there are $b_1' \in C_{b_1}$ and $b_2' \in C_{b_2}$ such that $b_1', b_2' \in D_a$. Therefore, $U_{b_1} \subseteq U_{b_1'} \subsetneq U_a$ and $U_{b_2} \subseteq U_{b_2'} \subsetneq U_a$. If we had $b_1' > a$, then $b_1 \in U_{b_1'}$ and the convexity of $U_{b_1'}$ would imply $a \in U_{b_1'}$ contradicting $U_{b_1'} \subsetneq U_a$. Hence, $b_1' < a$ and similarly $b_2' > a$. Since the D_i are disjoint, there is no $i \neq a$ such that $b_2' \in D_i$. Thus, $b_2' \notin Y$.

Let $c \in U_{b_2'}$ be some element with $b_2' < c$. The set U_c is non-empty. Pick some $c' \in U_c$ and note that $C_{c'}$ is finite. In particular, there is some $c'' \in C_{c'}$ such that $c'' \in D_c$. Hence, D_c is non-empty. Since D_c is finite, $\max(D_c)$ exists and $\max(D_c) \in Y$. Clearly, $a < b_2' < c \leq \max(D_c)$ and $a \in Z$, $\max(D_c) \in Y$. By the definition of Z as the rightmost convex subset of Y, this would imply $]a, \max(D_c)[\ \subseteq Z$. This is a contradiction to $b_2' \notin Y$.

Note that one can show with a similar proof that g cannot have maxima throughout the intervals.

□

Thus, defining the sets $A_{\phi_i\psi_j}, Y_{\phi_+\psi_+}, Y_{\phi_-\psi_-}, X_{const}$ as in the proof of Theorem 6.5 and repeating the same proof except for the part about the continuity of f and replacing the part that there are no minima or maxima throughout an interval by the claim, shows that $X_{dis} = Y_{\phi_+\psi_+} \cup Y_{\phi_-\psi_-} \cup A_{\phi_-\psi_+} \cup A_{\phi_+\psi_-} \cup A_{\phi_+\psi_{co}} \cup A_{\phi_-\psi_{co}} \cup A_{\phi_{co}\psi_+} \cup A_{\phi_{co}\psi_-}$ is a finite union of definable sets without interior. By Lemma 3.7, X_{dis} also has no interior. Hence, by local o-minimality, it is discrete and closed. Moreover, the sets X_{const} where f is constant, $X_+ = A_{\phi_+\psi_+} \setminus Y_{\phi_+\psi_+}$, where f is locally strictly increasing and locally continuous and $X_- = A_{\phi_-\psi_-} \setminus Y_{\phi_-\psi_-}$, where f is locally stricly decreasing and locally continuous, are definable and open. By construction, $I = X_{dis} \cup X_{const} \cup X_+ \cup X_-$.

□

For strongly locally o-minimal structures, we can again show strong monotonicity:

Corollary 6.12 ([40, Theorem 4.1]). *Let $\mathcal{M}$ be strongly locally o-minimal. Then, for every unary function $f : J \to M$ definable in $\mathcal{M}$ and every point $a \in M$, there exists an interval I around a such that $I \cap \operatorname{dom}(f)$ can be partitioned into a finite union of points and open intervals, on each of which f is constant, strictly increasing, or strictly decreasing.*

Proof. [24] First, we recall the statement and proof of [26, Theorem 9]: Every strongly locally o-minimal structure can be related to an o-minimal structure in the sense of the following two equivalent statements:

1. $\mathcal{M}$ is strongly locally o-minimal.

2. For every finite set of points $a_0, \dots, a_n \in M$, there are $b_0, \dots, b_n, c_0, \dots c_n \in M \cup \{\pm\infty\}$ with $b_i < c_i$ and $a_i \in \,]b_i, c_i[$ such that for $N = \,]b_0, c_0[\, \cup \bigcup_{i=1}^{n} [b_i, c_i[$, the structure $\mathcal{N} = (N, \{P_{N^n \cap X}\}_{n \in \mathbb{N}, X \in Def_{\mathcal{M}}^n})$ is o-minimal. Here, P_Y is a relation symbol denoting Y and $Def_{\mathcal{M}}^n$ is the set of definable subsets of M^n in the structure $\mathcal{M}$.

To show the equivalence, we show both implications:

1. $\Rightarrow$ 2. Let $a_0, \dots, a_n \in M$. By strong local o-minimality, there are $b_0, \dots, b_n \in M$, $c_0, \dots c_n \in M$ such that $a_i \in \,]b_i, c_i[$ and for every definable set $X \subseteq M$, $X \cap \,]b_i, c_i[$ is a finite union of points and intervals. Let $N = \,]b_0, c_0[\, \cup \bigcup_{i=1}^{n} [b_i, c_i[$ and $\mathcal{N} = (N, \{P_{N^n \cap X}\}_{n \in \omega, X \in Def_{\mathcal{M}}^n})$. Let $Y \subseteq N$ be definable in $\mathcal{N}$. As N is a definable subset of M in $\mathcal{M}$, it is easy to see that Y is indeed also a definable set in $\mathcal{M}$ and $Y = Y \cap N = (Y \cap \,]b_0, c_0[\, \cup \bigcup_{i=1}^{n} (Y \cap [b_i, c_i[\,))$. By strong local o-minimality, $Y \cap \,]b_i, c_i[$ is a finite union of intervals and points in $\mathcal{M}$. Thus, Y is also a finite union of intervals and points.

2. $\Rightarrow$ 1. Let $a \in M$. Let $b, c \in M$ be such that $a \in \,]b, c[$, $N = \,]b, c[$ and $\mathcal{N}$ be o-minimal. We show that the strong local o-minimality criterion holds for $]b, c[$: Let $X \subseteq M$ be definable in $\mathcal{M}$. By construction, $]b, c[\, \cap X$ is definable in $\mathcal{N}$. By o-minimality, $]b, c[\cap X$ is a finite union of intervals and points in $\mathcal{N}$. Since the intervals are also intervals in $\mathcal{M}$, the same holds for $]b, c[\, \cap X$ in $\mathcal{M}$.

Now, we apply this equivalence to show the corollary. Let $\mathcal{M}$ be strongly locally o-minimal. By the equivalence, for any $a \in M$, we can find $b, c \in M$ with $b < a < c$ such that the second condition holds for $N = \,]b, c[$. In particular, this implies that every definable proper subset of $]b, c[$ has an infimum and a supremum. Recall that by Theorem 6.11, $\mathcal{M}$ has local monotonicity. By the proof of Proposition 6.9, the corollary follows.

□

Moreover, for uniformly locally weakly o-minimal structures, we can additionally show a different version of the monotonicity theorem: If the image is bounded by a sufficiently small interval, the function has local monotonicity and the local continuity property in a local uniform parameterized way.

Theorem 6.13 (Parameterized Local Monotonicity Theorem, [15, Corollary 3.8])**.** *Let $\mathcal{M} = (M, <, \dots)$ be uniformly locally weakly o-minimal of the second kind. Let $A \subseteq M$ be open and definable and let $P \subseteq M^n$ be a definable subset. Let $f : A \times P \to M$ be*

[24] Here, we choose to do a different proof than the one given in the reference since the result already follows from two of our previously proven statements.

a definable function. For every $(a, b, p) \in M \times M \times M^n$, every sufficiently small open intervals I and J with $a \in I$ and $b \in J$ and every sufficiently small open box B with $p \in B$, the following assertion holds true:

There exists a partition of $f^{-1}(J) \cap (I \times B)$ into pairwise disjoint definable sets X_{fin}, X_-, X_+, X_{const} satisfying the following conditions for every $c \in B$:

1. *$\pi_1(X_{fin} \cap (f^{-1}(J) \cap (I \times \{c\})))$ is a finite set,*
2. *$\pi_1(X_{const} \cap (f^{-1}(J) \cap (I \times \{c\})))$ is a finite union of open convex sets and $f(-, c)$ is locally constant on the set,*
3. *$\pi_1(X_- \cap (f^{-1}(J) \cap (I \times \{c\})))$ is a finite union of open convex sets and $f(-, c)$ is locally strictly decreasing and continuous on the set,*
4. *$\pi_1(X_+ \cap (f^{-1}(J) \cap (I \times \{c\})))$ is a finite union of open convex sets and $f(-, c)$ is locally strictly increasing and continuous on the set.*

Proof. Let $P \subseteq M^n$ be a definable subset and $A \subseteq M$ be open and definable. Let $f : A \times P \to M$ be a definable function and let $(a, b, p) \in M \times M \times M^n$. Note that for every $c \in B \setminus P$, $f^{-1}(J) \cap (I \times \{c\}) = \emptyset$ and the same assertions trivially hold for arbitrary X_{fin}, X_-, X_+, X_{const}. Thus, it is sufficient to show the conditions for $c \in B \cap P$.

Step 1. There exist $I', J' \subseteq M, B' \subseteq M^n$ such that for every I, J with $a \in I \subseteq I', b \in J \subseteq J'$, every box $B \subseteq B'$ and every $c \in B$, the set $\pi_1(f^{-1}(J) \cap (I \times \{c\}))$ is a finite union of points and open convex sets.

Let $X \subseteq M^{n+3}$ be the definable set $X := \{(x, y_1, y_2, z) \in A \times M \times M \times P : (y_1 < f(x, z) < y_2)$. By uniform local weak o-minimality of the second kind, there exist an interval $I' \subseteq A$ with $a \in I'$, intervals J_1, J_2 with $b \in J_1 \cap J_2$ and an open box B' with $p \in B'$ such that the definable set $I \cap X_{(b_1,b_2,c)}$ is a finite union of points and open convex sets for all $b_1 \in J_1, b_2 \in J_2$ and $c \in B'$. Pick $b_1 \in J_1$ and $b_2 \in J_2$ such that $b_1 < b < b_2$. Let $J' =]b_1, b_2[$. Then, we have

$$\pi_1(f^{-1}(J') \cap (I' \times \{c\})) = \{x \in I' : b_1 < f(x, c) < b_2\} = X_{(b_1,b_2,c)} \cap I'$$

for all $c \in B$. Thus, $\pi_1(f^{-1}(J') \cap (I' \times \{c\}))$ is a finite union of points and open convex sets for all $b_1 \in J_1, b_2 \in J_2$ and $c \in B'$. It is easy to see that the claim also holds for all subintervals $I \subseteq I', J \subseteq J'$ and subsets $B \subseteq B'$.

Step 2. There exist intervals $I_1 \subseteq I', J_1 \subseteq J'$ and a box $B_1 \subseteq B'$ such that there exists a partition $f^{-1}(J_1) \cap (I_1 \times B_1) = X'_f \cup X_{const} \cup X_{nn}$ into pairwise disjoint definable sets and for every $c \in B_1$,

1. $\pi_1(X'_f \cap (f^{-1}(J) \cap (I \times \{c\})))$ is a finite set,
2. $\pi_1(X_{const} \cap (f^{-1}(J) \cap (I \times \{c\})))$ is a finite union of open convex sets and $f(-, c)$ is locally constant on the set,

3. $\pi_1(X_{nn} \cap (f^{-1}(J) \cap (I \times \{c\})))$ IS A FINITE UNION OF OPEN CONVEX SETS AND $f(-,c)$ IS LOCALLY INJECTIVE[25] ON THE SET.

Begin with $I = I'$, $J = J'$ and $B = B'$. In the following, we shrink I, J, B several times instead of introducing new subintervals for the ease of notation. It is easily checked that all the statements shown throughout this part still remain true for the shrunken sets I, J, B, as these statements always also hold for subintervals and boxes which are a subset of B.

$$X'_{const} := \Big\{(x,z) \in I \times B : (x,z) \in f^{-1}(J) \wedge \Big(\exists\, x_1, x_2 \ \Big(x_1 < x < x_2 \\ \wedge \forall\, x' \ \big(x_1 < x' < x_2 \to \big((x',z) \in f^{-1}(J) \to (f(x,z) = f(x',z))\big)\big)\Big)\Big)\Big\},$$

$$X_{const} := \big\{(x,z) \in X'_{const} : x \in ((\overset{\circ}{X'_{const}})_z)\big\}$$

and $\pi_1(X_{const} \cap (f^{-1}(J) \cap (I \times \{c\})))$ are definable sets.[26] By uniform local weak o-minimality of the second kind, we can shrink J, I, B such that $\pi_1(X_{const} \cap (f^{-1}(J) \cap (I \times \{c\})))$ is a finite union of points and open convex sets for all $c \in B$. By the definition of X_{const}, it even has to be a finite union of open convex sets and $f(-,c)$ is locally constant on $\pi_1(X_{const} \cap (f^{-1}(J) \cap (I \times \{c\})))$.

Note that, by construction, any isolated points of $((I \times \{c\}) \cap f^{-1}(J))$ are contained in X'_{const}: By denseness, we can find $x_1, x_2 \in I$ with $x_1 < x < x_2$ such that $]x_1, x_2[\cap (f^{-1}(J))_c = \{x\}$.

Define the definable sets

$$E_X = \{(x,z) \in I \times B : x \in \operatorname{bd}((X_{const})_z)\},$$

$$Y' = (f^{-1}(J) \cap (I \times B)) \setminus (E_X \cup X_{const}).$$

Possibly shrinking I and B , $\pi_1(E_X \cap (I \times \{c\}))$ is finite for all $c \in B$. Thus, $\pi_1(E_X \cap (f^{-1}(J) \cap (I \times \{c\})))$ is finite. Moreover, by definition, $\pi_1(Y' \cap (I \times \{c\}))$ is a finite union of (not necessarily open) convex sets for all $c \in B$, again possibly shrinking I and B. Define the definable sets

$$E_Y = \{(x,z) \in I \times B : x \in \operatorname{bd}(Y'_z)\},$$

$$Y = Y' \setminus E_Y.$$

Possibly shrinking I and B, $\pi_1(E_Y \cap (I \times \{c\}))$ is finite and $\pi_1(Y \cap (I \times \{c\}))$ is a finite union of open convex sets for all $c \in B$.

[25] A function $g : I \to M$ is called locally injective if, for every $x \in I$, there exists an open interval I' such that $x \in I' \subseteq I$ and the restriction of g to I' is injective.

[26] As a clarification, since the notation is not very clear here: In the definition of X_{const}, x is contained in the interior of the fiber, and not only in the fiber of the interior.

The set of sets of pairs of points where $f(-,z)$ maps to the same value defined by

$$F = \{(x,x',z) \in I \times I \times P : f(x,z) = f(x',z) \wedge (x,z) \in Y\}$$

is also definable and we can shrink I and B such that the fiber $F_{(b,c)} = \{x \in I : (x,b,c) \in F\}$ is a finite union of points and open convex sets for all $b \in I, c \in B$. Assuming this, $F_{(b,c)}$ must be finite: $F_{(b,c)}$ cannot have interior since this would imply that $f(-,z)$ would be constant there but $F_{(b,c)} \cap (X'_{const})_c = \emptyset$. Thus, $(f^{-1}(a))_c \cap Y_c$ is finite for every $c \in B$ and $a \in J$.

We define

$$X_{nn} = \Big\{(x,z) \in Y : \exists\, x_1, x_2 \in I \; \Big((x_1 < x < x_2) \wedge f(-,z) \text{ is injective on }]x_1,x_2[\Big)\Big\}.$$

We show that $(X_{nn})_z$ is dense in Y_z for any $z \in B$. Let $c \in B$ and $a_1 \in Y_c$ be arbitrary. Since Y_c is a finite union of open convex sets, there is $a_2 \in Y_c$ with $a_1 < a_2$ such that the interval $I_c = \,]a_1,a_2[\, \subseteq Y_c$. We define the map

$$g_c : f(I_c,c) \to \overline{I_c},\; y \mapsto \begin{cases} \min\,(f^{-1}(y))_c \cap I_c & \text{if } (f^{-1}(y))_c \cap I_c \neq \emptyset \\ a_1 & \text{else.} \end{cases}$$

By finiteness of $(f^{-1}(a))_c \cap Y_c$, this map is well defined and definable. By uniform local weak o-minimality of the second kind, there is a_3 such that either $]a_1,a_3[\, \cap g_c(f(I_c,c)) = \emptyset$ or $]a_1,a_3[\, \subseteq g_c(f(I_c,c))$. The first case is impossible: Let $u \in \,]a_1,a_3[$ and $v = f(u,c)$. Then, $g_c(v) \in \,]a_1,u] \subseteq \,]a_1,a_3[$. Thus, for any $a_1 \in Y_c$ there is a $a_3 \in Y_c, a_1 < a_3$ such that $f(-,c)$ is injective on $]a_1,a_3[$, implying $]a_1,a_3[\, \subseteq (X_{nn})_c$. We conclude that $(X_{nn})_c$ is dense in Y_c.

By uniform local weak o-minimality of the second kind, there exist a subinterval of I and a box contained in B such that $(X_{nn})_c$ and $(Y \setminus X_{nn})_c$ are finite unions of points and open convex sets on the subinterval for any c in the box. We shrink I and B such that this condition holds true for I, B. Then, $(Y \setminus X_{nn})_c$ is finite and $(X_{nn})_c$ is a finite union of open convex sets, as $(X_{nn})_c$ is dense in Y_c. Defining $X'_f = (Y \setminus X_{nn}) \cup E_X \cup E_Y$, the claim holds for the current J, I, B.

Step 3. Any definable function $g : I_g \to \overline{\overline{M}}$ on some interval I_g cannot have a local minimum at every $x \in I_g$ or a local maximum at every $x \in I_g$.

This is proven as Claim 1 in the proof of Theorem 6.11.

Step 4. There exist intervals $I_2 \subseteq I_1$, $J_2 \subseteq J_1$ and a box $B_2 \subseteq B_1$ such that there exists a partition $X_{nn} = X''_f \cup X'_+ \cup X'_-$ into pairwise disjoint definable sets and for every $c \in B$,

1. $\pi_1(X''_f \cap (f^{-1}(J) \cap (I \times \{c\})))$ is a finite set,
2. $\pi_1(X'_- \cap (f^{-1}(J) \cap (I \times \{c\})))$ is a finite union of open convex sets and $f(-,c)$ is locally strictly decreasing on the set,

3. $\pi_1(X'_+ \cap (f^{-1}(J) \cap (I \times \{c\})))$ IS A FINITE UNION OF OPEN CONVEX SETS AND $f(-,c)$ IS LOCALLY STRICTLY INCREASING ON THE SET.

Let $a \in X_{nn}$ and $c \in B$. As $f(-,c)$ is locally injective on $\pi_1(X_{nn} \cap (I \times \{c\}))$, we can find an interval I_a around a on which $f(-,c)$ is injective. Thus, for any $a' \in I_a$, we have $f(a',c) > f(a,c)$ or $f(a',c) < f(a,c)$. By uniform local weak o-minimality of the second kind, we can find $x_1 < x < x_2$ such that one of the two possibilities holds for all $a_1 \in]x_1, x[$ and one of the two possibilities holds for all $a_2 \in]x, x_2[$. Therefore, one can partition $X_{nn} = X'''_- \cup X'''_+ \cup X_{max} \cup X_{min}$ into the following pairwise disjoint definable sets:

$$\begin{aligned} X'''_- = \Big\{(x,z) \in X_{nn} : \exists\, x_1, x_2 \,\Big(&x_1 < x < x_2 \\ &\wedge \forall\, x' \,\Big(\big((x_1 < x' < x) \to (f(x',z) > f(x,z) \wedge (x',z) \in X_{nn})\big) \\ &\wedge \big((x < x' < x_2) \to (f(x',z) < f(x,z) \wedge (x',z) \in X_{nn})\big)\Big)\Big)\Big\} \end{aligned}$$

$$\begin{aligned} X'''_+ = \Big\{(x,z) \in X_{nn} : \exists\, x_1, x_2 \,\Big(&x_1 < x < x_2 \\ &\wedge \forall\, x' \,\Big(\big((x_1 < x' < x) \to (f(x',z) < f(x,z) \wedge (x',z) \in X_{nn})\big) \\ &\wedge \big((x < x' < x_2) \to (f(x',z) > f(x,z) \wedge (x',z) \in X_{nn})\big)\Big)\Big)\Big\} \end{aligned}$$

$$\begin{aligned} X_{max} = \Big\{(x,z) \in X_{nn} : \exists\, x_1, x_2 \,\Big(&x_1 < x < x_2 \\ &\wedge \forall\, x' \,\Big(\big((x_1 < x' < x) \to (f(x',z) > f(x,z) \wedge (x',z) \in X_{nn})\big) \\ &\wedge \big((x < x' < x_2) \to (f(x',z) > f(x,z) \wedge (x',z) \in X_{nn})\big)\Big)\Big)\Big\} \end{aligned}$$

$$\begin{aligned} X_{min} = \Big\{(x,z) \in X_{nn} : \exists\, x_1, x_2 \,\Big(&x_1 < x < x_2 \\ &\wedge \forall\, x' \,\Big(\big((x_1 < x' < x) \to (f(x',z) < f(x,z) \wedge (x',z) \in X_{nn})\big) \\ &\wedge \big((x < x' < x_2) \to (f(x',z) < f(x,z) \wedge (x',z) \in X_{nn})\big)\Big)\Big)\Big\} \end{aligned}$$

By the previous step, $\pi_1(X_{max} \cap I \times \{c\})$ and $\pi_1(X_{min} \cap I \times \{c\})$ are finite for every $c \in B$, if we eventually shrink I, J and B.

Now, we can again remove the boundary of X'''_- and X'''_+ in each fiber since it is clearly a finite set if we shrink I, B accordingly. Define

$$\begin{aligned} E &= \{(x,z) \in X'''_- : x \in \mathrm{bd}((X'''_-)_z)\} \cup \{(x,z) \in X'''_+ : x \in \mathrm{bd}((X'''_+)_z)\}, \\ X''_+ &= X'''_+ \setminus E, \\ X''_- &= X'''_- \setminus E. \end{aligned}$$

Now, define the definable sets

$$X'_+ = \{(x,z) \in X''_+ : f(-,z) \text{ is locally strictly increasing at } (x,z)\},$$
$$X'_- = \{(x,z) \in X''_- : f(-,z) \text{ is locally strictly decreasing at } (x,z)\}.$$

Consider the set $E_+ \coloneqq X''_+ \setminus X'_+$: Let $c \in B$. Note that every $x \in \pi_1(E_+ \cap I \times \{c\})$ is contained in at least one of the following two sets:

$$\chi_0 \coloneqq \{x \in X''_+ : \forall x_0\ ((x_0 < x) \to (\exists s,t\ ((x_0 < s < t < x) \wedge (f(s) \geq f(t)))))\}$$
$$\chi_1 \coloneqq \{x \in X''_+ : \forall x_1\ ((x < x_1) \to (\exists s,t\ ((x < s < t < x_1) \wedge (f(s) \geq f(t)))))\}.$$

In the next part, we show that χ_0 has empty interior.[27] Towards a contradiction, assume χ_0 has interior and let I_χ be some interval in this interior. Define g to be the definable function

$$g : I_\chi \to \overline{\overline{M}}, x \mapsto \inf(\{x\} \cup \{y \in I_\chi : \forall t \in [y,x[\quad (f(t) < f(x))\}).$$

Let $a \in I_\chi$ be arbitrary. By local o-minimality, there are b_1, b_2 with $b_1 < a < b_2$ such that either $g(y_1) > g(a)$ for all $y_1 \in]b_1, a[$ or $g(y_1) \leq g(a)$ for all $y_1 \in]b_1, a[$ and either $g(y_2) > g(a)$ for all $y_2 \in]a, b_2[$ or $g(y_2) \leq g(a)$ for all $y_2 \in]a, b_2[$. Since $I_\chi \subseteq X''_+$, we have $g(x) < x$ for all $x \in I_\chi$. Without loss of generality, we can assume $g(a) \leq b_1$.

Suppose $g(a) \geq g(t)$ for all $t \in [b_1, a[$. Since $a \in \chi_0$, we can find $s, t \in]b_1, a[$ with $s < t$ and $f(s) \geq f(t)$. Thus, by definition, $g(t) \geq s$ but then $g(t) \geq s > a_0 \geq g(a)$. Hence, $g(a) < g(t)$ for all $t \in [b_1, a[$. Similarly, since $b_2 \in \chi_0$, we can assume that $g(a) < g(t)$ for all $t \in]a, b_2]$. Therefore, a is a local minimum. But since a was chosen arbitrarily, g has local minima throughout I_χ, which is a contradiction to Step 3. Thus, χ_0 has empty interior. With a similar proof, one can show the same for χ_1.

By Lemma 3.7, $\pi_1(E_+ \cap I \times \{c\})$ has empty interior. This implies that $\pi_1(E_+ \cap I \times \{c\})$ is finite for every $c \in B$ if we choose I and B to be sufficiently small. By a similar proof, the same holds for $\pi_1(E_- \cap I \times \{c\})$. We can conclude that if we define $X''_f \coloneqq E_+ \cup E_- \cup E \cup X_{max} \cup X_{min}$, the claim holds for X''_f, X'_- and X'_+.

Step 5. There exist intervals $I_3 \subseteq I_2$, $J_3 \subseteq J_2$, a box $B_3 \subseteq B_2$ and partitions $X'_+ = X_+ \cup F_+$ and $X'_- = X_- \cup F_-$ into disjoint definable sets such that, for every $c \in B$,

1. $\pi_1(X_- \cap (f^{-1}(J) \cap (I \times \{c\})))$ is a finite union of open convex sets and $f(-,c)$ is locally strictly decreasing and continuous on the set,
2. $\pi_1(X_+ \cap (f^{-1}(J) \cap (I \times \{c\})))$ is a finite union of open convex sets and $f(-,c)$ is locally strictly increasing and continuous on the set,
3. $\pi_1(F_+ \cap (I \times \{c\}))$ and $\pi_1(F_- \cap (I \times \{c\}))$ are finite sets.

[27] This part is similar to the proof of Theorem 6.5.

First, we define the following definable sets:

$$F_+ = \{(x,z) \in X'_+ : f(-,z) \text{ is not continuous at } x\}$$
$$F_- = \{(x,z) \in X'_- : f(-,z) \text{ is not continuous at } x\}$$

Moreover, define $X_+ = X'_+ \setminus F_+$ and $X_- = X'_- \setminus F_-$. It is sufficient to show the finiteness of $(F_+)_c$ and $(F_-)_c$ for all $c \in B$, for the rest of the claim to be immediate.

Towards a contradiction, assume $(F_+)_c$ has interior for some $c \in B$. Since $f(-,c)$ is locally strictly increasing on $(F_+)_c \subseteq (X'_+)_c$, there is an interval I_F in the interior of $(F_+)_c$ such that f is strictly increasing on I_F. Note that $f(I_F, c)$ is a definable set. Possibly shrinking I_F, we can assume $f(I_F, c)$ to be a finite union of open convex sets and points. Since $f(-,c)$ is strictly increasing, $f(I_F, c)$ cannot be finite. Thus, it contains an open convex set. Pick $a < b$ in the open convex set and notice that f is an order preserving bijection on the interval between the points $f^{-1}(a)$ and $f^{-1}(b)$ and, hence, continuous. This is a contradiction to $f(-,z)$ not being continuous at x for every $(x,z) \in F_+$. Thus, $(F_+)_c$ does not have interior. The proof for $(F_-)_c$ is similar.

Combining all the statements from Steps 1 to 5, the statement of the theorem is immediate. □

Remark 6.14. As shown in [15, Theorem 3.6] with a similar proof, if we drop the continuity in the previous theorem the following two similar statements are true:

1. The same parameterized monotonicity result as in the previous theorem holds for all functions $f : A \times P \to \overline{\overline{M}}$.
2. If $\mathcal{M}$ is uniformly locally weakly o-minimal of the first kind, the same parameterized monotonicity result as in the previous theorem with $J = M$ and $B = M^n$ holds for all functions $f : A \times P \to \overline{\overline{M}}$.

Finally, we show that some of the tameness properties discussed in this section indeed do not generally hold in the broader setting of arbitrary locally o-minimal structures. To this end, we explicitly discuss two examples.

First, we consider strong local monotonicity.

Proposition 6.15 ([40, Proposition 2.13, Proposition 3.6])**.** *There exist locally o-minimal structures without strong local monotonicity.*

Proof. Recall the structure from Example 10:

$$\mathcal{N} = (N, <_{lex}, f),$$
$$N := \{(a,0) : a \in \mathbb{Q}^+\} \cup \{(a,b) : a \in \mathbb{Q}^- \setminus \mathbb{Z}^-, b \in \mathbb{Q}\} \cup \{(a,0) : a \in \mathbb{Z}_0^-\},$$
$$f : N^- \to N, (a,b) \mapsto (-a,0).$$

Then, f is a definable function not having strong local monotonicity, since it is locally constant but attains infinitely many different values on every interval around a negative integer as a domain.

□

Secondly, in the next proposition, we discuss an example without the local continuity property. Recall that in the Parameterized Monotonicity Theorem, we only consider functions which map to a bounded image. The example in the proof of the next proposition nicely visualizes that this is necessary for the continuity part of the statements.

Proposition 6.16 ([38, Example 3.2])**.** *There exist strongly locally o-minimal structures with definable functions that are nowhere continuous.*

Proof. Recall the structure $(\mathbb{Q}^2, <_{lex}, f)$ from Example 8. The structure is strongly locally o-minimal, but $f : \mathbb{Q}^2 \to \mathbb{Q}^2, (a, b) \mapsto (b, 0)$ is nowhere continuous. □

local monotonicity	local o-minimality & univariate $*$-continuity property; uniform local weak o-minimality of the 2. kind
local continuity property	local o-minimality & univariate $*$-continuity property
strong local monotonicity	local o-minimality & definable completeness; strong local o-minimality
parameterized local monotonicity and continuity	uniform local weak o-minimality of the 2. kind

Table 2: Summary of the presented tameness results in this section. Confer with Figures 3 and 4 to check for which other notions this implies the same results trivially since they are implied by one of the notions mentioned here.

6.2.2 Cell Decomposition and Decomposition into Submanifolds

Now, that we have established some local monotonicity results, we consider another, even stronger, geometric tameness property, called definable cell decomposition. Again, this property holds for o-minimal structures and in this section we evaluate if there are similar results for the notions of local o-minimality.

Definition 6.17 (Definable Cell Decomposition, [10, Definition 1.5])**.** Let $n \in \mathbb{N}_{>0}$ and $i_1, \ldots, i_n \in \{0, 1\}$. Then, $(i_1, \ldots, i_n)$-*cells* are definable subsets of M^n defined inductively as follows:

- A (0)-cell is a point in M and a (1)-cell is an open interval in M.
- An $(i_1, \ldots, i_n, 0)$-cell is the graph of a definable continuous function defined on an $(i_1, \ldots, i_n)$-cell. An $(i_1, \ldots, i_n, 1)$-cell is a definable set of the form $\{(x, y) \in C \times M : f(x) < y < g(x)\}$, where C is an $(i_1, \ldots, i_n)$-cell and f and g are definable continuous functions defined on C with $f < g$.[28]

[28]Here, $f < g$ means that for every $x \in C$, we have $f(x) < g(x)$.

The sequence $(i_1, \ldots, i_n)$ of ones and zeroes is called the *type of an* $(i_1, \ldots, i_n)$*-cell.* A definable set C_i is called a *cell* if there is some type $(i_1, \ldots, i_n)$ such that C_i is an $(i_1, \ldots, i_n)$-cell. An *open cell* is a $(1, 1, \ldots, 1)$-cell. The *dimension* $\dim_{cell}(C)$ *of an* $(i_1, \ldots, i_n)$*-cell* C is defined by $\dim_{cell}(C) = \sum_{j=1}^{n} i_j$.

A *definable cell decomposition of an open box* is defined inductively as follows:

- A definable cell decomposition of $B \subseteq M$ is a partition $B = \bigcup_{i=1}^{m} C_i$ into finitely many cells $C_i \subseteq M$ (i. e. points and intervals).
- A definable cell decomposition of $B \subseteq M^{n+1}$ is a partition $B = \bigcup_{i=1}^{m} C_i$ into finitely many cells $C_i \subseteq M^{n+1}$ such that $\pi_{\leq n}(B) = \bigcup_{i=1}^{m} \pi_{\leq n}(C_i)$ is a definable cell decomposition of $\pi_{\leq n}(B) \subseteq M^n$.

Consider a finite family $\{A_\lambda\}_{\lambda \in \Lambda}$ of definable subsets of B. *A definable cell decomposition of* B *partitioning* $\{A_\lambda\}_{\lambda \in \Lambda}$ is a definable cell decomposition of B such that the definable sets A_λ are unions of cells for all $\lambda \in \Lambda$.[29]

For o-minimal structures, there exists the following well known decomposition theorem:

Theorem 6.18 (Definable Cell Decomposition Theorem, [7, Theorem 2.11 of Chapter 3])**.** *Let* $\mathcal{M}$ *be o-minimal. Let* n *be an arbitrary positive integer. Let* $\{A_\lambda\}_{\lambda \in \Lambda}$ *be a finite family of definable subsets of* M^n*. There exists a definable cell decomposition of* M^n *partitioning the finite family* $\{A_\lambda\}_{\lambda \in \Lambda}$*.*

Note, this theorem clearly also implies a local version of it. If we have a definable box $B \subseteq M^n$ with $\bigcup_{\lambda \in \Lambda} A_\lambda \subseteq B$, we can apply the theorem to get a cell decomposition $\mathcal{C}$ and take $\mathcal{C}' = \{C \cap B : C \in \mathcal{C}\}$ as a cell decomposition partitioning B. Thus, we have the following corollary:

Corollary 6.19. *Let* $\mathcal{M}$ *be o-minimal. Let* n *be an arbitrary positive integer and* $B \subseteq M^n$ *a box. Let* $\{A_\lambda\}_{\lambda \in \Lambda}$ *be a finite family of definable subsets of* B*. There exists a definable cell decomposition of* B *partitioning the finite family* $\{A_\lambda\}_{\lambda \in \Lambda}$*.*

It is immediate that, for any structure, definable cell decomposition, as presented in Theorem 6.18, implies o-minimality. Thus, we cannot hope to prove the same result for locally o-minimal structures which are not o-minimal. Instead, we hope for a local version similar to Corollary 6.19. Indeed, such a result can be shown, as in the following main theorem regarding local cell decomposition:

Theorem 6.20 (Local Definable Cell Decomposition Theorem, [26, Proposition 13], [16, Theorem 4.2])**.** *Let* $\mathcal{M}$ *be a strongly locally o-minimal structure or a definably complete uniformly locally o-minimal structure of the second kind. Let* n *be an arbitrary positive integer. Let* $\{A_\lambda\}_{\lambda \in \Lambda}$ *be a finite family of definable subsets of* M^n*. For every point* $a \in M^n$*, there exist an open box* $B \ni a$ *and a definable cell decomposition of* B *partitioning the finite family* $\{B \cap A_\lambda : \lambda \in \Lambda \text{ and } B \cap A_\lambda \neq \emptyset\}$*.*

[29] In other literature this is sometimes also called a cell decomposition adapted to or compatible with $\{A_\lambda\}_{\lambda \in \Lambda}$.

Proof for strongly locally o-minimal structures. Recall from the proof of Corollary 6.12 that the following are equivalent:

1. $\mathcal{M}$ is strongly locally o-minimal.
2. For every $a_0, \ldots, a_n \in M$ finite set of points in M, there are $b_0, \ldots, b_n, c_0, \ldots c_n \in M \cup \{\pm\infty\}$ with $b_i < c_i$ and $a_i \in]b_i, c_i[$ such that for $N =]b_0, c_0[\cup \bigcup_{i=1}^{n} [b_i, c_i[$, the structure $\mathcal{N} = (N, \{P_{N^n \cap X}\}_{n \in \mathbb{N}, X \in Def_{\mathcal{M}}^n})$ is o-minimal.

Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a finite family of definable subsets of M^n and let $a = (a_0, \ldots, a_n) \in M^{n+1}$. Let $b_i, c_i \in M$ with $b_i < c_i$ and $a_i \in]b_i, c_i[$ such that for $N =]b_0, c_0[\cup \bigcup_{i=1}^{n} [b_i, c_i[$, the structure $\mathcal{N} = (N, \{P_{N^n \cap X}\}_{n \in \mathbb{N}, X \in Def_{\mathcal{M}}^n})$ is o-minimal. Let $B_M \subseteq M^n$ be the box $]b_1, c_1[\times \cdots \times]b_n, c_n[$ in the structure $\mathcal{M}$ and $B_N \subseteq N^n$ be the box $]b_1, c_1[\times \cdots \times]b_n, c_n[$ in the structure $\mathcal{N}$. Let A'_λ be the preimage of A_λ under the canonical embedding $\mathrm{Id} : N \to M, x \mapsto x$. Note that the sets $A''_\lambda = B_N \cap A'_\lambda$ are definable in $\mathcal{N}$. By Corollary 6.19, there is a definable cell decomposition of B_N partitioning $\{A''_\lambda\}_{\lambda \in \Lambda}$. It is easy to check, that mapping these sets to M via the canonical embedding results in a cell decomposition of B_M partitioning $\{B_M \cap A_\lambda\}$. □

Proof for definably complete, uniformly locally o-minimal structures of the second kind. [30] To prove the statement, we simultaneously show the following three assertions with an induction:

(CD)$_n$ Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a finite family of definable subsets of M^n and $a \in M^n$. There exist an open box B with $a \in B$ and a definable cell decomposition of B partitioning the finite family $\{B \cap A_\lambda : \lambda \in \Lambda \text{ and } B \cap A_\lambda \neq \emptyset\}$.

(PC)$_n$ Let $A \subseteq M^n$ be a definable subset, $f : A \to M$ be a definable function and $a \in M^n, b \in M$ be arbitrary. There is some open interval J_1 with $b \in J_1$ such that for every interval $J \subseteq J_1$ with $b \in J$, there exist an open box B with $a \in B$ and a definable cell decomposition of B partitioning $f^{-1}(J) \cap B$ such that the function f is continuous on every cell contained in $f^{-1}(J) \cap B$.

(UF)$_n$ Let $X \subseteq M^{n+1}$ be a definable and $a \in M, b \in M^n$ be arbitrary. There exist an open interval I around a, an open box B with $b \in B$ and a positive integer N such that, for every $y \in B$, the definable set $X_y \cap I$ contains an interval or $|X_y \cap I| \leq N$.

Induction Start. We have that (CD)$_1$ follows directly from local o-minimality and (PC)$_1$ follows from the Parameterized Local Monotonicity Theorem (Theorem 6.13), as any convex set in a definably complete structure is an interval.

Part 1 of the Induction Step. We show that the assertions (UF)$_n$ hold true for all positive integers n assuming the assertions (PC)$_n$, (CD)$_n$ and (UF)$_m$ for all $m < n$. Fix $a \in M$ and $b \in M^n$.

First, we show that the condition (UF)$_n$ is equivalent to

[30] Here, parts of the proof are similar to the reference, but we also modified larger segments in order not to need the statements from Corollaries 6.23 and 6.24 for the proof.

$(\mathrm{UF}')_n$ Let X be a definable subset of M^{n+1} such that the fiber X_y is empty or a discrete set for every $y \in M^n$. For every $a \in M$ and $b \in M^n$, there exist an open interval I containing the point a, an open box B with $b \in B$ and a positive integer N such that $|X_y \cap I| \leq N$ for every $y \in B$.

It is immediate from the definition that $(\mathrm{UF})_n$ implies $(\mathrm{UF}')_n$. To show the other direction, let X be a definable subset of M^{n+1}, let $a \in M$ and $b \in M^n$. By uniform local o-minimality of the second kind, there is an open box $b \in B \subseteq M^n$ and an open interval $a \in I \subseteq M$ such that $X_y \cap I$ is a finite union of open intervals and points for every $y \in B$. Define Y to be the set of the isolated points and the endpoints of the maximal intervals in all these fibers, i. e. the following set:

$$Y = \{(x, y) \in I \times B : x \in \mathrm{bd}(X_y)\}$$

Since Y_y is finite for all $y \in B$, we can apply $(\mathrm{UF}')_n$ and shrink I and B such that for all $y \in B$, we have $|Y_y \cap I| \leq N$ for some positive integer N. It follows from the definition of Y that X_y either contains an interval or $|X_y \cap I| \leq N$. Thus, $(\mathrm{UF})_n$ holds true.

Thus, it is sufficient to show $(\mathrm{UF}')_n$. Let X be a definable subset of M^{n+1} such that for every $y \in M^n$, the fiber X_y is empty or a discrete set. Let $a \in M, b \in M^n$.

For this proof, we call a point $(y, x) \in \overline{I} \times B$ *normal* if there exists an open subinterval I' of I and an open box B' such that $y \in I'$, $x \in B'$ and $(I' \times B') \cap X = \emptyset$ or $(I' \times B') \cap X = \Gamma'(f)$ for some definable continuous function $f : B' \to I'$.[31]

We consider the definable sets

$$\begin{aligned}
\mathcal{A}_I^+ &= \{x \in M^n : (y, x) \text{ is not normal for some } y \in \overline{I} \text{ with } y > a\},\\
\mathcal{A}_I^- &= \{x \in M^n : (y, x) \text{ is not normal for some } y \in \overline{I} \text{ with } y \leq a\} \text{ and}\\
\mathcal{N}_I &= M^n \setminus (\mathcal{A}_I^+ \cup \mathcal{A}_I^-).
\end{aligned}$$

Claim 1. For any definably connected subset C of $\mathcal{N}_I$, there exists a finite family $\{f_i : C \to \overline{I}\}_{i=1}^k$ of definable continuous functions such that $f_i < f_{i+1}$ for all $1 \leq i \leq k-1$ and $X \cap (I \times C) = \bigcup_{i=1}^k \Gamma'(f_i)$.

Proof of Claim 1. Let $c \in C$ be arbitrary. Define $k := |X_c \cap I|$ and let $y_1, \dots, y_k \in M$ be such that $X_c \cap I = \{y_1, \dots, y_k\}$ and $y_i < y_{i+1}$ for all $1 \leq i \leq k-1$. Let f_i be a continuous definable function with maximal definably connected domain $D_i \ni c$ attaining $f_i(c) = y_i$. As $c \in C \subseteq \mathcal{N}_I$, there is a continuous definable function attaining $f_i(c) = y_i$. Suppose there was some $x \in C \cap \mathrm{fr}(D_i)$ not in the domain of f_i. Let $y \in M$ such that $(y, x) \in \overline{\Gamma'(f_i)} \cap (\overline{I} \times \{x\})$. As $x \in \mathcal{N}_I$, (y, x) is normal. But $(I' \times B') \cap X \neq \emptyset$ for all $I' \ni y$ and $B' \ni x$ as $(y, x) \in \overline{\Gamma'(f_i)}$. Thus, there is a continuous function $g : B' \to J'$ such that $X \cap (J' \times B') = \Gamma'(g)$ for some open interval $J' \ni y$ and some

[31] The notation $\Gamma'(f)$ refers to the set $\{(y', x') \in I' \times B' : y' = f(x')\}$.

open box $B' \ni x$. In particular, $g = f$ on $B' \cap D_i$. Therefore, we can extend f_i to be a continuous function on the definably connected set $D_i \cup \{x\}$ by

$$f'_i(z) := \begin{cases} f_i(z) & \text{if } z \in D_i \\ g(z) & \text{else.} \end{cases}$$

This is a contradiction to the maximality of D_i. Thus, the domain of f_i contains C.

Moreover, we have $f_i < f_{i+1}$ on all of C for these functions. Assume not. Then, there exist some i with $1 \leq i < k$ and a point $q \in C$ such that $f_i(q) \geq f_{i+1}(q)$. The definable sets $\{x \in C : f_i(x) = f_{i+1}(x)\}$, $\{x \in C : f_i(x) < f_{i+1}(x)\}$ and $\{x \in C : f_i(x) > f_{i+1}(x)\}$ are open by the definition of $\mathcal{N}_I$. The definably connected set C is equal to the disjoint union of the definable open sets $\{x \in C : f_i(x) = f_{i+1}(x)\}, \{x \in C : f_i(x) < f_{i+1}(x)\}$ and $\{x \in C : f_i(x) > f_{i+1}(x)\}$. But at least two of them are not empty, contradicting that C is definably connected. □

Claim 2. There is some interval $I \ni a$ and some box $B \ni b$ and a cell decomposition of B partitioning $\mathcal{A}_I^+ \cap B$ and $\mathcal{A}_I^- \cap B$ such that all cells have empty interior.

Proof of Claim 2. First, suppose that for every $I \ni a$, $\mathcal{A}_I^- \cap B$ has a cell with non-empty interior for every box $B \ni b$. Define the definable function

$$\beta_I : \mathcal{A}_I^- \to \overline{I}, x \mapsto \sup\{y \in \overline{I} : y \leq a \text{ and } (y, x) \text{ is not normal}\}.$$

By definable completeness, this function is well-defined.

Furthermore, define the definable sets $\mathcal{B}_I^+, \mathcal{B}_I^-$ and the definable functions γ_I^+, γ_I^- by

$$\begin{aligned} \mathcal{B}_I^- &= \{x \in \mathcal{A}_I^- : \exists y \in \overline{I}\ (y < \beta_I(x) \wedge (y, x) \in X)\}, \\ \mathcal{B}_I^+ &= \{x \in \mathcal{A}_I^- : \exists y \in \overline{I}\ (y > \beta_I(x) \wedge (y, x) \in X)\}, \end{aligned}$$

$$\begin{aligned} \gamma_I^- &: \mathcal{B}_I^- \to \overline{I}, x \mapsto \sup\{y \in \overline{I} : y < \beta_I(x) \wedge (y, x) \in X\}, \\ \gamma_I^+ &: \mathcal{B}_I^+ \to \overline{I}, x \mapsto \inf\{y \in \overline{I} : y > \beta_I(x) \wedge (y, x) \in X\}. \end{aligned}$$

These functions are well-defined and nowhere equal to β as X_x is discrete for all $x \in M^n$.

Apply $(\mathrm{PC})_n$. There is a box $B \ni b$ and some interval J with $a \in J \subsetneq I$ and definable cell decompositions $\mathcal{C}_\beta, \mathcal{C}_{\gamma^+}, \mathcal{C}_{\gamma^-}$ of B partitioning $(\beta_I^{-1}(J) \cap B)$, $((\gamma_I^+)^{-1}(J) \cap B)$ and $((\gamma_I^-)^{-1}(J) \cap B)$ respectively, such that β_I is continuous on every cell of $\mathcal{C}_\beta$, γ_I^+ is continuous on every cell of $\mathcal{C}_{\gamma^+}$ and γ_I^- is continuous on every cell of $\mathcal{C}_{\gamma^-}$.

Note, $\mathfrak{D} = \{x \in \mathcal{A}_I^- : (\beta_I(x), x) \in X\}$ is a definable set.

By $(\mathrm{CD})_n$, possibly shrinking B, there is a cell decomposition $\mathcal{C}$ of B partitioning $\mathfrak{D}$ and all cells of $\mathcal{C}_\beta, \mathcal{C}_{\gamma^+}$ and $\mathcal{C}_{\gamma^-}$.

Then, β_I, γ_I^+ and γ_I^- are continuous on every cell of $\mathcal{C}$ that they are defined on. Since they are nowhere equal, this implies that for every $C \in \mathcal{C}$ and $x \in C \cap \mathfrak{D}$, there is a box $B_C \ni x$ and an interval $I_C \ni \beta(x)$ such that $\Gamma'(\gamma^+_{J'}) \cap I_C \times B_C = \emptyset$ and $\Gamma'(\gamma^-_{J'}) \cap I_C \times B_C = \emptyset$. Hence, $\Gamma'(\beta_I) \cap (I_C \times B_C) = X \cap (I_C \times B_C)$.

By our assumption, there is some open cell $C \in \mathcal{C}$ with $C \subseteq \mathcal{A}^-_I$. By definition, C is either contained in or disjoint to the sets that $\mathcal{C}$ partitions.

Let $C \subseteq \mathfrak{D}$. Then, all $(\beta(x), x)$ are normal for $x \in C$ as $(I' \times B') \cap X = \Gamma'(\beta)$ for sufficiently small I', B'. This contradicts the definition of β.

Let $C \subseteq (\mathcal{A}^-_{J'}) \setminus \mathfrak{D}$. Then, all $(\beta(x), x)$ are normal for $x \in C$ as $(I' \times B') \cap X = \emptyset$ for sufficiently small I', B'. This also contradicts the definition of β.

Hence, both cases result in a contradiction.

An analogous proof for $\mathcal{A}^+_I$, using

$$\beta_I : \mathcal{A}^+_I \to \overline{I}, x \mapsto \inf\{y \in \overline{I} : y \leq a \text{ and } (y, x) \text{ is not normal}\}$$

and replacing $\mathcal{A}^-_I$ by $\mathcal{A}^+_I$ everywhere, shows the second part of the claim. □

Thus, we have shown both claims. To prove $(\mathrm{UF}')_n$, first notice, that by $(\mathrm{CD})_n$, there is some cell decomposition $\mathcal{D}$ of some box $B \ni b$ partitioning $\mathcal{N}_I$, $\mathcal{A}^+_I$, $\mathcal{A}^-_I$ and $\pi_{\geq 2}(X)$.

We can apply $(\mathrm{UF}')_m$ for some $m < n$ to $X \cap (M \times D)$ for every cell D which has no interior, potentially shrinking I and B. To be more precise, we take the projection $\pi_{\neq l}$ forgetting a coordinate l where the index i_l of the cell equals 0. Then, $(\pi_{\neq l}(X \cap (M \times D)))_y$ is empty or a discrete set for every $y \in B$. Apply $(\mathrm{UF}')_{n-1}$ to $\pi_{\neq l}(X \cap (M \times D))$. Potentially shrinking I and B, there is some N_D such that $|(\pi_{\neq l}(X \cap (M \times D)))_y \cap I| \leq N_D$ for all $y \in \pi_{\neq l}(B)$. This implies $|(X \cap (M \times D))_y \cap I| \leq N_D$ for all $y \in B$. Equivalently, $|X_y \cap I| \leq N_D$ for all $y \in D$. By Claim 2, one can choose $\mathcal{D}$ and B such that all cells $C \in \mathcal{D}$ with interior are not a subset of $\mathcal{A}^+_I$ or $\mathcal{A}^-_I$. Thus, they are contained in $\mathcal{N}_I$.

Let $C \in \mathcal{D}$ be some open cell. By definable completeness, cells are definably connected.[32] Hence, there is some k_C such that $X \cap (I \times C) = \bigcup_{i=1}^{k} \Gamma'(f_i)$ for some f_i. By Claim 1, $|X_y \cap I| \leq k_C$ for all $y \in C$.

As the cell decomposition is finite, we can set $N = \max\{k_C, N_D : C, D \in \mathcal{D}\}$. With this definition, N fulfills $(\mathrm{UF}')_n$ and, therefore, $(\mathrm{UF})_n$ holds.

Part 2 of the Induction Step. In this step, we show that $(\mathrm{PC})_n$ holds, assuming $(\mathrm{CD})_n$ and $(\mathrm{PC})_{n-1}$ for all $n > 1$.

Let $A \subseteq M^n$ be a definable subset and $f : A \to M$ be a definable function. If $A \cap B$ has no interior for some box $B \ni b$, applying $(\mathrm{CD})_n$ to $A \cap B$ results in a cell decomposition with no cell having interior. For all cells without interior, there is a coordinate projection onto $n-1$ coordinates which restricted to the cell is a homeomorphism onto

[32]confer [7, Proposition 2.9 in Chapter 3]

its image.[33] Thus, by $(\mathrm{PC})_{n-1}$, we can assume that f is continuous the projections of the cells. Mapping the resulting cells back gives the desired cell decomposition.

Thus, without loss of generality, we can assume A be an open set and by the same reasoning, it is sufficient to show that there is a cell decomposition such that f is continuous on every open cell.

Let $a \in M, b = (b_1, \dots, b_n) \in M^n$. For every i in $\{1, \dots, n\}$, by the Parameterized Local Monotonicity Theorem (Theorem 6.13), there is a box $B_i \subseteq M^{n-1}$, an interval $I_i \subseteq M$ with $(b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n, b_i) \in B_i \times I_i$ and some interval $J_i \ni a$ such that $f(c_1, \dots, c_{i-1}, -, c_{i+1}, \dots, c_n) : I \cap (f^{-1}(J_i))_{(c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n)} \to J_i$ is continuous everywhere except for a finite set $F_{i,c}$ for every $c = (c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n) \in B_i$. Here, we slightly deviate from our definition of the fiber and by $I \cap (f^{-1}(J_i))_{(c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n)}$, we denote $\{x_i \in I : (c_1, \dots, c_{i-1}, x_i, c_{i+1}, \dots, c_n) \in f^{-1}(J_i)\}$.

Let

$$B_i' = \{(c_1, \dots, c_n) \in M^n : (c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n) \in B_i \wedge c_i \in I_i\},$$

$B = \bigcap_{i=1}^n B_i'$, $J = \bigcap_{i=1}^n J_i$ and $S_i = \{x \in F_{i,c} : c \in B_i\}$ for $1 \leq i \leq n$. Clearly, $b \in B$ and B is a non-empty box, $a \in J$ and J is an interval. Applying $(\mathrm{CD})_n$ and possibly shrinking B, there is a cell decomposition $\mathcal{C}$ of $B \cap f^{-1}(J)$ partitioning $\{S_i\}_{i=1,\dots,n}$. The fibers of S_i are finite. Therefore, S_i cannot have interior, as the fibers would have interior as well. This implies, that any cell $C \in \mathcal{C}$ with $C \subseteq S_i$ cannot have interior. By construction, f is continuous at every $x \in (B \cap f^{-1}(J)) \setminus (\bigcup_{i=1}^n S_i)$. Thus, f is continuous on every cell with interior.

Part 3 of the Induction Step. Finally, we show that $(\mathrm{CD})_{n+1}$ holds, assuming $(\mathrm{CD})_n$, $(\mathrm{UF})_n$ and $(\mathrm{PC})_n$ for all $n \geq 1$.

Let $a = (a_1, \dots, a_{n+1}) \in M^{n+1}$ and let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a finite family of definable subsets of M^{n+1}. Let $B \subseteq M^{n+1}$ be a sufficiently small box with $a \in B$.[34]

Define the definable set

$$Y = \bigcup_{\lambda \in \Lambda} \{(x, y) \in M^n \times M : (x, y) \in B, y \in \mathrm{bd}((A_\lambda \cap B)_x)\}.$$

The fiber $(\{(x, y) \in M^n \times M : (x, y) \in B, y \in \mathrm{bd}((A_\lambda \cap B)_x)\})_z$ cannot have interior for any $z \in M^n$, since the boundary of any set cannot have interior. By uniform local o-minimality of the second kind, there is some box $B_\lambda \subseteq M^n$ and some interval $I_\lambda \subseteq M$ with $(a_1, \dots, a_n) \in B_\lambda, a_{n+1} \in I_\lambda$ such that $I_\lambda \cap \{y \in M : (c, y) \in B, y \in \mathrm{bd}((A_\lambda \cap B)_c)\}$ is finite for all $c \in B_\lambda$. In particular, we can define the box $B_1 = \bigcap_{\lambda \in \Lambda} B_\lambda$ and the interval $I_1 = \bigcap_{\lambda \in \Lambda} I_\lambda$. Then, $(a_1, \dots, a_n) \in B_1, a_{n+1} \in I_1$ and $Y_c \cap I_1$ is finite for all $c \in B_1$. We apply $(\mathrm{UF})_n$ to $Y \cap B_1 \times I_1$. There are a positive integer N, some interval I_2 with $a_{n+1} \in I_2 \subseteq I_1$ and some box B_2 with $(a_1, \dots, a_n) \in B_2 \subseteq B_1$ such that $|Y_c \cap I_2| \leq N$ for all $c \in B_2$. Possibly shrinking B, we can assume $B \subseteq B_2 \times I_2$.

[33] We do so, by essentially removing a coordinate where the index in the type of the cell is 0.

[34] Again, we specify what sufficiently small means throughout the proof.

Define $C_i = \{x \in B : |Y_x \cap I_2| = i\}$ and let $f_{i,1}, \ldots, f_{i,i} : C_i \to I_2$ be definable functions with $f_{i,j} < f_{i,j+1}$ and $Y_x \cap I = \{f_{i,1}(x), \ldots, f_{i,i}(x)\}$ for all $1 \leq i \leq N$ and $1 \leq j \leq i-1$.

Applying $(\mathrm{PC})_n$ to each $f_{i,j}$, there are cell decompositions $\mathcal{C}_{i,j}$ of C_i, possibly shrinking I_2, B_2 and B further. Note, $f_{i,j}$ is continuous on each cell of $\mathcal{C}_{i,j}$. Define the following finite families of definable sets

$$C_{\lambda,i,j} = \{x \in C_i : f_{i,j}(x) \in (A_\lambda)_x\} \text{ and } D_{\lambda,i,j} = \{x \in C_i :]f_{i,j}(x), f_{i,j+1}(x)[\subseteq (A_\lambda)_x\}.$$

Applying $(\mathrm{CD})_n$, possibly shrinking B_2 and B again, there is a cell decomposition $\mathcal{C}$ partitioning all cells of $\mathcal{C}_{i,j}$ and the sets $C_{\lambda,i,j}$ and $D_{\lambda,i,j}$ for all $1 \leq i, j \leq n$ and $\lambda \in \Lambda$. Let $b_1, b_2 \in M$ be such that $I_2 =]b_1, b_2[$. Then, the set

$$\big\{]b_1|_C, f_{i,1}|_C[, \ldots,]f_{i,j}|_C, f_{i,j+1}|_C[, \ldots,]f_{i,i}|_C, b_2|_C[, \Gamma(f_{i,1}), \ldots, \Gamma(f_{i,i}) : C \in \mathcal{C} \big\}$$

is a definable cell decomposition of B partitioning $\{B \cap A_\lambda : \lambda \in \Lambda, B \cap A_\lambda \neq \emptyset\}$.

Here, $]f|_C, g|_C[$ refers to the set $\{(x, y) \in C \times M : f(x) < y < g(x)\}$. □

For almost o-minimal structures, we can deduce the following slightly stronger corollary with the same proof as for the strongly o-minimal case:

Corollary 6.21. *Let $\mathcal{M}$ be an almost locally o-minimal structure. Let n be an arbitrary positive integer. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a finite family of definable subsets of M^n. For every bounded open box $B \subseteq M^n$, there exists a definable cell decomposition of B partitioning the finite family $\{B \cap A_\lambda : \lambda \in \Lambda$ and $B \cap A_\lambda \neq \emptyset\}$.*

Proof. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a finite family of definable subsets of M^n and $B =]b_1, c_1[\times \cdots \times]b_n, c_n[\subseteq M^n$ be some bounded open box. Let $N =]b_1, c_1[\cup \bigcup_{i=2}^n [b_i, c_i[$ and recall the structure $\mathcal{N}$ defined in the proof of Corollary 6.12. By the definition of almost o-minimality, $\mathcal{N} = (N, \{P_{N^n \cap X}\}_{n \in \mathbb{N}, X \in Def^n_{\mathcal{M}}})$ is o-minimal. Let $B_N \subseteq N^n$ be the box $]b_1, c_1[\times \cdots \times]b_n, c_n[$ in the structure $\mathcal{N}$. Let A'_λ be the preimage of A_λ under the canonical embedding $\mathrm{Id} : N \to M, x \mapsto x$. Note that the sets $A''_\lambda = B_N \cap A'_\lambda$ are definable in $\mathcal{N}$.

By Corollary 6.19, there is a definable cell decomposition of B_N partitioning $\{A''_\lambda\}_{\lambda \in \Lambda}$. It is easy to check, that mapping these sets to M via the canonical embedding results in a cell decomposition of B_M partitioning $\{B_M \cap A_\lambda : \lambda \in \Lambda$ and $B_M \cap A_\lambda \neq \emptyset\}$. □

If we only consider definably complete structures this theorem cannot be generalized any further:

Proposition 6.22 ([16, Corollary 4.1])**.** *A definably complete locally o-minimal structure admits local definable cell decomposition if and only if it is a uniformly locally o-minimal structure of the second kind.*

Proof. One direction was shown in the previous theorem, the other direction follows from the definitions: Let $X \subseteq M^{n+1}$ be definable and let $a \in M, b \in M^n$. There is an open box $(a, b) \in B \subseteq M^{n+1}$ and a definable cell decomposition $\mathcal{C}$ of B partitioning $X \cap B$. Note that, by definition, there is a finite set $\mathcal{C}_X \subseteq \mathcal{C}$ such that $\bigcup_{C \in \mathcal{C}_X} C = X \cap B$. Define the interval $I = B_b \subseteq M$ and the box $B' = B_a \subseteq M^n$. We have $X_c \cap I = \bigcup_{C \in \mathcal{C}_X} C_c$ for all $c \in B'$. As C_c is a fiber of a cell, it is either empty, a point or an interval. Uniform local o-minimality of the second kind follows. □

The next two statements follow directly from $(\mathrm{CD})_n$ and $(\mathrm{PC})_n$ and are quite useful tools for further investigations. In [16] these are shown for all uniformly locally o-minimal structures of the second kind, not only definably complete ones, with a technical proof not relying on cell decomposition.

Corollary 6.23 ([16, Theorem 3.3]). *Let $\mathcal{M}$ be a strongly locally o-minimal structure or a definably complete, uniformly locally o-minimal structure of the second kind. Let $X \subseteq M^n$ be a definable set with interior. For every finite partition $X = X_1 \cup \cdots \cup X_m$ into definable subsets $X_i \subseteq M^n$, there is at least one set X_i with interior.*

Proof. For cells, this statement can be easily checked. By $(\mathrm{CD})_n$, there is some cell decomposition of some box B in the interior of X partitioning $X_1, \ldots, X_m$ that must contain at least one cell that has interior. By definition, this cell is contained in some X_i. Consequently, this X_i has interior. □

Corollary 6.24 ([16, Theorem 3.4]). *Let $\mathcal{M}$ be a strongly locally o-minimal structure or a definably complete, uniformly locally o-minimal structure of the second kind. Let $B \subseteq M^n$ be an open box, let $a \in M$ and let $f : B \to M$ be a definable function. There exists an interval I with $b \in I \subseteq M$ such that for every interval J with $b \in J \subseteq I$, $f^{-1}(J)$ has empty interior or there is an open box $B_2 \subseteq f^{-1}(J)$ such that f is continuous on B_2.*

Proof. By $(\mathrm{PC})_n$, we have some cell decomposition such that f is continuous on all cells. If $f^{-1}(J)$ has interior, then some cell has interior as well. Every open cell contains some box. □

As discussed before, if we want a decomposition of X into only finitely many sets we cannot hope for a decomposition into as well behaved sets as cells. But instead of weakening the restriction of finiteness, if we use a relaxed definition of "good-shaped" sets, we can get a decomposition for $*$-locally weakly o-minimal structures into finitely many "good-shaped" sets in the following sense:

Definition 6.25 (Normal/Quasi-Special Submanifolds, [13, Definition 4.1], [15, Definition 4.25]). Let $\mathcal{M}$ be an arbitrary expansion of a dense linear order without endpoints. Let $\{X_i\}_{i=1}^m$ be a finite family of definable subsets of M^n. Let $X \subseteq M^n$ be definable and $\pi : M^n \to M^d$ be a coordinate projection.

A point $x \in X$ is called (X,π)-*normal* if there is an open box $B \subseteq M^n$ with $x \in B$ such that $B \cap X$ is the graph of a continuous map defined on $\pi(B)$.[35]

A definable set $X \subseteq M^n$ is called a π-*normal submanifold* or simply a *normal submanifold* if every point in X is (X,π)-normal.

A definable set $X \subseteq M^n$ is a π-*quasi-special submanifold* or simply a *quasi-special submanifold* if $\pi(X)$ is a definable open set and, for every $x \in \pi(X)$, there exists an open box $U \subseteq M^d$ with $x \in U$ such that: For every $y \in X \cap \pi^{-1}(x)$, there exist an open box $V \subseteq M^n$ with $y \in V$ and a definable continuous map $\tau : U \to M^n$ such that $\pi(V) = U, \tau(U) = X \cap V$ and the composition $\pi \circ \tau$ is the identity map on U.

A *decomposition of M^n into normal submanifolds partitioning* $\{X_i\}_{i=1}^m$ is a finite family of normal submanifolds[36] $\{C_i\}_{i=1}^N$ such that $\bigcup_{i=1}^N C_i = M^n$, $C_i \cap C_j = \emptyset$ when $i \neq j$ and either C_i has an empty intersection with X_j or is contained in X_j for every $1 \leq i \leq N$ and $1 \leq j \leq m$. A decomposition $\{C_i\}_{i=1}^N$ into normal submanifolds satisfies the *frontier condition* if the closure of every normal submanifold C_i is the union of a subfamily of the decomposition.

A *decomposition of M^n into quasi-special submanifolds partitioning* $\{X_i\}_{i=1}^m$ is a finite family of quasi-special submanifolds[36] $\{C_i\}_{i=1}^N$ such that $\bigcup_{i=1}^N C_i = M^n$, $C_i \cap C_j = \emptyset$ when $i \neq j$ and C_i has an empty intersection with X_j or is contained in X_j for every $1 \leq i \leq m$ and $1 \leq j \leq N$. A decomposition $\{C_i\}_{i=1}^N$ of M^n into quasi-special submanifolds satisfies the *frontier condition* if the closure of every quasi-special manifold C_i is the union of a subfamily of the decomposition.

For the next proposition, we need the definition of the local naive dimension:

Definition 6.26 (Local Naive Dimension, $\dim_{LN}(X)$)**.** Let n be a positive integer and $X \subseteq M^n$ be a definable set. The local naive dimension of X, $\dim_{LN}(X)$, is the largest $m \leq n$ such that there exist a coordinate projection $\pi : M^n \to M^m$ and a point $a \in M^n$ such that the definable set $\pi(B \cap X)$ has a nonempty interior for any open box B containing the point a. For the empty set, $\dim_{LN}(\emptyset) = -\infty$.

Every structure that has local cell decomposition also has a decomposition into normal submanifolds:

Proposition 6.27. *Let $\mathcal{M}$ be locally o-minimal and admit local definable cell decomposition and $\{X_i\}_{i=1}^m$ be a finite family of definable subsets of M^n. Then, there exists a decomposition $\{S_j\}_{j=1}^k$ of M^n into normal submanifolds partitioning $\{X_i\}_{i=1}^m$ satisfying the frontier condition. Furthermore, the number N of normal submanifolds is not greater than a constant N determined only by m and n.*

Proof. [37]

[35]To be more precise, it is the graph of a continuous map after permuting the coordinates such that π is the projection onto the first d coordinates.

[36]The different submanifolds can be submanifolds for different projections here.

[37]The main structure of this proof is inspired by the proof of [15, Theorem 4.26]. However, the proof of Step 1 differs significantly from the proof given there.

Step 1. Let X be a definable subset of M^n. There exists a family $\{C_i\}_{i=1}^N$ of pairwise disjoint normal submanifolds with $X = \bigcup_{i=1}^N C_i$ and $N \leq 2^n$.

For this proof, define the full dimension of a definable subset X of M^n to be $\dim_{full}(X) = (d, e)$ with $d = \dim_{LN}(X)$ and e be the number of coordinate projections $\pi : M^n \to M^d$ such that there exists some $x \in X$ such that for every box $B \ni x$, $\pi(X \cap B)$ has a non-empty interior. Clearly, e is bounded for fixed n, d. Thus, we can prove the claim by induction over the full dimension of X, where the pairs (d, e) are ordered by the lexicographic order.

For $d = \dim_{LN}(X) = 0$, X itself is a normal submanifold. Let $x = (x_1, \dots, x_n) \in X$, let $i \in \{1, \dots, n\}$ be arbitrary and let $B \ni x$ be some box such that $\pi_i(B \cap X)$ has empty interior. By local o-minimality, we can find some $a_i, b_i \in M$ such that $]a_i, b_i[\cap \pi_i(X) = \{x_i\}$. Thus, if we choose a_i, b_i in that way for all i, we have $B \cap X = \{x\}$ for $B =]a_1, b_1[\times \cdots \times]a_n, b_n[$. Therefore, x is (X, π) normal with $\pi : M^n \to M^0$ and $d = 0$.

For the induction step, let X be arbitrary and let $(d, e) = \dim_{full}(X)$ with $d = \dim_N(X) > 0$. Suppose the assertion holds for all $(d', e') < (d, e)$. Let $\pi : M^n \to M^d$ be a coordinate projection such that the interior of $\pi(X)$ is non-empty (i. e. a projection witnessing $\dim_{LN}(X) = d$). Define $G \coloneqq \{x \in X : x \text{ is } (X, \pi)\text{-normal}\}$ and $W = X \setminus G$. It is easy to check, that for each $x \in G$ there is some open neighbourhood U of x such that $G \cap U = X \cap U$. Thus, each $x \in G$ is (G, π)-normal and G is a normal submanifold.

Suppose towards a contradiction that there exists some $x \in W$ such that for every box $B \ni x$, $\pi(W \cap B)$ has a non-empty interior.

By local definable cell decomposition, there is a box $B \ni x$ and a cell decomposition $\mathcal{C}$ of B partitioning $W \cap B$ and $X \cap B$. Note that $\dim_{LN}(X)$ implies that X cannot contain any cell with a dimension larger than d. Since $\pi(W \cap B)$ has a non-empty interior, $W \cap B$ contains a $(\underbrace{1, \dots, 1}_{d \text{ times}}, \underbrace{0, \dots, 0}_{n-d \text{ times}})$-cell C_W.

Claim 1. Every $x \in C_W$ is also contained in $\mathrm{fr}_{cell}(C_X)$[38] for some cell $C_X \in \mathcal{C}$ with $C_X \subseteq X$ and $C_X \neq C_W$.

Proof of Claim 1. Suppose there was some $x \in C_W$ such that $x \notin \mathrm{fr}_{cell}(C_X)$ for all cells $C_X \in \mathcal{C}$ with $C_X \subseteq X$ and $C_X \neq C_W$. Then, we can find a box $B' \subseteq B$ with $x \in B'$ and $B' \cap C_X = \emptyset$ for all cells $C_X \in \mathcal{C}$ with $C_X \subseteq X$ and $C_X \neq C_W$. Let $C_X \neq C_W$ be an arbitrary cell with $C_X \subseteq X$. By the definition of a cell decomposition, we have $C_X \cap C_W = \emptyset$, thus $x \notin C_X$. Note that $B_1 = M^n \setminus \overline{C_X}$ is an open set with $x \in B_1$ by the assumption. Moreover, $B \cap B_1$ is an open subset with $x \in B \cap B_1 \subseteq B$. Since $\mathcal{M}$ is a dense linear order, the definition of open sets implies that we can find a box $B' \subseteq B \cap B_1 \subseteq B$ with $x \in B'$. By construction, $B' \cap C_X = \emptyset$. Since there are only finitely many cells, we can recursively do the same construction for all cells $C_X \neq C_W$.

[38] By the frontier of a cell, fr_{cell}, we denote the preimage of the frontier of the cell projected onto the maximal number of coordinates such that the projection image is open.

Therefore, we can indeed find a box B' with $x \in B'$ and $X \cap B' = C_W \cap B'$. Moreover, $\pi(C_W) \cap \pi(B')$ is an open set containing $\pi(x)$, thus, also containing a box $B'' \ni \pi(x)$. Define the box $B''' := B'' \times \pi_{\geq d+1}(B')$. By construction, $x \in B'''$ and $X \cap B''' = C_W \cap B'''$. However, by the definition of cells $C_W \cap B'''$ is the graph of a continuous function on $\pi(C_W \cap B''') = \pi(B''')$. This implies that x is (X, π)-normal, contradicting $x \in W$. Thus, every $x \in C_W$ is contained in the frontier of some other cell of X. □

However, it is clear from the definitions of cells, that the projection onto d coordinates of the frontier of any d dimensional cell cannot have interior. By Corollary 6.23, we can deduce that the finite union of the projections of the frontiers of cells contained in X, namely $\bigcup_{C \in \mathcal{C}} \pi(\mathrm{fr}_{cell}(X \cap C))$, cannot have interior. This is a contradiction, since C_W is contained in this set by Claim 1. Thus, there exists no $x \in W$ such that for every box $B \ni x$, $\pi(W \cap B)$ has a non-empty interior.

Hence, we can apply the induction hypothesis to W.

By construction, in the resulting decomposition, there is at most one normal submanifold for each possible projection. Thus, we have a decomposition into at most 2^n normal submanifolds.

Step 2. Let $\{X_i\}_{i=1}^m$ be a definable subset of M^n. There exists a partition of M^n into normal submanifolds $\{C_i\}_{i=1}^N$ partitioning $\{X_i\}_{i=1}^m$ and satisfying the frontier condition. Furthermore, the number of normal submanifolds is not greater than a constant N determined only by m and n.

In the following, we construct a finite family of definable sets $Z_i \subseteq M^n$ which are pairwise disjoint such that $\bigcup_i Z_i = M^n$ and such that each set X_i and the boundary of each set $\mathrm{bd}(X_i)$ is equal to the finite union of a combination of other sets. Applying the statement of the first step to each of these sets finishes the proof. It is easy to check, that the number of the sets Z_i is uniquely determined by m and n.

Define $X_{m+j} := \mathrm{bd}(X_j)$ for each $j \in \{1, \ldots, m\}$ and $X_{2m+1} := M^n \setminus \bigcup_{i=1}^{2m} X_i$. Define $Z_0 = X_{2m+1}$. For every $l \in \{1, \ldots, 2m\}$ and every combination of indices $I = \{i_1, \ldots, i_l\} \subseteq 1, \ldots, 2m$, define a set $Z_k = \bigcap_{i \in I} X_i \setminus (\bigcup_{i \notin I} X_i)$ for some index k if $\bigcap_{i \in I} X_i \setminus (\bigcup_{i \notin I} X_i)$ is not equal to the empty set. Note that by construction, all these sets are disjoint and each set is either contained in X_i or disjoint to X_i for all $i \in \{1, \ldots, 2m\}$. By combinatorics, there are finitely many combinations to create such sets and the number of such sets N_k is bounded by a constant uniquely determined by n and m. □

Moreover, there is a decomposition into normal submanifolds for all $*$-locally weakly o-minimal structures enjoying the $*$-continuity property. To show this, we need some preparation.

Lemma 6.28 ([15, Lemma 4.17]). *Let $\mathcal{M}$ be a $*$-locally o-minimal structure with the univariate $*$-continuity property. Let $X \subseteq M^n$ be a definable set. If $\pi_{\leq(n-1)}(X)$ has* *non-empty interior and X_x has non-empty interior for every $x \in \pi_{\leq(n-1)}(X)$, then X has interior.*

Lemma 6.29 ([15, Lemma 4.18]). *Let $\mathcal{M}$ be a $*$-locally o-minimal structure with the univariate $*$-continuity property. Let $X \subseteq M^n$ be a definable set with interior. For any finite partition $X = X_1 \cup \dots \cup X_m$ into definable subsets $X_i \subseteq M^n$, there is at least one set X_i with interior.*

Proof of Lemma 6.28 and Lemma 6.29. By induction on m, we only need to show Lemma 6.29 for $m = 2$.

For the proof of Lemma 6.28 and Lemma 6.29, we do a simultaneous induction.

The induction start of Lemma 6.29 holds by Lemma 3.7.

Next, we prove that Lemma 6.28 holds for n if Lemma 6.29 holds for $n-1$. Let $X \subseteq M^n$ be a definable subset such that $\pi_{\leq(n-1)}(X)$ has interior and X_x has interior for every $x \in \pi_{\leq(n-1)}(X)$. The same holds for $X' \coloneqq \{(x,y) \in M^{n-1} \times M : x \in \pi_{\leq(n-1)}(X) \wedge y \in \overset{\circ}{(X_x)}\}$[39]. Note that $X' = \{(x,y) : x \in \pi_{\leq(n-1)}(X') \wedge y \in X'_x\}$. Let $c \in M$. By the induction hypothesis, either $Z = \{(x,y) \in X' : y < c\} \subseteq M^n$ or $Z_2 = \{(x,y) \in X' : y > c\}$ has interior. We present the proof for Z, the proof for Z_2 is similar and can be found in [15, Lemma 4.17].

The following two functions are definable:

$$\begin{aligned} f :& \pi_{\leq(n-1)}(Z) \to \overline{\overline{M}}, \\ & x \mapsto \sup(Z_x), \end{aligned}$$

$$\begin{aligned} f' :& \pi_{\leq(n-1)}(Z) \to \overline{\overline{M}} \cup \{-\infty\}, \\ & x \mapsto \inf\{y \in M : \forall t \in M \ ((f'(x) < t < f(x)) \to (x,t) \in Z)\}. \end{aligned}$$

By $*$-local weak o-minimality and the construction of Z, we have $f'(x) < f(x)$ for all $x \in \pi_{\leq(n-1)}(Z)$. We define the following definable subset of X:

$$Y \coloneqq \{(x,y) \in \pi_{\leq(n-1)}(Z) \times M : f'(x) < y < f(x)\}$$

If the set $\{x \in \pi_{\leq(n-1)}(Z) : f'(x) = -\infty\}$ has interior, then clearly Y has interior. Suppose not. By the induction hypothesis and since $\pi_{\leq(n-1)}(Z)$ has interior, $\{x \in \pi_{\leq(n-1)}(Z) : f'(x) \neq -\infty\}$ has some interior $Z_g \subseteq M^{n-1}$. Finally, we show that this set has interior Z_{conti} such that f and f' are continuous on Z_{conti}. This then implies that $Y' \coloneqq \{(x,y) \in Z_{conti} \times M : f'(x) < y < f(x)\}$ is an open set and Y has interior.

Again, by the induction hypothesis, either the subset of Z_g where f and f' are continuous at every point or one of the sets where one of the functions is discontinuous at

[39] As a clarification, in this definition, y is an element of the interior of the fiber not only the fiber of the interior.

every point, has to have interior. Suppose towards a contradiction, that the definable set of points where f is discontinuous has some interior Z_d. The proof for f' is similar.

For each coordinate i_j and each $x = (x_1, \dots, x_{n-1}) \in Z_d$, we define a function $f_{i_j,x}$: $\pi_{i_j}(Z_d) \to \overline{\overline{M}}, y \mapsto f(x_1, \dots, x_{i_j-1}, y, x_{i_j+1}, \dots, x_{n-1})$.

By [7, Lemma 2.16 in Chapter 3] and the induction hypothesis, this is equivalent to one of the following sets having interior for some coordinate i_j:

$$D_{i_j} \coloneqq \{x \in Z_d : f_{i_j,x} \text{ is not continuous at } \pi_{i_j}(x)\}$$
$$G_{i_j} \coloneqq \{x \in Z_d : f_{i_j,x} \text{ is not strongly monotone for some interval around } \pi_{i_j}(x)\}$$

Note that $f_{i_j,x} = f_{i_j,x_2}$ if $\pi_{\neq i_j}(x) = \pi_{\neq i_j}(x_2)$. Thus, D_{i_j} or G_{i_j} having interior implies that there is some function $f_{i_j,x}$ which is discontinuous on an interval or nowhere locally monotone on an interval. Both contradicts Theorem 6.5.

This concludes the first part of the induction step.

Finally, we prove that Lemma 6.29 holds for n, assuming Lemma 6.28 and Lemma 6.29 hold for all $l < n$. Suppose $X \subseteq M^n$ has non-empty interior. Define the sets

$$U = \{x \in \pi_{\leq(n-1)}(X) : X_x \text{ has non-empty interior}\},$$
$$U_1 = \{x \in \pi_{\leq(n-1)}(X_1) : X_{1x} \text{ has non-empty interior}\},$$
$$U_2 = \{x \in \pi_{\leq(n-1)}(X_2) : X_{2x} \text{ has non-empty interior}\}.$$

For every y in the interior of X, $\pi_{\leq(n-1)}(y)$ is in the interior of U. Thus, U has non-empty interior.

Clearly, $X_{1x} \cup X_{2x} = X_x$ for every $x \in \pi(X)$. Thus, X_x has interior if and only if X_{1x} or X_{2x} has interior by the induction start. Therefore, $U = U_1 \cup U_2$.

By the induction hypothesis for Lemma 6.29, either U_1 or U_2 has non-empty interior. Without loss of generality, we can assume that U_1 has non-empty interior. By the induction hypothesis for Lemma 6.28, this implies that X_1 has non-empty interior. □

Definition 6.30 (Naive Dimension, $\dim_N(X)$, [22, Definition 5.1.1])**.** Let n be a positive integer and $X \subseteq M^n$ be a definable set. The *naive dimension of* X, $\dim_N(X)$, is the largest $m \leq n$ such that there is a coordinate projection $\pi : M^n \to M^m$ with $\pi(X)$ having nonempty interior. For the empty set, $\dim_N(\emptyset) = -\infty$.

Theorem 6.31 (Decomposition into Normal Submanifolds, [15, Theorem 4.26])**.** *Let be a ∗-locally weakly o-minimal structure $\mathcal{M} = (M, <, \dots)$ with the univariate ∗-continuity property. Let $\{X_i\}_{i=1}^m$ be a finite family of definable subsets of M^n. There exists a decomposition $\{C_i\}_{i=1}^N$ into normal submanifolds partitioning $\{X_i\}_{i=1}^m$ and satisfying the frontier condition. Furthermore, the number of normal submanifolds is not greater than a constant N uniquely determined only by m and n.*

Proof. [40]

STEP 1. FOR EVERY DEFINABLE $X \subseteq M^n$, THERE EXISTS A FAMILY $\{C_i\}_{i=1}^N$ OF PAIRWISE DISJOINT NORMAL SUBMANIFOLDS WITH $X = \bigcup_{i=1}^N C_i$ AND $N = 2^n$.

For this proof, define the full dimension of a definable subset $X \subseteq M^n$ to be $\dim_{full}(X) = (d, e)$ with $d = \dim_N(X)$ and e be the number of coordinate projections $\pi : M^n \to M^d$ such that $\pi(X)$ has a nonempty interior. Clearly, e is bounded for fixed n, d. Thus, we can prove the claim by induction over the full dimension of X, where the pairs (d, e) are ordered by the lexicographic order.

For $d = \dim_N(X) = 0$, X itself is a normal submanifold. Let $x = (x_1, \ldots, x_n) \in X$, let $i \in \{1, \ldots, n\}$ be arbitrary. Since $\pi_i(X)$ has no interior, by local o-minimality (which is implied by $*$-local weak o-minimality), we can find some $a_i, b_i \in M$ such that $]a_i, b_i[\cap \pi_i(X) = \{x_i\}$. Thus, if we choose a_i, b_i in that way for all i, we have $B \cap X = \{x\}$ for $B =]a_1, b_1[\times \cdots \times]a_n, b_n[$. Therefore, x is (X, π) normal with $\pi : M^n \to M^0$ and $d = 0$.

For the induction step, let X be arbitrary and let $(d, e) = \dim_{full}(X)$ with $d = \dim_N > 0$. Suppose the assertion holds for all $(d', e') < (d, e)$. Let $\pi : M^n \to M^d$ be a coordinate projection such that the interior of $\pi(X)$ is non-empty (i. e. a projection witnessing $\dim_N(X) = d$). Define $G :=\{x \in X : x \text{ is } (X, \pi)\text{-normal}\}$ and $W = X \setminus G$. It is easy to check, that for each $x \in G$, there is some open neighborhood U of x such that $G \cap U = X \cap U$. Thus, each $x \in G$ is (G, π)-normal and G is a normal submanifold.

Suppose towards a contradiction that $\pi(W)$ has interior.

Moreover, suppose towards a contradiction, there is some interior $W' \subseteq \pi(W)$ such that for each $x \in W'$, there is some coordinate j that π is not projecting on such that $\pi_j(X \cap \pi^{-1}(x))$ has interior. In particular, since there are only finitely many coordinates, applying Lemma 6.29, there is some coordinate j such that $\{x \in \pi(X) : \pi_j(X \cap \pi^{-1}(x)) \text{ has interior}\}$ has interior. By Lemma 6.28, the projection image of X of the projection onto the coordinates that π is projecting on and additionally the j coordinate has interior contradicting $\dim_N(X) = d$. Thus, there is no interior $W' \subseteq \pi(W)$ such that for each $x \in W'$ there is some coordinate j, that π is not projecting on, such that $\pi_j(X \cap \pi^{-1}(x))$ has interior.

By Lemma 6.29 and local o-minimality, there is some interior $W_g \subseteq \pi(W)$ such that $\pi_j(X \cap \pi^{-1}(x))$ is discrete for every j and every $x \in W_g$.

Let $c \in M$. We define a map: $g : W_g \to \left(\overline{\overline{M}}\right)^{n-d}$ coordinate-wise. For $j \in \{1, \ldots, n-d\}$ suppose $(g(x))_i$ is defined for all $i < j$. Let $x \in W_g$. Define

[40] Again, the main structure of this proof is inspired by the proof of [15, Theorem 4.26]. However, the proof of Step 1 differs significantly from the proof given there.

$$G_{1,j} \coloneqq \{y \in M : (((g(x))_1, \dots, (g(x))_{j-1}, y) \in \pi_{\geq d+1}(\pi_{\leq d+j}(X \cap \pi^{-1}(x)))) \wedge (y < c)\}$$
$$G_{2,j} \coloneqq \{y \in M : (((g(x))_1, \dots, (g(x))_{j-1}, y) \in \pi_{\geq d+1}(\pi_{\leq d+j}(X \cap \pi^{-1}(x)))) \wedge (y > c)\}$$

$$(g(x))_j = \begin{cases} \sup(G_{1,j}) & \text{if the set } G_{1,j} \text{ is non-empty,} \\ \inf(G_{2,j}) & \text{else.} \end{cases}$$

By $*$-local weak o-minimality, we have that either $G_{1,j}$ is non-empty and finite or $G_{1,j}$ is empty and $G_{2,j}$ non-empty and finite on an interval around $(g(x))_j$. Therefore, $(g(x))_j \in M$ and $g : W_g \to M^{n-d}$. Moreover, $(x, g(x)) \in W$ for every $x \in W_g$ by the construction.

With Lemma 6.29, Theorem 6.5 and [7, Lemma 2.16 in Chapter 3], we can find some interior $Z \subseteq W_g$ on which the definable functions $(g(x))_1, \dots, (g(x))_{n-d}$ are continuous. In particular, g is continuous on Z.

For each $j \in \{1, \dots, n-d\}$, we can again define functions:

$$\begin{aligned} h_{j,1} :& Z \to \overline{\overline{M}} \cup \{-\infty\}, x \mapsto \sup\{y \in M : y \in \pi_j(X \cap \pi^{-1}(x)) \wedge y < (g(x))_j\}\} \\ h_{j,2} :& Z \to \overline{\overline{M}} \cup \{\infty\}, x \mapsto \inf\{y \in M : y \in \pi_j(X \cap \pi^{-1}(x)) \wedge y > (g(x))_j\}\} \end{aligned}$$

Again, by discreteness and $*$-local weak o-minimality, $h_{j,1}(x) < (g(x))_j < h_{j,2}(x)$ follows. By Lemma 6.29, Theorem 6.5 and [7, Lemma 2.16 in Chapter 3] and possibly shrinking Z, we can assume that $h_{j,1}$ and $h_{j,2}$ are continuous. We can define the following non-empty open set:

$$B' \coloneqq Z \times]h_{1,1}(x), h_{1,2}(x)[\times \cdots \times]h_{n-d,1}(x), h_{n-d,2}(x)[\subseteq M^n.$$

By the construction of $h_{j,i}$, we have $B' \cap X = \{(x, g(x)) : x \in Z\}$. Pick some $x \in Z$. Since B' is open, there is some box $B \subseteq B'$ with $(x, g(x)) \in B$. This implies that $(x, g(x))$ is (X, π)-normal. This is a contradiction to $x \in Z' \subseteq W$.

Thus, we have that $\pi(W)$ has empty interior and we can apply the induction hypothesis to W.

By construction, we have that there is at most one normal submanifold for each possible projection in the resulting decomposition. Thus, we have a decomposition into at most 2^n normal submanifolds.

Step 2. Let $\{X_i\}_{i=1}^m$ be a definable subset of M^n. There exists a partition of M^n into normal submanifolds $\{C_i\}_{i=1}^N$ partitioning $\{X_i\}_{i=1}^m$ and satisfying the frontier condition. Furthermore, the number N of normal submanifolds is not greater than a number uniquely determined only by m and n.

The proof is identical to Step 2 from Proposition 6.27. □

In the case of definably complete, locally o-minimal structures, a decomposition into normal submanifolds is already a quasi-special submanifold by [13, Theorem 4.2], resulting in the following corollary:

Corollary 6.32 (Decomposition into Quasi-Special Submanifolds, [13, Theorem 4.5])**.** *Let $\mathcal{M} = (M, <, \dots)$ be definably complete and locally o-minimal. Let $\{X_i\}_{i=1}^m$ be a finite family of definable subsets of M^n. There exists a decomposition $\{C_i\}_{i=1}^N$ of M^n into quasi-special submanifolds partitioning $\{X_i\}_{i=1}^m$ satisfying the frontier condition. Furthermore, the number N of quasi-special submanifolds is not greater than a number uniquely determined only by m and n.*

Proof. Note, that every π-normal submanifold X is a π-quasi-special submanifold if for all $y \in X$, we can choose the open box $B^y \subseteq M^n$ from the definition of a normal point in such a way that for all $y_1, y_2 \in X$: $\pi(y_1) = \pi(y_2)$ implies $\pi(B^{y_1}) = \pi(B^{y_2})$.

In the following, we show that we can choose the boxes B^y in such a way by induction over the number k of coordinates that π projects on. Possibly permuting coordinates, we can assume that π projects onto the first k coordinates.

Induction Hypothesis. For all $y \in X$, we can choose the open box $B^y \subseteq M^n$ from the definition of a normal point in such a way that for all $y_1, y_2 \in X$, $\pi_{\leq k}(y_1) = \pi_{\leq k}(y_2)$ implies $\pi_{\leq l}(B^{y_1}) = \pi_{\leq l}(B^{y_2})$.

Induction Start. For $l = 0$ the statement is trivial.

Induction Step. Let $1 < l \leq k$ and suppose the statement holds for $l - 1$.

Now, for every $y \in X$, let B^y be boxes such that the induction hypothesis holds. Let $y \in X$ and let $a, b \in M$ be some bounds $a < \pi_l(y) < b$. We consider the definable set $X_y = \{y_1 \in X : \pi_{\neq l}(y_1) = \pi_{\neq l}(y)\}$ and the definable maps

$$\begin{aligned}\phi_+ :& X_y \to M \\ & y_1 \mapsto \inf(\{b\} \cup \{x \in]\pi_l(y_1), \infty[: \exists\, y_2\ (\pi_l(y_2) = x \wedge \pi_{\neq l}(y_2) \in \pi_{\neq l}(B^{y_1}))\},\end{aligned}$$

$$\begin{aligned}\phi_- :& X_y \to M \\ & y_1 \mapsto \sup(\{a\} \cup \{x \in]-\infty, \pi_l(y_1)[: \exists\, y_2\ (\pi_l(y_2) = x \wedge \pi_{\neq l}(y_2) \in \pi_{\neq l}(By_1))\}.\end{aligned}$$

The maps are well defined by definable completeness. The set X_y is discrete and $\phi_-(y_1) < \pi_l(y_1) < \phi_+(y_1)$, since X is a π-normal submanifold. Hence, the set X_y has no interior and is closed by local o-minimality. This implies that $\phi_-(X_y)$ and $\phi_+(X_y)$ have no interior, by Theorem 6.5. By local o-minimality, these sets must then be discrete and closed. In particular, $\sup(\phi_-(X_y)) < \pi_l(y_1) < \inf(\phi_+(X_y))$. Therefore, the boxes

$$B'_{y_1} := \left\{ z \in M^n : \pi_{\neq l}(z) \in \pi_{\neq l}(B_{y_1}) \wedge \pi_l(z) \in]\sup(\phi_-(X_y)), \inf(\phi_+(X_y))[\, \right\}$$

have the desired property. We can do a similar construction for every y corresponding to another set X_y.

□

Remark 6.33. Note that this implies that any definably complete locally o-minimal structure which is not uniformly o-minimal of the second kind, would not have cell decomposition but a decomposition into quasi-special submanifolds. We cannot provide such an example so far and only suspect that such a structure exists.

local cell decomposition	uniform local o-minimality of the 2. kind & definable completeness; strong local o-minimality
no local cell decomposition	no uniform local o-minimality of the 2. kind & definable completeness
normal submanifold decomposition	uniform local o-minimality of the 2. kind & definable completeness; strong local o-minimality; $*$-local weak o-minimality & $*$-continuity
quasi special submanifold decomposition	definable completeness & local o-minimality

Table 3: Summary of the presented tameness results in this section. Confer with Figures 3 and 4 to check for which other notions this implies the same results trivially since they are implied by one of the notions mentioned here.

6.2.3 Dimensions

The concept of dimension is a tool to describe the size of sets in some way. There are several different reasonable definitions for dimension. Since there is not necessarily a metric available in arbitrary dense linear orders, we only consider topological notions of dimension here, considering the topology induced by the linear order of the structure $\mathcal{M}$. In this section, we introduce some notions of topological dimension and evaluate in which settings these coincide with another. Moreover, some of the tame properties that these notions have for some notions of local o-minimality, are presented.

Recall Definitions 6.26 and 6.30 from the previous section. The *naive dimension* of a definable subset $X \subseteq M^n$ is defined by: $\dim_N(X)$ is the largest $m \leq n$ such that there is a coordinate projection $\pi : M^n \to M^m$ such that $\pi(X)$ has nonempty interior. The *local naive dimension* of a definable subset $X \subseteq M^n$ is defined by: $\dim_{LN}(X)$ is the largest $m \leq n$ such that there exist a coordinate projection $\pi : M^n \to M^m$ and a point $a \in M^n$ such that the definable set $\pi(B \cap X)$ has a nonempty interior for every open box B containing the point a.

Fact 6.34. *Let n be a positive integer and $X \subseteq M^n$ be a definable set. Then,* $\dim_N(X) \geq \dim_{LN}(X)$.

Proof. Suppose there is a coordinate projection, such that the definable set $\pi(B \cap X)$ has a nonempty interior for any open box B containing the point a. We have $\pi(B \cap X) \subseteq \pi(X)$. Thus, $\pi(X)$ has non-empty interior. It follows that $\dim_N(X) \geq \dim_{LN}(X)$. □

These notions are not equivalent for every definable set of arbitrary locally o-minimal structures:

Example 12. Let $\mathcal{M} = (\mathbb{Q}^2, <_{lex}, f)$ with $f : \mathbb{Q}^2 \to \mathbb{Q}^2, (a, b) \mapsto (b, 0)$ be the strongly locally o-minimal structure from Example 8. Let $X = \{(x, y) \in (\mathbb{Q}^2) \times (\mathbb{Q}^2) : y = f(x)\}$ be the graph of f. Then, $\dim_N(X) = 1$ but $\dim_{LN}(X) = 0$.

Proof. Clearly, $\pi_1(X) = \mathbb{Q}^2$ and thus $\dim_N(X) \geq 1$. Moreover, the graph of a function cannot have interior, implying $\dim_N(X) = 1$.

For every $((a_1, a_2), (b_1, b_2)) \in (\mathbb{Q}^2) \times (\mathbb{Q}^2)$, there are $(c_1, c_2), (c_3, c_4), (d_1, d_2), (d_3, d_4) \in (\mathbb{Q}^2) \times (\mathbb{Q}^2)$, with $c_1 = d_1 = a_1$, $c_3 = d_3 = b_1$, $c_2 < a_2 < d_2$ and $c_4 < b_2 < d_4$.

Define the box $B :=](c_1, c_2), (d_1, d_2)[\times](c_3, c_4), (d_3, d_4)[$. Then, by construction, either $B \cap X = \emptyset$ or $B \cap X = \{((a_1, b_1), (b_1, 0))\}$. Since the point $((a_1, a_2), (b_1, b_2))$ was chosen arbitrarily, $\dim_{LN}(X) \leq 0$ follows. As X is clearly non-empty, this implies $\dim_{LN}(X) = 0$. □

However, in the setting of $*$-locally weakly o-minimal structures with the univariate $*$-continuity property, these notions are equivalent. Moreover, they are also equivalent to the following definition of dimension:

Definition 6.35 ($\dim(X)$)**.** Let n be a positive integer and $X \subseteq M^n$ be a definable set. The *dimension of* X*,* $\dim(X)$*,* is the largest m such that there exists an open box $B \subseteq M^m$ and a definable continuous injective map $f : B \to X$ which is homeomorphic onto its image. For the empty set, $\dim(\emptyset) = -\infty$.

Proposition 6.36 ([15, Proposition 4.14, Corollary 4.22])**.** *The equality* $\dim_N(X) = \dim_{LN}(X) = \dim(X)$ *holds true for every $*$-locally weakly o-minimal structure $\mathcal{M}$ with the univariate $*$-continuity property and every definable $X \subseteq M^n$.*

Proof. [41] The case $X = \emptyset$ is immediate from the definitions. Let $X \neq \emptyset$:

$\dim_N(X) = \dim_{LN}(X)$: Suppose towards a contradiction that there is some definable set $X \subseteq M^n$ with $d = \dim_N(X) > \dim_{LN}(X)$. Let π be a projection witnessing $\dim_N(X) = d$. Let Y equal the interior of $\pi(X)$. Then, $Y \subseteq M^d$ is a non-empty open set by the assumption.

The next part of the proof is similar to the proof of Theorem 6.31.

Suppose, towards a contradiction, there is some interior $Y' \subseteq Y$ such that for each $x \in Y'$, there is some coordinate j, that π is not projecting on, such that $\pi_j(X \cap$

[41] Some ideas in the proof are inspired by the proof of [15, Proposition 4.14, Corollary 4.22]. However, the proof differs significantly from the proof given there.

$\pi^{-1}(x))$ has interior. In particular, since there are only finitely many coordinates, applying Lemma 6.29, there is some coordinate j such that $\{x \in \pi(X) : \pi_j(X \cap \pi^{-1}(x))$ has interior$\}$ has interior. By Lemma 6.28, the projection image of X of the projection onto the coordinates that π is projecting on and additionally the j coordinate has interior contradicting $\dim_N(X) = d$.

By Lemma 6.29 and local o-minimality, there is some interior $Y_g \subseteq Y$ such that $\pi_j(X \cap \pi^{-1}(x))$ is discrete for every j and every $x \in Y_g$.

Let $c \in M$. We define a map: $g : Y_g \to \left(\overline{\overline{M}}\right)^{n-d}$ coordinate-wise. For $j \in \{1, \dots, n-d\}$, suppose $(g(x))_i$ is defined for all $i < j$. Let $x \in Y_g$. Define

$$G_{1,j} \coloneqq \{y \in M : (((g(x))_1, \dots, (g(x))_{j-1}, y) \in \pi_{\geq d+1}(\pi_{\leq d+j}(X \cap \pi^{-1}(x)))) \wedge (y < c)\},$$
$$G_{2,j} \coloneqq \{y \in M : (((g(x))_1, \dots, (g(x))_{j-1}, y) \in \pi_{\geq d+1}(\pi_{\leq d+j}(X \cap \pi^{-1}(x)))) \wedge (y > c)\},$$

$$(g(x))_j = \begin{cases} \sup(G_{1,j}) & \text{if the set } G_{1,j} \text{ is non-empty,} \\ \inf(G_{2,j}) & \text{else.} \end{cases}$$

By $*$-local weak o-minimality, either one of the sets $G_{1,j}$, $G_{2,j}$, that is not empty, is finite on an interval around $(g(x))_j$. This implies $(g(x))_j \in M$ and $g : Y_g \to M^{n-d}$. Moreover, by the construction, $(x, g(x)) \in X$ for every $x \in Y_g$.

With Lemma 6.29, Theorem 6.5 and [7, Lemma 2.16 in Chapter 3], we can find some interior $Z \subseteq Y_g$ on which the definable functions $(g(x))_1, \dots, (g(x))_{n-d}$ are continuous. In particular, g is continuous on Z.

Let $z \in Z$ and $B \subseteq M^n$ be an arbitrary box with $(z, g(z)) \in B$. By continuity of g, $A \coloneqq Z \cap \pi(\{(x, g(x)) : x \in Y_g\} \cap B)$ is an open set with $z \in A$. Since this is a subset of $X_B \coloneqq \pi(X \cap B)$, X_B has non-empty interior. Hence, $\dim_{LN}(X) \geq d$. This is a contradiction to the assumption $d = \dim_N(X) > \dim_{LN}(X)$.

By Fact 6.34, we can deduce $\dim_N(X) = \dim_{LN}(X)$.

$\dim_N(X) = \dim(X)$: Let n be a positive integer, $X \subseteq M^n$ be a definable set, $B \subseteq M^m$ an open box and $f : B \to X$ a definable continuous injective map which is homeomorphic onto its image. In particular, f is injective and continuous.

The statement $\dim_N(X) \geq \dim_N(B)$ is trivial for $\dim_N(B) = 0$.

By induction, we can assume $\dim_N(f((y, \pi_{\geq 2}(B)))) \geq \dim_N(B) - 1$ for all $y \in \pi_1(B)$. By Lemma 3.7, possibly shrinking $\pi_1(B)$, we can assume the sets $f((y, \pi_{\geq 2}(B)))$ all have interior for the same projection onto $d \coloneqq \dim_N(B) - 1$ coordinates. Without loss of generality, assume these are the first d coordinates.

Now, suppose towards a contradiction, that $\dim_N(X) = d$. Pick some interval $]y_1, y_2[\subseteq \pi_1(B)$, some $y_3 \in]y_1, y_2[$ and some $y' \in \pi_{\geq 2}(B)$. By injectivity and continuity, there must be some coordinate i such that $\pi_i(f(]y_1, y_2[, y'))$ is an open convex set. In particular, $\pi_1(f(y_3, y'))$ is contained in this interior. If $i \leq d$, this interior would have an non-empty, open cut with $\pi_1(f((y, \pi_{\geq 2}(B))))$, contradicting the injectivity. If $i > d$, then by Lemma 6.28, the projection onto the first d and the i-th coordinate of X has interior. This contradicts $\dim_N(X) = d$. Thus, $\dim_N(X) \geq \dim_N(B) = m$.

For the other direction, suppose X is a definable set with $\dim_N(X) = d$. Thus, there is some projection π projecting onto d coordinates such that $\pi(X)$ has interior and some open box $B \subseteq \pi(X) \subseteq M^d$. Define a map $g : B \to \left(\overline{\overline{M}}\right)^{n-d}$ similar to the construction in the proof of $\dim_N(X) = \dim_{LN}(X)$:

Possibly shrinking B, we can assume that $\pi_j(X \cap \pi^{-1}(x))$ is discrete for every j and every $x \in B$. Define g coordinate-wise as follows:

$$G_{1,j} \coloneqq \{y \in M : (((g(x))_1, \ldots, (g(x))_{j-1}, y) \in \pi_{\geq d+1}(\pi_{\leq d+j}(X \cap \pi^{-1}(x)))) \wedge (y < c)\}$$
$$G_{2,j} \coloneqq \{y \in M : (((g(x))_1, \ldots, (g(x))_{j-1}, y) \in \pi_{\geq d+1}(\pi_{\leq d+j}(X \cap \pi^{-1}(x)))) \wedge (y > c)\}$$

$$(g(x))_j = \begin{cases} \sup(G_{1,j}) & \text{if the set } G_{1,j} \text{ is non-empty,} \\ \inf(G_{2,j}) & \text{else.} \end{cases}$$

By $*$-local weak o-minimality, either one of the sets $G_{1,j}, G_{2,j}$, that is not empty, is finite on an interval around $(g(x))_j$. This implies $(g(x))_j \in M$ and $g : Y_g \to M^{n-d}$. Moreover, by the construction, $(x, g(x)) \in X$ for every $x \in B$.

The map $f : B \to X, x \mapsto (x, g(x))$ is definable and injective. By Lemma 6.29, Theorem 6.5 and [7, Lemma 2.16 in Chapter 3], possibly shrinking B, we can assume that g is continuous. The inverse of f is the projection onto the first d and thus, well-defined, definable and continuous. Therefore, f is a definable continuous injective map which is homeomorphic onto its image and $\dim(X) \geq d$.

□

Remark 6.37. In [15], $\dim_N(X) = \dim_{LN}(X)$ and $\dim_N(X) = \dim(X)$ are shown in more general settings using some technical definitions.

There is one more commonly used definition for dimension specifically for structures admitting local definable cell decomposition.

Definition 6.38 ($\dim_C(X)$)**.** Let $X \subseteq M^n$ be a definable set. Let $\mathcal{M}$ admit local definable cell decomposition. Then, $\dim_C(X)$ is the largest $m \leq n$ such that for some local cell decomposition, X contains an $(i_1, \ldots, i_n)$-cell D with $\dim_{cell}(D) = \sum_{j=1}^{n} i_j = m$. For the empty set, $\dim_C(\emptyset) = -\infty$.

Proposition 6.39 ([16, Corollary 5.3])**.** *Let $\mathcal{M}$ be a locally o-minimal structure which admits local definable cell decomposition. Let $X \subseteq M^n$ be a definable set. Then,*

$$\dim(X) = \dim_C(X) = \dim_{LN}(X).$$

Proof. The case $X = \emptyset$ is immediate from the definitions. For $X \neq \emptyset$:

$\dim_C(X) = \dim_{LN}(X)$: For any $(i_1, \ldots, i_n)$-cell C, any $a \in C$ and any open box $B \ni a$, let π be the projection onto the coordinates j with $i_j = 1$. Then, $\pi(B \cap X)$ is an open neighborhood of $\pi(a)$. Therefore, $\dim_C(X) \leq \dim_{LN}(X)$.

Let a and π be such that $\pi(a)$ is in the interior of $\pi(B \cap X)$ for all boxes B with $a \in B$. By local definable cell decomposition, there is a box $B \ni a$ and a cell decomposition $\mathcal{C}$ of B partitioning X. By Corollary 6.23, as $\pi(B \cap X)$ has interior, there must be a cell C such that $\pi(C)$ has interior. Thus, $\dim_C(X) \geq \dim_{LN}(X)$.

$\dim_C(X) = \dim(X)$: Let $m = \dim(X)$. Let $B \subseteq M^m$ be an open box and $f : B \to X$ be a homeomorphic map onto its image. It is easy to check that f and f^{-1} map cells onto cells of the same dimension. Since B is an open box, there must be a local cell decomposition of some subset of B containing an open cell D. Thus, $\dim_C(X) \geq \dim_{cell}(f(D)) = \dim_{cell}(D) = m$.

Suppose towards a contradiction, that for some local cell decomposition, X contains a $(j_1, \ldots, j_n)$-cell D' with $\dim_{cell}(D') = l \geq m + 1$. Let π be the projection on all coordinates i with $j_i = 1$. Clearly, $\pi(D) \subseteq M^l$ is open and contains some box B'. But then, $(\pi|_{D'})^{-1}|_{B'} : B' \to X$ is a definable continuous injective map which is homeomorphic onto its image.

□

If we consider structures admitting local definable cell decomposition and in this setting equivalent notions $\dim, \dim_C$ and $\dim_{LN}$ of dimension, then we can prove the following tame properties of dimension:

Proposition 6.40 ([16, Lemma 5.1, Corollary 5.4, Theorem 5.6])**.** *Consider a locally o-minimal structure $\mathcal{M}$ which admits local definable cell decomposition. Let $X, Y \subseteq M^n$, $Z \subseteq M^m$ be definable non-empty sets. Let $\sigma : M^n \to M^n, (x_1, \ldots, x_n) \mapsto (x_{\overline{\sigma}(1)}, \ldots, x_{\overline{\sigma}(n)})$ be a coordinate permutation. The following assertions hold true:*

1. $X \subseteq Y$ *implies* $\dim(X) \leq \dim(Y)$,
2. $\dim(X) = \dim(\sigma(X))$,
3. $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$,
4. $\dim(X \times Z) = \dim(X) + \dim(Z)$,
5. $\dim(\mathrm{fr}(X)) < \dim X$.

Proof. By Proposition 6.39, it is sufficient to show the properties for one of the definitions of dimension $\dim(X), \dim_C(X), \dim_{LN}(X)$.

1. Follows from the definition of $\dim(X)$.
2. Follows from the definition of $\dim_{LN}(X)$.

3. Follows from the definition of $\dim_C(X)$.

4. For every $(i_1, \ldots, i_n)$-cell $C \subseteq X$ and $(j_1, \ldots, j_m)$-cell $D \subseteq Z$, the set $C \times D \subseteq X \times Z$ is a $(i_1, \ldots, i_n, j_1, \ldots, j_m)$-cell. Thus, $\dim_{LN}(X) + \dim_{LN}(Z) \leq \dim_{LN}(X \times Z)$. Similarly, for any $(i_1, \ldots, i_n, j_1, \ldots, j_m)$-cell in $X \times Z$, the coordinate projection onto the first n coordinates results in a $(i_1, \ldots, i_n)$-cell in X and the coordinate projection onto the last m coordinates results in a $(j_1, \ldots, j_m)$-cell in Z. Thus, $\dim_{LN}(X) + \dim_{LN}(Z) \geq \dim_{LN}(X \times Z)$.

5. As X is non-empty, $\dim(X) \geq 0$. Thus, the statement is true for $\text{fr}(X) = \emptyset$. Since $\dim(X) = \dim_C(X)$, if $\dim(\text{fr}(X)) \geq \dim(X)$, there is a local cell decomposition of some box B and a cell $C \subseteq (\text{fr}(X) \cap B)$ with $\dim_{cell}(C) = \dim(\text{fr}(X)) \geq \dim(X) \geq \dim(X \cap B)$. It is easy to check, that intersecting a cell with a box results either in the empty-set or a cell of the same dimension. Thus, it is sufficient to show that for every $a \in M^n$ and every sufficiently small box $B \ni a$, $\dim(\text{fr}(X) \cap B) < \dim(X \cap B)$ if $\text{fr}(X) \cap B \neq \emptyset$.

   Proof by induction on $\dim(X)$:

   Induction Start. For $\dim(X) = 0$, we have $\text{fr}(X) = \emptyset$. Thus, the statement is immediate.

   Induction Hypothesis. For all Y with $0 \leq \dim(Y) < \dim(X)$, every $a \in M^n$ and every sufficiently small box $B \ni a$ with $\text{fr}(Y) \cap B \neq \emptyset$, we have $\dim(\text{fr}(Y) \cap B) < \dim(Y \cap B)$.

   Induction Step. Let $\dim(X) > 0$ and suppose that the induction hypothesis holds. Let $a \in M^n$ and $B \ni a$ be a sufficiently small box. Let σ_i be the coordinate permutation

   $$\sigma_i : M^n \to M^n, (x_1, \ldots, x_n) \mapsto (x_i, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

   Let $I_1, \ldots, I_n$ be open intervals such that $B = I_1 \times \cdots \times I_n$ and let $J_i \subseteq M^{n-1}$ be the open boxes $J_i = I_1 \times \cdots \times I_{i-1} \times I_{i+1} \times \cdots \times I_n$.

   Consider the definable sets

   $$F_i = \{x \in I_i : (\text{fr}(\sigma_i(X)))_x \cap J_i \neq \text{fr}((\sigma_i(X))_x) \cap J_i\}.$$

   Step 1 of the Proof of the Induction Step. $\dim(F_i) \leq 0$ *for all* $i \in \{1, \ldots, n\}$ *and* B *sufficiently small.*

   By the second statement of this proposition, we only have to consider the case $i = 1$.

   Define the definable sets $U_{(a,b)} =]a_1, b_1[\times \cdots \times]a_{n-1}, b_{n-1}[$ and

   $$I_{U_{(a,b)}} = \{x \in I_1 : ((\text{fr}(X))_x \setminus \text{fr}((X)_x)) \cap U_{(a,b)} \neq \emptyset \text{ and } X_x \cap U_{(a,b)} = \emptyset\}$$

for all $a, b \in J_1$ with $(a, b) \in J' \coloneqq \{(a, b) \in J_1 \times J_1 : \bigwedge_{i=1}^{n-1} a_i < b_i\}$. Moreover, define

$$D = \bigcup_{(a,b)\in J_2} (I_{U_{(a,b)}} \times \{(a, b)\}) \subseteq M^{2n-1}.$$

Since we choose B sufficiently small, we can assume that we have a definable cell decomposition of $I_1 \times J_1 \times J_1$ partitioning the definable set D.

First, we can show that $\dim(I_{U_{(a,b)}}) \leq 0$ for all $a, b \in M^{n-1}$ such that $(a, b) \in J'$. Towards a contradiction, suppose some $I_{U_{(a,b)}}$ contains interior. Pick some interval $Z_1 \subseteq I_{U_{(a,b)}}$ contained in this interior.

By the definition of $I_{U_{(a,b)}}$, $X_x \cap U_{(a,b)} = \emptyset$ for all $x \in Z_1$, and this implies $X \cap (Z_1 \times U_{(a,b)}) = \emptyset$. As $(Z_1 \times U_{(a,b)})$ is an open set, $\overline{X} \cap (Z_1 \times U_{(a,b)}) = \emptyset$. This contradicts $\big((\mathrm{fr}(X))_x \setminus \mathrm{fr}(X_x)\big) \cap U_{(a,b)} \neq \emptyset$ for $x \in Z_1$.

Secondly, we show $\dim(D) \leq 2(n-1)$. Suppose not, then $\dim(D) = 2n-1$. In particular, the cell decomposition partitioning D contains an open cell $C_D \subseteq D$. Let $(y', a', b') \in I_1 \times J_1 \times J_1$ be some point in the interior of the cell. As $(y', a', b') \in D$, we have $a_i < b_i$ for $1 \leq i < n$. By the definition of cells, we have $\dim(C_D) = \dim((C_D)_{(a',b')}) + \dim(\pi_{\geq 2}(C_D))$. By the first statement of this proposition, we can deduce: $\dim(C_D) = \dim((C_D)_{(a',b')}) + \dim(\pi_{\geq 2}(C_D)) \leq \dim(I_{U_{(a',b')}}) + \dim(\pi_{\geq 2}(C_D)) \leq 0 + \dim(\pi_{\geq 2}(C_D)) \leq 2n-2$. This is a contradiction, to C_D being an open cell. Thus, $\dim(D) \leq 2(n-1)$.

Third, for every $x \in F_1$, we show $\dim(D_x \cap (J_1 \times J_1)) \geq 2(n-1)$. Let $x \in F_1$ be arbitrary. By the definition of F_1, there is some $y \in \big((\mathrm{fr}(X))_x \setminus (\mathrm{fr}(X_x))\big) \cap J_1$. In particular, $y \in (\mathrm{fr}(X))_x$, which implies $y \notin X_x$, and $y \notin \mathrm{fr}(X_x)$. This implies there is some open box around y which has an empty intersection with X_x. Therefore, we can find $(a, b) \in J'$ such that $y \in U_{(a,b)} \subseteq J_1$ and $x \in I_{U_{(a,b)}}$. For all $a', b' \in J_1$ with $a_i < a'_i < y_i < b'_i < b_i$, we have $y \in U_{(a',b')} \subseteq U_{(a,b)}$ and, thus, $x \in I_{U_{(a',b')}}$. In conclusion, $(a', b') \in D_x$ for all (a', b') with $a_i < a'_i < y_i$ and $y_i < b'_i < b_i$. Hence, $D_x \cap (J_1 \times J_1)$ contains an open box and $\dim(D_x \cap (J_1 \times J_1)) \geq 2(n-1)$.

Suppose F_1 has interior. Then, there is some interval $Z_1 \subseteq I_1$ contained in this interior. However, recall, we have some cell decomposition of $I_1 \times J_1 \times J_1$ partitioning D and that the fiber of this cell decomposition is again a cell decomposition of $J_1 \times J_1$ partitioning D_x. For every $x \in Z_1$, $\dim(D_x \cap (J_1 \times J_1)) \geq 2(n-1)$. Therefore, there must be some cell C in the cell decomposition with $\dim(C_x) \geq 2(n-1)$. As there are only finitely many cells, there has to be some cell with $\dim(C_x) \geq 2(n-1)$ for infinitely many $x \in Z_1$. Thus, $\pi_1(C)$ is infinite. As it is a cell, it must be some interval and $\dim(\pi_1(C)) = 1$. Recall that for a cell C with $x \in \pi_1(C)$, $\dim(C) = \dim((C)_x) + \dim(\pi_1(C))$. Therefore, $\dim(C) \geq 2(n-1) + 1$.

By the first property of this proposition, $2(n-1) \geq \dim(D \cap (I_1 \times J_1 \times J_1)) \geq \dim(D \cap (F_1 \times J_1 \times J_1))$. This is a contradiction.

Step 2 of the Proof of the Induction Step. For any sufficiently small box $B \ni a$ with $\mathrm{fr}(X) \cap B \neq \emptyset$*,* $\dim(\mathrm{fr}(X) \cap B) < \dim(X \cap B)$*.*

Again, choosing B sufficiently small, we can assume, that there is a definable cell decomposition of B partitioning $X \cap B$ and $(\mathrm{fr}(X)) \cap B$.

Set $H_i = M^{i-1} \times F_i \times M^{n-i}$ and $H = \bigcap_{i=1}^{n} H_i = F_1 \times \cdots \times F_n$. By the forth statement of this proposition, we have

$$\dim(H) = \dim(F_1) + \cdots + \dim(F_n) \leq 0 + \cdots + 0 = 0.$$

Moreover, $\mathrm{fr}(X) \subseteq H \cup \bigcup_{i=1}^{n}((\mathrm{fr}(X)) \setminus H_i)$. By the third and fourth statement of this proposition,

$$\dim((\mathrm{fr}(X)) \cap B) \leq \dim(H) + \max_{1 \leq i \leq n} \dim(((\mathrm{fr}(X)) \setminus H_i) \cap B).$$

Thus, it is sufficient to show

$$\dim(((\mathrm{fr}(X)) \setminus H_i) \cap B) < \dim(X \cap B)$$

for $(\mathrm{fr}(X)) \cap B \neq \emptyset$ and $i \in \{1, \ldots, n\}$ to prove Step 2. By the second statement of this proposition, we only need to consider the case $i = 1$.

Note, $B = I_1 \times J_1$ and

$$\big((\mathrm{fr}(X)) \setminus H_1\big) \cup B = \bigcup_{x \in I_1 \setminus F_1} \Big(\{x\} \times \big((\mathrm{fr}(X_x)) \cap J_1\big)\Big) := A$$

by the definition of H_1.

By the induction hypothesis,

$$\dim((\mathrm{fr}(X_x)) \cap J_1) < \dim(X_x \cap J_1) \text{ if } (\mathrm{fr}(X_x)) \cap J_1 \neq \emptyset.$$

In the following, we show that this implies $\dim(A) < \dim(X \cap B)$.

Let C be a cell contained in A with $\dim(C) = \dim(A)$.

If $\dim(\pi_1(C)) \leq 0$, then for all $y \in \pi_1(C)$, we have $\dim(C) \leq \dim(A_y) = \dim((\mathrm{fr}(X_y)) \cap J_1)$. Moreover, $\dim((\mathrm{fr}(X_y)) \cap J_1) < \dim(X_y \cap J_1) \leq \dim(X \cap B)$ since $y \in \pi_1(C)$ implies $(\mathrm{fr}(X_y)) \cap J_1) \neq \emptyset$. In conclusion, $\dim(A) = \dim(C) < \dim(X \cap B)$.

For $\dim(\pi_1(C)) = 1$: Note that for every $y \in \pi_1(C)$, we have $\dim(C_y) < \dim(X_y)$. Let $\mathcal{C} = \{C^i\}_{i=1}^{N}$ be the cell decomposition of B partitioning $X \cap B$ and $(\mathrm{fr}(X)) \cap B$. Then, the fibers of the cells are a cell decomposition of the fiber of B partitioning the fibers of $X \cap B$. Thus, for every $y \in \pi_1(C)$, there must be some cell $C^i \subseteq X \cap B$ with $\dim((C^i)_y) = \dim(X_y \cap J_1) > \dim(C_y)$. Since there are only finitely many cells and $\pi_1(C)$ has interior, there must be one cell C^j fulfilling this inequality for infinitely many $y \in \pi_1(C)$. Thus, $\pi_1(C^j)$ also has interior and for some $y \in \pi_1(C) \cap \pi_1(C^j)$, we have $\dim(C^j) = \dim(\pi_1(C^j)) + \dim((C^j)_y) > 1 + \dim((C)_y) = \dim(\pi_1(C)) + \dim((C)_y) = \dim(C)$. As C^j is a cell contained in $X \cap B$, we can conclude $\dim(A) = \dim(C) < \dim(X \cap B)$.

Thus, indeed $\dim(((\mathrm{fr}(X)) \setminus H_i) \cap B) = \dim(A) < \dim(X \cap B)$ which finishes the proof of Step 2.

Therefore, the induction step holds and the statement is true for all definable non-empty $X \subseteq M^n$.

□

For the naive dimension the following statements are generally true for any dense linear order without endpoints $\mathcal{M} = (M, <, \dots)$. In particular, this implies that they also equivalently hold for the dimension dim and the local naive dimension $\dim_{LN}$ (cf. Proposition 6.36) for definable subsets of $*$-locally weakly o-minimal structures enjoying the univariate $*$-continuity property.

Proposition 6.41 ([15, Proposition 4.2])**.** *Let $\mathcal{M} = (M, <, \dots)$ be an expansion of a dense linear order without endpoints. Let $X \subseteq Y \subseteq M^n, Z \subseteq M^m$ be definable non-empty sets. Let $\sigma : M^n \to M^n, (x_1, \dots, x_n) \mapsto (x_{\overline{\sigma}(1)}, \dots, x_{\overline{\sigma}(n)})$ be a coordinate permutation. The following assertions hold true.*

1. $\dim_N(X) \leq \dim_N(Y)$,
2. $\dim_N(X) = \dim_N(\sigma(X))$,
3. $\dim_N(X \times Z) = \dim_N(X) + \dim_N(Z)$.

Proof. The first two statements are immediate from the definition of $\dim_N$.

Let $\{i_1, \dots, i_l\} \subseteq \{1, \dots, n\}, \{j_1, \dots, j_k\} \subseteq \{1, \dots, m\}$ be some sets of coordinates and $\pi_{i_1,\dots,i_l,j_1,\dots,j_k} : M^{n+m} \to M^{l+k}$, $\pi_{i_1,\dots,i_l} : M^n \to M^l$ and $\pi_{j_1,\dots,j_k} : M^m \to M^k$ be the corresponding coordinate projections on these coordinates.

The set $\pi_{i_1,\dots,i_l,j_1,\dots,j_k}(X \times Z)$ has interior if and only if both, $\pi_{i_1,\dots,i_l}(X)$ and $\pi_{j_1,\dots,j_k}(Z)$, have interior. By the definition of $\dim_N$, the statement follows.

□

Moreover, if we consider the naive dimension and equivalently the local naive dimension for $*$-locally weakly o-minimal structures enjoying the univariate $*$-continuity property (cf. Proposition 6.36), we can additionally show the following nice properties:

Proposition 6.42 ([15, Corollary 4.23])**.** *Let $\mathcal{M} = (M, <, \dots)$ be a $*$-locally weakly o-minimal structure enjoying the univariate $*$-continuity property. Let X and Y be nonempty definable subsets of M^n, $Z \subseteq M^m$ and let $f : X \to M^m$ be a definable map. The following assertions hold true:*

1. $\dim_N(X \cup Y) = \max\{\dim_N(X), \dim_N(Y)\}$.

2. *The notation* $\mathrm{Disc}(f)$ *denotes the set of points at which the map* f *is discontinuous. The inequality* $\dim_N(\mathrm{Disc}(f)) < \dim_N(X)$ *holds true.*

3. $\dim_N(\mathrm{fr}(X)) < \dim_N(X)$.

4. *Let,* $d \in \mathbb{N}$ *and* $\phi : X \to Z$ *be a definable surjective map with* $\dim_N(\phi^{-1}(y)) = d$ *for all* $y \in Z$*. Then,* $\dim_N(X) = \dim_N(Z) + \dim_N(\phi^{-1}(y))$ *for all* $y \in Z$*.*

5. $\dim_N(f(X)) \leq \dim_N(X)$.

Proof.

1. $\dim_N(X \cup Y) \geq \max\{\dim_N(X), \dim_N(Y)\}$ follows from the first statement of Proposition 6.41. For the other direction, let π be a projection witnessing $\dim_N(X \cup Y) = d$. Thus, $\pi(X \cup Y) = \pi(X) \cup \pi(Y)$ has interior. By Lemma 6.29, either $\pi(X)$ or $\pi(Y)$ has interior.

2. By Theorem 6.31, there is a decomposition of X into normal submanifolds partitioning $D(f)$. One can show that by the finiteness of the partition, for each submanifold C of the highest dimension, there is some $x \in C$ and some box $B \ni x$ such that $X \cap B = C \cap B$. Thus, f is continuous at $\pi_{\leq n}(x)$. This implies that C cannot be contained in $\mathrm{Disc}(f)$. Thus, $\dim_N(X) = \dim_N(C) > \dim_N(\mathrm{Disc}(f))$.

3. This is an immediate consequence of the previous statement: Let $c \neq d \in M$. Define
$$f : \mathrm{cl}(X) \to M, x \mapsto \begin{cases} c & \text{if } x \in X \\ d & \text{else.} \end{cases}$$
Then, $\mathrm{fr}(X) \subseteq \mathrm{Disc}(f)$.

4. We choose not to present the extensive proof of this property here. For a proof confer [15, Corollary 4.23].

5. This is a consequence of the first and the fourth statement. Note, we define an injective f' with the same image and the domain being a subset of X with a similar construction as in the proof of Theorem 6.31.

□

Remark 6.43. Originally, the properties of Proposition 6.42 are actually proven in a slightly more general setting exactly stating which technical properties are needed in order for the statements to hold. The interested reader may refer to the reference [15, Corollary 4.23].

Moreover, by [15, Corollary 4.24], the fourth statement of Proposition 6.42 holds for a $*$-locally weakly o-minimal structure just in the here presented case of a structure with univariate $*$-continuity property. For structures without the univariate $*$-continuity property, the equality is false.

Lastly, we mention two more definitions for dimension which coincide with the local naive dimension in the case of $*$-locally weakly o-minimal structures with the univariate $*$-continuity property.

Definition 6.44 (Pillay's Dimension Rank, [15, Definition 4.33])**.** Let $\mathcal{M}$ be a $*$-locally weakly o-minimal structure with the univariate $*$-continuity property and $X \subseteq M^n$ a definable set. Define $D(X)$ by:

- $D(\emptyset) = -\infty$ and for $X \neq \emptyset$, $D(X) \geq 0$.
- If $D(X) \geq \alpha$ for all ordinals $\delta > \alpha$, then $D(X) \geq \delta$.
- If there exists a definable closed $Y \subseteq X$ such that Y has empty interior in X[42] and $D(Y) \geq \alpha$, then $D(X) \geq \alpha + 1$.

Set $D(X) = \alpha$, if $D(X) \geq \alpha$ but not $D(X) \geq \alpha + 1$. Set $D(X) = \infty$, if $D(X) \geq \alpha$ for all α.

Proposition 6.45 ([15, Proposition 4.34])**.** *Let $\mathcal{M}$ be a $*$-locally weakly o-minimal structure with the univariate $*$-continuity property and $X \subseteq M^n$ a definable set. Then, $D(X) = \dim_N(X)$.*

Proof. For the empty set, the statement is immediate. We prove the statement by an induction over $d = \dim_N(X)$. For $d = 0$, X is discrete and closed and every non-empty definable subset of X has non-empty interior in X.

Let $d > 0$.

Let Y be a definable subset of X with $\dim_N(Y) = \dim_N(X)$. Recall that $\dim_N = \dim_{LN}$ and thus there is some $x \in M^n$ such that for every box $B \ni y$, we have $\dim_N(Y \cap B) = \dim_N(X \cap B)$. Let π be a projection witnessing the dimension of $Y \cap B$. Then, we can shrink B such that $\pi(Y \cap B) = \pi(B)$. With a similar construction as in the proof of Theorem 6.31 one can define functions h_1, h_2 with which we can find some box $B' \subseteq B$ such that $\pi(Y \cap B') = \pi(B') = \pi(B' \cap X)$. Thus, Y has interior in X. Therefore, every definable closed $Y \subseteq X$ has dimension $\dim_N(Y) < \dim_N(X)$ and by the induction hypothesis $D(Y) = \dim_N(Y)$, implying $D(X) \leq \dim_N(X)$.

By Theorem 6.31, there is a decomposition of X into normal submanifolds. Let X_1 be a π-normal submanifold of dimension d. Define $Z_1 = \overline{X \setminus X_1} \cap X_1$. By Proposition 6.42, $\dim_N(Z_1) \leq \dim_N(\mathrm{fr}(X)) < \dim_N(X)$. Since the concatenation of two projections is again a projection, $\dim_N(\pi(Z_1)) < d$. Thus, by Lemma 6.29, $\dim_N(\pi(X_1) \setminus \pi(Z_1)) = d$ and $\pi(X_1) \setminus \pi(Z_1)$ has interior. Pick a box B' in this interior and a box B in $\pi^{-1}(B')$. By the definition of Z_1, we have $X \cap B = X_1 \cap B$.

Let $y \in X_1 \cap B$. Since X_1 is a normal submanifold, there is a box $B_2 \subseteq B$ with $y \in B_2$ such that X_1 is the graph of a continuous function defined on $\pi(B_2)$. Let $C' \subseteq \pi(B_2)$ be a non-empty closed box with $\pi(y) \in C$. Let $C = \{x \in C' : \pi_1(x) = \pi_1(\pi(y))\}$.

[42] i. e. with regards to the subspace topology

Then, C is closed, definable and $\dim_N(C) = d - 1$. By construction, C has empty interior in X. Thus, $D(X) \geq \dim_N(X)$.

□

Definition 6.46 (Discrete Closure, Rank, [15, Definition 4.28, Definition 4.30]). Let $\mathcal{M} = (M, \dots)$ be a structure and let $A \subseteq M$. The *discrete closure* of A, $\mathrm{discl}_{\mathcal{M}}(A)$, is the set of all $x \in M$ which are contained in a discrete and closed set that is definable over A (without other parameters).

The *rank of* $S \subseteq M^n$ *over* $A \subseteq M$ is defined as

$$\mathrm{rk}^{\mathrm{discl}}_{\mathcal{M}}(S/A) = \max\{\mathrm{rk}^{\mathrm{discl}}(\{a_1, \dots, a_n\}/A) : (a_1, \dots, a_n) \in S\}.$$

The rank of S over A with regard to some theory T, ${\mathrm{rk}_T}^{\mathrm{discl}}(S/A) = \mathrm{rk}^{\mathrm{discl}}_{\mathbb{M}}(S/A)$ is defined to be the rank in some monster model $\mathbb{M}$.

Proposition 6.47 ([15, Theorem 3.2, Theorem 3.5]). *Let $\mathcal{M} = (M, \dots)$ be a $*$-locally weakly o-minimal structure with the univariate $*$-continuity property. Let $T = Th(\mathcal{M})$ be the theory of the model. Let $A \subseteq M$ and X be a subset of M^n definable over A (without other parameters).*

The pair (M, discl) is a pregeometry.

Moreover, $\dim_N(X) = \mathrm{rk}^{discl}_T(X/A)$.

We choose not to present the proof here, but the reader can find a very detailed proof in the Appendix of [15].

$*$-local weak o-minimality & $*$-continuity	$\dim(X) = \dim_N(X) = \dim_{LN}(X) = D(X) = \mathrm{rk}^{discl}_T(X/A)$ and all of these have the properties stated in Propositions 6.41 and 6.42.
uniform local o-minimality of the 2. kind & definable completeness; strong local o-minimality	$\dim(X) = \dim_C(X) = \dim_{LN}(X)$ and all of these have the properties stated in Proposition 6.40.

Table 4: Summary of the presented results in this section. Confer with Figures 3 and 4 to check for which other notions this implies the same results trivially since they are implied by one of the notions mentioned here.

7 Locally O-Minimal Groups and Fields

In this section, we are not considering arbitrary dense linear orders but only expansions of dense, linear ordered groups and fields. These restricted settings lead to additional tameness results. Expansions of groups and fields have been the subject of intensive study not only in the locally o-minimal setting but also with substantial results for less restrictive notions like d-minimality. The review of tameness beyond o-minimality for groups or for fields would by itself be an interesting topic for an extensive study but is exceeding the scope of this thesis.

Therefore, we shortly review some of the specific results for locally o-minimal groups and fields and put them into the context of this thesis, but leave out proofs and details. This section is only intended as an outlook to familiar results of the ones presented in previous sections and the interested reader is encouraged to consider the given references for greater insight into this topic.

7.1 Groups

In this subsection, we focus solely on locally o-minimal groups and tameness results specific to these.

First, consider locally o-minimal archimedian structures.

Definition 7.1 (Archimedean, [40])**.** Let $\mathcal{M} = (M, <, +, 0, \dots)$ be an expansion of an ordered group. $\mathcal{M}$ is called *archimedean* if for every $a, b \in M$ with $0 < a < b$ there is some positive integer n such that $na > b$. Here, na denotes $\underbrace{a + \cdots + a}_{n \text{ times}}$.

Proposition 7.2 ([40, Theorem 6.3])**.** *Let $\mathcal{M}$ be an archimedean locally o-minimal ordered group. Then, $\mathcal{M}$ is abelian and divisible. In particular, $\mathcal{M}$ is o-minimal.*

Thus, in the case of archimedean groups local o-minimality coincides with o-minimality. As we want to investigate tameness beyond o-minimality, we are only interested in results for non-archimedean locally o-minimal groups. For these, there are also some stronger versions of the tameness results presented in the previous sections.

Regarding elementary equivalence, recall that almost o-minimality is not preserved under elementary equivalence. Moreover, by [10, Corollary 4.30]), a structure elementarily equivalent to an almost o-minimal expansion of an ordered group is a uniformly locally o-minimal structure of the first kind.

Regarding geometric tameness, first, we revisit local monotonicity and continuity. While there are no notable better results for the definitions of local monotonicity and continuity that we introduced so far, note that the definability of addition naturally induces a definable metric on the structure. Thus, one can consider uniform continuity:

Proposition 7.3 ([12, Corollary 2.8])**.** *Let $\mathcal{M} = (M, <, +, 0, \dots)$ be a definably complete expansion of an ordered group. Let C be a definable, closed and bounded set. Then, every definable continuous function $f : C \to M$ is uniformly continuous.*

In particular, by Corollary 6.7, for every definably complete expansion of an ordered group $\mathcal{M} = (M, <, +, 0, \dots)$, the proposition implies that every definable function $f : I \to M$ is locally uniformly continuous everywhere except for a discrete set.

Secondly, we consider cell decomposition. For almost o-minimal expansions of ordered groups there is a slightly stronger version of Corollary 6.21 allowing for a uniform choice of cell decompositions with the same types for arbitrary parameters from an unbounded set of parameters.

Theorem 7.4 (Uniform local definable cell decomposition, [10, Theorem 1.7])**.** *Let $\mathcal{M} = (M, <, 0, +, \dots)$ be an almost o-minimal expansion of an ordered group. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a finite family of definable subsets of M^{m+n}, let $R \in M$ be an arbitrary element and $B = \,]-R, R[^n$.*

Then, there exists a finite partition into definable sets $M^m \times B = \bigcup_{i=1}^k X_i$ such that $B = \bigcup_{i=1}^k (X_i)_b$ is a definable cell decomposition of B for every $b \in M^m$ and either $X_i \cap A_\lambda = \emptyset$ or $X_i \subseteq A_\lambda$ for every $1 \leq i \leq k$ and $\lambda \in \Lambda$.

Furthermore, the type of the cell $(X_i)_b$ is independent of the choice of b if $(X_i)_b \neq \emptyset$.

Third, recall that we also discussed decompositions into normal and quasi-special submanifolds. For definably complete expansions of ordered groups, there is even a decomposition into special submanifolds, which is a stronger version of the decomposition into quasi-special submanifolds also requiring that the boxes V_y can be chosen pairwise disjoint.

Definition 7.5 (Special Submanifold, [11, Definition 3.1])**.** Let $\mathcal{M}$ be a dense linear order without endpoints and let $\pi = \pi_{\leq d} : M^n \to M^d$. Let $Y \subseteq M^n$ be a definable subset. Y is called a *$\pi \circ \tau$-special submanifold* (or simply *special submanifold*) if there is a coordinate permutation $\tau : M^n \to M^n, (x_1, \dots, x_n) \mapsto (x_{\tau(1)}, \dots, x_{\tau(n)})$ such that the definable set $X = \tau(Y)$ has the following properties:

For every $x \in M^d$, there exist an open box U in M^d containing the point x and a family $\{V_y\}_{y\in X_x}$ of mutually disjoint open boxes in M^n such that

1. $\pi(V_y) = U$ for all $y \in X_x$,
2. $(X \cap \pi^{-1}(U)) \subseteq (\bigcup_{y\in X_x} V_y)$,
3. $V_y \cap X$ is the graph of a continuous map defined on U for each $y \in X_x$.

Definition 7.6 (Decomposition into Special Submanifolds, [11, Definition 3.18])**.** Let $\mathcal{M}$ be a dense linear order without endpoints and let $\{X_i\}_{i=1}^m$ be a finite family of definable subsets of M^n. A *decomposition of M^n into special submanifolds partitioning* $\{X_i\}_{i=1}^m$ is a finite family of pairwise disjoint special submanifolds $\{C_i\}_{i=1}^N$ such that $\bigcup_{i=1}^N C_i = M^n$ and for every i, j: Either C_i has an empty intersection with X_j or it is contained in X_j.

A decomposition $\{C_i\}_{i=1}^N$ of M^n into special submanifolds satisfies the *frontier condition* if the closure of any special manifold $\mathrm{cl}(C_i)$ is the union of a subfamily of the decomposition.

Theorem 7.7 ([11, Theorem 3.19]). *Let $\mathcal{M} = (M, <, 0, +, \dots)$ be a definably complete locally o-minimal expansion of an ordered group. Let $\{X_i\}_{i=1}^m$ be a finite family of definable subsets of M^n. There exists a decomposition $\{C_i\}_{i=1}^N$ of M^n into special submanifolds partitioning $\{X_i\}_{i=1}^m$ and satisfying the frontier condition. Furthermore, the number of special submanifolds is bounded by a constant only depending on m and n.*

Remark 7.8. On a side note, there is an even stronger version of this theorem: [11, Theorem 3.22] states that this decomposition can be chosen such that the special submanifolds have tubular neighborhoods as defined in [11, Definition 3.21].

Moreover, the reader interested in another definition of "good-shaped" sets, can find a different approach in [10, Definition 4.19, Theorem 4.22], where so called multi-cells are discussed and a finite decomposition for almost o-minimal expansions of ordered groups are shown. We omit the details here.

Finally, there are also some other results not relating to our results from the previous chapters. In [12] the reader can find a comprehensive discussion of the tameness of the topology of definably complete locally o-minimal expansions of ordered groups and definable topological groups. The reader can find several interesting results there, which are beyond the scope of this thesis.

7.2 Groups with Bounded Multiplication

Definition 7.9 (Definable Bounded Multiplication, [21, Definition 1.3]). An expansion $\mathcal{M} = (M, <, 0, +, \dots)$ of a linear ordered group has *definable bounded multiplication compatible to* $+$ if there exist an element $1 \in M$ and a map $\cdot : M \times M \to M$ such that

1. the tuple $(M, <, 0, 1, +, \cdot)$ is an ordered field,
2. for any bounded open interval I, the restriction $\cdot|_{I \times I}$ is definable in M.

We simply say that M has definable bounded multiplication if the addition in consideration is clear from the context.

In Proposition 3.11, we proved that every uniformly locally o-minimal expansion of the second kind of an ordered field is o-minimal. For the more general setting of expansions of ordered abelian groups having definable bounded multiplication, there is the following result:

Proposition 7.10 ([21, Proposition 3.2]). *A uniformly locally o-minimal expansion of the second kind of an ordered abelian group having definable bounded multiplication is almost o-minimal. In particular, it is definably complete.*

Regarding definable functions in expansions of dense linear ordered groups having definable bounded multiplication, there are additional nice properties. In [21, Theorem 4.10, Proposition 4.5 and Proposition 4.6] the interested reader can find versions of the Łojasiewicz's Inequality and Michael's Selection Theorem for definably complete locally o-minimal expansions of an ordered group having definable bounded multiplication. Moreover, the following result is also proven in the same publication:

Proposition 7.11 ([21, Weak Tietze Extension Theorem, Theorem 4.7])**.** *Let $\mathcal{M}$ be a definably complete locally o-minimal expansion of an ordered group having definable bounded multiplication. Let S be a definable, closed and bounded subset of M^n. Then, every definable continuous function $f : S \to M$ has a definable continuous extension $\overline{f} : M^n \to M$.*

Moreover, there are again also more results like e. g. [18, Theorem 1.4] which considers definable topologies for definably complete uniformly locally o-minimal expansion of ordered abelian groups.

7.3 Fields

Throughout this subsection, let $\mathcal{K} = (K, <, +, \cdot, 0, 1, \dots)$ be an expansion of a locally o-minimal field.

First, recall that by Proposition 3.11, any uniformly locally o-minimal ordered field $\mathcal{K}$ is o-minimal. Thus, it is only interesting to consider locally o-minimal structures which are not uniformly locally o-minimal of the second kind.

Secondly, every definably complete field $\mathcal{K}$ is type complete by the following proposition:

Proposition 7.12 ([19, Proposition 2.3.(8)])**.** *Let $\mathcal{K}$ be a definably complete locally o-minimal expansion of an ordered field. For any definable subset X of K, there exists $r_1, r_2 \in K$ such that either $]r_1, \infty[\subseteq X$ or $X \cap]r_1, \infty[= \emptyset$ and either $]-\infty, r_2[\subseteq X$ or $X \cap]\infty, r_2[= \emptyset$.*

Thus, a reader interested in definably complete locally o-minimal expansions of an ordered field should also consider results for type complete structures like in [37].

By [9], one of the following two cases has to hold for definable functions in definably complete locally o-minimal ordered fields:

Proposition 7.13 ([9, Theorem 5.18])**.** *Let $\mathcal{K}$ be a definably complete locally o-minimal ordered field. Then, $\mathcal{K}$ is either power-bounded, or it is exponential. Being exponential means that $\mathcal{K}$ defines an exponential function. K is called power bounded, if for every ultimately non-zero definable function $f : K \to K$ there exist $c \in K \setminus \{0\}$ and r in the field of exponents of K, such that $f \sim cx^r$.*

Note that in fields, we can define differentiability. Thus, it is reasonable to consider differentiability as an additional tameness property. For definably complete locally o-minimal expansions of ordered fields, there is a decomposition into special submanifolds which are additionally differentiable.

Definition 7.14 ($\mathcal{C}^r$-Submanifolds, [19, Definition 2.10])**.** Let $Y \subseteq M^n$ be a definable subset. Y is called a *$\pi \circ \tau$-special $\mathcal{C}^r$-submanifold* (or simply *special $\mathcal{C}^r$-submanifold*) if there is a coordinate permutation $\tau : M^n \to M^n, (x_1, \ldots, x_n) \mapsto (x_{\tau(1)}, \ldots, x_{\tau(n)})$ such that the definable set $X = \tau(Y)$ has the following properties:

For every $x \in M^d$, there exist an open box U in M^d containing the point x and a family $\{V_y\}_{y \in X_x}$ of mutually disjoint open boxes in M^n such that

- $\pi(V_y) = U$ for all $y \in X_x$,
- $(X \cap \pi^{-1}(U)) \subseteq (\bigcup_{y \in X_x} V_y)$,
- $V_y \cap X$ is empty or the graph of a continuous $\mathcal{C}^r$-map defined on U for each $y \in X_x$.

A decomposition of K^n into special $\mathcal{C}^r$-submanifolds partitioning $\{X_i\}_{i=1}^m$ is a finite family of pairwise disjoint $\mathcal{C}^r$-special submanifolds $\{C_i\}_{i=1}^N$ such that $\bigcup_{i=1}^N C_i = M^n$ and for every i, j: either C_i has an empty intersection with X_j or it is contained in X_j.

Proposition 7.15 ([19, Proposition 2.11])**.** *Let $\mathcal{K}$ be a definably complete locally o-minimal expansion of an ordered field. Let r be a nonnegative integer. Let $\{X_i\}_{i=1}^m$ be a finite family of definable subsets of K^n.*

Then, there exists a decomposition of K^n into special $\mathcal{C}^r$-submanifolds partitioning $\{X_i\}_{i=1}^m$. In addition, the number of special $\mathcal{C}^r$-submanifolds is bounded by a function of m and n.

Remark 7.16. Again, on a side note, there are also versions with tubular neighborhoods in [19, Theorem 2.14, Theorem 3.9].

Moreover, in [19, Theorem 3.10, Theorem 4.5, Theorem 4.12], the reader can also find some explicit tameness results shown for the notion of $\mathcal{C}^r$-submanifolds.

Another topic discussed in [9, Theorem 5.25] and [20] are definable groups in definably complete locally o-minimal ordered fields. These also have several tame properties, but we will not go into detail here.

On a last note, there is a version of the well-known Definable Positivstellensatz for definably complete locally o-minimal expansions of ordered fields:

Proposition 7.17 (Definable Positivstellensatz, [19, Theorem 3.14])**.** *Let $\mathcal{K}$ be a definably complete locally o-minimal expansion of an ordered field. Let $f_1, \ldots, f_k$ be definable $\mathcal{C}^r$ functions on K^n such that the set $S = \{x \in K^n : \wedge_{i=1}^k f_i(x) \geq 0\}$ is not empty. Let g be a definable $\mathcal{C}^r$ function on K^n. The following assertions hold true:*

1. *If $g \geq 0$ on S, there exist definable $\mathcal{C}^r$ functions $p, v_0, \ldots, v_k$ on K^n such that $p^{-1}(0) \subseteq g^{-1}(0)$ and $p^2 g = v_0^2 + \sum_{i=1}^k v_i^2 f_i$.*
2. *If $g > 0$ on S, there exist definable $\mathcal{C}^r$ functions $v_0, \ldots, v_k$ on K^n such that $g = v_0^2 + \sum_{i=1}^k v_i^2 f_i$.*

8 Conclusion and Outlook

While local o-minimality alone seems to be too weak a condition to imply any significant tameness results, it is an overarching classification containing several slightly stronger notions like $*$-local weak o-minimality, weak uniform local o-minimality of the second kind and locally o-minimal structures with the univariate $*$-continuity property. For these notions, the desired local versions of the geometric tameness properties from o-minimality do indeed hold.

One inherent downside of the study of local o-minimal structures instead of o-minimal ones, is that there are only local tameness results. While the local monotonicity and cell decomposition results have some nice applications for dimension theory, they are missing the global finiteness of o-minimality which is a powerful tool since it allows for inductive arguments. However, the results presented here definitely justify to consider many of the notions discussed as geometrically tame.

Due to historic developments in the field, various different notions of local o-minimality and familiar notions have been introduced. Most notable, the ones for which the most general versions of important tameness have been shown, being almost o-minimality, strong local monotonicity, uniform local weak o-minimality of the second kind and $*$-local weak o-minimality.[43] Moreover, while there are some examples proving that some of the notions are not equivalent, there are still many cases without any example. For example, it would be interesting to know if there is an example of a $*$-locally weak o-minimal structure which is not uniformly weakly o-minimal of the second kind.

In conclusion, there are various interesting tameness results for local o-minimality, in more restrictive settings like expansions of groups and fields and even for arbitrary dense linear orders. Thus, this is an important notion for tameness beyond o-minimality.

[43]While DC, TC, weak o-minimality and the $*$-continuity property are all also fundamental notions with important implications, these are rather tools and related study areas and not so central for local o-minimality, thus, not included here.

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