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Equivariant Sheaves on Flag Varieties

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The aim of our talk was to give an algebraic description of the Borel-equivariant derived category of sheaves on the flag variety of a connected reductive algebraic group.

Let G be a complex algebraic group acting on a complex variety X . We introduced the G -equivariant (bounded, constructible) derived category $\mathcal{D}_{G,c}^b(X)$ of sheaves of real or complex vector spaces on X (see [BL94]). It carries the perverse t-structure with heart the category of G -equivariant perverse sheaves. If G acts with finitely many orbits, there are only finitely many simple objects in this heart; we denote their direct sum by \mathcal{JC} . The extension algebra of this object is

$$\mathrm{Ext}(\mathcal{JC}) := \bigoplus_{n \in \mathbb{N}} \mathrm{Hom}(\mathcal{JC}, \mathcal{JC}[n]).$$

We view this graded algebra as a differential graded (dg) algebra with differential $d = 0$.

Let \mathcal{A} be a dg algebra. We defined the derived category $\mathcal{D}(\mathcal{A})$ of \mathcal{A} (see e.g. [Kel98]). The perfect derived category $\mathrm{Perf}(\mathcal{A})$ of \mathcal{A} is the thick subcategory of $\mathcal{D}(\mathcal{A})$ generated by \mathcal{A} (i. e. the smallest full triangulated subcategory that

contains \mathcal{A} and is closed under taking direct summands). Its objects are precisely the compact objects in $\mathcal{D}(\mathcal{A})$.

The following conjecture of Soergel and Lunts (cf. [Lun95]) relates the geometric category $\mathcal{D}_{G,c}^b(X)$ and the algebraic category $\text{Perf}(\text{Ext}(\mathcal{J}\mathcal{C}))$: If a complex reductive group G acts on a projective variety X with finitely many orbits, there is an equivalence of triangulated categories

$$\mathcal{D}_{G,c}^b(X) \cong \text{Perf}(\text{Ext}(\mathcal{J}\mathcal{C})).$$

This conjecture (or a similar statement) is known to be true for a connected Lie group acting on a point ([BL94, 12.7.2]), for a torus acting on an affine or projective normal toric variety ([Lun95]), and for a complex semisimple adjoint group acting on a smooth complete symmetric variety (in the sense of de Concini and Procesi) ([Gui05]). We recently became aware of a related result for the loop rotation equivariant derived Satake category of the affine loop Grassmannian in [BF08]. Our main result is:

Theorem 1 ([Sch08]). Let G be a complex connected reductive affine algebraic group, $B \subset G$ a Borel subgroup, and $X = G/B$ the flag variety. Then there is an equivalence of triangulated categories

$$\mathcal{D}_{B,c}^b(X) \cong \text{Perf}(\text{Ext}(\mathcal{J}\mathcal{C})).$$

We conclude with some remarks:

- Note that $\mathcal{D}_{B,c}^b(X)$ is equivalent to $\mathcal{D}_{G,c}^b(G \times_B X)$ or $\mathcal{D}_{G,c}^b(X \times X)$ by the induction equivalence. Hence our result fits into the setting of the conjecture.
- The perverse t-structure on $\mathcal{D}_{B,c}^b(X)$ corresponds to a t-structure on the perfect derived category $\text{Perf}(\text{Ext}(\mathcal{J}\mathcal{C}))$ that can be described for a more general class of dg algebras (see [Sch08a]). This yields an algebraic description of the category of B -equivariant perverse sheaves on X .
- The algebra $\text{Ext}(\mathcal{J}\mathcal{C})$ is isomorphic to the endomorphism algebra of the B -equivariant hypercohomology of $\mathcal{J}\mathcal{C}$ ([Soe01]); this hypercohomology can be described using Soergel’s bimodules or the moment graph picture ([BM01]). In particular, the category $\mathcal{D}_{B,c}^b(X)$ depends only on the corresponding combinatorial data.
- The non-equivariant analog of this theorem is also true. In fact, we prove the theorem as a limit of equivalences that are similar to the non-equivariant analog. For the proof of the non-equivariant version we need the formality of a carefully constructed dg algebra; to obtain this formality we use mixed Hodge modules and purity results on intersection cohomology complexes.

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MV-polytopes/cycles and affine buildings

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1. BASIC NOTATIONS AND DEFINITIONS

We want to give a combinatorial construction of MV-polytopes. This is done by using the LS-gallery model by Gaussent and Littelmann [1], a discrete and building-theoretic version of Littelmann’s path model. This gives a construction of MV-polytopes alternativ to the one given by Kamnitzer in [2] and [3] and independent of the type of the algebraic group. We start by fixing the basic notations.

Notation 1.1. By G we denote a complex, simply-connected, semi-simple algebraic group. We fix $B \subset G$ a Borel subgroup, $T \subset B$ a maximal torus, and denote by W its Weyl group. In addition we denote by B^- the Borel subgroup opposite to B , i.e., the Borel subgroup such that $B \cap B^- = T$, and by U^- its unipotent radical. Finally we denote by $\mathcal{O} = \mathbb{C}[[t]]$ the ring of formal power series and by $\mathcal{K} = \mathbb{C}((t))$ its field of fraction, the field of formal Laurent series.

Using these we have a number of associated objects.

Notation 1.2. Let us denote by $\mathcal{G} = G(\mathcal{K})/G(\mathcal{O})$ the affine Grassmannian, by X^\vee the coweight lattice of G , and by X^\vee_+ the dominant coweights.

Let us now look at the basic geometric set-up:

$$X^{\vee\mathcal{C}} \xrightarrow{i} \mathcal{G}^{\mathcal{C}} \longrightarrow \mathbb{P}(V) \xrightarrow{\mu} X^\vee \otimes \mathbb{R}.$$

The inclusion i is an inclusion as T -fixed points and we denote the image of a coweight λ by t^λ , $\mathbb{P}(V)$ is a projective space over a suitable representation of the affine Kac-Moody group $\hat{\mathcal{L}}(G)$ corresponding to G , and the map μ is its usual moment map.