Exercises, Algebra I (Commutative Algebra) – Week 5

Exercise 22. (Annihilator, 2 pts)
Let $m_1, \ldots, m_k \in M$ be a set of generators of M.

1. Let $\frac{a}{s} \in \mathrm{Ann}(S^{-1}M)$. For any i we have $\frac{a}{s}\frac{m_i}{1} = 0$; thus there is a $t_i \in S$, such that $t_i(am_i) = 0$. In particular $t_i a \in \mathrm{Ann}(m_i)$. Thus for $t_a = \Pi_{i=1}^k t_i$, we get $tam_i = 0$ for any i i.e. (since $(m_i)_i$ generate M) $ta \in \mathrm{Ann}(M)$. Thus $\frac{a}{s} = \frac{t_a a}{t_a s} \in S^{-1}\mathrm{Ann}(M)$ i.e. $\mathrm{Ann}(S^{-1}M) \subset S^{-1}\mathrm{Ann}(M)$.
   Conversely, if $\frac{a}{s} \in S^{-1}\mathrm{Ann}(M)$, with $a \in \mathrm{Ann}(M)$, then for any $\frac{m}{t} \in S^{-1}M$, $\frac{a}{s} \cdot \frac{m}{t} = \frac{am}{st} = \frac{0}{st} = 0$. Thus $\frac{a}{s} \in \mathrm{Ann}(S^{-1}M)$, proving that $\mathrm{Ann}(S^{-1}M) = S^{-1}\mathrm{Ann}(M)$.

2. If $S^{-1}M = 0$ then for each i, $\frac{m_i}{1} = 0 \in S^{-1}M$ i.e. there is a $s_i \in S$ such that $s_i m_i = 0 \in M$. Set $s = \Pi_{i=1}^k s_i \in S$. Then $sm_i = 0$ for any i thus ($(m_i)_i$ generate M) $sm = 0 \in M$ for any $m \in M$ i.e. $s \in \mathrm{Ann}(M)$. So $s \in S \cap \mathrm{Ann}(M)$.
   Conversely, if $s \in S \cap \mathrm{Ann}(M)$, since $sm = 0$ for any $m \in M$ and $t \in S$ (by definition) $\frac{m}{t} = 0 \in S^{-1}M$ i.e. $S^{-1}M = 0$.

Exercise 23. (Nakayama lemma, 3 points)
Let us denote $Q = \mathrm{Coker}(N \to M)$. Since M is finitely generated and Q is a quotient of M, Q is also finitely generated (for example by the image of a set of generators of M).
Tensoring the exact sequence $N \to M \xrightarrow{\pi} Q \to 0$ by $A/\mathfrak{a}$ we get the exact sequence $N/\mathfrak{a}N \to M/\mathfrak{a}M \xrightarrow{\pi \otimes \mathrm{id}_{A/\mathfrak{a}}} Q/\mathfrak{a}Q \to 0$. Thus $Q/\mathfrak{a}Q$ is the cokernel of $N/\mathfrak{a}N \to M/\mathfrak{a}M$, which is 0 by assumption. So $Q = \mathfrak{a}Q$.

1. Since $\mathfrak{a} \subset \mathfrak{R}$, Nakayama lemma (iii) yields $Q = 0$ i.e. $N \to M$ is surjective.

2. In this case, Nakayama lemma (ii) provides a $b = 1 + a \in A$, with $a \in \mathfrak{a}$, such that $bQ = 0$. In particular, $bq = 0$ for any $q \in Q$. Since b is invertible in A_b, we get that $\frac{q}{b^i} = 0$ in Q_b for any $q \in Q$ and $i \geq 0$ i.e. $Q_b = 0$. But tensoring the exact sequence $N \to M \to Q \to 0$ by A_b we get the exact sequence $N_b \to M_b \to Q_b \to 0$ i.e. $Q_b = 0$ is the cokernel of $N_b \to M_b$. Hence the claimed surjectivity.

3. Define a homomorphism of A-modules $g : \oplus_{i=1}^n Ae_i \to M$ by (extend linearly) $e_i \mapsto m_i$. By assumption, $g \otimes \mathrm{id}_{A/\mathfrak{a}} : \oplus_{i=1}^n A/\mathfrak{a}e_i \to M/\mathfrak{a}M$ is surjective. Then by the previous question, there is a $b = 1 + a$, with $a \in \mathfrak{a}$, such that $g \otimes \mathrm{id}_{A_b} : \oplus_{i=1}^n A_b e_i \to M_b$ is surjective i.e. $\frac{m_1}{1}, \ldots, \frac{m_n}{1}$ generate M_b as A_b-module.

Exercise 24. (Non-zero divisors as multiplicative set, 3 points)

1. Let $a \in \ker(A \to S^{-1}A)$; we have $\frac{a}{1} = 0$ in $S^{-1}A$ i.e. there is a $s \in S$ such that $sa = 0$ in A. So if $a \neq 0$, s is a zero-divisor. Contradiction. So $a = 0$. Hence the injectivity of $A \to S^{-1}A$.
   For a multiplicative set $S \subsetneq S'$ containing S, consider the localization $g : A \to S'^{-1}A$. Pick a $s' \in S' \backslash S$. By definition of S, $s' \in A$ is a zero-divisor. Thus there is a $A \ni a \neq 0$ such that $s'a = 0$ in A. So we get $g(a) = \frac{a}{1} = 0$. i.e. g is not injective.

Solutions to be handed in before Monday May 11, 4pm.

2. If $\frac{a}{s} \in S^{-1}A$ is not a zero-divisor then for any $S^{-1}A \ni \frac{b}{s'} \neq 0$, with $b \in A$ and $s' \in S$, $\frac{a}{s}\frac{b}{s'} \neq 0$. Since $A \to S^{-1}A$ is injective (according to the first question), for any $A \ni b \neq 0$, $\frac{b}{1} \neq 0$; in particular $\frac{ab}{s} \neq 0$ i.e. for any $s' \in S$, $s'ab \neq 0$ in A. As a result we get that for any $A \ni b \neq 0$ $ab \neq 0$ i.e. a is not a zero divisor. Thus $a \in S$ and $\frac{a}{s}\frac{s}{a} = 1$ in $S^{-1}A$.

3. Under the assumption of this question, we have $S \subset A^*$ and since a unit cannot be a zero divisor ($A \neq 0$), we actually have $A^* = S$. Using the first question, we only have to check that $f : A \to S^{-1}A$ is surjective: for $\frac{a}{s} \in S^{-1}A$, since $s \in S = A^*$, consider $s^{-1}a \in A$; since $ss^{-1}a - a = 0$, we get $f(s^{-1}a) = \frac{s^{-1}a}{1} = \frac{a}{s}$ in $S^{-1}A$.

Exercise 25. (Flat scalar extensions, 5 points)

1. $\mathbb{Z} \to \mathbb{F}_p$: $\mathbb{F}-p$ is not flat over $\mathbb{Z}$ as shown by the inclusion $f : \mathbb{Z} \hookrightarrow \mathbb{Z}$, $k \mapsto pk$. Tensoring with $\mathbb{F}_p$, we get that $f \otimes \mathrm{id}_{\mathbb{F}_p} : \mathbb{F}_p \to \mathbb{F}_p$, is $k \mapsto pk$ which is the 0 map, in particular it is not injective.

2. $\mathbb{Z} \to \mathbb{Q}$: $\mathbb{Q}$ is a flat $\mathbb{Z}$-module: notice first that $\mathbb{Q} \simeq Z_{(0)}$, the localization at the prime ideal $(0) \subset \mathbb{Z}$. Indeed for an injective homomorphism of $\mathbb{Z}$-modules $f : M \hookrightarrow M'$ let $\sum_{i=1}^{n} m_i \otimes \frac{p_i}{q_i} \in \ker(f \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Q}})$; we have

$$\sum_{i=1}^{n} m_i \otimes \frac{p_i}{q_i} = \sum_i m_i \otimes \frac{p_i}{\Pi_{\ell=1}^{n} q_\ell}\Pi_{k\neq i}q_k = \sum_i m_i p_i \Pi_{k\neq i}q_k \otimes \frac{1}{\Pi_\ell q_\ell} = (\sum_i m_i p_i \Pi_{k\neq i}q_k) \otimes \frac{1}{\Pi_\ell q_\ell}$$

and $f(\sum_i m_i p_i \Pi_{k\neq i}q_k) \otimes \frac{1}{\Pi_\ell q_\ell} = 0 \in M' \otimes \mathbb{Q}$. Now since $M' \otimes \mathbb{Q} \simeq M' \otimes \mathbb{Z}_{(0)} \simeq M'_{(0)}$, $M'_{(0)} \ni 0 = f(\sum_i m_i p_i \Pi_{k\neq i}q_k) \otimes \frac{1}{\Pi_\ell q_\ell} = \frac{f(\sum_i m_i p_i \Pi_{k\neq i}q_k)}{\Pi_\ell q_\ell}$ means that there is a $b \in \mathbb{Z}\backslash\{0\}$ such that $f(b\sum_i m_i p_i \Pi_{k\neq i}q_k) = bf(\sum_i m_i p_i \Pi_{k\neq i}q_k) = 0 \in M'$. As f is injective, we get $b\sum_i m_i p_i \Pi_{k\neq i}q_k = 0 \in M$. In particular, $\frac{\sum_i m_i p_i \Pi_{k\neq i}q_k}{\Pi_\ell q_\ell} = 0 \in M_{(0)}$. Thus $f \otimes \mathrm{id}_{\mathbb{Z}}$ is injective.

3. $A \to A[x]$: by definition, $A[x]$ is a free A-module ($(x^i)_{i\in\mathbb{N}}$ being a basis) so it is in particular flat.

4. Actually the question is trivial (as noticed by G.Andreychev) since $\mathbb{Q}[x,y]/(y^2 - x)$ is a $\mathbb{Q}$-vector space, as such it is a free $\mathbb{Q}$-module. So it is flat over $\mathbb{Q}$ ans since $\mathbb{Q}$ is flat over $\mathbb{Z}$, we get, by Proposition 5.6, that $\mathbb{Q}[x,y]/(y^2 - x)$ is flat over $\mathbb{Z}$.
The question is more interesting for $\mathbb{Z} \to \mathbb{Z}[x,y]/(y^2-x)$: Let us prove that $\mathbb{Z}[x,y]/(y^2-x)$ is a flat $\mathbb{Z}$-module. This ring homomorphism can be decomposed as

$$\mathbb{Z} \to \mathbb{Z}[x] \to \mathbb{Z}[x][y] \simeq \mathbb{Z}[x,y] \to \mathbb{Z}[x,y]/(y^2 - x)$$

the last homomorphism being the quotient by the principal ideal of $\mathbb{Z}[x,y]$ generated by $y^2 - x$. We have just seen that $\mathbb{Z}[x]$ is a flat $\mathbb{Z}$-module. Now, since Euclidean division by monic polynomials works in $A[y]$ for any ring A, we have:

$$\begin{array}{l}\text{Let } A \neq 0 \text{ be a ring and } a \in A, \text{ then } \varphi : A^2 \to A[y]/(y^2 - a),\ (b,c) \mapsto b\overline{y} + c \\ \text{is an isomorphism of } A\text{-modules}\end{array} \qquad (*)$$

(see the proof below) Applying this remark to $A = \mathbb{Z}[x]$ and $a = x$, we get that $\mathbb{Z}[x,y]/(y^2 - x)$ is a free (thus flat) $\mathbb{Z}[x]$-module. As a conclusion (Proposition 5.6), $\mathbb{Z}[x,y]/(y^2 - x)$ is a flat $\mathbb{Z}$-module.

Beweis. Notice that (even if A happened to have zero-divisors) for any non-zero polynomial $f = \sum_{i=0}^{n} b_i y^i \in A[y]$, ith $b_n \neq 0$ $\deg((y^2 - a)f) = 2 + \deg(f)$ since its leading term is $b_n y^{n+2} \neq 0$. So, let $(b,c) \in \ker(\varphi)$ we have $by + c \in (y^2 - a)$ in $A[y]$; but any

non-zero polynomial in $(y^2 - a)$ has degree at least 2. Thus $by + c = 0 \in A[y]$ i.e. $(b, c) = (0, 0)$, proving that φ is injective.
Now let us prove by induction that any polynomial $f \in A[y]$ can be written $f = (y^2-a)g+h$ where $g, h \in A[y]$ and $\deg(h) < 2$. It is clear for polynomial of degree 0 and 1. Now let $k > 0$ such that the property is true for polynomials of degree at most k. Given $f = \sum_{i=0}^{k+1} b_i y^i \in A[x]$ of degree $k+1$ (i.e. $b-k+1 \neq 0$), $f' = f - b_{k+1}y^{k-1}(y^2-a) \in A[y]$ has degree < 0 so by our induction hypothesis, there are $g, h \in A[y]$ with $\deg(h) < 2$, such that $f' = (y^2 - a)g + h$. So we get $f = (y^2 - a)(g + b_{k+1}y^{k-1}) + h$ and $\deg(h) < 2$. Thus by induction, the property is true.
So let $\overline{f} \in A[y]/(y^2 - a)$ and $f \in A[y]$ in its preimage. By the above property, we can write $f = (y^2 - a)g + h$ for some $g, h \in A[y]$ with $\deg(h) < 2$. In particular $\overline{f} = h \mod (y^2 - a)$. Writing $h = by + c$, we get $\varphi(b, c) = \overline{f}$ proving the surjectivity of φ. □

Exercise 26. (Localization, 4 points)

1. We have $1 = 1 + x_2 \cdot 0 \in S$, and for $f_1(x_1) + x_2g_1(x_1),\ f_2(x_1) + x_2g_2(x_1) \in S$, with $f_1 \neq 0,\ f_2 \neq 0$, we have

$$(f_1(x_1)+x_2g_1(x_1))(f_2(x_1)+x_2g_2(x_1)) = f_1f_2+x_2(f_1g_2+f_2g_1)+x_2^2(g_1g_2) = f_1f_2+x_2(f_1g_2+f_2g_1)$$

in A and $f_1f_2 \neq 0$ since they belongs to the integral domain $k[x_1] \subset A$. So $(f_1(x_1) + x_2g_1(x_1))(f_2(x_1) + x_2g_2(x_1)) \in S$ i.e. S is a multiplicative set.
Since we have a ring isomorphism $k[x_1][x_2] \simeq k[x_1, x_2]$, and and inclusion of rings $k[x_1] \subset k(x_1)$, we have an induced ring homomorpism $\alpha : A \to k(x_1)[x_2]/(x_2^2)$. For $f + x_2g \in S$, we have

$$(f + x_2g)\frac{f - x_2g}{f^2} = \frac{f^2 - x_2^2g^2}{f^2} = \frac{f^2}{f^2} = 1$$

thus $\alpha(S)$ is contained in the group of invertible elements of $k(x_1)[x_2]/(x_2^2)$. Now let $\varphi : A \to B$ be a ring homomorphism such that $g(S) \subset B^*$. Define $\overline{\varphi} : k(x_1)[x_2]/(x_2^2) \to B$ by $\frac{f(x_1)+x_2g(x_2)}{h(x_1)} \mapsto (\varphi(h(x_1)))^{-1}\varphi(f(x_1) + x_2g(x_2))$. It is well-defined since $k[x_1]\backslash\{0\} \subset S$ so its image under φ is contained B^*; moreover any other representative of a given $\frac{f(x_1)+x_2g(x_1)}{h(x_1)}$ is of the form $\frac{h'(x_1)f(x_1)+x_2h'(x_1)g(x_1)}{h'(x_1)h(x_1)}$ for some $h' \neq 0$ and using that φ is a ring homomorphism

$$\begin{aligned}(\varphi(h'(x_1)h(x_1)))^{-1}\varphi(h'(x_1)(f(x_1) + x_2g(x_1))) &= \varphi(h(x_1))^{-1}\varphi(h'(x_1))^{-1}\varphi(h'(x_1))\varphi(f(x_1) + x_2g(x_1)) \\ &= (\varphi(h(x_1)))^{-1}\varphi(f(x_1) + x_2g(x_2)).\end{aligned}$$

One check that $\overline{\varphi}$ is a ring homomorphism the same way

$$\begin{aligned}\overline{\varphi}(\frac{f_1 + x_2g_1}{h_1} + \frac{f_2 + x_2g_2}{h_2}) &= \overline{\varphi}(\frac{h_2(f_1 + x_2g_1) + h_1(f_2 + x_2g_2)}{h_1h_2}) \\ &= \varphi(h_1h_2)^{-1}\varphi(h_2(f_1 + x_2g_1) + h_1(f_2 + x_2g_2)) \\ &= \varphi(h_2)^{-1}\varphi(h_1)^{-1}(\varphi(h_2)\varphi(f_1 + x_2g_1) + \varphi(h_1)\varphi(f_2 + x_2g_2)) \\ &= \varphi(h_1)^{-1}\varphi(f_1 + x_2g_1) + \varphi(h_2)^{-1}\varphi(f_2 + x_2g_2) \\ &= \overline{\varphi}(\frac{f_1 + x_2g_1}{h_1}) + \overline{\varphi}(\frac{f_2 + x_2g_2}{h_2})\end{aligned}$$

and

$$\begin{aligned}\overline{\varphi}(\frac{f_1 + x_2g_1}{h_1} \cdot \frac{f_2 + x_2g_2}{h_2}) &= \overline{\varphi}(\frac{(f_1 + x_2g_1)(f_2 + x_2g_2)}{h_1h_2}) \\ &= \varphi(h_1h_2)^{-1}\varphi((f_1 + x_2g_1)(f_2 + x_2g_2)) \\ &= \varphi(h_1)^{-1}\varphi(f_1 + x_2g_1)\varphi(h_2)^{-1}\varphi(f_2 + x_2g_2) \\ &= \overline{\varphi}(\frac{f_1 + x_2g_1}{h_1}) \cdot \overline{\varphi}(\frac{f_2 + x_2g_2}{h_2})\end{aligned}$$

finally $\overline{\varphi}(1) = \overline{\varphi}(\frac{1}{1}) = \varphi(1)^{-1}\varphi(1) = 1$. And a direct calculation shows that $\varphi = \overline{\varphi} \circ \alpha$. Moreover if $\beta : k(x_1)[x_2]/(x_2^2) \to B$ is a ring homomorphism satisfying $\varphi = \beta \circ \alpha$. Then for any $h \in k[x_1]\backslash\{0\} \subset S$, $\beta(\alpha(h)) = \beta(h) = \varphi(h) = \overline{\varphi}(h)$ from which we also see that $\beta(h)$ is invertible and since $1 = \beta(1) = \beta(\frac{h}{h}) = \beta(\frac{1}{h})\beta(h)$ we have $\beta(\frac{1}{h}) = \beta(h)^{-1}$. We also have $\beta(\alpha(f + x_2 g)) = \beta(f + x_2 g) = \varphi(f + x_2 g) = \overline{\varphi}(f + x_2 g)$. Thus

$$\beta(\frac{f + x_2 g}{h}) = \beta(\frac{1}{h})\beta(f + x_2 g) = \beta(h)^{-1}\beta(f + x_2 g) = \varphi(h)^{-1}\varphi(f + x_2 g) = \overline{\varphi}(\frac{f + x_2 g}{h})$$

Hence the uniqueness of such $\overline{\varphi}$. As a conclusion α satisfies the universal property of the localization; so it is isomorphic to the localization of A with respect to S.

2. Look at the first projection $p_1 : A \times B \to A$ which is a ring homomorphism satisfying $p_1(S) = \{1\} \subset A^*$. Let $g : A \times B \to C$ be a ring homomorphism such that $g(S) \subset C^*$. Since $(1,0)^2 = (1,0)$, we get $g((1,0)) = g((1,0)^2) = g((1,0))^2$ in C which, as $g((1,0))$ is invertible, yields $g((1,0)) = 1$.
   Now, define a map $f : A \to C$ by $a \mapsto g((a,0))$. It is well-defined and it is a ring homomorphism: $f(1) = g((1,0)) = 1$ by the above discussion.
   For any $a, a' \in A$, using that g is a ring homomorphism, we get:

$$f(a + a') = g((a + a', 0)) = g((a,0) + (a',0)) = g((a,0)) + g((a',0)) = f(a) + f(a')$$

   and $f(aa') = g((aa',0)) = g((a,0)(a',0)) = g((a,0))g((a',0)) = f(a)f(a')$.
   To see that $g = f \circ p_1$ it is sufficient to prove that $g((0,b)) = 0$ for any $b \in B$ (since $g((a,b)) = g((a,0) + (0,b)) = g((a,0)) + g((0,b)))$; but for any $b \in B$, $(0,b)(1,0) = (0,0)$ so that (g ring homomorphism) $0 = g((0,0)) = g((0,b))g((1,0)) = g((0,b)) \cdot 1$.
   Let us prove the uniqueness of f: let $h : A \to C$ be a ring homomorphism satisfying $g = h \circ p_1$. For $a \in A$, we have $h(a) = h(p_1((a,0))) = g((a,0)) = f(a)$; thus $f = h$. So $p_1 : A \times B \to A$ is the localization with respect to S.

3. ($\Rightarrow$) Assume $M \to S^{-1}M$ is bijective. Let $s \in S$. If $M \xrightarrow{s\cdot} M$ is not injective, then there is a $m \in M\backslash\{0\}$ such that $sm = 0 \in M$. But this means that $\frac{m}{1} = 0 \in S^{-1}M$ i.e. that $M \to S^{-1}M$ is not injective. Contradiction. So for any $s \in S$, $M \xrightarrow{s\cdot} M$ is injective.
   Now, let us prove the surjectivity of the homomorphisms $M \xrightarrow{s\cdot} M$. Take a $s \in S$. Given a $m \in M$, since $M \to S^{-1}M$ is surjective, there is a $n \in M$ such that $\frac{n}{1} = \frac{m}{s}$ in $S^{-1}M$ which means that there is a $s' \in S$, such that $s'(sn - m) = 0 \in M$. But by the above discussion $M \xrightarrow{s'\cdot} M$ is injective; thus $sn = m$ i.e. $M \xrightarrow{s\cdot} M$ is surjective.

   ($\Leftarrow$) If $m \in \ker(M \to S^{-1}M)$, then $\frac{m}{1} = 0 \in S^{-1}M$ i.e. there is a $s \in S$ such that $sm = 0 \in M$. But since $M \xrightarrow{s\cdot} M$ is injective, we get $m = 0$ i.e. $M \to S^{-1}M$.
   Now, consider $\frac{m}{s} \in S^{-1}M$. By sujectivity of $M \xrightarrow{s\cdot} M$, we can find a $n \in M$ such that $m = sn \in M$. We then have $\frac{n}{1} = \frac{m}{s} \in S^{-1}M$. thus $M \to S^{-1}M$ is surjective.