

# Six operations on dg enhancements of derived categories of sheaves and applications

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**Abstract:** We lift Grothendieck’s six functor formalism for derived categories of sheaves on ringed spaces over a field to differential graded enhancements. Two applications concerning homological smoothness of derived categories of schemes are given.

## 1 Motivation

Grothendieck’s six functor formalism in the topological setting concerns the six functors  $\otimes^{\mathbf{L}}$ ,  $\mathbf{R}\mathcal{H}om$ ,  $\mathbf{L}\alpha^*$ ,  $\mathbf{R}\alpha_*$ ,  $\mathbf{R}\alpha_!$ ,  $\alpha^!$  between derived categories of sheaves on ringed spaces and their relations. It takes place in the 2-multicategory  $\mathrm{TRCAT}_{\mathbf{k}}$  of triangulated  $\mathbf{k}$ -categories (the prefix “multi” takes care of functors with several inputs like  $\otimes^{\mathbf{L}}$  and  $\mathbf{R}\mathcal{H}om$ ). The relevant objects are the derived categories  $\mathbf{D}(X)$  of sheaves of  $\mathcal{O}$ -modules on ringed spaces  $(X, \mathcal{O})$ . The six functors  $\otimes^{\mathbf{L}}$ ,  $\mathbf{R}\mathcal{H}om$ ,  $\mathbf{L}\alpha^*$ ,  $\mathbf{R}\alpha_*$ ,  $\mathbf{R}\alpha_!$ ,  $\alpha^!$  are 1-morphisms between these objects. The relations between these functors are encoded in two ways: first, by 2-morphisms like  $\mathrm{id} \rightarrow \mathbf{R}\alpha_*\mathbf{L}\alpha^*$  and 2-isomorphisms like  $\mathbf{R}\alpha_*\mathbf{R}\mathcal{H}om(-, \alpha^!(-)) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om(\mathbf{R}\alpha_!(-), -)$ ; second, by commutative diagrams constructed from these 2-morphisms: they encode for example that  $(\mathbf{L}\alpha^*, \mathbf{R}\alpha_*)$  is a pair of adjoint functors or that  $(\mathbf{D}(X), \otimes^{\mathbf{L}})$  is a symmetric monoidal category.

Nowadays, triangulated categories are often replaced by suitable differential graded (dg) enhancements because some useful constructions can be performed with dg categories but not with triangulated categories. Therefore it is natural to ask whether Grothendieck’s six functor formalism lifts to the level of dg enhancements.

We give an affirmative answer to this question if we fix a field  $\mathbf{k}$  and work with  $\mathbf{k}$ -ringed spaces, i. e. pairs  $(X, \mathcal{O})$  consisting of a topological space  $X$  and a sheaf  $\mathcal{O}$  of  $\mathbf{k}$ -algebras on  $X$ .

**Theorem 1** *Let  $\mathbf{k}$  be a field. Then Grothendieck’s six functor formalism for  $\mathbf{k}$ -ringed spaces lifts to dg  $\mathbf{k}$ -enhancements.*

The precise meaning of Theorem 1 is explained in the rest of this extended abstract. For proofs and more details we refer the reader to [Sch15a]. The symbol  $\mathbf{k}$  always denotes a field.

## 2 User’s guide to the enhanced six functor formalism

### 2.1 Enhancements considered

Let  $\underline{\mathbf{C}}(X)$  denote the dg  $\mathbf{k}$ -category of complexes of sheaves on the  $\mathbf{k}$ -ringed space  $(X, \mathcal{O})$ . The dg  $\mathbf{k}$ -subcategory  $\mathbb{I}(X)$  of h-injective complexes of injective sheaves is the dg  $\mathbf{k}$ -enhancement of  $\mathbf{D}(X)$  we consider. It is a pretriangulated dg  $\mathbf{k}$ -category with translation

and the obvious functor from its homotopy category  $[\underline{\mathbb{L}}(X)]$  to  $\mathbf{D}(X)$  is an equivalence of triangulated  $\mathbf{k}$ -categories.

## 2.2 Key ingredient: dg $\mathbf{k}$ -enriched resolutions

**Theorem 2** (Existence of dg  $\mathbf{k}$ -enriched h-injective resolutions) *There is a **dg  $\mathbf{k}$ -functor**  $\mathfrak{i}: \underline{\mathbf{C}}(X) \rightarrow \underline{\mathbb{L}}(X)$  together with a dg  $\mathbf{k}$ -natural transformation  $\text{id} \rightarrow \mathfrak{i}$  whose evaluation  $E \rightarrow \mathfrak{i}E$  at each object  $E$  is a quasi-isomorphism.*

The proof of Theorem 2 uses enriched model category theory. The assumption that  $\mathbf{k}$  is a field is crucial. Theorem 2 does not hold for  $\mathbf{k}$  replaced by the integers. The proof that it fails for the ringed space  $(\text{pt}, \mathbb{Z})$  boils down to the following lemma.

**Lemma 3** *There is no **additive functor***

$$\text{Mod}(\mathbb{Z}) \rightarrow \text{Mod}(\mathbb{Z})^{[1]}$$

*mapping an abelian group  $A$  to a monomorphism  $A \hookrightarrow I_A$  into an injective abelian group  $I_A$ . Here  $\text{Mod}(\mathbb{Z})^{[1]}$  denotes the arrow category of abelian groups.*

*Proof.* Consider  $T = \mathbb{Z}/2\mathbb{Z}$ . Then  $\text{id}_T$  is mapped to the identity  $\text{id}_{T \hookrightarrow I_T} = (\text{id}_T, \text{id}_{I_T})$  of  $T \hookrightarrow I_T$ . Additivity and  $0 = 2\text{id}_T$  imply  $0 = 2\text{id}_{I_T}$ . But  $I_T$  is divisible and we obtain the contradiction  $I_T = 0$ .  $\square$

Fix a dg  $\mathbf{k}$ -functor  $\mathfrak{i}$  as in Theorem 2. The induced functor  $[\mathfrak{i}]: [\underline{\mathbf{C}}(X)] \rightarrow [\underline{\mathbb{L}}(X)]$  on homotopy categories factors to an equivalence

$$(1) \quad \overline{[\mathfrak{i}]}: \mathbf{D}(X) \xrightarrow{\sim} [\underline{\mathbb{L}}(X)]$$

of triangulated  $\mathbf{k}$ -categories.

## 2.3 Lifts of the six functors

Given a morphism  $\alpha: Y \rightarrow X$  of  $\mathbf{k}$ -ringed spaces we define the dg  $\mathbf{k}$ -functor  $\underline{\alpha}_*$  as the composition

$$\underline{\alpha}_*: \underline{\mathbb{L}}(Y) \xrightarrow{\alpha_*} \underline{\mathbf{C}}(X) \xrightarrow{\mathfrak{i}} \underline{\mathbb{L}}(X).$$

This dg  $\mathbf{k}$ -functor lifts the derived functor  $\mathbf{R}\alpha_*$  in the sense that the diagram

$$(2) \quad \begin{array}{ccc} \mathbf{D}(Y) & \xrightarrow{\mathbf{R}\alpha_*} & \mathbf{D}(X) \\ \overline{[\mathfrak{i}]} \downarrow \sim & & \overline{[\mathfrak{i}]} \downarrow \sim \\ [\underline{\mathbb{L}}(Y)] & \xrightarrow{[\underline{\alpha}_*]} & [\underline{\mathbb{L}}(X)] \end{array}$$

of triangulated  $\mathbf{k}$ -categories commutes up to a canonical 2-isomorphism.

Using similar techniques, we lift the functors  $\mathbf{L}\alpha^*$ ,  $\otimes^{\mathbf{L}}$ ,  $\mathbf{R}\mathcal{H}om$ ,  $\mathbf{R}\alpha_!$ ,  $\alpha^!$  to dg  $\mathbf{k}$ -functors  $\underline{\alpha}^*$ ,  $\underline{\otimes}$ ,  $\underline{\mathcal{H}om}$ ,  $\underline{\alpha}_!$ ,  $\underline{\alpha}^!$ . The definitions of  $\underline{\alpha}^*$  and  $\underline{\otimes}$  use dg  $\mathbf{k}$ -enriched h-flat resolutions.

## 2.4 Key definition: the 2-multicategory $\text{ENH}_k$

Let  $\text{DGCAT}_k$  denote the 2-multicategory of dg  $k$ -categories. The objects of the 2-multicategory  $\text{ENH}_k$  we want to define are the dg  $k$ -categories  $\mathbb{L}(X)$  introduced above.

Given objects  $\mathbb{L}(X)$  and  $\mathbb{L}(Y)$  let  $\tau: F' \rightarrow F$  be a morphism in  $\text{DGCAT}_k(\mathbb{L}(X), \mathbb{L}(Y))$ ,

$$\begin{array}{ccc} & F & \\ & \curvearrowright & \\ \mathbb{L}(Y) & \xleftarrow{\quad} & \mathbb{L}(X) \\ & \curvearrowleft & \\ & F' & \end{array}$$

We say that  $\tau$  is an objectwise homotopy equivalence if  $\tau_I: F'(I) \rightarrow F(I)$  is a homotopy equivalence (or, equivalently, a quasi-isomorphism) for all  $I \in \mathbb{L}(X)$ . An equivalent condition is that the induced morphism  $[\tau]: [F'] \rightarrow [F]$  in  $\text{TRCAT}_k(\mathbb{L}(X), \mathbb{L}(Y))$  is an isomorphism.

Now we complete the definition of  $\text{ENH}_k$ : the morphism category  $\text{ENH}_k(\mathbb{L}(X), \mathbb{L}(Y))$  is defined as the localization of  $\text{DGCAT}_k(\mathbb{L}(X), \mathbb{L}(Y))$  with respect to the class of all objectwise homotopy equivalences. A similar definition applies if several source objects are involved.

Obviously, mapping a dg  $k$ -category to its homotopy category induces a functor

$$(3) \quad [-]: \text{ENH}_k \rightarrow \text{TRCAT}_k.$$

This functor reflects 2-isomorphisms.

It is possible to go back from  $\text{ENH}_k$  to  $\text{DGCAT}_k$ : Any 2-morphism  $\rho: F \rightarrow G$  in  $\text{ENH}_k$  can be represented by a roof  $F \xleftarrow{\tau} F' \xrightarrow{\sigma} G$  of 2-morphisms in  $\text{DGCAT}_k$  where  $\tau$  is an objectwise homotopy equivalence. Moreover,  $\rho$  is a 2-isomorphism if and only if, in any such representing roof,  $\sigma$  is an objectwise homotopy equivalence.

## 2.5 Lifts of the relations

Explicit zig-zags of dg  $k$ -natural transformations define a 2-morphism  $\text{id} \rightarrow \underline{\alpha}_* \alpha^*$  and a 2-isomorphism  $\underline{\alpha}_* \mathcal{H}om(-, \alpha^!(-)) \xrightarrow{\sim} \mathcal{H}om(\underline{\alpha}_!(-), -)$  in  $\text{ENH}_k$  whose images under (3) coincide modulo the equivalences (1) and the canonical 2-isomorphisms (cf. (2)) with the 2-morphism  $\text{id} \rightarrow \mathbf{R}\alpha_* \mathbf{L}\alpha^*$  and the 2-isomorphism  $\mathbf{R}\alpha_* \mathbf{R}\mathcal{H}om(-, \alpha^!(-)) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om(\mathbf{R}\alpha_!(-), -)$ . Similarly, we lift all standard 2-(iso)morphisms between compositions of the six functors to 2-(iso)morphisms in  $\text{ENH}_k$ . Moreover, we show that  $(\underline{\alpha}^*, \underline{\alpha}_*)$  is a pair of adjoint 1-morphisms in  $\text{ENH}_k$  and that  $(\mathbb{L}(X), \otimes)$  is a symmetric monoidal object of  $\text{ENH}_k$ . Other relations encoded by commutative diagrams lift similarly.

## 2.6 Relation to the homotopy category of dg $k$ -categories

Let  $X$  and  $Y$  be  $k$ -ringed spaces. Results by Toën in [Toë07], [Toë11, 4.1, Prop. 1] show that the obvious map defines a bijection

$$(4) \quad \text{Isom ENH}_k(\mathbb{L}(X), \mathbb{L}(Y)) \xrightarrow{\sim} \text{Hom}_{\text{Ho}(\text{dgcats}_k)}(\mathbb{L}(X), \mathbb{L}(Y))$$

where the left hand side denotes isomorphism classes of objects in  $\text{ENH}_k(\mathbb{L}(X), \mathbb{L}(Y))$ , and  $\text{Ho}(\text{dgcats}_k)$  denotes the homotopy category of dg  $k$ -categories, i. e. the localization of the category  $\text{dgcats}_k$  of dg  $k$ -categories with respect to the class of quasi-equivalences. This bijection says that the 2-multicategory  $\text{ENH}_k$  contains finer information than the homotopy category of dg  $k$ -categories.

## 2.7 Some remarks

The formalism involving the four functors  $\otimes^{\mathbf{L}}$ ,  $\mathbf{R}\mathcal{H}om$ ,  $\mathbf{L}\alpha^*$ ,  $\mathbf{R}\alpha_*$  lifts more generally to  $\mathbf{k}$ -ringed topoi.

Our techniques should apply to many other contexts, e. g. dg modules over dg  $\mathbf{k}$ -categories, quasi-coherent sheaves on  $\mathbf{k}$ -schemes,  $\mathcal{D}$ -modules on  $\mathbf{k}$ -schemes, sheaves of modules over other non-commutative structure sheaves of  $\mathbf{k}$ -algebras,  $\overline{\mathbb{Q}}_\ell$ -sheaves on pro-étale sites. Results on some of these topics will be explained in future work.

## 3 Applications

The following results will appear in [Sch15b]. They are based on previous work of Valery Lunts in [Lun10] and joint results in [LS14]. Recall that a dg  $\mathbf{k}$ -category  $\underline{\mathcal{A}}$  is homologically  $\mathbf{k}$ -smooth if the (right) dg  $\underline{\mathcal{A}} \otimes_{\mathbf{k}} \underline{\mathcal{A}}^{\text{op}}$ -module  $\underline{\mathcal{A}}$  (the “diagonal bimodule”) is a compact object of the derived category of dg  $\underline{\mathcal{A}} \otimes_{\mathbf{k}} \underline{\mathcal{A}}^{\text{op}}$ -modules.

**Theorem 4** *Let  $X$  be a separated scheme of finite type over a field  $\mathbf{k}$ . Then  $X$  is smooth over  $\mathbf{k}$  (in the sense of algebraic geometry) if and only if  $D_{\text{perf}}(X)$  is homologically  $\mathbf{k}$ -smooth (which by definition means that its dg  $\mathbf{k}$ -enhancement  $\mathbb{L}_{\text{perf}}(X)$  is homologically  $\mathbf{k}$ -smooth).*

We hope to extend this result to suitable Deligne-Mumford stacks over a field.

**Theorem 5** *Let  $X$  be a separated scheme of finite type over a perfect field  $\mathbf{k}$ . Then  $D_{\text{coh}}^{\mathbf{b}}(X)$  is homologically  $\mathbf{k}$ -smooth (which by definition means that its dg  $\mathbf{k}$ -enhancement  $\mathbb{L}_{\text{coh}}^{\mathbf{b}}(X)$  is homologically  $\mathbf{k}$ -smooth).*

## References

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