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Summer term 2020

Exercises, Algebra I (Commutative Algebra) – Week 14

Exercise 69. (Regular sequences vs. system of parameters)
Assume $(A, \mathfrak{m})$ is a Noetherian local Cohen–Macaulay ring of dimension d and $a_1, \ldots, a_d \in \mathfrak{m}$. Show that $a_1, \ldots, a_d$ is a regular sequence if and only if $a_1, \ldots, a_d$ is a sequence of parameters.

Exercise 70. (Regular sequences and dimension)
Adapt the arguments in the proof of Lemma 19.9 to solve Exercise 19.10: Any M-regular sequence $a_1, \ldots, a_r$ satisfies $\dim(M/(a_1, \ldots, a_r)M) + r = \dim(M)$.

Exercise 71. (Hausdorff)
Prove the assertion of (vi) in Remark 19.1: The topology on an A-module M induced by a filtration $M \supset M_1 \supset M_2 \supset \cdots$ is Hausdorff if and only if $\bigcap M = (0)$.

Exercise 72. (E_8-singularity)
Let k be an algebraically closed field and A the localization of $k[x, y, z]/(x^2 + y^3 + z^5)$ at the maximal ideal $\mathfrak{m} = (x, y, z)$. Show that A is factorial but not regular. (This is essentially the only normal surface singularity which is factorial.) Proceed in three steps:

(i) Show that $z \in A$ is a prime element.

(ii) The ring homomorphism $k[X, Y] \to A_z$, given by $X \mapsto x/z^3$ and $Y \mapsto y/z^2$ is injective and there exists a (unique) $t \in k[X, Y]$ with image $1/z$. Then $k[X, Y]_t \cong A_z$.

(iii) Conclude from A_z factorial that A is also factorial.

Exams: Please, make sure to read the following instructions (German, English) by July 24^{th}.

Please try to do these exercise. Although there will probably no time to discuss these, they are a useful preparation for the exam.