Exercise Sheet 11

Discussed on 07.07.2021

Problem 1. Let k be an algebraically closed field and C a proper smooth connected curve over k. We will assume that the relative Picard functor $\mathrm{Pic}^0_{C/k}$ is representable by a k-scheme which is locally of finite type. The goal is to show that $\mathrm{Pic}^0_{C/k}$ is an abelian variety of dimension $g := g(C)$.

(a) Let X be a scheme. Show that there is a canonical bijection

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times).$$

Hint: Use without proof that

$$H^1(X, \mathcal{O}_X^\times) = \varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times).$$

Here the filtered colimit on the right-hand side is taken over all open coverings $\mathfrak{U} : X = \bigcup_i U_i$ of X and $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times)$ denotes Čech cohomology (see Stacks Section 01ED). Show that $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times)$ is in canonical bijection to the set of line bundles $\mathcal{L}$ on X such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$ for all i.

(b) Show that the tangent space of $\mathrm{Pic}^0_{C/k}$ at 0 equals $H^1(C, \mathcal{O}_C)$ and thus has dimension g.

Hint: Recall that the tangent space of $\mathrm{Pic}^0_{C/k}$ is computed as $\ker(\mathrm{Pic}^0_{C/k}(k[\epsilon]) \to \mathrm{Pic}^0_{C/k}(k))$, where $k[\epsilon] := k[T]/T^2$. Note that $C_{k[\epsilon]}$ has the same topological space as C, but has structure sheaf $\mathcal{O}_{C_{k[\epsilon]}} = \mathcal{O}_C[\epsilon]$. Now consider the cohomology sequence associated to the exact sequence of sheaves $1 \to 1 + \epsilon\mathcal{O}_C \to \mathcal{O}_C[\epsilon]^\times \to \mathcal{O}_C^\times \to 1$ on $|C|$.

(c) Show that $\mathrm{Pic}^0_{C/k}$ is smooth over k.

Hint: Recall the lifting criterion for smoothness (sheet 2): It is enough to show that for every k-algebra A with ideal $I \subset A$ such that $I^2 = 0$, the map $\mathrm{Pic}^0_{C/k}(A) \twoheadrightarrow \mathrm{Pic}^0_{C/k}(A/I)$ is surjective. To show this, argue similar to (b): Note that C_A and $C_{A/I}$ have the same topological space and there is a short exact sequence $1 \to 1 + f^*I \to \mathcal{O}_A \to \mathcal{O}_{A/I} \to 1$ of sheaves on $|C_A|$, where $f\colon C_A \to \operatorname{Spec} A$ is the projection. Look at the associated long exact sequence of cohomology and use that $H^2(C_A, f^*I) = 0$ because C is a curve.

(d) Fix a point $P \in C(k)$. Show that there is a map $\varphi\colon C^g \to \mathrm{Pic}^0_{C/k}$ which on k-points is given by $(P_1, \dots, P_g) \mapsto \mathcal{O}_C([P_1] + \cdots + [P_g] - g[P])$.

(e) Prove that the map φ is surjective. Deduce that $\mathrm{Pic}^0_{C/k}$ is proper and connected.

Hint: For the first part, you need to show that every line bundle $\mathcal{L}$ on C of degree 0 is of the form $\mathcal{O}_C([P_1] + \cdots + [P_g] - g[P])$ for some $P_1, \dots, P_g \in C(k)$. Apply Riemann-Roch to $\mathcal{L} \otimes \mathcal{O}_C(g[P])$ to deduce that there is a non-zero map $\mathcal{O}_C \to \mathcal{L} \otimes \mathcal{O}_C(g[P])$; then look at the quotient of that map.