Exercise Sheet 12

Discussed on 14.07.2021

Problem 1. (a) Consider the situation of the "local ring lemma" from lecture 22: A is a 2-dimensional noetherian local integral domain and $a, b \in A$ are such that $A/(a,b)$ is artinian. Show that if A is additionally normal, then $(A/a)[b] = 0$.

Hint: Recall that $A = \bigcap_{\mathfrak{p}} A_{\mathfrak{p}}$, where the intersection on the right ranges over all prime ideals $\mathfrak{p} \subset A$ of height 1. Note that each $A_{\mathfrak{p}}$ is a discrete valuation ring and hence induces a valuation $v_{\mathfrak{p}}$ on A.

(b) Let k be a field, let X be a 2-dimensional proper normal variety over k and let $Z_1, Z_2 \subset X$ be effective Cartier divisors whose intersection has dimension 0. Show that

$$\mathcal{O}_X(Z_1) \cdot \mathcal{O}_X(Z_2) = \operatorname{len}(Z_1 \cap Z_2).$$

Here $\operatorname{len}(Z_1 \cap Z_2)$ denotes the length (i.e. dimension) of the coordinate ring of the affine scheme $Z_1 \cap Z_2$ over k.

(c) (Bezout's Theorem) Let k be a field, let $F_1, F_2 \in k[x, y, z]$ be homogeneous polynomials and denote $Z_i := V_+(F_i) \subset \mathbb{P}^2_k$. If $Z_1 \cap Z_2$ has dimension 0, show that

$$\operatorname{len}(Z_1 \cap Z_2) = \deg F_1 \cdot \deg F_2.$$

Hint: Recall that $\mathcal{O}_{\mathbb{P}^2}(Z_i) = \mathcal{O}_{\mathbb{P}^2}(\deg F_i)$.

Problem 2. Let $k = \bar{k}$ be an algebraically closed field.

(a) Let $f\colon X \to Y$ be a homomorphism of abelian varieties over k. Endow $f(X)$ with the reduced scheme structure. Show that f factors through $f(X)$ and that $f(X)$ is itself an abelian variety.

Hint: Recall that a group variety over an algebraically closed field is smooth if and only if it is reduced (cf. exercise sheet 2).

(b) Let $K := \ker(G \to H)$ be the kernel of a homomorphism of finite commutative k-group schemes. Show that $\deg G = \deg H \cdot \deg K$ if any only if $H = G/K$ if and only if $G \to H$ is flat and surjective.

Hint: The action of K on G is free. Hence the map $G \to G/K$ has locally free of degree $\deg K$ (see lecture on quotients). Also, $G \to H$ factors over $G/K \to H$.

(c) Use without proof that group varieties in characteristic 0 are smooth.

Let $f\colon X \to Y$ be a surjective homomorphism of abelian varieties over k. Prove that $\ker(f)$ is an abelian variety if and only if for all $n \in \mathbb{Z}_{\geq 1}$, the map $X[n] \to Y[n]$ is flat and surjective.

Hint: Let $K^0 \subseteq K := \ker(f)$ be the connected component containing 0 with reduced scheme structure. Prove that K^0 is an abelian variety. Use (b) to show that $K^0[n] = K[n]$ if and only if $X[n] \to Y[n]$ is surjective.

Show next that connectedness of K is equivalent to $K^0[n](k) = K[n](k)$ for all n.

For smoothness in case $\mathrm{char}(k) = p$, argue that it is enough to see that K is regular at 0, i.e. that $\dim_k \mathrm{Lie}(K) = \dim K$. Prove and use now the observation that $\mathrm{Lie}(K^0) = \mathrm{Lie}(K^0[p])$.

(d) Let $f_1 \colon X_1 \to Y$ and $f_2 \colon X_2 \to Y$ be surjective homomorphisms of abelian varieties over k. Show that $X_1 \times_Y X_2$ is an abelian variety if and only if $X_1[n] \times X_2[n] \to Y[n]$ is flat and surjective for all $n \in \mathbb{Z}_{\geq 1}$.