1 Introduction

My research, broadly speaking, lies at the juncture of algebraic geometry and topology. My work is organised around studying algebraic varieties that arise as solutions to moduli problems (e.g. moduli space of curves, Hurwitz spaces, moduli space of morphisms between certain varieties, configuration spaces, smooth sections of a $\mathfrak{g}^r_d$, among other things). Quite often these spaces are reduced finite type schemes defined over $\mathbb{Z}$, and can be studied from three deeply connected points of view:

i **Topology:** Their base change to $\mathbb{C}$ gives them the structure of a complex algebraic variety, and thus, give rise to (compactly supported) singular cohomology groups equipped with a mixed Hodge structure.

ii **Geometry:** When we change the base to $\overline{\mathbb{F}}_q$, $q = p^d$ for some prime p, $d > 0$, the resulting varieties are equipped with the Frobenius action Frob_q. Associated to such setup are:

- the (compactly supported) étale cohomology group equipped with a Galois action $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$.
- the weights given by the eigenvalues of Frob_q.

iii **Arithmetic:** The set of $\mathbb{F}_q$ of the given variety, which can be realised as the set of fixed points of Frob_q.

The following lists my major results in this direction to date.

1. **Moduli space of polynomial maps on $\mathbb{A}^1$ with prescribed ramification.** In 'Cohomology of polynomial maps on $\mathbb{A}^1$ with prescribed ramification' ([Ban20]) I study the moduli spaces $Simp^m_n$ of degree $n+1$ morphisms $\mathbb{A}^1_K \to \mathbb{A}^1_K$ with "ramification length $< m$"over an algebraically closed field K, where we introduce the notion of the ramification length of a morphism to quantify the complexity in its ramification behaviour. To put it in context, orthogonal to my result is a plethora of papers concerning the (co)homology of the moduli space of polynomials with a prescribed order of zeroes (see e.g [Arn69, Arn70]), Napolitano ([Nap98]) etc. most of which use the Leray-Serre spectral sequence for inclusion, a method that fails when we are interested in recording the ramification behaviour. In fact, our results should be viewed in the spirit of the long standing open problem of understanding the topology of the Hurwitz space. The irreducibility of the Hurwitz space is a classical result proved in [Cle72], [RW06], and the cohomology of Hurwitz schemes with Galois group (satisfying certain criterion) has been addressed in the beautiful paper of Ellenberg-Venkatesh-Westerland ([EVW16]) but the topology of subvarieties of the Hurwitz scheme corresponding specific *ramification loci* is almost completely unknown. My result in [Ban20] is that of stability of the cohomology (singular, with $\mathbb{Q}$ coefficients, as well as étale with $\mathbb{Q}_\ell$ coefficients) of these Hurwitz spaces satisfying certain ramification conditions, as the degree of the cover grows.
2. **Computing cohomology via symmetric semisimplicial filtration.** In [Ban21a], inspired by Deligne's use of the simplicial theory of hypercoverings in defining mixed Hodge structures ([Del75]), we replace the indexing category Δ by the *symmetric simplicial category* ΔS and study (a class of) ΔS-hypercoverings, which we call *spaces admitting symmetric (semi)simplicial filtration*. For ΔS-hypercoverings we construct a spectral sequence, somewhat like the Čech-to-derived category spectral sequence. The advantage of working on ΔS is that all of the combinatorial complexities that come with working on Δ are bypassed, giving simpler, unified computation of (in some cases, stable) singular cohomology (with $\mathbb{Q}$ coefficients) as well as étale cohomology (with $\mathbb{Q}_\ell$ coefficients) of the moduli space of degree n maps $C \to \mathbb{P}^r$, C a smooth projective curve of genus g (this was already computed in [Seg79], [FW16] using scanning maps, a method that only works over $\mathbb{C}$), unordered configuration spaces for any Hausdorff topological space (see [Tot96], [FWW19] and the references therein), the moduli space of smooth sections of a fixed $\mathfrak{g}^r_d$ that is m-very ample for some m etc.
3. **Certain subspaces of $Sym^n(\mathbb{P}^1)$ and Occam's razor for Hodge structures.** In [VMW13] Vakil and Wood made several conjectures on the topology of symmetric powers of geometrically irreducible varieties based on their computations on motivic zeta functions. Two of those conjectures are about subspaces of $Sym^n(\mathbb{P}^1)$. In [Ban21b], I disprove one of them, and prove a stronger form of the other, thereby obtaining (counter)examples to the principle of Occam's razor for Hodge structures. The proof is almost a direct consequence of [Ban21a, Theorem 1].

2 Research program: Hurwitz spaces vs. its cousins– $\mathscr{M}_g$ and locally closed subvarieties of $Sym^n(\mathbb{P}^1)$

Let $\mathscr{H}_{n,g}$ denote the scheme that parametrizes degree n morphisms from smooth projective curves of genus g to $\mathbb{P}^1$. For a positive integer m, define the subvarieties $\mathscr{H}^m_{n,g} \subset \mathscr{H}_{n,g}$ of morphisms with "total ramification $< m$" in the following way. Let $\phi \in \mathscr{H}_{n,g}$ and $\mathrm{Branch}(\phi) \subset \mathbb{P}^1$ denote the branch locus of ϕ. A point $b \in \mathrm{Branch}(\phi)$ determines a set of not necessarily distinct integers via

$$B_b(\phi) := \{v_\phi(a) : a \in \phi^{-1}(b)\},$$

where $v_\phi(a)$ denotes the *valuation* of ϕ at a. Let $l(B_b(\phi)) := \sum_{e \in B_b(\phi)}(e-1)$ be the *ramification length of ϕ over b*. The *total ramification length of ϕ* is

$$length(\phi) := \sum_{b \in \mathrm{Branch}(\phi)} \big(l(B_b(\phi)) - 1\big).$$

Let

$$\mathscr{H}^m_{n,g} := \{\phi \in \mathscr{H}^m_{n,g} : length(\phi) < m\},$$

and in particular when $m = 1$ we get what is called the *simply branched Hurwitz space* $\mathscr{H}^s_{n,g}$, and

$$\mathscr{H}_{n,g} = \sqcup_{m \geq 1}(\mathscr{H}^m_{n,g} - \mathscr{H}^{m-1}_{n,g}).$$

So the more general problem is to understand the topology of $\mathscr{H}^m_{n,g}$ for all m. From the very outset, we see that $\mathscr{H}_{n,g}$ is related to $\mathscr{M}_g$, *the moduli space of smooth projective curves of genus* g and $Sym^{2n+2g-2}(\mathbb{P}^1)$, the scheme parametrizing (unordered) sets of $2n+2g-2$ points in $\mathbb{P}^1$, via elements in $\mathscr{H}_{n,g}$ being mapped to the source curve, and to the branch locus in $\mathbb{P}^1$ respectively. For example, when $m = 1$, we have the following:

$$\begin{tikzcd}
 & \mathscr{H}^s_{n,g} \arrow[dl, "\text{source curve}"'] \arrow[dr, "\text{branch points}"] & \\
\mathscr{M}_g & & UConf_{2n+2g-2}(\mathbb{P}^1)
\end{tikzcd}$$

where $UConf_{2n+2g-2}(\mathbb{P}^1)$ denotes the scheme parametrizing (unordered) sets of $2n+2g-2$ distinct points in $\mathbb{P}^1$. Momentarily considering all schemes as varieties over $\mathbb{C}$ equipped with the complex-analytic topology, we make an important observation: even though we have considerable knowledge of the topology of $\mathscr{M}_g$ (e.g. the famous Harer stability theorem, see [Har85]) and $UConf_{2n+2g-2}(\mathbb{P}^1)$ (see e.g. [Arn69]), the topology of $\mathscr{H}^s_{n,g}$ is completely unknown (unless $n = 2$ i.e. the case of hyperelliptic curves where, crudely speaking, it all boils down to the study of $UConf_{2n+2g-2}(\mathbb{P}^1)$).

The Hurwitz scheme has an interdisciplinary flavour: other than algebraic geometry, the study of Hurwitz schemes interacts with-

i. topology, via the categorical equivalence between smooth projective curves over $K = \mathbb{C}$ and compact Riemann surfaces; and

ii. arithmetic geometry, via the categorical equivalence between smooth projective curves over $K = \mathbb{F}_q$ and finitely generated field extensions of $\mathbb{F}_q$ of transcendence degree 1.

Project 2.1. *Fix $m > 0$. Understand the topology of $\mathscr{H}^m_{n,g}$. In particular, develop a ramification-stratification dictionary, understand $H^*(\mathscr{H}^m_{n,g}(\mathbb{C}); \mathbf{Q})$ in terms of this dictionary and the stable rational cohomology of $\mathscr{M}_g$. Infer how the topological, algebro-geometric and arithmetic properties of the Hurwitz schemes are related as n grows.*

Orthogonal to this project are results for low (fixed) values of n (namely $n \leq 5$) that compute the rational Chow ring (see [PV15] for $n = 3$, [CL21] for $n = 4, 5$), stability questions are in terms of the genus. They exploit explicit descriptions of branched covers of low degrees developed by Casnati-Ekedahl (see [CE96]), and therefore cannot be generalised to higher degrees.

In [Ban20] I consider the case of $g = 0$, and only those morphisms which are totally ramified at ∞. The reason for considering this case is that it extracts almost all key players that go into the study of $\mathscr{H}^m_{n,g}$- the next section digs deeper into this. The following summarises my results to this date.

Results. Fix an algebraically closed field K. The Hurwitz scheme $\mathscr{H}_{n+1,0}$ contains M_n as Zariski closed strata defined by the locus of morphisms $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ satisfying (a) $\phi(\infty) = \infty$, and (b) $v_\phi(\infty) = (n+1)$. Mimicking the construction of $\mathscr{H}^m_{n,g}$, define

$$Simp^m_n := \{\phi \in \mathscr{H}_{n+1,0} : v_\phi(\infty) = n+1, \phi(\infty) = \infty, \sum_{b \in \mathrm{Branch}(\phi) - \{\infty\}} \big(l(B_b(\phi)) - 1\big) < m\},$$

or equivalently $Simp_n^m = \mathcal{H}_{n+1,0}^{m+n} \cap \mathrm{M}_n$. When $m=1$, we get the locus of *simply-branched polynomials* and denote it by $Simp_n$. Let $\imath(N)$ denotes the number of partitions of a positive integer N. Let $\mathbf{c}: \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined via

$$\mathbf{c}(m) = \sum_{k\geq 1}\Big(\sum_{\substack{\sum_{i=1}^k n_i = m,\\ n_1 \leq \ldots \leq n_k}} \imath(n_1+1)\ldots\imath(n_k+1)\Big).$$

By H^i (respectively $H^i_{ét}$) we will mean singular (respectively étale) cohomology. If V is a $\mathbb{Q}_\ell$ vector space and if $m \in \mathbb{Z}$ then we let $V(m)$ denote the m^{th} *Tate twist of* V.

Theorem 1. *(Banerjee, [Ban20]) Let $m,n \geq 1$. Then the following hold.*

1. *(*Stabilisation of rational cohomology.*)For all $n \geq 3m$:*

$$H^i(Simp_n^m(\mathbb{C});\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for } i=0,\\ \mathbb{Q}^{\oplus \mathbf{c}(m)} & \text{for } i=m,\\ 0 & \text{otherwise.}\end{cases}$$

2. *(*Lack of stability for étale cohomology in positive characteristics.*) Let κ be a field satisfying $char\,\kappa > n+1$ or $char\,\kappa = 0$. Then for all $n \geq 3m$, we have the following isomorphism of $Gal(\overline{\kappa}/\kappa)$-representations:*

$$H^i_{ét}(Simp^m_{n\,/\overline{\kappa}};\mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell(0) & \text{for } i=0,\\ \mathbb{Q}_\ell(-m)^{\oplus \mathbf{c}(m)} & \text{for } i=m,\\ 0 & \text{otherwise,}\end{cases}$$

whenever ℓ is prime to $char\,\kappa$.

Paired with the Grothendieck-Lefschetz fixed point theorem we get:

Corollary 2. *Let $m,n \geq 1$ and let $q=p^d$, where p is a prime and $d \geq 1$. Then*

$$\#Simp_n^m(\mathbb{F}_q) = q^n - \mathbf{c}(m)q^{n-m}$$

for all $n < p-1$ and $m \leq \frac{n}{3}$.

The proof is based on: (i) relocating the problem to the space of ordered ramification points, which we call the *ramification cover*, (ii) constructing a poset that encodes stratification of the ramification cover determined by the ramification types, and exploit the property of *shellability* of the poset (see [Wac06] for shellability and generalities on posets), (iii) studying the geometry of each strata, and (iv) finding a suitable resolution of the proper pushforward of the structure sheaf of the open stratum whose cohomology we want to compute, in terms of locally constant sheaves supported on the closed strata, and lastly (v) working out the resulting spectral sequence. In some cases, I appeal to the Weil conjectures to transfer the topological data to the arithmetic side, in other cases, standalone arithmetic considerations are called for.

Remark 2.2. *The lack of stability for étale cohomology in positive characteristics:* In contrast to the stability of rational cohomology, on the algebraic side, the étale cohomology groups $H^i_{ét}(Simp^m_{n\,/\overline{\kappa}});\mathbb{Q}_\ell)$ do not stabilize when $char\,\kappa > 0$- a divergence from other comparable stability results (see [EVW16], and [FWW19]). For $n,m \geq 0$, if κ satisfies $char\,\kappa > n+1$ and $n \geq 3m$, I compute $H^i_{ét}(Simp^m_{n\,/\overline{\kappa}};\mathbb{Q}_\ell)$ as $Gal(\overline{\kappa}/\kappa)$-representations and we see that $i=m$ is the only non-zero survivor when $i>0$. One of the offenders are the Artin-Schreier examples. Indeed, the moduli space of polynomials $f \in \overline{\mathbb{F}}_p[x]$ of degree n that are *unramified* as self-maps of $\mathbb{A}^1_{\overline{\mathbb{F}}_p}$ is nonempty when (in fact, if and only if) n is a prime power. The fact that $H^i_{ét}(Simp^m_{n\,/\overline{\mathbb{F}}_p};\mathbb{Q}_\ell)$ does not stabilize is a manifestation of Abhyankar's philosophy: that prime-to-p situation mimics the characteristic 0 picture, else, every type of cover that can possibly occur, indeed occurs (in [DH17, Section 3], the authors rightfully refer to the latter half of the philosophy as an instance of *Murphy's law*).

Theorem 1 is a special instance of the following problems, the answers to which are entirely unknown.

Problem 2.3. *Fix $m>0$. Compute $H^i(\mathcal{H}^m_{n,0};\mathbf{Q})$. Or, for a fixed $i \geq 0$, does there exist $f(i)$, a function of i such that for $n \geq f(i)$, we have $H^i(\mathcal{H}^m_{n,0};\mathbf{Q}) \cong H^i(\mathcal{H}^m_{n+1,0};\mathbf{Q})$? If yes, can we find a formula for $f(i)$?*

Problem 2.4. *Fix $m>0$. Compute $H^i(\mathcal{H}^m_{n,g};\mathbf{Q})$. Or, for a fixed $i \geq 0$, and $g \geq 3i-2$ (this range of g is for Harer's stability to hold) does there exist $f(i)$, a function of i such that for $n \geq f(i)$, we have $H^i(\mathcal{H}^m_{n,g};\mathbf{Q}) \cong H^i(\mathcal{H}^m_{n+1,g};\mathbf{Q})$? If yes, can we find a formula for $f(i)$?*

As an aside, entirely different from these cohomology questions is a result in an upcoming result on the unirationality of a subspace of $\mathscr{H}_{4,g}$ characterised by the fact that a generic cover has Galois group D_4.

Theorem A. *(Banerjee) Let $D_4 = \langle \tau, \sigma : \tau^2 = \sigma^4 = 1, \tau\sigma = \sigma^3\tau \rangle$ be the dihedral group of 8 elements. Let $\mathscr{H}^4(a,b,c)$ denote the Hurwitz space whose generic element is a branched cover with Galois group D_4, where a branched cover has a points whose ramification corresponds to the conjugacy class of τ, b points correspond to the conjugacy class of σ^2, and c points correspond to the conjugacy class of $\tau\sigma$. Then $\mathscr{H}^4(a,b,c)$ is unirational.*

The proof is based on Casnati-Ekedahl's structure theorem paired with constraints imposed by the pushfoward of the structure sheaf of the covers as representations of D_4.

2.1 Simply-branched polynomials vs. configuration spaces

The theme of this subsection can be summed up as follows:

"*$Simp_n$ and $UConf_n(\mathbb{C})$ look deceptively similar.*"

Work on $Simp_n$ would be incomplete if its similarity (or lack of it) with $UConf_n(\mathbb{C})$ is not completely explored.

Main idea. Let's take a slight detour and recall Arnol'd work. We have the ordered configuration space $PConf_n(\mathbb{C})$ as a natural S_n-cover of $UConf_n(\mathbb{C})$. Looking at $UConf_n(\mathbb{C})$ as squarefree polynomials over $\mathbb{C}$, note that $PConf_n(\mathbb{C})$ a.k.a the 'root cover' consists of ordered tuples of roots of the squarefree polynomials in $UConf_n(\mathbb{C})$, i.e $PConf_n(\mathbb{C}) = \mathbb{C}^n - \cup_{i<j} Z_{ij}$, where Z_{ij}, the 'fat diagonals' are divisors defined by

$$Z_{ij} = \left\{(z_1, \ldots, z_n) : z_i = z_j\right\}.$$

Arnol'd showed, among other things, that $H^*(PConf_n(\mathbb{C}); \mathbb{Z})$ is generated by logarithmic one-forms along Z_{ij}.

In [Ban20] I make use of something analogous to the root cover, which I called the ramification cover. In essence, it is a natural (branched) S_n-cover to keep track of the set of ordered ramification points. We have a branched covering map $\pi : X_n \to \mathrm{M}_n$ where

$$\mathrm{M}_n = \{\text{degree } n+1 \text{ polynomials}\}/Aut(\mathbb{A}^1) \cong \mathbb{A}^n$$

and X_n is the of ordered ramification points of elements in M_n with multiplicities dictated by the *differential length*, which equals ramification index minus 1. [1]. Note that $X_n \cong \mathbb{A}^n$. A very natural question to study the S_n representations of $H^*(U_n; \mathbb{Q})$, where $U_n = \pi^{-1} Simp_n$. In fact

$$X_n - U_n = \bigcup_{1 \le i < j \le n} T_{ij} \bigcup \bigcup_{1 \le i < j \le n} D_{ij}$$

where

$$T_{ij} = \left\{(z_1, \ldots, z_n) : z_i = z_j\right\}, \text{ and}$$
$$D_{ij} = \left\{(z_1, \ldots, z_n) : \frac{\Big(I_\circ \pi(z_1, \ldots, z_n)\Big)(z_i) - \Big(I_\circ \pi(z_1, \ldots, z_n)\Big)(z_j)}{(z_i - z_j)^3} = 0\right\}$$

Here, $\pi(z_1, \ldots, z_n) := (x - z_1) \ldots (x - z_n)$ and for any polynomial f, let $I(f)$ denotes antiderivative (which, of course, is defined up to a constant, but that's fine for us since we're going modulo $Aut(\mathbb{A}^1)$ anyway.) So D_{ij} accounts for when two ramification points map to the same branch point, whereas T_{ij}, when ramification index is ≥ 3 at some point. Furthermore, it so happens that even though the divisors D_{ij} are highly singular, when one goes modulo S_{n-2} by permuting the $n-2$ coordinates of D_{ij} save i and j, the image is an affine space. In the proof of [Ban20, Theorem A] we see that $H^1(Simp_n; \mathbb{Q})$ has cohomology classes 'coming from' two sets of divisors: the D_{ij} and T_{ij}. Comparing this result with Arnol'd's gives us a very natural conjecture.

Define $R(n)$ to be the exterior $\mathbb{Q}$-algebra generated by ω_{ij} and η_{ij} for $1 \le i < j \le n$, satisfying

$$\omega_{kl}\omega_{lm} + \omega_{lm}\omega_{km} + \omega_{km}\omega_{kl} = 0,$$

and

$$\eta_{kl}\eta_{lm} + \eta_{lm}\eta_{km} + \eta_{km}\eta_{kl} = 0.$$

It is not unreasonable to expect the following.

[1] The differential length goes by other similar names, like, for example, *length, different* etc. Our definition holds only for tamely ramified morphisms, which always the case for when the base field has characteristic 0. For a general definition, see, e.g [Har77, Proposition 2.2].

Problem 2.5. *Let $n \geq 3$. Then $H^*(U_n;\mathbb{Q}) \cong R(n)$ with the isomorphism given by*

$$d\Big(\log(z_i - z_j)\Big) \mapsto \omega_{ij}$$

and

$$d\Big(\log\big(I_\circ\pi(z_1,\dots,z_n))(z_i) - I_\circ\pi(z_1,\dots,z_n)(z_j)\big)\Big) \mapsto \eta_{ij}$$

In fact, this leads us to consider the following path, largely inspired by the work of Church-Ellenberg-Farb (see [CEF15]).

Problem 2.6. *Decompose $H^*(U_n;\mathbb{C})$ into irreducible S_n representations.*

Needless to say, an answer to the above question will give us a wide range of answers. For example, one can answer questions about the topology of various incidence varieties of the following type. More precisely, let

$$Simp_{r,n} := \{(f(x),\{b_1,\dots,b_r\}) \in Simp_n \times UConf_r(\mathbb{C}) : b_1,\dots,b_r \text{ are branch points of } f(x)\}.$$

Then a natural problem to consider is the following.

Problem 2.7. *Let $r,n \geq 0$. Compute $H^*(Simp_{r,n};\mathbf{Q})$, and subsequently $\#Simp_{r,n}(\mathbb{F}_q)$.*

Note that Problem 2.6 would follow from Problem 2.5 as a corollary because the varieties $Simp_{r,n}$ are just intermediate covering spaces of $Simp_n$, obtained by quotienting U_n by subgroups of S_n.

Finally, note that $UConf_n(\mathbb{C}) = K(B_n,1)$ where B_n denotes the Artin braid group. Therefore, $Simp_n$ is a $K(\pi,1)$ as well, and $\pi_1(Simp_n)$ is a finite index subgroup of B_n. Similarly, $\pi_1(U_n)$ is a finite index subgroup of the pure braid group $PB_n = \pi_1(PConf_n(\mathbb{C}))$.

Problem 2.8. *Compute $\pi_1(Simp_n)$ and $\pi_1(U_n)$.*

One key step in Arnol'd's computation of $H^*(PConf_n(\mathbb{C});\mathbb{Z})$ is using the forgetful map $PConf_n(\mathbb{C}) \to PConf_{n-1}(\mathbb{C})$ and studying the action of PB_{n-1} on the homology of the fibres (the triviality of this action is used in Arnol'd's spectral sequence argument). So if one intends to prove Problem 2.6 by modelling a strategy based on Arnol'd's argument, finding an answer to Problem 2.5 would be crucial.

One must also make sense of the forgetful map $U_{n+1} \to U_n$- which, if we think about it, is quite interesting. In the configuration spaces picture, the forgetful map is literally forgetting a point, or pulling out factors from a polynomial. But when we consider ramification of polynomials as opposed to zeroes, forgetting a ramification point makes no geometric sense! In other words, suppose $f(x)$ is a simply-branched polynomial of degree n, and a is a root of f, then there is no guarantee if $g(x) = \frac{f(x)}{x-a}$ is a simply-branched polynomial.

Problem 2.9. *Describe the forgetful map $U_{n+1} \to U_n$.*

3 Research program: Filtration of cohomology via objects over crossed simplicial groups

Motivation. Much of this project arose from an attempt to understand the (stable) (co)homology in examples like the unordered configuration space (on any Hausdorff topological space, or any quasiprojective algebraic variety over a field), the moduli space of morphisms of a fixed degree from a variety to $\mathbb{P}^n$, smooth sections of a linear system over a variety, and many others, under a universal framework– that of being objects over ΔS, the *symmetric simplicial category*. Now, the theory of simplicial spaces forms the core of Verdier's theory of hypercoverings and the subsequent vast generalisations in Deligne's theory of cohomological descent. So I take a chip off the same block and recast it by replacing the indexing category Δ (commonly known as the simplicial category) by ΔS (which we call the *symmetric simplicial category*, and which contains Δ as a subcategory). The main advantage of rewording (parts of) the theory of hypercovering over ΔS is that, as opposed to Δ, morphisms in ΔS are, by definition, those of Δ, with the permutation maps on the objects $[n] = \{0,1,\dots,n\}$ thrown in (see [Ban21a, Definition 2.10] and also [FL91]). The category ΔS was first introduced as a part of the concept of *crossed simplicial groups* developed independently by Fiedorowicz-Loday ([FL91]) and Krasauskas ([Kra87]). For ΔS-hypercoverings we construct a spectral sequence, somewhat like the Čech-to-derived category spectral sequence, that abuts to the cohomology of the moduli space under consideration.

In particular, if one's goal is, for example, to compute (stable) (co)homology of moduli spaces (which often come in families indexed by a parameter, say n) that are naturally quotients of spaces equipped with permutation actions by $\{S_n\}$, then ΔS, by encoding the permutation action in the indexing category itself, gives us a precise tool to entirely bypass all the combinatorial complexities that naturally form a part of Δ.

Setup. I introduce the notion of *spaces admitting symmetric (semi)simplicial filtration* defined by a set of axioms (see [Ban21a, Definition 2.10]). Roughly speaking, given a family of spaces $\{X_n\}_{n\in\mathbb{N}}$ we say $\{X_n\}$ (or X_n) admits symmetric semisimplicial filtration by a space M if:

1. there exists $e \in \mathbb{N}$ (which we call '*filter gap*') such that for each $n \in \mathbb{N}$, the spaces $\{T_p = M^p \times X_{n-ep}\}$ form a $\Delta_{\mathrm{inj}}S$ object,
2. the face maps of $T_\bullet$ are proper finite morphisms satisfying certain conditions (see Definition **??**).

We call $U_n := X_n - f_0(M \times X_{n-e})$ the '*unfiltered* or *zeroth stratum*' of X_n. Before stating my main result to this end, we set up some notations and conventions.

Result. For a graded vector space V, let $V^{(r)}$ denote its r^{th} graded component, and let $V^{\mathrm{odd}} := \oplus_{j\in\mathbb{Z}} V^{(2j+1)}$ and $V^{\mathrm{even}} := \oplus_{j\in\mathbb{Z}} V^{(2j)}$ denote the odd and even graded subspaces of V, respectively. Throughout this paper, by a *space* we mean a locally-compact Hausdorff topological space or a quasi-projective algebraic variety over some field. By a *morphism* we mean a continuous map of topological spaces or a morphism of algebraic varieties. For a $\mathbb{Z}$-scheme X we continue to denote its base change to any algebraically closed field K by X; in turn we mean $H^q(X;\mathbf{Q})$ (respectively, $H^q_c(X;\mathbf{Q})$) to stand for both the singular cohomology $H^q(X(\mathbb{C});\mathbb{Q})$ (respectively, $H^q_c(X(\mathbb{C});\mathbb{Q})$, singular cohomology with compact support) as well as the étale cohomology $H^q_{ét}(X(K);\mathbb{Q}_\ell)$ (respectively, $H^q_{ét,c}(X(K);\mathbb{Q}_\ell)$, étale cohomology with compact support), ℓ coprime to char K. My main theorem, and the basis of computation of the (stable) cohomology in all the examples mentioned in the next subsection, is the following.

Theorem 3 (Banerjee: *Cohomology of spaces admitting symmetric semisimplicial filtration*)**.** *Let M and $\{X_n\}_{n\in\mathbb{N}}$ be locally compact connected Hausdorff topological spaces. Suppose that $\{X_n\}_{n\in\mathbb{N}}$ admits a semisimplicial filtration by powers of M, with face maps given by*

$$f_i : M^p \times X_{n-ep} \to M^{p-1} \times X_{n-e(p-1)}.$$

Let $e > 0$ be the filter gap and $\{U_n\}$ the zeroth strata. Then there exists a spectral sequence

$$E_1^{p,q} = \bigoplus_{l+m=q} \bigoplus_{i+j=p} \left(Sym^i H_c^{odd}(M;\mathbb{Q}) \otimes \Lambda^j H_c^{even}(M;\mathbb{Q})\right)^{(l)} \otimes H_c^m(X_{n-ep};\mathbb{Q}) \implies H_c^{p+q}(U_n;\mathbb{Q}) \tag{3.1}$$

where the differentials are given by alternating sum of the pullbacks on cohomology induced by the face maps:

$$d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$$
$$d_1^{p,q} := \sum_{i=0}^{p-1} (-1)^i f_i^*.$$

If $\{X_n\}$ and M are quasi-projective algebraic varieties over a field K, then there is a spectral sequence of $Gal(\overline{K}/K)$-representations

$$E_1^{p,q} = \bigoplus_{l+m=q} \bigoplus_{i+j=p} \left(Sym^i H_{ét,c}^{odd}(M_{/\overline{K}};\mathbb{Q}_\ell) \otimes \Lambda^j H_{ét,c}^{even}(M_{/\overline{K}};\mathbb{Q}_\ell)\right)^{(l)} \otimes H_{ét,c}^m(X_{n-ep}{}_{/\overline{K}};\mathbb{Q}_\ell)$$
$$\implies H_{ét,c}^{p+q}(U_{n/\overline{K}};\mathbb{Q}_\ell)$$

where ℓ is coprime to char K*, and the differentials are exactly the same as above.* □

An interesting remark in this regard is the following observation.

Remark 3.1. *Here is a brief intuitive reason behind why one might expect Theorem 3 to hold. Note that zeroth stratum U_n from [Ban21a, Definition 2.10] can be thought of as the space of 'M-indecomposables', intuitively speaking. If we interpret our theorems as trying to compute the cohomology of 'M-indecomposables' in X_n, which in a sense they are, and if we make the (rather big) assumption that indecomposables in a topological sense is 'the same as' indecomposables in cohomology, then,*

$$\mathrm{Ext}_{H^*(SymM)}(H^*(X_*), k)$$

should be computing the 'derived indecomposables' of $H^*(X_*)$ as a $H^*(SymM)$-module. The associated graded of this is

$$\mathrm{Ext}_{H^*(SymM)}(k,k) \otimes H^*(X_*) \cong Sym^*(H^*(\Sigma M)) \otimes H^*(X_*)$$

where ΣM denotes the suspension of M, and this is precisely what our E_1 page spectral sequence in Theorem 3 looks like.

Without getting into the nitty-gritty of the axioms that define spaces admitting symmetric semisimplicial filtration (for which, see [Ban21a, Definition 2.10]), let me paint a picture of spaces admitting symmetric semisimplicial filtration, and ΔS objects in general, through some examples.

3.1 Examples of spaces admitting symmetric semisimplicial filtration, and related results.

There are many examples of families of spaces admitting a symmetric semisimplicial filtration (and thus satisfying the hypothesis of Theorem 3), including, but not limited to the following.

1. **The n^{th}-symmetric powers of a space X.** Let $X_n = Sym^n X$. Define

$$\begin{aligned} f_i : X^{p+1} \times Sym^{n-2(p+1)}X &\to X^p \times Sym^{n-2p}X \\ (a_0, \ldots, a_p), \{b_1, \ldots, b_{n-2p}\} &\mapsto (a_0, \ldots, \hat{a}_i, \ldots, a_p), \{a_i, a_i, b_1, \ldots, b_{n-2p}\} \end{aligned} \tag{3.2}$$

where $(a_0, \ldots, a_p)$ denotes an ordered $(p+1)$-tuple of elements in X, and $\{b_1, \ldots, b_p\}$ denotes an unordered p-tuple and $\hat{a}_i$ stands for a_i (the $(i+1)^{th}$ entry) removed. It is easy to check that with these morphisms as face maps, the semisimplicial space $\{X^p \times Sym^{n-2p}X\}$ naturally forms a $\Delta_{\text{inj}}S$-space. The zeroth stratum $U_n = UConf_n(X)$ is the unordered configuration space of n distinct points in X. Therefore as an immediate consequence of Theorem 3 I obtain the following:

Result. ([Ban21a, Corollary 6]) There exists an E_1 page spectral sequence which reads as:

$$\begin{aligned} E_1^{p,q} = \bigoplus_{i+j=p} \bigoplus_{l+m=q} \left(Sym^i H_c^{odd}(X;\mathbb{Q}) \otimes \Lambda^j H_c^{even}(X;\mathbb{Q}) \right)^{(l)} \otimes H_c^m(Sym^{n-2p}X;\mathbb{Q}) \\ \implies H_c^{p+q}(UConf_n X;\mathbb{Q}). \end{aligned} \tag{3.3}$$

with differentials given by the alternating sum of the pullbacks on cohomology with compact supports induced by the face maps:

$$\begin{aligned} &\big((\alpha_1 \cdots \alpha_i) \otimes (\beta_1 \wedge \cdots \wedge \beta_j)\big) \otimes \big((\beta'_1 \cdots \beta'_{j'}) \otimes (\alpha'_1 \wedge \cdots \wedge \alpha'_{i'})\big) \mapsto \\ \sum_{1 \le r < s \le i'} (-1)^{r+s} &\big((\alpha_1 \cdots \alpha_i(\alpha'_r + \alpha'_s)) \otimes (\beta_1 \wedge \cdots \wedge \beta_j)\big) \otimes \big((\beta'_1 \cdots \beta'_{j'}) \otimes (\alpha'_1 \wedge \cdots \wedge \widehat{\alpha'_r} \cdots \wedge \widehat{\alpha'_s} \wedge \cdots \alpha'_{i'})\big) \end{aligned}$$

where $i+j=p$, $i'+j'=n-2p$ and $\alpha_1, \ldots, \alpha_i, \alpha'_1, \ldots, \alpha'_{i'} \in H_c^{odd}(X)$ and $\beta_1, \ldots, \beta_j, \beta'_1, \ldots, \beta'_{j'} \in H_c^{even}(X)$.

2. **The moduli space of $(r+1)$-tuples of monic polynomials of degree n.** Let $Poly^{n,r+1}$ be the space of $(r+1)$-tuples of monic degree n homogeneous polynomials in one variable over an algebraically closed field K, and let $Poly_v^{n,r+1}$ be the locus of those r-tuples having no common roots of multiplicity $\ge v$. Then $Poly^{n,r+1}$ admits a symmetric semisimplicial filtration by $\mathbb{A}^1$. Indeed, we have a semisimplicial space given by $\{(\mathbb{A}^1)^p \times Poly^{n-pv,r+1}\}_{0 \le p \le n}$ with face maps defined by

$$\begin{aligned} f_i : (\mathbb{A}^1)^{(p+1)} \times Poly^{n-(p+1)v,r+1} &\to (\mathbb{A}^1)^p \times Poly^{n-pv,r+1} \\ (a_0, \ldots, a_p), (P_1(z), \ldots, P_r(z)) &\mapsto (a_0, \ldots, \hat{a}_i, \ldots, a_{p-1}), \big((z-a_i)^v P_1(z), \ldots, (z-a_i)^v P_r(z)\big) \end{aligned}$$

which not only form a $\Delta_{\text{inj}}S$ object, but also satisfy the added axioms of [Ban21a, Definition 2.10]. The spectral sequence in Theorem 3 degenerates on E_1 and gives us:

Result. ([Ban21a, Corollary 7]) Over $\mathbb{C}$, the following holds:

$$H^i(Poly_v^{n,r+1};\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = 0, 2v(r+1)-3 \\ 0 & \text{otherwise} \end{cases}.$$

Furthermore, there exists an isomorphism of $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$- representations:

$$H^i(Poly_v^{n,r+1}(\overline{\mathbb{F}_q});\mathbf{Q}_\ell) \cong \begin{cases} \mathbf{Q}_\ell(0) & i = 0 \\ \mathbf{Q}_\ell((v-n)(r+1)-1) & i = 2v(r+1)-3 \\ 0 & \text{otherwise} \end{cases},$$

ℓ coprime to q, thus recovering the cohomology part of Farb-Wolfson's [FW16, Theorem 1.2], and in the special case of $v=1$ this an algebro-geometric and arithmetic analogue of Segal's [Seg79, Propositions 1.1 and 1.2].

3. **The moduli space of degree n maps $C \to \mathbb{P}^r$, $\mathrm{Mor}_n(C, \mathbb{P}^r)$.** Let C be a smooth projective curve of genus $g \ge 0$ defined over an algebraically closed field K and let $J(C)$ denote the Jacobian of C. Let $Pic^n(C)$, which is (noncanonically, by a translation) isomorphic to $J(C)$, denote the space of degree n line bundles on C. A degree n map $C \to \mathbb{P}^r$ is determined by

   i. a choice of a line bundle $L \in Pic^n(C)$

ii. sections $s_0, \cdots, s_r \in H^0(C, L)$ having no common zeroes

whence we have

$$\begin{aligned} C &\to \mathbb{P}^r \\ x &\mapsto [s_0(x) : \cdots : s_r(x)]. \end{aligned}$$

Let $\mathrm{Mor}_n(C, \mathbb{P}^r)$ denote the moduli space of all degree n maps $C \to \mathbb{P}^r$. Define X_n by

$$X_n := \{L, [s_0 : \ldots : s_r] : L \in Pic^n(C), s_i \in H^0(C, L) \text{ for all } i\}.$$

When $n \geq 2g$ (for $g \geq 2$, even $n \geq 2g-1$ works for our purposes), by the Riemann-Roch theorem $\dim H^0(C, L) = n - g + 1$ for all $L \in Pic^n(C)$, which makes X_n the projectivization of a rank $(r+1)(n-g+1)$ vector bundle on $Pic^n(C)$ and $\mathrm{Mor}_n(C, \mathbb{P}^r) \subset X_n$ is Zariski open dense: [2]

$$\begin{tikzcd} \mathbb{P}(H^0(C, L)^{r+1}) \cong \mathbb{P}^{(n-g+1)(r+1)-1} \arrow[r, hook] & X_n \arrow[d] \\ & Pic^n(C) \end{tikzcd}$$

Now X_n has a natural stratification given by the number of common zeroes of an $(r+1)$-tuple of global sections of some degree n line bundle, which in turn shows, by [Ban21a, Definition 2.10] that X_n admits a symmetric semisimplicial filtration by C, with filter gap $e = 1$ and the zeroth stratum $\mathrm{Mor}_n(C, \mathbb{P}^r)$. An easy consequence of Theorem 3 (more precisely, [Ban21a, Theorem 2]) is the following.

Result. (Banerjee, [Ban21a, Theorem 3]) Let me denote a vector space spanned by $\{a_1, \ldots, a_k\}$ over $\mathbf{Q}$ by $\mathbf{Q}\{a_1, \ldots, a_k\}.n \geq 2g$, and let $r \leq n - g$. Let $n_0 := n - 2g$. Then there exists a second quadrant spectral sequence, which converges to $H^*(\mathrm{Mor}_n(C, \mathbb{P}^r); \mathbf{Q})$ an an algebra, which has the following description. The E_2 term is a bigraded algebra that collapses on $E_2^{-p,q}\Big|_{p \leq n_0}$. Furthermore, $E_2^{-p,q}\Big|_{p \leq n_0}$ is a quotient of the graded commutative $\mathbf{Q}$-algebra

$$H^*(J(C); \mathbf{Q})[h]/h^r \otimes \wedge \mathbf{Q}\{t\} \otimes Sym\mathbf{Q}\{\alpha_1, \ldots, \alpha_{2g}\},$$

where $H^i(J(C); \mathbf{Q})$ has degree $(0, i)$, h has degree $(0, 2)$, t has degree $(-1, 2r+2)$ and α_i has degree $(-1, 2r+1)$ for all i, modulo elements of degree $(-i, j)$ with $i > n_0$. Furthermore this is a spectral sequence of mixed Hodge structures, with $\mathbf{Q}\{\alpha_1, \ldots, \alpha_{2g}\}$ and $\mathbf{Q}\{t\}$ each carrying a pure Hodge structure of weight $2(r+1)$, and h is of type $(1, 1)$.

When $C = \mathbb{P}^1$, the Jacobian of $\mathbb{P}^1$ is just a point, and I show that ([Ban21a, Corollary 4]):

$$H^*(\mathrm{Mor}_n(\mathbb{P}^1, \mathbb{P}^r); \mathbf{Q}) \cong \mathbf{Q}[h]/h^r \otimes \wedge \mathbf{Q}\{t\}$$

where t has cohomological degree $(2r+1)$. Furthermore, over a field κ, with algebraic closure $\overline{\kappa}$, we have an isomorphism of $Gal(\overline{\kappa}/\kappa)$-representations:

$$H^i_{\acute{e}t}(\mathrm{Mor}_n(\mathbb{P}^1, \mathbb{P}^r); \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell(-j) & i = 2j, 0 \leq j \leq r-1 \\ \mathbb{Q}_\ell(-(j+1)) & i = 2j+1, r \leq j \leq 2r-1 \\ 0 & \text{otherwise.} \end{cases}$$

The difference between my computation and prior similar results by Segal, and subsequent expansions by Farb-Wolfson (see [Seg79], [FW16]) is that they use scanning map, which works only in the analytic topology over $\mathbb{C}$. Whereas I give a characteristic-free algebraic approach which is particularly useful to keep track of the weights.

4. **The moduli space of basepoint free linear systems on C.** The moduli space of nondegenrate degree n maps $C \to \mathbb{P}^r$ i.e. those for which the image is not contained in a hyperplane in $\mathbb{P}^r$, is freely acted upon by PGL_{r+1} and the quotient is the *moduli space of basepoint free linear systems on C of degree n and rank r*, which we denote by $Lin^r_n(\mathbb{P}^1)$. The space $Lin^r_n(\mathbb{P}^1)$ also naturally forms the zeroth stratum of the space of all linear systems of rank r and degree n; the latter naturally admits a symmetric semisimplicial filtration by C. In the special case of $C = \mathbb{P}^1$, using a similar method as above, in an upcoming project, Claudio Gómez-Gonzáles and I show, among other things:

[2] For $n \leq 2g-2$ the description of X_n as the projectivisation of a vector bundle on $Pic^n(C)$ no longer holds; it has been the subject of intense study for decades, under the name of Brill-Noether theory (see [ACGH]).

Theorem 4. *Let r and n be positive integers such that $n \geq r+2$. Then the following hold:*

(a) Over $\mathbb{C}$ we have an isomorphism of graded $\mathbb{Q}$-algebras:

$$H^*(Lin_n^r(\mathbb{P}^1)(\mathbb{C});\mathbb{Q}) \cong H^*(G(r-1,n-1)(\mathbb{C});\mathbb{Q})$$

where $G(r-1,n-1)$ denotes the Grassmannian of $(r-1)$-planes in $(n-1)$-dimensional vector space. Furthermore, it is an isomorphism of mixed Hodge structures.

(b) Let κ be a field with $\operatorname{char}\kappa = 0$ or coprime to d. Then we have an isomorphism of $Gal(\overline{\kappa}/\kappa)$-representations:

$$H^*_{\acute{e}t}(Lin_n^r(\mathbb{P}^1)(\overline{\kappa});\mathbb{Q}_\ell) \cong H^*_{\acute{e}t}(G(r-1,n-1)(\overline{\kappa});\mathbb{Q}_\ell) \tag{3.4}$$

where ℓ is coprime to $\operatorname{char}\kappa$.

This naturally leads to the following interesting question.

Problem 3.2. *Is the $\mathbb{Q}$-algebra isomorphism induced by an actual map between $Lin_n^r(\mathbb{P}^1)$ and $G(r-1,n-1)$? Are they homotopy equivalent?*

Alternatively, fix a vector space V_0 of codimension 2 in $\Gamma(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1})$ and consider the rational map given by

$$\phi : G(r+1,\Gamma(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(n))) \dashrightarrow G(r-1,n-1)$$
$$V \mapsto V \cap V_0.$$

It is easy to show that the open dense subvariety, which we denote by U and where ϕ is a morphism, is an affine space bundle on $G(r-1,n-1)$. Is there a sequence of flips and flops that can relate $Lin_n^r(\mathbb{P}^1)$ and U?

Result. Another application of Theorem 3 in a subsequent paper titled "On the cohomology of certain subspaces of $Sym^n(\mathbb{P}^1)$ and Occam's razor for Hodge structures" (see [Ban21b]) is the following. I disprove Conjecture H' of [VMW13].

Theorem 5. *We have*

$$\lim_{n\to\infty} H^i(w_{1^n 23}(\mathbb{P}^1_{\mathbb{C}});\mathbb{Q}) = \begin{cases} 1 & \text{for } i = 0,1,2, \\ 2 & \text{for } i > 2. \end{cases}$$

Furthermore, $H^i(w_{1^n 22}(\mathbb{P}^1);\mathbb{Q})$ is pure of weight $-2i$ and Hodge type $(-i,-i)$ for all i.

The following corollary to Theorem 5 is a refinement of the statement of Conjecture G' of [VMW13].

Corollary 6. *We have*

$$\lim_{n\to\infty} \dim H^i(w_{1^n 22}(\mathbb{P}^1_{\mathbb{C}});\mathbb{Q}) \cong \begin{cases} 1 & \text{for } i = 0, 2k+1, k \geq 0, \\ 2 & \text{for } i = 4k, k \geq 1 \\ 0 & \text{for } i = 4k+2, k \geq 0. \end{cases}$$

Furthermore, $H^i(w_{1^n 22}(\mathbb{P}^1);\mathbb{Q})$ is pure of weight $-2i$ and Hodge type $(-i,-i)$ for all $0 \leq i \leq n+2$.

It should be observed here that spaces admitting symmetric semisimplicial filtration arise as a special type of $\Delta_{\text{inj}}S$ object. In particular, unlike the previous examples, the following is an instance where the cohomology computation is heavily dependent on the $\Delta_{\text{inj}}S$ structure of the moduli space under consideration; however, the space itself does not satisfy the conditions of admitting a symmetric semisimplicial filtration by a fixed space.

The moduli space of smooth sections of a $\mathfrak{g}^r_d$. A *linear series* (or *system*) is a vector subspace of the vector space of global sections of a line bundle on a smooth projective curve. A linear system V on a smooth projective curve X is called a $\mathfrak{g}^r_d$ if $V \subset H^0(X,L)$, where L is a degree d line bundle on X and V is a complex $(r+1)$ dimensional vector space. A $\mathfrak{g}^r_d$, say V, is *m-very ample* if for every effective divisor $\xi \in X$ of degree $(m+1)$, we have that

$$\dim \mathcal{V}(-\xi) = r+1-(m+1),$$

where $\mathcal{V}(-\xi) := H^0(X, L\otimes\mathcal{O}(-\xi)) \cap \mathcal{V}$. Though the following result cannot be obtained as a corollary to Theorem 1, the basic principles of the proof of Theorem 1 hold almost verbatim to prove the following on the (stable) cohomology of the moduli space of smooth sections of a $\mathfrak{g}^r_d$ that is m-very ample.

Theorem 7. *Let X be a smooth projective curve of genus g over $\mathbb{C}$. Let $\mathscr{V}$ be a linear system on X of type $\mathfrak{g}^r_d$; moreover let $\mathscr{V}$ be m-very ample. Define $\mathscr{V}^\circ \subset \mathscr{V}$ to be the locus of smooth sections in $\mathscr{V}$. Then for all $i \leq \frac{m-1}{2}$ the following holds:*

$$H^i(\mathscr{V}^\circ;\mathbb{Q}) \cong \begin{cases} Sym^{p-2}H^1(X;\mathbb{Q})(-(p-1)) \oplus Sym^p H^1(X;\mathbb{Q})(-p) & i = 2p \\ Sym^{p-1}H^1(X;\mathbb{Q})(-(p-1)) \oplus Sym^p H^1(X;\mathbb{Q})(-(p+1)) & i = 2p+1. \end{cases}$$

In my knowledge, Tommasi ([Tom]) is also studying the moduli space of smooth sections of a line bundle over a smooth projective curve with the goal of computing some stable cohomology, sans the notion of m-very ampleness.

Problem 3.3. *Generalize Theorem 7 to higher dimensional varieties. More specifically, given that there are at least three different ways to generalise m-very ampleness on algebraic curves to higher dimensional smooth projective varieties, as shown by Beltrametti, Francia, Sommese in [MBS x], compare the stability results obtained corresponding each of the notions of m-very ampleness.*

Furthermore, in an upcoming project I show the following.

Problem 3.4. *Given an m very ample line bundle L on a smooth projective variety X, the moduli space of nonzero smooth global sections of L on X is stably homologous to the moduli space of basepoint free framed linear systems of L of a fixed rank.*

3.2 Future directions: hypercovers as objects on crossed simplicial groups

The main definition and the key result of Fiedorowicz-Loday that form not only the basis of my recent projects like [Ban21a] and [Ban21b], but also much of my plans and projects in the immediate future are as follows.

Definition 3.5. *(Definition 1.1, Fiedorowicz-Loday) A sequence of groups G_n, $n > 0$, is a* crossed simplicial group *if it is equipped with the following structure. There is a small category ΔG, which is part of the structure, such that: (a) the objects of ΔG are $[n]$, $n > 0$, (b) ΔG contains Δ as a subcategory, (c) $Aut_{\Delta G}([n]) = G^{op}$ (opposite group of G_n), (d) any morphism in ΔG, can be uniquely written as a composite $\phi \circ g$ where $\phi \in \mathrm{Hom}_\Delta([m], n)$ and $g \in G^{op}_m$ (whence the notation ΔG).*

In fact, Conne's cyclic category is the special case when $G_n = C_{n+1}$, the cyclic group on $(n+1)$ elements. All results in the previous subsection (i.e. all my results so far) have dealt with the case of $G_n = S_{n+1}$, the symmetric group on $(n+1)$ elements. One of their main results is that there are surprisingly few crossed simplicial groups out there! In fact, they show the following.

Theorem 3.6. *([FL91, Theorem 3.6], Classification of crossed simplicial groups.) For any crossed simplicial group G_* there exists an exact sequence (unique up to isomorphism) of crossed simplicial groups*

$$1 \to G'_* \to G_* \to G''_* \to 1$$

such that G'_* is a simplicial group and G''_* is one of the following crossed simplicial groups: 1, C_* (the cyclic groups), S_* (the symmetric groups), $\mathbb{Z}/2$, D_* (the dihedral groups), $\mathbb{Z}/2 \times S_*$ and H_* (the hyperoctahedral groups).

This leads to vast field of possible questions of the following flavour.

Project 3.7. *Find 'natural' moduli spaces that are objects over ΔG, where G_* is a crossed simplicial group. In particular, can we formulate statements on (co)homological stability à la Theorem 3?*

One situation where we can expect an affirmative answer is also the most expected, and in a sense a trivial one: that of subspaces of $Sym^n X$ that are invariant under a chosen crossed simplicial group. To the best of my knowledge, the only piece of work so far that uses some instances of crossed simplicial groups to describe certain moduli spaces (that of ribbon graphs corresponding certain Riemann surfaces) is by Dyckherhoff and Kapranov ([DK15]); their focus has been on understanding the combinatorial models on marked surfaces.

In particular, much remains to be explored in terms of casting the simplicial theory of hypercovers through the lens of crossed simplicial groups.

References

[ACGH] Enrico Arbarello, Maurizio Cornalba, Phillip Griffiths, and Joseph Daniel Harris. *Geometry of Algebraic Curves: Volume I.* Springer; 1st ed. 1985, Corr. 2nd printing 2007 edition (May 15, 2007).

[Arn69] V.I. Arnol'd. The cohomology of the colored braid group. *Mat. Zametki, 5:227–231*, 1969.

[Arn70] V.I. Arnol'd. On some topological invariants of algebraic functions. *Tr. Mosc. Mat. Obsc., pages 27–46*, 1970.

[Ban20] Oishee Banerjee. Cohomology of the space of polynomial maps on $\mathbb{A}^1$ with prescribed ramification. *Advances in Mathematics 359, 106881*, 2020.

[Ban21a] Oishee Banerjee. Filtration of cohomology via symmetric semi-simplicial spaces. *arXiv:1909.00458*, 2021.

[Ban21b] Oishee Banerjee. On the cohomology of certain subspaces of $Sym^n(\mathbb{P}^1)$ and occam's razor for hodge structures. *Research in the Mathematical Sciences volume 8, Article number: 25*, April 2021.

[CE96] G. Casnati and T. Ekedahl. overs of algebraic varieties. i. a general structure theorem, covers of degree3, 4 and enriques surfaces,. *Journal of Algebraic Geometry, no. 3, 439–460. MR 1382731*, 1996.

[CEF15] Thomas Church, Jordan Ellenberg, and Benson Farb. Fi-modules and stability for representations of symmetric groups. *Duke Mathematical Journal 164*, 2015.

[CL21] Samir Canning and Hannah Larson. Intersection theory on low-degree hurwitz spaces. *arXiv:2103.09902*, 2021.

[Cle72] A. Clebsch. Zur theorie der riemann'schen flachen. *Mathematische Annalen, 6:216–230*, 1872.

[Del75] Pierre Deligne. Théorie de hodge : lll. *Publ. Math. IHES 44, pp. 6–77.*, 1975.

[DH17] R. Pries K. Stevenson D. Harbater, A. Obus. Abhyankar's conjectures in galois theory: currentstatus and future directions. *Bull. Amer. Math. Soc.*, 2017.

[DK15] Tobias Dyckerhoff and Mikhail Kapranov. Crossed simplicial groups and structured surfaces. *Stacks and categories in geometry, topology, and algebra Volume 643 of Contemp. Math.page 37–110.*, 2015.

[EVW16] Jordan S. Ellenberg, Akshay Venkatesh, and Craig Westerland. Homological stability for hurwitz spaces and the cohen-lenstra conjecture over function fields. *Annals of Mathematics*, 2016.

[FL91] Zbigniew Fiedorowicz and Jean-Louis Loday. Crossed simplicial groups and their associated homology. *Transactions of the American Mathematical Society, Volume 326, 57-87*, July, 1991.

[FW16] Benson Farb and Jesse Wolfson. Topology and arithmetic of resultants, I: spaces of rational maps. *Topology and arithmetic of resultants, I, New York Jour. of Math. 22*, 2016.

[FWW19] Benson Farb, Jesse Wolfson, and Melanie Matchett Wood. Coincidences of homological densities, predicted by arithmetic. *Advances in Mathematics, vol. 352, pp. 670-716.*, 2019.

[Har77] Robin Hartshorne. *Algebraic Geometry*. Springer, 1977.

[Har85] John L. Harer. Stability of the homology of the mapping class groups of orientable surfaces. *Annals of MathematicsSecond Series, Vol. 121, No.2 , pp. 215-249*, 1985.

[Kra87] R. L. Krasauskas. Skew simplicial groups. *Litovsk. Math. Sb. 27:89-99*, 1987.

[MBS x] P. Francia M. Beltrametti and A.J Sommese. On reider's method and higher order embeddings. *Duke math. j. 58.2, pp. 425–439*, 1989, doi: 10.1090/s0894- 0347-01-00368-x.

[Nap98] F. Napolitano. Topology of complements of strata of the discriminant of polynomials. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics Volume 327, Issue 7, Pages 665-670*, 1998.

[PV15] Anand Patel and Ravi Vakil. On the chow ring of the hurwitz space of degree three covers of p^1. *Annali della Scuola normale superiore di Pisa, Classe di scienze*, 2015.

[RW06] Matthieu Romagny and Stefan Wewers. Hurwitz spaces. *Śeminaires & Congrés 13, p. 313–341*, 2006.

[Seg79] Graeme Segal. The topology of spaces of rational functions. *Acta Math. 143, no. 1-2, 39–72*, 1979.

[Tom] Orsola Tommasi. in preparation.

[Tot96] Burt Totaro. Configuration spaces of algebraic varieties. *Topology 35, no. 4, 1057–1067*, 1996.

[VMW13] Ravi Vakil and Melanie Matchett-Wood. Discriminants in the grothendieck ring of varieties. *https://arxiv.org/pdf/1208.3166.pdf*, 2013.

[Wac06] Michelle L. Wachs. Poset topology: Tools and applications. *Geometric Combinatorics*, 2006.