

SCRIPT FOR THE COURSES ALGEBRAIC TOPOLOGY I + II (WS 24/25 AND SS 25): BASIC INTRODUCTION TO HOMOTOPY THEORY

LÜCK, WOLFGANG

ABSTRACT. This manuscript is based on the lecture courses *Algebraic Topology I* from the winter term 24/25 and *Algebraic Topology II* from the summer term 25. It consists of an introduction to homotopy theory starting with the basic definitions about homotopy groups and ending with Serre's proof that the stable stems are finite except in dimension zero.

1. INTRODUCTION

This manuscript is based on the lecture courses *Algebraic Topology I* from the winter term 24/25 and *Algebraic Topology II* from the summer term 25. These are the fourth and fifth course in a series of all together five courses on topology. The first three were *Introduction to Topology and Geometry*, *Topology I*, and *Topology II*, which I presented in the Summer term 23, the winter 23/24 and the summer term 24.

The table of contents shall give an overview about the material presented in these courses. Key words are: homotopy groups, cofibrations, fibrations, Whitehead Theorems, Hurewicz Theorem, Excision Theorem due to Blakers and Massey, Freudenthal's Suspension Theorem, stable homotopy theory, stable stems, spectra, Eilenberg MacLane spaces, bordism theory, the Pontrjagin Thom construction, spectral sequences, and Serre's Theorem about the finiteness of stable stems.

1.1. Prerequisites. One does not need all the material of the courses *Introduction to Topology and Geometry*, *Topology I*, and *Topology II* but at least the following:

- Topological spaces;
- Fundamental groups;
- CW-complexes;
- Coverings;
- Chain complexes and modules over a ring;
- Singular and cellular (co-)homology including the Universal Coefficient Theorem.
- Basics about smooth manifolds;
- Basics about bundles and vector bundles;

No previous knowledge about homotopy theory is required except for the fundamental group. In the course *Topology I*, and *Topology II* I just covered [17].

1.2. Acknowledgements. I thank Dominik Kirstein and Christian Kremer who were assistants for the courses, the tutors of the exercises, and the students. I received many valuable comments about the manuscript from them.

CONTENTS

1. Introduction	1
-----------------	---

Date: WS 24/25 + SS 25.

1.1.	Prerequisites	1
1.2.	Acknowledgements	1
2.	Basic definitions and properties of homotopy groups	4
2.1.	Review of the fundamental group	4
2.2.	Basic definitions and the group structure on homotopy groups	6
2.3.	Functorial properties of homotopy groups	8
2.4.	Homotopy groups and coverings	9
2.5.	The long exact sequence of a pair and a triple	10
2.6.	Connectivity	13
2.7.	Homotopy groups and colimits	14
3.	Hopf's Degree Theorem	14
3.1.	Some basics about differential topology and the mapping degree	14
3.2.	The proof of Hopf's Degree Theorem	15
3.3.	The homotopy groups of the n -sphere in the degree $\leq n$	20
4.	The Cellular Approximation Theorem	21
5.	The Whitehead Theorem	24
6.	CW-Approximation	29
7.	The category of compactly generated spaces	33
7.1.	Open subsets	33
7.2.	The retraction functor k	34
7.3.	Mapping spaces, product spaces, and subspaces	34
7.4.	Basic features of the category of compactly generated spaces	34
8.	Cofibrations	35
8.1.	Basics about cofibrations	35
8.2.	Cofibrations and NDR-pairs	39
8.3.	Relative CW -complexes are cofibrations	41
8.4.	Well-pointed spaces	41
8.5.	Comparing pointed homotopy and homotopy	42
8.6.	The Homotopy Theorem for pushouts and cofibrations	43
8.7.	(Pointed) cylinders, cones and suspensions	46
8.8.	Turning a map into a cofibration	48
8.9.	The Cofiber Sequence	49
8.10.	Group structures on the Puppe Sequence	51
9.	Fibrations	51
9.1.	Basics about fibrations	51
9.2.	Turning a map into a fibration	53
9.3.	Homotopy Theorem for pullbacks and fibrations	54
9.4.	The fiber transport	55
9.5.	Homotopy equivalences and fibrations	55
9.6.	The Fiber Sequence	56
9.7.	Group structures on pointed sets associated to the Fiber Sequence	58
9.8.	The adjunction between suspension and loop spaces	59
9.9.	Locally trivial bundles are fibrations	59
9.10.	Duality between cofibrations and fibrations	60
10.	The long exact homotopy sequence associated to a fibration	60
10.1.	The homotopy sequence	60
10.2.	The Hopf fibration	62
10.3.	Homotopy groups of loop spaces	62
10.4.	Homotopy groups of classifying spaces BG	63
10.5.	On the homotopy groups of some classical Lie groups	63
11.	The Excision Theorem of Blakers-Massey	64
11.1.	The statement of the Excision Theorem of Blakers-Massey	64

11.2.	The proof of the Excision Theorem of Blakers-Massey	65
11.3.	The Excision Theorem for $n = 0$	70
11.4.	Some applications of the Excision Theorem of Blakers-Massey	71
11.5.	The Freudenthal Suspension Theorem	74
11.6.	Stable homotopy groups	76
12.	The Hurewicz Theorem	77
12.1.	The Hurewicz homomorphism	77
12.2.	The Hurewicz Theorem	78
12.3.	The relative Hurewicz Theorem	80
12.4.	Applications of the Hurewicz Theorem	81
13.	Moore spaces	82
14.	Eilenberg-MacLane spaces	86
15.	Postnikov towers	89
16.	Spectra	90
16.1.	Basics about spectra	90
16.2.	Homology and cohomology theories for pointed spaces and pairs	92
16.3.	The homology and cohomology theory assigned to a spectrum	95
16.4.	Brown's Representation Theorem	99
16.5.	Basics about vector bundles	100
16.6.	Thom spaces and Thom spectra	102
16.7.	Topological K -theory	103
16.8.	Outlook	104
17.	The Pontrjagin-Thom Construction	105
17.1.	ξ -bordism	105
17.2.	The Pontrjagin-Thom construction of ξ -bordism	106
17.3.	The Pontrjagin-Thom construction and bordism for stable systems of bundles	109
17.4.	Unoriented bordism	110
17.5.	The unoriented bordism ring	113
17.6.	Conventions about orientations	114
17.7.	Oriented bordism	115
17.8.	The oriented bordism ring	115
17.9.	Framed bordism	116
18.	The Atiyah-Hirzebruch spectral sequence for homology for arbitrary filtrations	119
19.	Basic notions and facts about homological spectral sequences	124
20.	The Atiyah-Hirzebruch spectral sequence for homology	127
20.1.	Chern characters	132
21.	(Co)homology with local coefficients	134
21.1.	Modules over a category	134
21.2.	(Co)homology for local coefficient systems coming from a fibration	135
21.3.	Poincaré duality for non-orientable closed manifolds	138
22.	(Co)homology theories over a space	141
22.1.	Homology theories over a space	141
22.2.	Cohomology theories over a space	143
22.3.	The Leray-Hirsch Theorem	144
22.4.	The Thom Isomorphism	145
23.	The construction of the Leray-Serre spectral sequence	147
24.	Some applications of the Leray-Serre spectral sequence	150
25.	Naturality properties of spectral sequences	155
26.	Basic notions and facts about cohomological spectral sequences	156
26.1.	The convergence problem cohomology	159

26.2.	Outlook: The Completion Theorem of Atiyah and Segal	161
26.3.	Multiplicative structures	162
27.	Edge homomorphisms	163
28.	Applications of the cohomological Leray-Serre spectral sequence	166
28.1.	The Gysin sequence	166
28.2.	The first Chern class of a principal S^1 -bundle	168
28.3.	A variation of the Leray-Hirsch Theorem	168
28.4.	An example concerning the Leray-Hirsch Theorem 28.6	170
28.5.	Computation of the singular cohomology ring with rational coefficients of $U(n)$	172
28.6.	Singular Cohomology of ΩS^3	172
28.7.	Computation of π_1^s	175
28.8.	The rational singular cohomology of $K(\mathbb{Z}, n)$	176
29.	Serre classes	180
29.1.	Basics about Serre classes	180
29.2.	Some applications using spectral sequences	181
30.	The Hurewicz Theorem modulo a Serre class	185
31.	The Whitehead Theorem modulo a Serre ideal	186
32.	(Stable) Homotopy groups of spheres	188
32.1.	Finite generation and rational computations	188
32.2.	p -coprimary computations	191
33.	Exercises part I	196
34.	Exercises part II	200
	References	206

2. BASIC DEFINITIONS AND PROPERTIES OF HOMOTOPY GROUPS

2.1. Review of the fundamental group. We briefly recall the notion and the basic properties of the *fundamental group* $\pi_1(X, x)$ of a pointed space (X, x)

Let $X = (X, x)$ be a *pointed space*, i.e., a topological space X with an explicit choice of a so called *base point* $x \in X$. Denote by I the unit interval $[0, 1]$. A *loop at x in X* is a map of pairs $w: (I, \partial I) \rightarrow (X, \{x\})$. Elements in $\pi_1(X, x)$ are homotopy classes of loops at x in X . Note that this means that two loops $w, w': (I, \partial I) \rightarrow (X, \{x\})$ are homotopic if there is a homotopy $h: I \times I \rightarrow X$ such that $h(s, 0) = w(s)$, $h(s, 1) = w'(s)$, and $h(0, t) = h(1, t) = x$ hold for all $s, t \in I$. Given two loops v, w at x in X , we get a new loop $v * w$ by putting

$$v * w(s) = \begin{cases} v(2s) & \text{if } s \in [0, 1/2]; \\ w(2s - 1) & \text{if } s \in [1/2, 1]. \end{cases}$$

The group structure on $\pi_1(X, x)$ is given by the formula $[v] \cdot [w] = [v * w]$. The unit element is given by the constant loop $c_x: (I, \partial I) \rightarrow (X, \{x\})$ sending $s \in I$ to x and the inverse of $[w]$ is given by $[w^-]$ for $w^-: (I, \partial I) \rightarrow (X, \{x\})$, $s \mapsto w(1 - s)$.

Here are some basic properties of the fundamental group:

- A pointed map $f: (X, x) \rightarrow (Y, y)$ induces a group homomorphism

$$\pi_1(f, x): \pi_1(X, x) \rightarrow \pi_1(Y, y), \quad [w] \mapsto [f \circ w]$$

which depends only on the pointed homotopy class of f ;

- We get a functor from the category of pointed spaces to the category of groups;

- Given pointed spaces (X_i, x_i) for $i = 0, 1$, we get from the two projections $\text{pr}_i: (X_0 \times X_1, (x_0, x_1)) \rightarrow (X_i, x_i)$ for $i = 0, 1$ an isomorphism

$$\begin{aligned} \pi_1(\text{pr}_0, (x_0, x_1)) \times \pi_1(\text{pr}_1, (x_0, x_1)) &: \pi_1(X_0 \times X_1, (x_0, x_1)) \\ &\xrightarrow{\cong} \pi_1(X_0, x_0) \times \pi_1(X_1, x_1); \end{aligned}$$

- Let $p: X \rightarrow Y$ be a covering. Choose $x \in X$ and put $y = p(x)$. Then the induced map $\pi_1(p, x): \pi_1(X, x) \rightarrow \pi_1(Y, y)$ is injective.

If p is actually a G -covering for the group G and X is path connected, then we obtain an exact sequence of groups

$$1 \rightarrow \pi_1(X, x) \xrightarrow{\pi_1(p, x)} \pi_1(Y, y) \xrightarrow{\partial} G \rightarrow 1;$$

- The mapping degree induces an isomorphism $\pi_1(S^1) \xrightarrow{\cong} \mathbb{Z}$;
- There is a *Seifert-van Kampen Theorem*. It allows to read off a presentation of the fundamental group from the 2-skeleton X_2 and implies that the inclusion $X_2 \rightarrow X$ induces an isomorphism $\pi_1(X_2, x) \rightarrow \pi_1(X, x)$ for any choice of base point $x \in X$. In particular $\pi_1(X, x)$ vanishes if X is a CW -complex which has no 1-cells. Moreover, $\pi_1(\bigvee_{i=1}^r S^1, x)$ is the free group of rank r . So in general $\pi_1(X)$ is not abelian. Actually any group occurs as $\pi_1(X, x)$ for a 2-dimensional path connected CW -complex X ;
- We get a functor T_1 from the fundamental groupoid $\Pi(X)$ to the category of groups by sending an object in $\Pi(X)$ which is a point $x \in X$ to $\pi_1(X, x)$. A morphism $[u]: x \rightarrow y$ in $\Pi(X)$ is a homotopy class $[u]$ relative endpoints of paths $u: I \rightarrow X$ from x to y . It is sent to the group homomorphism $T_1([u]): \pi_1(X, x) \rightarrow \pi_1(X, y)$ mapping $[w]$ to $[u^{-1} * w * u]$. Recall that the composite of the morphism $[u]: x \rightarrow y$ and $[v]: y \rightarrow z$ in $\Pi(X)$ is given by $[v] \circ [u] = [u * v]$. One easily checks $T_1([v] \circ [u]) = T_1([v]) \circ T_1([u])$. Recall that there is a canonical isomorphism of $\pi_1(X, x)$ to the opposite of the group $\text{aut}_{\Pi(X)}(x)$;
- Consider two maps $f_0, f_1: X \rightarrow Y$. Let $h: X \times I \rightarrow Y$ be a homotopy between f_0 and f_1 . Choose a base point x and put $y_i = f_i(x)$ for $i = 0, 1$. Let $u: I \rightarrow Y$ be the path from y_0 to y_1 given by $u(t) = h(x, t)$. We obtain a group isomorphism $T_1([u]): \pi_1(Y, y_0) \xrightarrow{\cong} \pi_1(Y, y_1)$ and the following diagram of groups commutes

$$(2.1) \quad \begin{array}{ccc} & & \pi_1(Y, y_0) \\ & \nearrow^{\pi_1(f_0, x_0)} & \downarrow \cong T_1([u]) \\ \pi_1(X, x) & & \pi_1(Y, y_1) \\ & \searrow_{\pi_1(f_1, x_1)} & \end{array}$$

Now consider a *pointed pair* (X, A, x) , i.e., a topological pair (X, A) together with a choice of a base point $x \in A$. Define the set $\pi_1(X, A, x)$ as the set of homotopy classes relative $\{0\}$ of maps of pairs $w: (I, \partial I) \rightarrow (X, A)$ satisfying $w(0) = x$, or, equivalently, of homotopy classes of maps of triads $(I; \{0\}, \{1\}) \rightarrow (X, \{x\}, A)$. Note that $w(1)$ is not necessarily equal to x and is only required to lie in A . If $A = \{x\}$, then $\pi_1(X, A, x)$ agrees with $\pi_1(X, x)$. In general there is no group structure on $\pi_1(X, A, x)$.

Define $\pi_0(X)$ as the *set of path components* of X . Note that this is the same as the homotopy classes of maps $\{\bullet\} \rightarrow X$. If (X, x) is pointed map, we sometimes

write $\pi_0(X, x)$ instead of $\pi_0(X)$ to indicate that the set $\pi_0(X)$ has a preferred base point, namely the path component containing x .

Next we construct the (in some sense exact) sequence

$$(2.2) \quad \pi_1(A, x) \xrightarrow{\pi_1(i, x)} \pi_1(X, x) \xrightarrow{\pi_1(j, x)} \pi_1(X, A, x) \xrightarrow{\partial_1} \pi_0(A) \\ \xrightarrow{\pi_0(i)} \pi_0(X) \xrightarrow{\pi_0(j)} \pi_0(X, A) \rightarrow \{*\}.$$

The map $\pi_1(i, x)$ is the group homomorphism given by the inclusion $i: (A, x) \rightarrow (X, x)$. The map of sets $\pi_1(j, x): \pi_1(X, x) \rightarrow \pi_1(X, A, x)$ is the obvious map given by forgetting that $w(1) = x$ holds in connection with $\pi_1(X, x)$. The map ∂_1 sends $[w]$ represented by $w: (I, \partial I) \rightarrow (X, A)$ to the path component of A containing $w(1)$. The map of sets $\pi_0(i)$ sends the path component C of A to the path component D of X containing $i(C)$. The pointed set $\pi_0(X, A)$ is the quotient of the set $\pi_0(X)$ by collapsing the image of $\pi_0(i): \pi_0(A) \rightarrow \pi_0(X)$ to one element and $\pi_0(j)$ is the obvious projection.

This sequence is exact in the following sense. The image of $\pi_1(i, x)$ is the preimage under $\pi_1(j, x)$ of the element in $\pi_1(X, A, x)$ given by the constant map $c_x: I \rightarrow X$. The image of $\pi_1(j, x)$ is the preimage under ∂_1 of the path component of A containing x . The image of ∂_1 is the preimage under $\pi_0(i)$ of the path component of X containing x . The image of $\pi_0(i)$ is the preimage under $\pi_0(j)$ of the preferred base point in $\pi_0(X, A)$. The map $\pi_0(j)$ is surjective.

2.2. Basic definitions and the group structure on homotopy groups. Next we want to generalize the notion of the fundamental group to the notion of the homotopy group in degree n for all integers $n \geq 1$. The basic idea is to replace $I = [0, 1]$ and $\partial I = \{0, 1\}$ by the n -dimensional cube

$$I^n = \prod_{i=1}^n [0, 1] = \{(s_1, s_2, \dots, s_n) \mid s_i \in [0, 1]\}$$

where we define

$$\partial I^n = \{(s_1, s_2, \dots, s_n) \mid s_i \in I, \exists i \in \{1, 2, \dots, n\} \text{ with } s_i \in \{0, 1\}\}.$$

Given a pointed space X , we define the set $\pi_n(X, x)$ to be the set of homotopy classes $[f]$ of maps of pairs $f: (I^n, \partial I^n) \rightarrow (X, \{x\})$. Given two elements $[f]$ and $[g]$, we define their product $[f] \cdot [g]$ by the homotopy class of the map of pairs $f * g: (I^n, \partial I^n) \rightarrow (X, \{x\})$ defined by

$$(2.3) \quad f * g(s_1, s_2, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & \text{if } s_1 \in [0, 1/2]; \\ g(2s_1 - 1, s_2, \dots, s_n) & \text{if } s_1 \in [1/2, 1]. \end{cases}$$

The unit is given by the homotopy class $[c_x]$ of the constant map $c_x: (I^n, \partial I^n) \rightarrow (X, \{x\})$. The inverse of $[f]$ is the class $[f^-]$ for the map $f^-: (I^n, \partial I^n) \rightarrow (X, \{x\})$ sending $(s_1, s_2, \dots, s_n)$ to $(1 - s_1, s_2, \dots, s_n)$. The proof that this defines a group $\pi_n(X, x)$ called n -homotopy group of the pointed space (X, x) is the essentially the same as the one for $\pi_1(X)$. The construction above for $n = 1$ agrees with the definition of $\pi_1(X, x)$ presented in Subsection 2.1. If we define I^0 to be $\{\bullet\}$ and $\partial I^0 = \emptyset$, the definition of the set $\pi_0(X, x)$ above agrees with the definition of $\pi_0(X)$ as the set of path components of X . Recall that $\pi_0(X)$ has no group structure in general and the $\pi_1(X, x)$ is not necessarily commutative. However, the following lemma is true.

Lemma 2.4. *The group $\pi_n(X, x)$ is abelian for $n \geq 2$.*

Proof. The basic observation is that in the cube I^n for $n \geq 2$ there is enough room to show $[f] \cdot [g] = [g] \cdot [f]$. The desired homotopy is indicated for $n = 2$ by the following sequence of pictures:

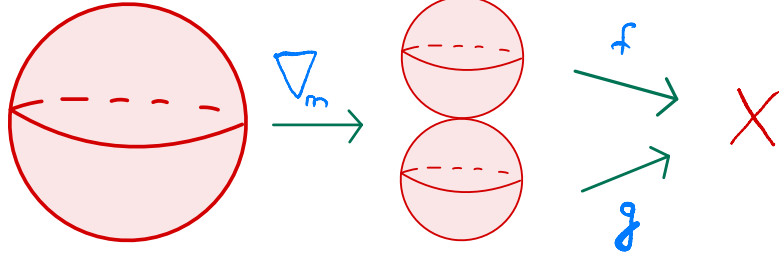


The homotopy begins by shrinking the domains of f and g to smaller subcubes of I^n , where the region outside these subcubes is mapped to the basepoint. After this has been done, there is room to slide the two subcubes around anywhere in I^n as long as they stay disjoint. Hence for $n \geq 2$ they can be slid past each other, interchanging their positions. Then to finish the homotopy, the domains of f and g can be enlarged back to their original size. The whole process can actually be done using just the coordinates s_1 and s_2 , keeping the other coordinates fixed. $\square$

Any map of pairs $f: (I^n, \partial I^n) \rightarrow (X, \{x\})$ factorizes in a unique way over the projection $\text{pr}: I^n \rightarrow I^n/\partial I^n$ to a pointed map $\bar{f}: (I^n/\partial I^n, \partial I^n/\partial I^n) \rightarrow (X, x)$. Obviously this is compatible with the notion of a homotopy of maps of pairs $(I^n, \partial I^n) \rightarrow (X, \{x\})$ and of a pointed homotopy of pointed maps $(I^n/\partial I^n, \partial I^n/\partial I^n) \rightarrow (X, x)$. There is an obvious homeomorphism of pairs $(I^n/\partial I^n, \partial I^n/\partial I^n) \rightarrow (S^n, \{s\})$ for the fixed base point $s = (1, 0, \dots, 0) \in S^n$. Hence we can interpret an element in $\pi_n(X, x)$ as a pointed homotopy of pointed maps $(S^n, s) \rightarrow (X, x)$. The multiplication in this picture is given as follows. Consider pointed maps $f_i: (S^n, s) \rightarrow (X, x)$ for $i = 0, 1$. Let $[f_0]$ and $[f_1]$ be their classes in $\pi_n(X, x)$. They define a pointed map $f_0 \vee f_1: (S^n \vee S^n, s) \rightarrow (X, x)$. Let

$$(2.5) \quad \nabla_n: S^n \rightarrow S^n \vee S^n$$

be the so-called *pinching map* which is obtained by collapsing the equator $S^{n-1} \subseteq S^n$ given by $\{(x_0, x_1, \dots, x_n) \in S^n \mid x_n = 0\}$ to a point. Then $[f_0] \cdot [f_1]$ is represented by the composite $f_0 \vee f_1 \circ \nabla_n$, as illustrated in the following picture

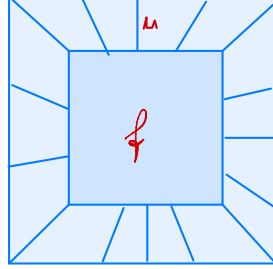


The interpretation in terms of pointed maps $(S^n, s) \rightarrow (X, x)$ is useful for some theoretical considerations and in connection with CW -complexes, whereas the picture in terms of maps of pairs $(I^n, \partial I^n) \rightarrow (X, \{x\})$ is better suited for some constructions and proofs, e.g., the proof of Lemma 2.4.

2.3. Functorial properties of homotopy groups. Obviously a map of pointed spaces $f: (X, x) \rightarrow (Y, y)$ induces a group homomorphism $\pi_n(f, x): \pi_n(X, x) \rightarrow \pi_n(Y, y)$ for $n \geq 1$ by composition. We get a functor from the category of pointed spaces to the category of abelian groups by sending (X, x) to $\pi_n(X, x)$ for $n \geq 2$, whereas for $n = 1$ we get a functor from the category of pointed spaces to the category of groups by sending (X, x) to $\pi_1(X, x)$ for $n = 1$. We get a functor from the category of topological spaces to sets by sending X to $\pi_0(X)$.

Obviously $\pi_n(f, x)$ depends only on the pointed homotopy class of f and $\pi_0(f)$ depends only on the homotopy class of f .

Next we construct for every $n \geq 2$ a functor T_n from $\Pi(X)$ to the category of abelian groups. It sends an object in $\Pi(X)$, which is a point x in X , to the abelian group $\pi_n(X, x)$. Consider a morphism $[u]: x \rightarrow y$ in $\Pi(X)$ represented by a path u in X from x to y . It is sent to the homomorphism of abelian groups $T_n([u]): \pi_n(X, x) \rightarrow \pi_n(X, y)$ defined as follows. Consider $[f] \in \pi_n(X, x)$ represented by the map $f: (I^n, \partial I^n) \rightarrow (X, \{x\})$. Consider a new map $uf: (I^n, \partial I^n) \rightarrow (X, \{x\})$ by shrinking the domain of f to a smaller concentric cube in I^n , then inserting the path u on each radial segment in the shell between this smaller cube and ∂I^n , as indicated in the picture below



We leave it to the reader to figure out the elementary proof that this definition is independent of all the choices and indeed yields a functor T_n from $\Pi(X)$ to the category of abelian groups.

Recall that there is a canonical isomorphism of $\pi_1(X, x)$ to the opposite of the group $\text{aut}_{\Pi(X)}(x)$. Hence we obtain from the functor T_n above the structure of a $\mathbb{Z}[\pi_1(X, x)]$ -module on $\pi_n(X, x)$ for $n \geq 2$. Recall that for $n = 1$ the functor T_1 is actually given by conjugation.

Consider two maps $f_0, f_1: X \rightarrow Y$. Let $h: X \times I \rightarrow Y$ be a homotopy between f_0 and f_1 . Choose a base point x and put $y_i = f_i(x)$ for $i = 0, 1$. Let $u: I \rightarrow Y$ be the path from y_0 to y_1 given by $u(t) = h(x, t)$. For $n \geq 2$ we obtain an isomorphism of abelian groups $T_n([u]): \pi_n(Y, y_0) \xrightarrow{\cong} \pi_n(Y, y_1)$ and the following diagram of abelian groups commutes

$$(2.6) \quad \begin{array}{ccc} & & \pi_n(Y, y_0) \\ & \nearrow \pi_n(f_0, x_0) & \downarrow \cong \quad T_n([u]) \\ \pi_n(X, x) & & \pi_n(Y, y_1) \\ & \searrow \pi_n(f_1, x_1) & \end{array}$$

A consequence of (2.1) and (2.6) is that a homotopy equivalence $f: X \rightarrow Y$ induces for every $x \in X$ and $n \geq 1$ a bijection $\pi_n(f, x): \pi_n(X, x) \xrightarrow{\cong} \pi_n(Y, f(x))$. Moreover, for a path connected space X the isomorphism class of $\pi_n(X, x)$ is independent of the choice of $x \in X$. Therefore we sometimes write $\pi_n(X)$ instead of $\pi_n(X, x)$.

Given pointed spaces (X_i, x_i) for $i = 0, 1$, we get from the two projections $\text{pr}_i: (X_0 \times X_1, (x_0, x_1)) \rightarrow (X_i, x_i)$ for $i = 0, 1$ a group isomorphism

$$\begin{aligned} \pi_n(\text{pr}_0, (x_0, x_1)) \times \pi_n(\text{pr}_1, (x_0, x_1)) &: \pi_n(X_0 \times X_1, (x_0, x_1)) \\ &\xrightarrow{\cong} \pi_n(X_0, x_0) \times \pi_n(X_1, x_1) \end{aligned}$$

for every $n \geq 1$.

2.4. Homotopy groups and coverings.

Theorem 2.7 (Homotopy groups and covering). *Let $p: X \rightarrow Y$ be a covering. Choose a base point $x \in X$ and put $y = p(x)$. Then for $n \geq 2$ the map induced by p*

$$\pi_n(p, x): \pi_n(X, x) \rightarrow \pi_n(Y, y)$$

is bijective.

Proof. Consider a map $f: S^n \rightarrow Y$ sending the base point s to y . Since $n \geq 2$ holds by assumption, S^n is simply connected. Hence the image of $\pi_1(f, x)$ is contained in the image $\pi_1(p, x)$. A standard theorem about coverings and liftings implies that we can find a lift $\tilde{f}: (S^n, s) \rightarrow (X, x)$ of f , i.e., a pointed map $\tilde{f}$ satisfying $p \circ \tilde{f} = f$. This shows that $\pi_n(p, x)$ is surjective for $n \geq 2$.

Injectivity follows from the standard theorem about lifting homotopies along coverings, the argument is the same as for the injectivity of $\pi_1(p, x)$. This standard theorem says that for a map $u: Z \rightarrow X$ and a homotopy $h: Z \times I \rightarrow Y$ with $h_0 = p \circ u$ we can find precisely one homotopy $\tilde{h}: Z \times I \rightarrow X$ with $p \circ \tilde{h} = h$ and $\tilde{h}_0 = u$. $\square$

Theorem 2.7 implies for a connected CW -complex X that for the universal covering $p: \tilde{X} \rightarrow X$ and any choice of base points $\tilde{x} \in \tilde{X}$ and $x \in X$ with $p(\tilde{x}) = x$ the map $\pi_n(p, \tilde{x}): \pi_n(\tilde{X}, \tilde{x}) \rightarrow \pi_n(X, x)$ is bijective for $n \geq 2$. If we additionally assume that $\tilde{X}$ is contractible, we get $\pi_n(X, x) = 0$ for $n \geq 2$. In particular we get for any base point $s \in S^1$ and $n \geq 1$

$$(2.8) \quad \pi_n(S^1, s) \cong \begin{cases} \mathbb{Z} & \text{if } n = 1; \\ \{1\} & \text{if } n \geq 2, \end{cases}$$

since the universal covering of S^1 is given by the map $\mathbb{R} \rightarrow S^1$ sending $t \in \mathbb{R}$ to $\exp(2\pi it)$.

2.5. The long exact sequence of a pair and a triple. Consider a pointed pair (X, A, x) , i.e., a pair of topological spaces (X, A) together with a base point $x \in A$. We can consider I^{n-1} as the subspace of I^n given by those points $(s_1, s_2, \dots, s_n)$ satisfying $s_n = 0$. Let J_{n-1} be the subspace of ∂I^n which is the closure of $\partial I^n \setminus I^{n-1}$ in ∂I^n . Explicitly we get

$$J_{n-1} = (\partial I^n \setminus I^{n-1}) \cup \partial I^{n-1} = \{(s_1, s_2, \dots, s_n) \in I^n \mid (\exists i \in \{1, 2, \dots, (n-1)\} \text{ with } s_i \in \{0, 1\}) \text{ or } (s_n = 1)\}.$$

Obviously $I_{n-1} \cup J_{n-1} = \partial I^n$ and $I_{n-1} \cap J_{n-1} = \partial I^{n-1}$. For $n \geq 1$ we define the set $\pi_n(X, A, x)$ as the set homotopy classes $[f]$ of maps of triples $f: (I^n, \partial I^n, J_{n-1}) \rightarrow (X, A, \{x\})$. For $n \geq 2$, this becomes a group by defining $[f_0] \cdot [f_1]$ by the class $[f_0 * f_1]$ for the maps of triples $f_0 * f_1: (I^n, \partial I^n, J_{n-1}) \rightarrow (X, A, \{x\})$ defined in (2.3). There is no reasonable group structure on $\pi_1(X, A, x)$. It is not hard to check that this group structure on $\pi_n(X, A, x)$ for $n \geq 2$ is well-defined and that the following result is true.

Lemma 2.9. *The group $\pi_n(X, A, x)$ is abelian for $n \geq 3$.*

Note that there is an obvious identification of $\pi_n(X, \{x\}, x)$ defined above and of $\pi_n(X, x)$ defined in Subsection 2.2.

Obviously we obtain a functor from the category of pointed pairs to the category of groups by $\pi_2(X, A, x)$ and a functor from the category of pointed pairs to the category of abelian groups by $\pi_n(X, A, x)$ for $n \geq 3$. If two maps $f_0, f_1: (X, A, x) \rightarrow (Y, B, y)$ of pointed pairs are homotopic as maps of pointed pairs, then $\pi_n(f_0, x) = \pi_n(f_1, x)$ holds for $n \geq 1$. Given a pair (X, A) , one can define a functor T_n from the fundamental groupoid $\Pi(A)$ of A to the category of groups or abelian groups

by assigning to a point $x \in A$ the homotopy group $\pi_2(X, A, x)$ or $\pi_n(X, A, x)$ for $n \geq 3$, the construction appearing in Subsection 2.3 for a space X carries directly over. In particular $\pi_n(X, A, x)$ inherits the structure of a $\mathbb{Z}[\pi_1(A, x)]$ -module for $n \geq 3$.

A map of triples $f: (I^n, \partial I^n, J_{n-1}) \rightarrow (X, A, \{x\})$ factorizes uniquely through the projection $\text{pr}: (I^n, \partial I^n, J_{n-1}) \rightarrow (I^n/J_{n-1}, \partial I^n/J_{n-1}, J_{n-1}/J_{n-1})$ to a map of pointed pairs $(I^n/J_{n-1}, \partial I^n/J_{n-1}, J_{n-1}/J_{n-1}) \rightarrow (X, A, x)$. There is a homeomorphism $(I^n/J_{n-1}, \partial I^n/J_{n-1}, \{J_{n-1}/J_{n-1}\}) \xrightarrow{\cong} (D^n, S^{n-1}, \{s\})$ of triples. Hence one can define $\pi_n(X, A, x)$ also the set of homotopy classes of pointed maps of pointed pairs $(D^n, S^{n-1}, s) \rightarrow (X, A, x)$. The multiplication in this picture is given as follows. Consider pointed maps of pointed pairs $f_i: (D^n, S^{n-1}, s) \rightarrow (X, A, x)$ for $i = 0, 1$. Let $[f_0]$ and $[f_1]$ be their classes in $\pi_n(X, A, x)$. They define a pointed map of pointed pairs $f_0 \vee f_1: (D^n \vee D^n, S^{n-1} \vee S^{n-1}, s) \rightarrow (X, A, x)$. Let

$$(2.10) \quad \nabla'_n: D^n \rightarrow D^n \vee D^n$$

be the so-called *pinching map* which is obtained by collapsing $D^{n-1} \subseteq D^n$ given by $\{(x_1, \dots, x_n) \in D^n \mid x_n = 0\}$ to a point. Note that ∇'_n is a map of pointed pairs $(D^n, S^{n-1}, s) \rightarrow (D^n \vee D^n, S^{n-1} \vee S^{n-1}, s)$ and its restriction to (S^{n-1}, s) is the pinching map defined in (2.5). Then $[f_0] \cdot [f_1]$ is represented by the composite $f_0 \vee f_1 \circ \nabla'_n$.

Define for $n \geq 2$ a group homomorphism $\partial_n: \pi_n(X, A, x) \rightarrow \pi_1(A, x)$ by sending the class $[f]$ of the map of pointed pairs $f: (D^n, S^{n-1}, s) \rightarrow (X, A, s)$ to the pointed homotopy class of maps of pointed spaces obtained by restriction to (S^{n-1}, s) . Let $i: A \rightarrow X$ and $j: X \rightarrow (X, A)$ be the canonical inclusions.

Theorem 2.11. *We obtain a long exact sequence of groups infinite to the left*

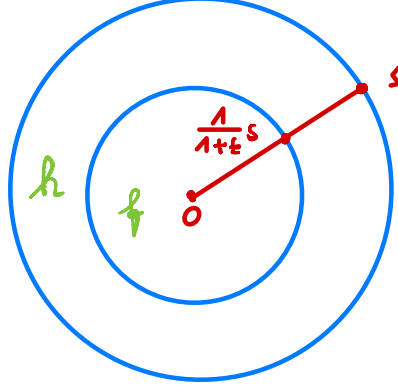
$$\begin{aligned} \cdots \xrightarrow{\partial_{n+2}} \pi_{n+1}(A, x) &\xrightarrow{\pi_{n+1}(i, x)} \pi_{n+1}(X, x) \xrightarrow{\pi_{n+1}(j, x)} \pi_{n+1}(X, A, x) \\ &\xrightarrow{\partial_{n+1}} \pi_n(A, x) \xrightarrow{\pi_n(i, x)} \pi_n(X, x) \xrightarrow{\pi_n(j, x)} \cdots \\ &\cdots \xrightarrow{\pi_2(j, x)} \pi_2(X, A, x) \xrightarrow{\partial_2} \pi_1(A, x) \xrightarrow{\pi_1(i, x)} \pi_1(X, x). \end{aligned}$$

Proof. We only show exactness at $\pi_n(X, A, x)$, the proofs at the other places are analogous. Consider a pointed map $f: (S^n, s) \rightarrow (X, x)$. The image of the class $[f]$ under the composite $\pi_n(X, x) \xrightarrow{\pi_n(j, x)} \pi_n(X, A, x) \xrightarrow{\partial_n} \pi_{n-1}(A, x)$ is by construction represented by the constant map $c_x: S^{n-1} \rightarrow A$ and hence zero. This shows $\text{im}(\pi_n(j, x)) \subseteq \ker(\partial_n)$. It remains to prove $\ker(\partial_n) \subseteq \text{im}(\pi_n(j, x))$.

Consider a map of pointed pairs $f: (D^n, S^{n-1}, s) \rightarrow (X, A, x)$ such that $[f]$ lies in the kernel of $\partial_n: \pi_n(X, A, x) \rightarrow \pi_{n-1}(A, x)$. Then the map of pointed spaces $f|_{S^{n-1}}: (S^{n-1}, s) \rightarrow (A, x)$ is nullhomotopic as pointed map of pointed spaces. Choose such a nullhomotopy $h: S^{n-1} \times I \rightarrow A$ with $h_0 = f|_{S^{n-1}}$ and $h_1 = c_x$ for the constant function. Note that $h(s, t) = x$ holds for $t \in I$. Define a homotopy $k: D^n \times I \rightarrow X$ as follows:

$$k(z, t) = \begin{cases} f((t+1)z) & \text{if } \|z\| \leq \frac{1}{1+t}; \\ h\left(\frac{z}{\|z\|}, 2\|z\| - \frac{2}{1+t}\right) & \text{if } \|z\| \geq \frac{1}{1+t}. \end{cases}$$

Roughly speaking, k_t is given on the disk $\frac{1}{1+t} \cdot D^n$ of radius $\frac{1}{1+t}$ by f with an appropriate scaling of z and on the annulus between $\frac{1}{1+t} \cdot S^{n-1}$ and S^{n-1} by the restriction of the homotopy h to $S^1 \times [2 - 2/(1+t), 1]$



We have $k(z, 0) = f(z)$ for $z \in D^n$, $k(s, t) = x$ for $t \in I$, $k(z, t) \in A$ for $z \in S^{n-1}$ and $t \in I$, and $k(z, 1) = x$ for $z \in S^{n-1}$. Hence k is a homotopy of pointed maps of pointed pairs $(D^n, S^{n-1}, s) \rightarrow (X, A, x)$ between $k_0 = f$ and k_1 . Therefore $[f] = [k_1]$ holds in $\pi_n(X, A, x)$. Since $k_1(z) = x$ holds for $z \in S^{n-1}$, the class $[k_1]$ lies in the image of $\pi_n(j, x): \pi_n(X, x) \rightarrow \pi_n(X, A, x)$. Hence we get $\text{im}(\pi_n(j, x)) = \ker(\partial_n)$. $\square$

Remark 2.12. Let G be any group. Then we can find a path connected pointed 2-dimensional CW -complex (A, x) with $\pi_1(A, x) \cong G$. Let X be the cone over A . Then we obtain a path connected pointed 3-dimensional CW -complex (X, A, x) such that $\pi_2(X, A, x) \cong \pi_1(A, x) \cong G$ holds by Theorem 2.11.

Remark 2.13. One can combine the exact sequences appearing in Theorem 2.2 and Theorem 2.11 to an exact sequence

$$(2.14) \quad \begin{aligned} \cdots \xrightarrow{\partial_{n+2}} \pi_{n+1}(A, x) \xrightarrow{\pi_{n+1}(i, x)} \pi_{n+1}(X, x) \xrightarrow{\pi_{n+1}(j, x)} \pi_{n+1}(X, A, x) \\ \xrightarrow{\partial_{n+1}} \pi_n(A, x) \xrightarrow{\pi_n(i, x)} \pi_n(X, x) \xrightarrow{\pi_n(j, x)} \cdots \xrightarrow{\pi_2(j, x)} \pi_2(X, A, x) \xrightarrow{\partial_2} \pi_1(A, x) \\ \xrightarrow{\pi_1(i, x)} \pi_1(X, x) \xrightarrow{\partial_1} \pi_0(A) \xrightarrow{\pi_0(i)} \pi_0(X) \xrightarrow{\pi_0(j)} \pi_0(X, A) \rightarrow \{*\} \end{aligned}$$

which is compatible with the group structures as long as these exist.

It is not hard to check that one obtains for a triple (X, B, A) and a base point $x \in A$ an exact sequence of the shape

$$(2.15) \quad \begin{aligned} \cdots \xrightarrow{\partial_{n+2}} \pi_{n+1}(B, A, x) \xrightarrow{\pi_{n+1}(i, x)} \pi_{n+1}(X, A, x) \xrightarrow{\pi_{n+1}(j, x)} \pi_{n+1}(X, B, x) \\ \xrightarrow{\partial_{n+1}} \pi_n(B, A, x) \xrightarrow{\pi_n(i, x)} \pi_n(X, A, x) \xrightarrow{\pi_n(j, x)} \\ \cdots \xrightarrow{\pi_2(j, x)} \pi_2(X, B, x) \xrightarrow{\partial_2} \pi_1(B, A, x) \xrightarrow{\pi_1(i, x)} \pi_1(X, A, x) \xrightarrow{\pi_1(j, x)} \pi_1(X, B, x) \\ \xrightarrow{\partial_1} \pi_0(B, A) \xrightarrow{\pi_0(i)} \pi_0(X, A) \xrightarrow{\pi_0(j)} \pi_0(X, B) \rightarrow \{*\} \end{aligned}$$

which is compatible with the group structures as long as these exist.

Remark 2.16 (Long exact homotopy sequence of a pointed map). Let $f: (X, x) \rightarrow (Y, y)$ be a map of pointed spaces. Denote by $\text{cyl}(f)$ its mapping cylinder. Note that

we obtain a pointed pair $(\text{cyl}(f), X, x)$. The canonical projection $\text{cyl}(f) \rightarrow Y$ is a homotopy equivalence and satisfies $\text{pr}(x) = y$. Hence it induces an isomorphism of groups $\pi_n(\text{pr}, x): \pi_n(\text{cyl}(f), x) \xrightarrow{\cong} \pi_n(Y, y)$ for $n \geq 1$ and a bijection $\pi_0(\text{cyl}(f)) \xrightarrow{\cong} \pi_0(Y)$. Define $\pi_n(f, x) = \pi_n(\text{cyl}(f), X, x)$ for $n \geq 1$. Let $\pi_0(f)$ be the quotient of $\pi_0(Y)$ obtained by collapsing the image of $\pi_0(f): \pi_0(X) \rightarrow \pi_0(Y)$. Then the long exact sequence of the pointed pair $(\text{cyl}(f), X, x)$ of (2.14) yields the long exact homotopy sequence of the map f

$$(2.17) \quad \cdots \xrightarrow{\partial_{n+2}} \pi_{n+1}(X, x) \xrightarrow{\pi_{n+1}(f, x)} \pi_{n+1}(Y, y) \rightarrow \pi_{n+1}(f, x) \\ \xrightarrow{\partial_{n+1}} \pi_n(X, x) \xrightarrow{\pi_n(f, x)} \pi_n(Y, y) \rightarrow \cdots \rightarrow \pi_2(f, x) \xrightarrow{\partial_2} \pi_1(X, x) \\ \xrightarrow{\pi_1(f, x)} \pi_1(Y, y) \xrightarrow{\partial_1} \pi_0(X) \xrightarrow{\pi_0(f)} \pi_0(Y) \rightarrow \pi_0(f) \rightarrow \{1\}.$$

Note that $\pi_n(f, x)$ can have two different meanings in the notation above.

2.6. Connectivity.

Definition 2.18 (Connectivity). A space X is called *0-connected* if $\pi_0(X)$ consists of one point, or, equivalently, X is path connected. It is called *n-connected* for $n \geq 1$ if X is path connected and $\pi_k(X, x)$ is trivial for every base point x and $1 \leq k \leq n$. It is called *∞ -connected* or *weakly contractible* if it is path connected and $\pi_k(X, x)$ is trivial for every base point x and $k \geq 1$.

A map $f: X \rightarrow Y$ is called *0-connected* if the induced map $\pi_0(f): \pi_0(X) \rightarrow \pi_0(Y)$ is surjective. It is called *n-connected* for $n \geq 1$, if the map $\pi_0(f): \pi_0(X) \rightarrow \pi_0(Y)$ is bijective and for every base point x the map $\pi_k(f, x): \pi_k(X, x) \rightarrow \pi_k(Y, f(x))$ is bijective for $1 \leq k < n$ and surjective for $k = n$. It is called *∞ -connected* or a *weak homotopy equivalence* if the map $\pi_0(f): \pi_0(X) \rightarrow \pi_0(Y)$ is bijective and for every base point x and $k \geq 1$ the map $\pi_k(f, x): \pi_k(X, x) \rightarrow \pi_k(Y, f(x))$ is bijective. Note that f is *n-connected* if and only if $\pi_0(f): \pi_0(X) \rightarrow \pi_0(Y)$ is surjective and the group $\pi_k(f, x)$ defined in Remark 2.16 is trivial for $1 \leq k \leq n$.

A pair (X, A) is called *n-connected* for $n \geq 0$ or $n = \infty$, if the inclusion $i: A \rightarrow X$ is *n-connected*. This is equivalent to the condition that $\pi_0(X, A)$ and $\pi_k(X, A, x)$ for $1 \leq k \leq n$ are trivial.

Remark 2.19. One easily checks that the following assertions are equivalent for a pointed space (X, x) and $n \geq 1$:

- $\pi_n(X, x)$ is trivial for any base point $x \in X$;
- Every map $S^n \rightarrow X$ is nullhomotopic;
- Every map $S^n \rightarrow X$ extends to a map $D^{n+1} \rightarrow X$.

This implies that the following assertions are equivalent for a space X and $n \geq 0$ or $n = \infty$:

- X is *n-connected*;
- Given any k with $0 \leq k \leq n$, every map $S^k \rightarrow X$ is nullhomotopic;
- Given any k with $0 \leq k \leq n$, every map $S^k \rightarrow X$ extends to a map $D^{k+1} \rightarrow X$.

Moreover, the following assertions are equivalent for a pair (X, A) and $n \geq 0$ or $n = \infty$:

- (X, A) is *n-connected*;
- Given any k with $0 \leq k \leq n$, every map $(D^k, S^{k-1}) \rightarrow (X, A)$ is homotopic relative S^{k-1} to a map $D^k \rightarrow A$;
- Given any k with $0 \leq k \leq n$, every map $(D^k, S^{k-1}) \rightarrow (X, A)$ is homotopic through such maps to a map $D^k \rightarrow A$;
- Given any k with $0 \leq k \leq n$, every map $(D^k, S^{k-1}) \rightarrow (X, A)$ is homotopic through such maps to a constant map $D^k \rightarrow A$.

2.7. Homotopy groups and colimits.

Theorem 2.20 (Homotopy groups and colimits). *Let X be a topological Hausdorff space with a sequence of closed subspaces $X_0 \subset X_1 \subseteq \dots \subseteq X$ such that X is the union of the X_i -s and carries the colimit topology.*

Then for every $x_0 \in X$ and $n \geq 1$ the canonical group homomorphism induced by the inclusions $j_k: X_k \rightarrow X$

$$\operatorname{colim}_{k \rightarrow \infty} \pi_n(j_k, x_0): \operatorname{colim}_{k \rightarrow \infty} \pi_n(X_k, x_0) \rightarrow \pi_n(X, x_0)$$

is bijective. Also the map of sets

$$\operatorname{colim}_{k \rightarrow \infty} \pi_0(j_k): \operatorname{colim}_{k \rightarrow \infty} \pi_0(X_k) \rightarrow \pi_0(X, x_0)$$

is bijective.

Proof. We first prove that for any compact subset $C \subseteq X$ there exists a natural number k with $C \subseteq X_k$. Suppose that for every $k \geq 0$ we have $C \not\subseteq X_k$. Then we can choose a sequence of $x_0, x_1, x_2, \dots$ in C and a strictly monotone increasing function $j: \mathbb{Z}^{\geq 0} \rightarrow \mathbb{Z}^{\geq 0}$ with $x_i \in X_{j(i)} \setminus X_{j(i-1)}$ for $i = 1, 2, \dots$. Put $S = \{x_0, x_1, x_2, \dots\}$. Obviously S is infinite. Let $T \subseteq S$ be any subset. Note that the intersection $T \cap X_k$ is finite and hence a closed subset of X_k for $k = 0, 1, 2, \dots$. Since X carries the colimit topology, T is closed in X . Hence S is a discrete subset of X . As C is compact and S is a closed subset of C , the set S is compact. As S is a discrete and compact set, it must be finite, a contradiction.

We only treat the case $n \geq 1$, the case $n = 0$ is analogous. Consider an element $[f] \in \pi_n(X, x_0)$ represented by a pointed map $f: (S^n, s) \rightarrow (X, x_0)$. Then image of f lies already in X_i for some $i \geq 0$. Hence $[f]$ lies in the image of the map $\pi_n(X_i, x_0) \rightarrow \pi_n(X, x_0)$ induced by the inclusion $X_i \rightarrow X$. This implies that $[f]$ lies in the image of $\operatorname{colim}_{k \rightarrow \infty} \pi_n(j_k, x_0): \operatorname{colim}_{k \rightarrow \infty} \pi_n(X_k, x_0) \rightarrow \pi_n(X, x_0)$. Hence this map is surjective. To prove injectivity, we consider an element $[g]$ in its kernel. There exists $i \geq 0$ and an element $[g'] \in \pi_n(X_i, x_0)$ such that the structure map $\pi_n(X_i, x_0) \rightarrow \operatorname{colim}_{j \rightarrow \infty} \pi_n(X_j, x_0)$ sends $[g']$ to $[g]$. The element $[g']$ lies in the kernel of the map $\pi_n(X_i, x_0) \rightarrow \pi_n(X, x_0)$ induced by the inclusion $X_i \rightarrow X$. If $g': (S^n, s) \rightarrow (X_i, x_0)$ is a representative of $[g']$, there is a nullhomotopy $h: S^n \times I \rightarrow X$ for it. The image of h lies already in X_j for some j with $i \leq j$. Hence the image of $[g']$ under the map $\pi_n(X_i, x_0) \rightarrow \pi_n(X_j, x_0)$ induced by the inclusion $X_i \rightarrow X_j$ is trivial. This implies that $[g]$ is trivial. $\square$

3. HOPF'S DEGREE THEOREM

In this section we give the proof of the following theorem.

Theorem 3.1 (Hopf's Degree Theorem). *Let M be a connected oriented closed smooth manifold of dimension $n \geq 1$. Then the degree defines a bijection*

$$\deg: [M, S^n] \rightarrow \mathbb{Z}.$$

3.1. Some basics about differential topology and the mapping degree.

Its proof needs some preparation. We recall some basic facts about differential topology and the mapping degree.

- Let M and N be smooth manifolds. Then a (continuous) map $f: M \rightarrow N$ is homotopic to a smooth map. If two smooth maps $M \rightarrow N$ are homotopic, then one can find a smooth homotopy between them.
- Let M and N be smooth manifolds and $L \subseteq N \setminus \partial N$ be a smooth submanifold without boundary. Then any smooth map $f: M \rightarrow N$ with $f(\partial M) \cap L = \emptyset$ is smoothly homotopic relativ ∂M to a map $g: M \rightarrow N$ which is *transversal* to L at every $x \in M$, i.e., we have either $f(x) \notin L$ or we

have $f(x) \in L$ and $T_x f(T_x M) + T_{f(x)} L = T_{f(x)} N$. If $\dim(M) + \dim(L) < \dim(N)$ holds, then f is transversal to L if and only if $f(M) \cap L = \emptyset$.

- If $L = \{y\}$ for $y \in N \setminus \partial N$, then we say that y is a *regular value* of f if f is transversal to $\{y\}$.
- Every smooth map $f: M \rightarrow N$ has a regular value $y \in N \setminus \partial N$. Actually the points in $N \setminus \partial N$ for which y is not a regular value has measure zero in N by the Theorem of Sard.

If $y \in N \setminus \partial N$ is a regular value of f , M is compact, and $\dim(M) = \dim(N)$, then $f^{-1}(y)$ is finite and for every $x \in f^{-1}(y)$ the differential induces an isomorphism $T_x f: T_x M \rightarrow T_y N$.

- Let $f: M \rightarrow N$ be a map of connected oriented compact smooth oriented manifolds of dimension n such that $f(\partial M) \subseteq \partial N$ holds. Let $y \in N \setminus \partial N$ be any regular value. For $x \in f^{-1}(y) \subseteq M \setminus \partial M$ the orientations on M and N yield orientations on the finite dimensional vector spaces $T_x M$ and $T_y N$. Define $\epsilon(x) \in \{\pm 1\}$ to be 1 if $T_x f: T_x M \xrightarrow{\cong} T_y N$ respects these orientations and to be -1 otherwise.

Recall degree of f is the natural number for which $H_n(f): H_n(M, \partial M) \rightarrow H_n(N, \partial N)$ sends $[M, \partial M]$ to $\deg(f) \cdot [N, \partial N]$. We get

$$(3.2) \quad \deg(f) = \sum_{x \in f^{-1}(y)} \epsilon(x).$$

This formula is well-known for $\partial M = \partial N = \emptyset$. The proof in this case extends directly to the more general case above. Or one considers the map of closed oriented manifolds $f \cup_{\partial f} f: M \cup_{\partial M} M \rightarrow N \cup_{\partial N} N$ for $\partial f: \partial M \rightarrow \partial N$ given by $f|_{\partial M}$.

- Let M be a smooth Riemannian manifold and $x \in M \setminus \partial M$. Then there is an $\epsilon > 0$, an open subset U of M containing x , and a diffeomorphism called *exponential map*

$$(3.3) \quad \exp_x: D_\epsilon^\circ T_x M := \{v \in T_x M \mid \|v\| < \epsilon\} \rightarrow U$$

such that the differential $T_0 \exp_x: T_0(T_x M) \rightarrow T_x M$ of $\exp_x$ at $0 \in T_x M$ becomes the identity under the canonical identification $T_0(T_x M) = T_x M$.

3.2. The proof of Hopf's Degree Theorem. We prove Hopf's Degree Theorem 3.1 by induction over the dimension $n = \dim(M)$. If $n = 1$, then M is diffeomorphic to S^1 and elementary covering theory shows that the degree induces a bijection $\deg: [S^1, S^1] \xrightarrow{\cong} \mathbb{Z}$. The induction step from $(n-1)$ to $n \geq 2$ is done as follows.

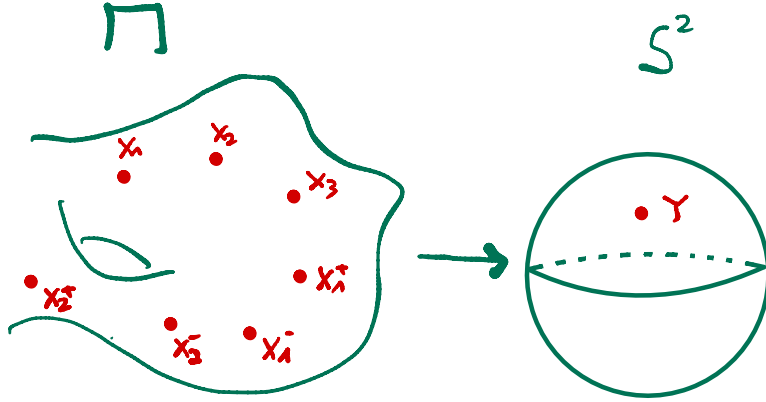
Fix $x \in M$ and an embedding $i: D^n \hookrightarrow M$ such that $i(0) = x$ holds and $T_0 i: T_0 D^n \xrightarrow{\cong} T_x M$ is compatible with the orientations coming from the standard orientation on D^n and the given orientation on M . Define the collapse map $c: M \rightarrow D^n/S^{n-1} \cong S^n$ by sending $i(x)$ for $x \in D^n$ to the element given by x in D^n/S^{n-1} and every point $y \in M \setminus i(D^n)$ to the point S^{n-1}/S^{n-1} in D^n/S^{n-1} . We conclude from (3.2) applied to the regular value $z \in D^n/S^{n-1} = S^n$ given by $0 \in D^n$ of c that $\deg(c) = 1$. Given any $d \in \mathbb{Z}$, there exists a selfmap $u_d: S^n \rightarrow S^n$ with $\deg(u_d) = d$. It can be constructed as the $(n-1)$ -fold suspension of the map $S^1 \rightarrow S^1$ sending z to z^d . Then $\deg(u_d \circ c) = d$. This shows that $\deg: [M, S^n] \rightarrow \mathbb{Z}$ is surjective.

In order to show that $\deg: [M, S^n] \rightarrow \mathbb{Z}$ is injective, we have to show that two smooth maps $f, g: M \rightarrow S^n$ with $\deg(f) = \deg(g)$ are homotopic. Since there is a diffeomorphism $u: S^n \rightarrow S^n$ with degree -1 and $\deg(u \circ f) = -\deg(f)$, we can assume in the sequel that $d = \deg(f) = \deg(g)$ satisfies $d \geq 0$.

We can change f and g up to homotopy and find $y \in S^n$ such that both f and g are smooth and have y as regular value. Then we can write

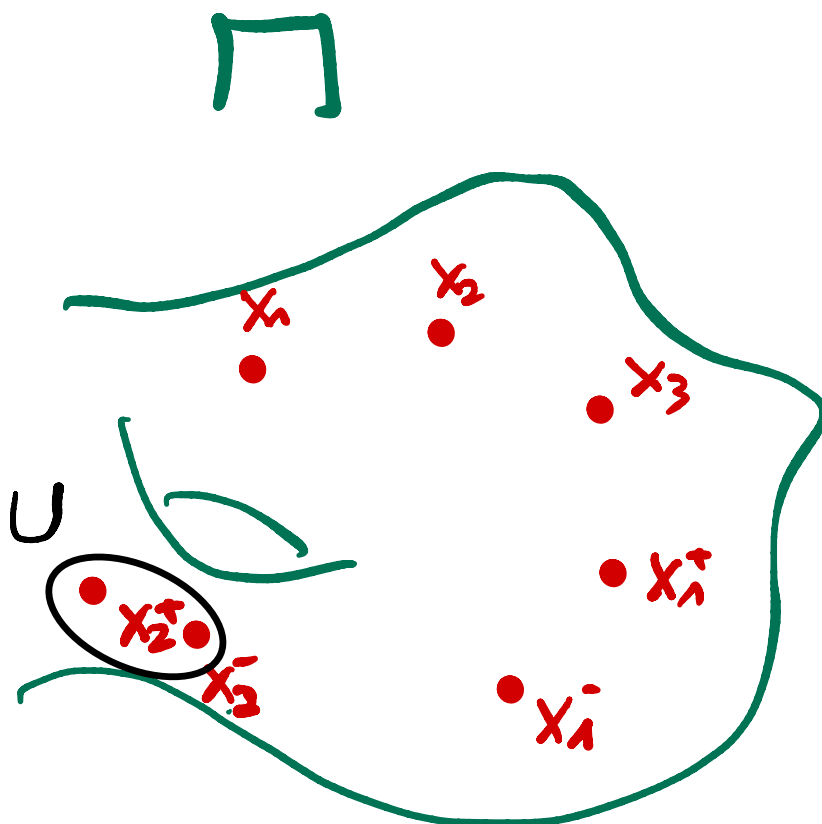
$$f^{-1}(y) = \{x_1, x_2, \dots, x_d\} \amalg \{x_1^+, x_1^-, \dots, x_m^+, x_m^-\}$$

for some $m \geq 0$ such that $\epsilon(x_i) = 1$ for $i = 1, 2, \dots, d$ and $\epsilon(x_j^\pm) = \pm 1$ holds for $j = 1, 2, \dots, m$.



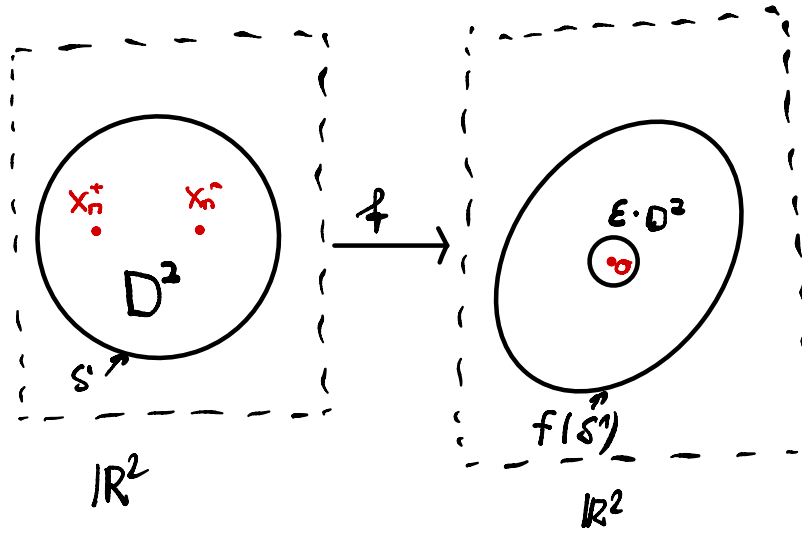
We next describe a procedure how to change f up to homotopy so that $m = 0$, or, equivalently $f^{-1}(y) = \{x_1, x_2, \dots, x_d\}$ holds. This will be done by an inductive procedure where we change f up to homotopy such that $m \geq 1$ becomes $(m - 1)$, in other words, we get rid of the points x_m^+ and x_m^- .

Choose an embedded arc in M joining x_m^+ and x_m^- that does not meet any of the other points in $f^{-1}(y)$. Let U be an open neighbourhood of x_m^- that is diffeomorphic to $\mathbb{R}^n$. Now perform a local homotopy of f along this arc to move x_m^- so close to x_m^+ such that x_m^- lies in U .

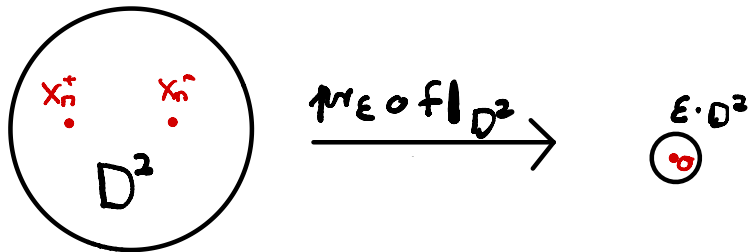


Hence it suffices to prove the following: Given a map $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that f is transversal to $0 \in \mathbb{R}^n$, the preimage $f^{-1}(0)$ consists of precisely two points x_0 and x_1 belonging to the interior of the disk $D^n \subseteq \mathbb{R}^n$, the differential $T_{x_0}f: T_{x_0}\mathbb{R}^n \rightarrow T_0\mathbb{R}^n$ is bijective and reverses the standard orientations, and the differential $T_{x_1}f: T_{x_1}\mathbb{R}^n \rightarrow T_0\mathbb{R}^n$ is bijective and preserves the standard orientations, then we can change f up to homotopy relative $\mathbb{R}^n \setminus D^n$ so that $f^{-1}(0)$ is empty.

Choose $\epsilon > 0$ so small that the image of $S^{n-1} \subseteq \mathbb{R}^n$ under f does not meet the interior of $\epsilon \cdot D^n$.



Let $\text{pr}_\epsilon : \mathbb{R}^n \rightarrow \epsilon \cdot D_n$ be the retraction that sends $x \in \mathbb{R}^n$ to $\frac{\epsilon}{\|x\|} \cdot x$ if $\|x\| \geq \epsilon$, and to x if $\|x\| \leq \epsilon$. Then $\text{pr}_\epsilon \circ f$ induces a map of compact oriented manifolds $(D^n, S^{n-1}) \rightarrow (\epsilon \cdot D^n, \epsilon \cdot S^{n-1})$. By inspecting the preimage of $0 \in \epsilon \cdot D^n$ we conclude from (3.2) that its degree is zero.



Since the following diagram commutes and the vertical maps given by boundary homomorphisms of pairs are isomorphism of infinite cyclic groups respecting the

fundamental classes

$$\begin{array}{ccc} H_n(D^n, S^{n-1}) & \xrightarrow{H_n(f)} & H_n(\epsilon \cdot D^n, \epsilon \cdot S^{n-1}) \\ \cong \downarrow & & \downarrow \cong \\ H_n(S^{n-1}) & \xrightarrow{H_n(f|_{S^{n-1}})} & H_n(\epsilon \cdot S^{n-1}) \end{array}$$

the induced map $(\text{pr}_\epsilon \circ f)|_{S^{n-1}}: S^{n-1} \rightarrow \epsilon \cdot S^{n-1}$ has degree zero and hence is nullhomotopic by the induction hypothesis. This implies that the map $f_0: S^{n-1} \rightarrow \mathbb{R}^n \setminus \{0\}$ induced by f is nullhomotopic and hence extends to a map $f_1: D^n \rightarrow \mathbb{R}^n \setminus \{0\}$. Let $f': \mathbb{R}^n \rightarrow \mathbb{R}^n \setminus \{0\}$ be the map whose restriction to D^n is f_1 and whose restriction to $\mathbb{R}^n \setminus D^n$ agrees with the restriction of f to $\mathbb{R}^n \setminus D^n$. We obtain a homotopy $h: f \simeq f'$ of maps $\mathbb{R}^n \rightarrow \mathbb{R}^n$ by $h(x, t) = t \cdot f'(x) + (1-t) \cdot f$ that is stationary outside the interior of D^n . Since the image of f' does not contain zero, the claim follows.

This argument applies also to g . If $d = 0$, then $\text{im}(f)$ and $\text{im}(g)$ are contained in the contractible subspace $S^n \setminus \{y\}$ of S^n and hence f and g are homotopic. It remains to consider the case $d \geq 1$. Then we can find finite subsets $\{x_1, x_2, \dots, x_d\}$ and $\{x'_1, x'_2, \dots, x'_d\}$ of M such that $f^{-1}(y) = \{x_1, x_2, \dots, x_d\}$ and $g^{-1}(y) = \{x'_1, x'_2, \dots, x'_d\}$ holds and the differentials $T_{x_i}f: T_{x_i}M \rightarrow T_y S^n$ and $T_{x'_i}g: T_{x'_i}M \rightarrow T_y S^n$ are orientation preserving isomorphisms for $i = 1, 2, \dots, d$. Now we can construct a diffeomorphism $a: M \rightarrow M$ which is homotopic to the identity and satisfies $w(x_i) = x'_i$ for $i = 1, 2, \dots, d$. Then g and $g' = g \circ a$ are homotopic, $f^{-1}(y) = g'^{-1}(y) = \{x_1, x_2, \dots, x_d\}$ and the differentials $T_{x_i}f: T_{x_i}M \rightarrow T_y S^n$ and $T_{x_i}g': T_{x_i}M \rightarrow T_y S^n$ are orientation preserving isomorphisms for $i = 1, 2, \dots, d$. It remains to show that f and g' are homotopic.

For this purpose we need the following construction. Let $u_0, u_1: \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n$ be linear $\mathbb{R}$ -isomorphisms which are orientation preserving. Then we can find a homotopy $h: \mathbb{R}^n \times I \rightarrow \mathbb{R}^n$ such that $h_0 = u_0$ and $h_1 = u_1$ holds and $h_t: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a orientation preserving automorphism for $t \in I$. This follows from the fact that $\{A \in GL_n(\mathbb{R}) \mid \det(A) > 0\}$ is path connected for $n \geq 1$. Define the homotopy

$$H: \mathbb{R}^n \times I \rightarrow \mathbb{R}^n, \quad (v, t) \mapsto \begin{cases} h_t(v) & \text{if } \|v\| \leq 1; \\ h_{(2-\|v\|) \cdot t}(v) & \text{if } 1 \leq \|v\| \leq 2; \\ u_0(v) & \text{if } \|v\| \geq 2. \end{cases}$$

Then we have

$$\begin{aligned} H_t^{-1}(0) &= 0 \quad \text{for } t \in I; \\ H_0 &= u_0; \\ H_t(v) &= u_0(v) \quad \text{for } t \in I \text{ and } \|v\| \geq 2; \\ H_1(v) &= \begin{cases} u_1(v) & \text{if } \|v\| \leq 1; \\ h_{(2-\|v\|)}(v) & \text{if } 1 \leq \|v\| \leq 2; \\ u_0(v) & \text{if } \|v\| \geq 2. \end{cases} \end{aligned}$$

So H is a homotopy between $H_0 = u_0$ and H_1 which is stationary on $\{v \in V \mid \|v\| \geq 2\}$ and satisfies $H_t^{-1}(0) = 0$ for $t \in I$ and $H_1(v) = u_1(v)$ for $\|v\| \leq 1$.

Using this construction and the exponential map (3.3), we can change g' by a homotopy to a map $g'': M \rightarrow S^n$, such that for $i = 1, 2, \dots, d$ there are disjoint embedded disks $D_i^n \subseteq M$ such that $0 \in D_i^n$ corresponds to x_i , $f|_{D_i^n} = g''|_{D_i^n}$ holds and we have $f^{-1}(y) = (g'')^{-1}(y) = \{x_1, x_2, \dots, x_d\}$. Let X be the complement in M of the disjoint union $\coprod_{i=1}^d D_i^n \setminus \partial S_i^{n-1}$. This is a manifold with boundary

$\partial X = \coprod_{i=1}^d S_i^{n-1}$ such that $f(X)$ and $g''(X)$ are contained in $S^n \setminus \{y\}$ and $f|_{\partial X} = g''|_{\partial X}$ holds. As $S^n \setminus \{y\}$ is contractible, the maps $f|_X$ and $g''|_X$ from X to S^n are homotopic relative ∂X . Recall that f and g'' agree on $\coprod_{i=1}^d D_i^n$. Hence f and g'' are homotopic as maps $M \rightarrow S^n$. This implies that the maps f and g from M to S^n are homotopic. This finishes the proof of Hopf's Degree Theorem 3.1.

3.3. The homotopy groups of the n -sphere in the degree $\leq n$.

Theorem 3.4. *We get for every $n \geq 1$*

$$\pi_k(S^n) \cong \begin{cases} \{0\} & k < n; \\ \mathbb{Z} & k = n. \end{cases}$$

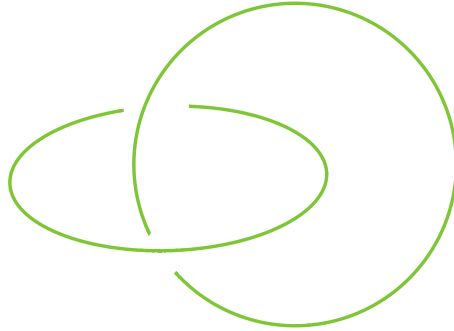
There is an explicit isomorphism $\mathbb{Z} \xrightarrow{\cong} \pi_n(S^n)$ which sends $1 \in \mathbb{Z}$ to $[\text{id}_{S^n}]$. Its inverse $\pi_n(S^n) \xrightarrow{\cong} \mathbb{Z}$ sends $[f]$ to the degree of f .

Proof. Suppose $k < n$. Let $f: S^k \rightarrow S^n$ be any map. Since we can change any map $f: S^k \rightarrow S^n$ up to homotopy into a smooth map transversal to $y \in S^n$, we can change f by a homotopy to map $S^n \rightarrow S^n \setminus \{y\}$. As $S^n \setminus \{y\}$ is contractible, f is nullhomotopic. This implies $\pi_k(S^n, s) = \{0\}$ for every $s \in S$.

The degree defines a bijection $\text{deg}: [S^n, S^n] \xrightarrow{\cong} \mathbb{Z}$ because of Hopf's Degree Theorem 3.1 for $n \geq 1$. By inspecting the proof of surjectivity of this map we see that the forgetful map $\pi_n(S^n, s) \rightarrow [S^n, S^n]$ is surjective. We conclude from (2.1) and (2.6) that the forgetful map $\pi_n(S^n, s) \rightarrow [S^n, S^n]$ is injective. $\square$

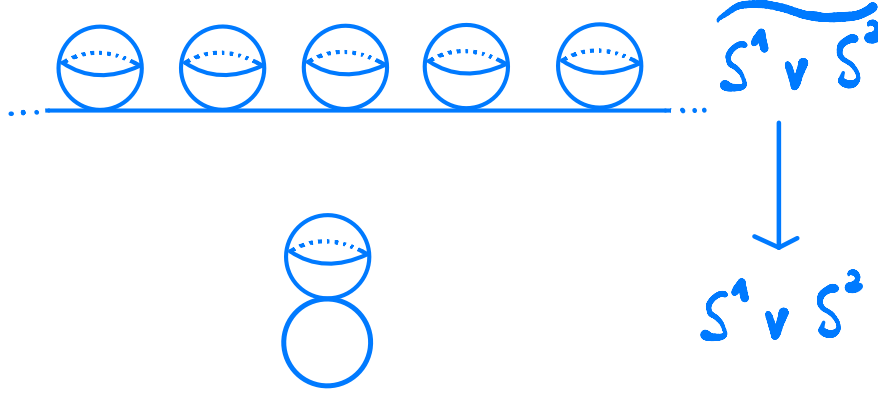
Example 3.5 (The Hopf map and $\pi_3(S^2)$). One may think that $\pi_k(S^n, s)$ vanishes for $k > n$ as $H_k(S^n)$ vanishes for $k > n$. But this is not true as the following example due to Hopf shows. We can think of S^3 as the subset of $\mathbb{C}^2$ given by $\{(z_1, z_2) \mid z_1 \bar{z}_1 + z_2 \bar{z}_2 = 1\}$. We get an S^1 -action on S^3 by $z \cdot (z_1, z_2) = (zz_1, zz_2)$. This action is free and the quotient space S^3/S^1 is homeomorphic to S^2 . Thus we get a projection $p: S^3 \rightarrow S^2$. We will later show that $\pi_3(S^2) \cong \mathbb{Z}$ holds with the class $[p]$ of p as generator, see Theorem 10.5.

One indication that $[p]$ is not zero in $\pi_3(S^2)$ is the observation that the preimages of the north and the south pole of S^2 are two embedded S^1 -s in S^3 which are linked.



Example 3.6 ($\pi_n(S^1 \vee S^n)$ is not finitely generated.).

Consider $X = S^1 \vee S^n$ for $n \geq 2$. Its universal covering $\tilde{X}$ is obtained from $\mathbb{R}$ by gluing to each element in $\mathbb{Z}$ a copy of S^n along the base point.



The map $\tilde{X} \rightarrow \bigvee_{i \in \mathbb{Z}} S^n$ given by collapsing $\mathbb{R}$ to point turns out to be a point homotopy equivalence. This can be seen by a direct inspection or follows from Lemm 8.25 and Theorem 8.28. Hence we conclude $\pi_n(X) \cong \pi_n(\tilde{X}) \cong \pi_n(\bigvee_{i \in \mathbb{Z}} S^n)$ from Theorem 2.7. For each $k \in \mathbb{Z}$ we have the pointed inclusion $j_k: S^n \rightarrow \bigvee_{i \in \mathbb{Z}} S^n$ of the k -th summand and the pointed projection $\text{pr}_k: \bigvee_{i \in \mathbb{Z}} S^n \rightarrow S^n$ onto the k -th summand. Obviously $\text{pr}_k \circ j_k$ is the identity and $\text{pr}_k \circ j_l$ is the constant map for $k \neq l$. Hence the map $\bigoplus_{i \in \mathbb{Z}} \pi_n(j_i): \bigoplus_{i \in \mathbb{Z}} \pi_n(S^n) \rightarrow \pi_n(\bigvee_{i \in \mathbb{Z}} S^n)$ is injective. As $\pi_n(S^n) \cong \mathbb{Z}$, the abelian group $\pi_n(S^1 \vee S^n)$ is not finitely generated.

Actually, we know that $\pi_n(S^1 \cup S^n)$ is a $\mathbb{Z}[\pi_1(S^1)]$ -module and it will turn out that it is $\mathbb{Z}[\pi_1(S^1)]$ -isomorphic to $\mathbb{Z}[\pi_1(S^1)]$.

Remark 3.7 (Outlook about $\pi_k(S^n)$ for $k > n$). It is an open (and extremely hard) problem to compute $\pi_k(S^n, s)$ for $2 \leq n < k$ in general. There is not even a formula known which might give the answer. There is no obvious pattern in the computations, one has carried out so far. At least one knows that $\pi_k(S^n)$ is finite for $k > n$ except for $\pi_{4i-1}(S^{2i})$ for $i \geq 1$ which is a direct sum of a copy of $\mathbb{Z}$ and some finite abelian group.

4. THE CELLULAR APPROXIMATION THEOREM

In this section we want to sketch the proof of the following theorem.

Theorem 4.1 (Cellular Approximation Theorem). *Let (X, A) be a CW-pair and Y be a CW-complex. Let $f: X \rightarrow Y$ be a map whose restriction $f|_A: A \rightarrow Y$ to A is cellular. Then f is homotopic relative A to a cellular map $X \rightarrow Y$.*

By a colimit argument one can reduce the proof of the Cellular Approximation Theorem 4.1 to the proof of following lemma.

Lemma 4.2. *Consider any $k \in \{0, 1, 2, \dots\}$. Let $f: X \rightarrow Y$ be a map of CW-complexes. Suppose that $f(X_{k-1}) \subseteq Y_{k-1}$ holds.*

Then we can change f up to homotopy relative X_{k-1} such that $f(X_k) \subseteq Y_k$ holds.

In order to arrange that $f(X_k) \subseteq Y_k$ holds, we must achieve for every closed k -dimensional cell e of X by a homotopy of $f|_e$ relative ∂e that e does not meet

any cell of Y of dimension $> k$. Note that each compact subset of Y meets only finitely many cells. Hence for a closed cell e of X of dimension k there are only finitely many closed cells $e_1, e_2, \dots, e_m$ of Y satisfying $f(e) \cap e_i \neq \emptyset$. Choose $\{i \in \{1, 2, \dots, m\} \mid \dim(e_i) > \dim(e)\}$. If such an i does not exist, we are already done for e . If such i exists, we can arrange that $\dim(e_i) \geq \dim(e_j)$ holds for all $j \in \{1, 2, \dots, m\}$ and we have to change $f|_e$ up to homotopy relative ∂e such that $f(e)$ meets only the cells $e_1, e_2, \dots, e_{i-1}, e_{i+1}, e_m$ of Y . Therefore it suffices to show the following lemma.

Lemma 4.3. *Consider $0 \leq k < l$. Let (W, V) be pair for which there exists a pushout*

$$\begin{array}{ccc} S^{l-1} & \xrightarrow{q} & V \\ \downarrow & & \downarrow \\ D^l & \xrightarrow{Q} & W. \end{array}$$

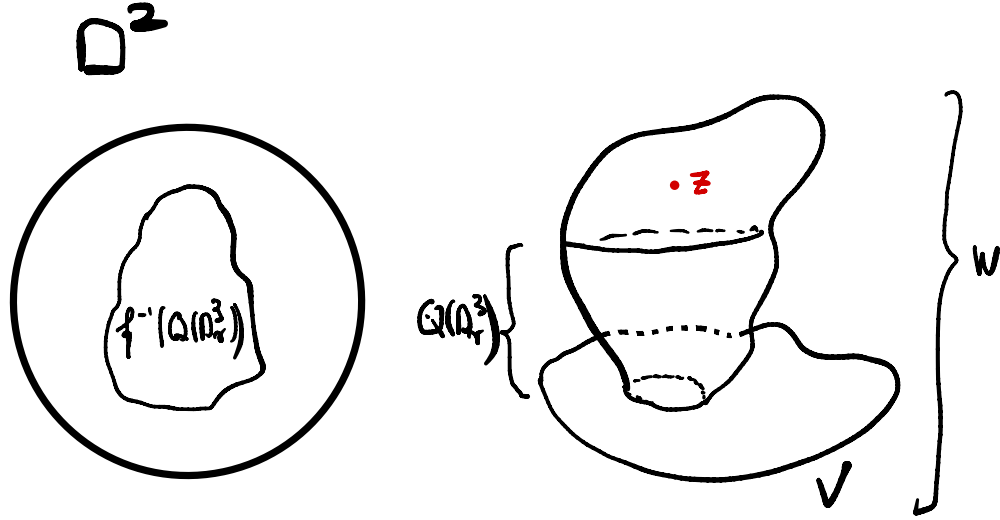
Consider any map $f: (D^k, S^{k-1}) \rightarrow (W, V)$.

Then f is homotopic relative S^{k-1} to a map $D^k \rightarrow V$.

Consider any point $z \in W \setminus V$. Then $(W \setminus \{z\}, V)$ is a strong deformation retraction, i.e., there exists a homotopy $h: W \setminus \{z\} \times I \rightarrow W \setminus \{z\}$ such that $h(y, 0) = y$ and $h(y, 1) \in V$ hold for $y \in W \setminus \{z\}$ and $h(y, t) = y$ holds for $y \in V$ and $t \in I$. Hence Lemma 4.3 follows from the next lemma.

Lemma 4.4. *Consider the situation of Lemma 4.3. Then there exists $z \in W \setminus V$ such that f is homotopic relative S^{k-1} to a map $D^k \rightarrow W \setminus \{z\}$.*

Sketch of proof. Fix $r \in (0, 1)$. Let $D_r^l \subseteq D^l$ be the open ball of radius r , i.e., $\{x \in D^l \mid \|x\| < r\}$. If $D^k \setminus f^{-1}(Q(D_r^l)) = \emptyset$, we are obviously done. Hence we can assume without loss of generality that $D^k \setminus f^{-1}(Q(D_r^l))$ is non-empty. Then one can arrange by an improved version of the Whitney Approximation Theorem that f is homotopy relative to $D^k \setminus f^{-1}(Q(D_r^l))$ to a map $g: (D^l, S^{l-1}) \rightarrow (W, V)$ such that the map induced by g from the open subset $f^{-1}(Q(D_r^l))$ of D^k to the open subset $Q(D_r^l)$ of W , which we can be equipped with the structure of a smooth manifold diffeomorphic to D_r^L , is smooth.



Since by Sard's Theorem this smooth map g has a regular value z and $k < l$, we get $g(D^k) \subseteq W \setminus \{z\}$. $\square$

This finishes the sketch of the proof of the Cellular Approximation Theorem 4.1.

Corollary 4.5. *Consider $n \geq 0$. Let (X, A) be a CW-pair such that all cells in $X \setminus A$ have dimension $> n$. Then (X, A) is n -connected. In particular (X, X_n) is n -connected for a CW-complex X .*

Proof. We only deal with the case, where A is non-empty. The proof for $A = \emptyset$ follows from the one, where $A = \{x\}$ for any zero-cell $\{x\} \in X$, since X is the disjoint union of its path components and every path component contains a zero-cell.

First we show that $\pi_0(f): \pi_0(A) \rightarrow \pi_0(X)$ is surjective for $n = 0$ and bijective for $n \geq 1$. Surjectivity follows from Cellular Approximation Theorem 4.1 applied to any map $\{\bullet\} \rightarrow X$ using the fact that $X_0 = A$ holds. Note for the sequel that any path component of a CW-complex must contain a zero-cell. By the Cellular Approximation Theorem 4.1 any path in X connecting two zero-cells in A is homotopic relative endpoints to a path in A as $X_1 = A$ holds if $n \geq 1$. This shows the bijectivity of $\pi_0(f)$ if $n \geq 1$.

It remains to show that $\pi_i(X, A, a) = \{1\}$ holds for any base point $a \in A$ and $i \in \{1, 2, \dots, n\}$. Since any path component of A contains a zero-cell, diagrams (2.1) and (2.6) imply that we can assume without loss of generality that a is a zero-cell of A . Consider an element $[f] \in \pi_i(X, A, a)$ given by a map of triples $f: (D^i, S^{i-1}, \{s\}) \rightarrow (X, A, \{a\})$. Equip S^{i-1} with the CW-structure consisting of precisely two cells, namely one 0-cell $\{s\}$ given by the base point s and one $(i-1)$ -cell. By the Cellular Approximation Theorem 4.1 the map $f|_{S^{i-1}}: S^{i-1} \rightarrow A$ is relative $\{s\}$ homotopic to cellular map. One easily checks that this implies that $f: (D^i, S^{i-1}, \{s\}) \rightarrow (X, A, \{a\})$ is homotopic as a map of triples to a map f' such that $f'|_{S^{i-1}}: S^{i-1} \rightarrow A$ is cellular. (This is a standard cofibration argument as we will see later, or done by direct inspection.) By the Cellular Approximation Theorem 4.1 the map f' is homotopic relative S^{i-1} to map $f'': (D^i, S^{i-1}) \rightarrow (X_i, A)$.

As $X_i = A$ holds and hence $\pi_1(X_i, X_i, a)$ is trivial by the long exact sequence of the pointed pair (X_i, X_i, a) , see Theorem 2.11, we conclude $[f] = [f'] = [f''] = 1$ in $\pi_i(X, A, a)$. $\square$

5. THE WHITEHEAD THEOREM

In this section we want to prove the following theorem.

Theorem 5.1 (Whitehead Theorem). *Let $f: Y \rightarrow Z$ be a map.*

(i) *Consider any $n \in \{0, 1, 2, \dots\}$. Then the following assertions are equivalent:*

(a) *The map induced by composition with f*

$$f_*: [X, Y] \rightarrow [X, Z], \quad [g] \mapsto [f \circ g]$$

is bijective for every CW-complex X of dimension $\dim(X) < n$ and is surjective for every CW-complex X of dimension $\dim(X) = n$;

(b) *The map $f: Y \rightarrow Z$ is n -connected;*

(ii) *The following assertions are equivalent:*

(a) *The map induced by composition with f*

$$f_*: [X, Y] \rightarrow [X, Z], \quad [g] \mapsto [f \circ g];$$

is bijective for every CW-complex X ;

(b) *The map $f: Y \rightarrow Z$ is a weak homotopy equivalence.*

Its proof needs some preparations.

Lemma 5.2. *Let Y be a space which is n -connected for some $n \in \{0, 1, 2, \dots\} \amalg \{\infty\}$. Let (X, A) be a relative CW-complex whose relative dimension $\dim(X, A)$ is less or equal to n .*

Then any map $f: A \rightarrow Y$ can be extended to a map $F: X \rightarrow Y$.

Proof. We construct for $k = -1, 0, 1, 2, \dots$ with $k \leq n$ maps $f_k: X_k \rightarrow Y$ such that $f_{-1}: X_{-1} = A \rightarrow Y$ is the given map f and we have $f_k|_{X_{k-1}} = f_{k-1}$ for $k \geq 0$. Then Lemma 5.2 is a consequence of the following argument. If $n < \infty$, then we can take $F = f_n$. If $n = \infty$, we define $F = \operatorname{colim}_{k \rightarrow \infty} f_k$ having in mind that by the definition of a CW-pair we have $X = \operatorname{colim}_{k \rightarrow \infty} X_k$.

The induction beginning $k = -1$ is trivial. The induction step from $(k-1)$ to k is done as follows. Choose a cellular pushout

$$\begin{array}{ccc} \coprod_{i \in I} S^{k-1} & \xrightarrow{\coprod_{i \in I} q_i} & X_{k-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I} D^k & \xrightarrow{\coprod_{i \in I} Q_i} & X_k. \end{array}$$

We conclude from the pushout property that we can construct f_k from f_{k-1} if for any $i \in I$ we can extend the composite $f_{k-1} \circ q_i: S^{k-1} \rightarrow Y$ to a map $D^k \rightarrow Y$. This can be done as Y is by assumption k -connected. $\square$

Lemma 5.3. *Let (Y, B) be a pair which is n -connected for some $n \in \{0, 1, 2, \dots\} \amalg \{\infty\}$. Let (X, A) be a relative CW-complex whose relative dimension $\dim(X, A)$ is less or equal to n .*

Then any map $f: (X, A) \rightarrow (Y, B)$ is homotopic relative A to a map $f: (X, A) \rightarrow (Y, B)$ with $g(X) \subseteq B$.

Proof. We construct for $k = -1, 0, 1, 2, \dots$ with $k \leq n$ a map

$$h_k: X_k \times I \cup_{X_k \times \{0\}} X \times \{0\} \rightarrow Y$$

such that the following conditions are satisfied:

- $h_{-1}: A \times I \cup_{A \times \{0\}} X \times \{0\} \rightarrow X$ sends (a, t) to $f(a)$ for $(a, t) \in A \times I$ and $(x, 0)$ to $f(x)$ for $x \in X$.
- We have $h_k(x, 0) = f(x)$ for $x \in X$;
- We have $h_k(x, 1) \in B$ for every $x \in X_k$
- For $0 \leq k \leq n$ we have $h_k|_{X_{k-1} \times I} = h_{k-1}|_{X_{k-1} \times I}$.

Then Lemma 5.3 is a consequence of the following argument. If $n < \infty$, then $h = h_n$ is the desired homotopy relative A from f to a map with image in B . Suppose $n = \infty$. Since $X = \text{colim}_{k \rightarrow \infty} X_k$, we get $X \times I = \text{colim}_{k \rightarrow \infty} (X_k \times I)$ and we obtain the desired homotopy h by $\text{colim}_{k \rightarrow \infty} h_k$.

The induction beginning $k = -1$ is trivial. The induction step from $(k-1)$ to k is done as follows. Choose a cellular pushout

$$\begin{array}{ccc} \coprod_{i \in I} S^{k-1} & \xrightarrow{\coprod_{i \in I} q_i} & X_{k-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I} D^k & \xrightarrow{\coprod_{i \in I} Q_i} & X_k. \end{array}$$

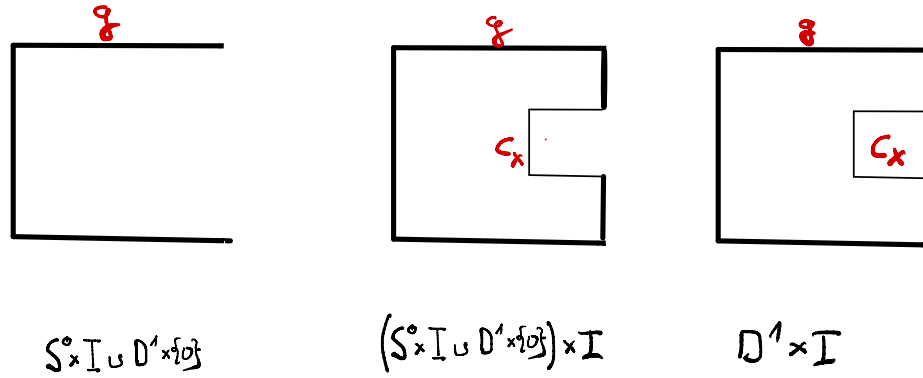
Then we obtain a pushout

$$\begin{array}{ccc} \coprod_{i \in I} S^{k-1} \times I \cup_{S^{k-1} \times \{0\}} D^k \times \{0\} & \xrightarrow{\coprod_{i \in I} q'_i} & X_{k-1} \times I \cup_{X_{k-1} \times \{0\}} X \times \{0\} \\ \downarrow & & \downarrow \\ \coprod_{i \in I} D^k \times I & \xrightarrow{\coprod_{i \in I} Q'_i} & X_k \times I \cup_{X_k \times \{0\}} X \times \{0\} \end{array}$$

where q'_i is given by $q_i \times \text{id}_I \cup_{q_i \times \text{id}_{\{0\}}} Q_i \times \text{id}_{\{0\}}$. We conclude from the pushout property that it suffices to construct for every $i \in I$ an extension of the map

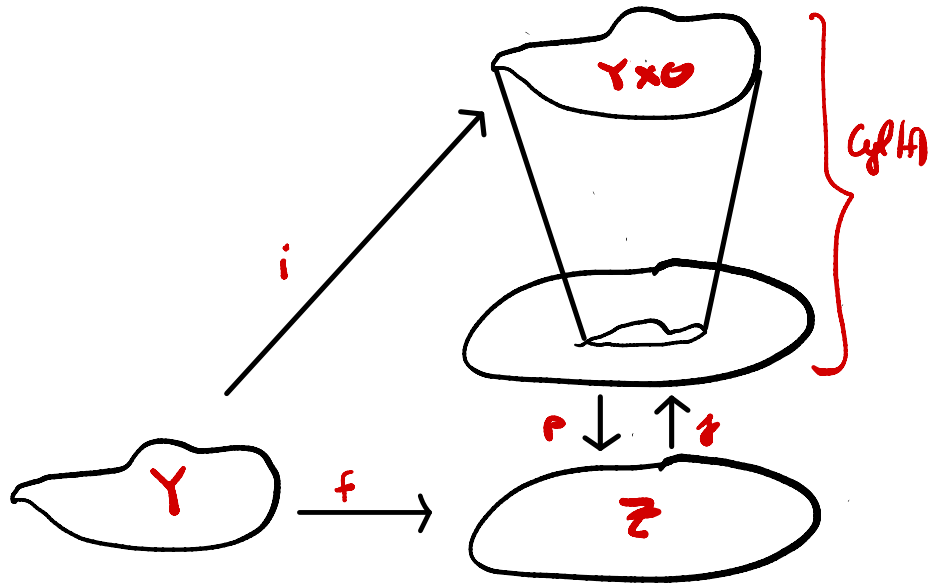
$$u = h_{k-1} \circ q'_i: S^{k-1} \times I \cup_{S^{k-1} \times \{0\}} D^k \times \{0\} \rightarrow Y$$

to a map $U: D^k \times I \rightarrow Y$ such that $g(D^k \times \{1\}) \subseteq B$ holds. Up to homeomorphism the pair $(S^{k-1} \times I \cup_{S^{k-1} \times \{0\}} D^k \times \{0\}, S^{k-1} \times \{1\})$ can be identified with (D^k, S^{k-1}) . So we can think of u as a map of triples $(D^k, S^{k-1}, \{s\}) \rightarrow (Y, B, \{x\})$ for $x = u(s)$. Hence it defines a element in $\pi_k(Y, B, x)$. As $\pi_k(Y, B, x)$ is by assumption trivial, there is a homotopy of maps of triples $(S^{k-1} \times I \cup_{S^{k-1} \times \{0\}} D^k \times \{0\}, S^{k-1} \times \{1\}, \{(s, 1)\}) \rightarrow (Y, B, \{x\})$ from u to the constant map c_x . Obviously the latter map extends to the constant map $c_x: D^k \times I, D^k \times \{1\}, \{(s, 1)\} \rightarrow (Y, B, \{x\})$. Hence we can extend u to a map $U: (D^k \times I, D^k \times \{1\}, \{s\}) \rightarrow (Y, B, \{x\})$.



This finishes the proof of Lemma 5.3. □

Proof of the Whitehead Theorem 5.1. Let $\text{cyl}(f)$ be the mapping cylinder of f . Let $i: X \rightarrow \text{cyl}(f)$ and $j: Y \rightarrow \text{cyl}(f)$ be the canonical inclusions and $p: \text{cyl}(f) \rightarrow Y$ be the canonical projection. Then $p \circ i = f$, $p \circ j = \text{id}_Y$, and $j \circ p \simeq \text{id}_{\text{cyl}(f)}$. Hence we can assume without loss of generality that $f: Y \rightarrow Z$ is an inclusion of pairs, otherwise replace the given $f: Y \rightarrow Z$ by $i: Y \rightarrow \text{cyl}(f)$.



$\implies$ (i)a and (ii)b $\implies$ (ii)b

The surjectivity of $f_*: [X, Y] \rightarrow [X, Z]$ follows for $\dim(X) \leq n$ directly from Lemma 5.3 applied to a map $g: (X, \emptyset) \rightarrow (Z, Y)$. Finally we prove the injectivity of f_* under the assumption that either $n = \infty$ or $\dim(X) < n < \infty$ holds. Consider $g_0, g_1: X \rightarrow Y$ and a homotopy $h: f \circ g_0 \simeq f \circ g_1$ of maps from X to Z . We obtain a map of pairs $(h, g_0 \amalg g_1): (X \times I, X \times \{0, 1\}) \rightarrow (Z, Y)$. This map is homotopic relative $X \times \{0, 1\}$ to a map $k: X \times I \rightarrow Y$ by Lemma 5.3 since $\dim(X \times I) \leq n$ holds. Obviously k is a homotopy of maps $X \rightarrow Y$ between g_0 and g_1 .

(i)a $\implies$ (i)b and (ii)a $\implies$ (ii)b The map $\pi_0(f): \pi_0(Y) \rightarrow \pi_0(Z)$ can be identified with the map $f_*: [\{\bullet\}, Y] \rightarrow [\{\bullet\}, Z]$. Hence the claim is true for $n = 0$. So it suffices to treat the case $n \in \mathbb{Z}^{\geq 1} \amalg \{\infty\}$. Then $\pi_0(f)$ is bijective. It remains to show for any $y \in Y$ that $\pi_k(f, y): \pi_k(Y, y) \rightarrow \pi_k(Z, f(y))$ is bijective for $1 \leq k < n$ and surjective for $1 \leq k \leq n$.

We begin with surjectivity for $1 \leq k \leq n$. Choose an index set I and a map $v: (S, s) \rightarrow (Z, f(y))$ for $S = \bigvee_{i \in I} S^k$ equipped with the obvious base point s such that $\pi_k(v, s): \pi_k(S, s) \rightarrow \pi_k(Z, f(y))$ is surjective. Then we can find by assumption a map $u: S \rightarrow Y$ such that $f \circ u$ is homotopic to v . For an appropriate path $w: [0, 1] \rightarrow Z$ from $u(s)$ to z , we obtain a commutative diagram

$$\begin{array}{ccc} \pi_k(Y, u(s)) & \xrightarrow{\pi_k(f, u(s))} & \pi_k(Z, f \circ u(s)) \\ \uparrow \pi_k(u, s) & & \cong \downarrow t_{[f \circ w]} \\ \pi_k(S, s) & \xrightarrow{\pi_k(v, s)} & \pi_k(Z, z). \end{array}$$

Next we show injectivity for $1 \leq k < n$. Choose an index set I and a map $u: (S, s) \rightarrow (Z, f(y))$ for $S = \bigvee_{i \in I} S^k$ equipped with the obvious base point s such that the sequence $\pi_k(S, s) \xrightarrow{\pi_k(u, s)} \pi_k(Y, y) \xrightarrow{\pi_k(f, y)} \pi_k(Z, f(y))$ is exact. The composite $f \circ u: S \rightarrow Z$ is nullhomotopic. Since S has dimension $\leq (n-1)$, the map $f_*: [S, Y] \rightarrow [S, Z]$ is bijective by assumption. Hence u is nullhomotopic. This implies that there is a path $w: [0, 1] \rightarrow Y$ from y to some point y' such that the composite $\pi_k(u, s): \pi_k(S, s) \rightarrow \pi_k(Y, y)$ with the isomorphism $t_{[w]}: \pi_k(Y, y) \xrightarrow{\cong} \pi_k(Y, y')$ is trivial. Hence $\pi_k(u, s): \pi_k(S, s) \rightarrow \pi_k(Y, y)$ is trivial. This implies that the kernel of $\pi_k(f, y)$ is trivial and hence that $\pi_k(f, y)$ is injective.

This finishes the proof of the Whitehead Theorem 5.1. $\square$

Corollary 5.4. *Let $f: X \rightarrow Y$ be a map of CW-complexes. Then f is a homotopy equivalence if and only if f is a weak homotopy equivalence.*

Proof. We conclude from the diagrams (2.1) and (2.6) that f is a weak homotopy equivalence if it is a homotopy equivalence. Suppose that f is a weak homotopy equivalence. Theorem 5.1 (ii) implies that $f_*: [Y, X] \rightarrow [Y, Y]$ is bijective. Let $a: Y \rightarrow X$ be map with $f_*([a]) = [f \circ a] = \text{id}_Y$. Then a is a weak homotopy equivalence. Theorem 5.1 (ii) again implies that $a_*: [X, Y] \rightarrow [X, X]$ is bijective. So we can choose a map $b: X \rightarrow Y$ with $[a \circ b] = [\text{id}_X]$. This implies $b \simeq f \circ a \circ b \simeq f$. Hence a is a homotopy inverse of f and in particular f is a homotopy equivalence. $\square$

Example 5.5 (S^∞). Define the real vector space $\mathbb{R}^\infty := \bigoplus_{i=1}^\infty \mathbb{R}$. It inherits a norm by

$$\|(x_1, x_2, x_3, \dots)\| = \sqrt{\sum_{i=1}^\infty x_i^2}.$$

In particular $\mathbb{R}^\infty$ inherits a metric and the structure of a topological space. We can identify the topological space $\mathbb{R}^n$ with the subspace consisting of points $(x_1, x_2, \dots)$ for which $x_i = 0$ for $i > n$ holds. Let $S^\infty \subseteq \mathbb{R}^\infty$ be the subspace consisting of points z satisfying $\|z\| = 1$. Then S^n can be identified with $S^\infty \cap \mathbb{R}^{n+1}$ for $n \geq 0$. Moreover, we get:

- (i) We have the nested sequence $S^0 \subseteq S^1 \subseteq S^2 \subseteq \dots \subseteq S^\infty$ such that S^∞ is the unions of the S^n -s. The colimit topology with respect to this filtration is not the subspace topology $S^\infty \subseteq \mathbb{R}^\infty$;
- (ii) S^∞ equipped with the colimit topology carries a CW -structure with S^n as n -skeleton;
- (iii) S^∞ equipped with the subspace topology does not carry the structure of a CW -complex;
- (iv) S^∞ equipped with the subspace topology is contractible;
- (v) S^∞ equipped with the colimit topology is contractible;
- (vi) Consider the identity $S^\infty \rightarrow S^\infty$, where we equip the domain with the colimit topology and the codomain with the subspace topology. Then this map is bijective and continuous and is a homotopy equivalence but is not a homeomorphism.

For $n \geq 1$ consider the element a_n in S^∞ whose i -th entry is $\sqrt{1 - n^{-1}}$ for $i = 1$, n^{-1} for $i = 2, \dots, n+1$, and is 0 for $i \geq (n+2)$. Let $A = \{a_n \mid n \geq 1\}$. Since the intersection of A with S^n is finite for $n \geq 1$, it is a closed subspace of S^∞ with respect to the colimit topology. Since $(1, 0, 0, \dots)$ does not belong to A and $\lim_{n \rightarrow \infty} a_n = (1, 0, 0, \dots)$ holds with respect to the metric above, A is not closed with respect to the subspace topology. This finishes the proof of assertion (i).

We leave the obvious proof of the assertion (ii) is left to the reader.

Assertion (iii) is proved as follows. Suppose that S^∞ with the subspace topology has a CW -structure. Since then S^∞ is a metrizable CW -complex, it must be locally compact by [9, Theorem B on page 81]. This implies there is an $\epsilon > 0$ such that the intersection of S^∞ with the closed ball of radius ϵ around $(1, 0, 0, \dots)$ is compact. Hence we can find $\delta > 0$ such that the sequence $(x_n)_{n \geq 1}$ given by $x_n = \sqrt{1 - \delta} \cdot e_1 + \sqrt{\delta} \cdot e_n$ with e_i the i -th element of the standard base belongs to the intersection of S^∞ with the closed ball of radius ϵ around $(1, 0, 0, \dots)$. Hence it has a subsequence which is a Cauchy sequence. Since this is not the case, we get a contradiction.

Next we prove assertion (iv). Let $s: S^\infty \rightarrow S^\infty$ be the shift map sending $(x_1, x_2, x_3, \dots)$ to $(0, x_1, x_2, x_3, \dots)$. Define

$$h: S^\infty \times I \rightarrow S^\infty, \quad x \mapsto \frac{t \cdot s(x) + (1-t) \cdot x}{\|t \cdot s(x) + (1-t) \cdot x\|}.$$

This is a homotopy between id_{S^∞} and s . Now consider the homotopy

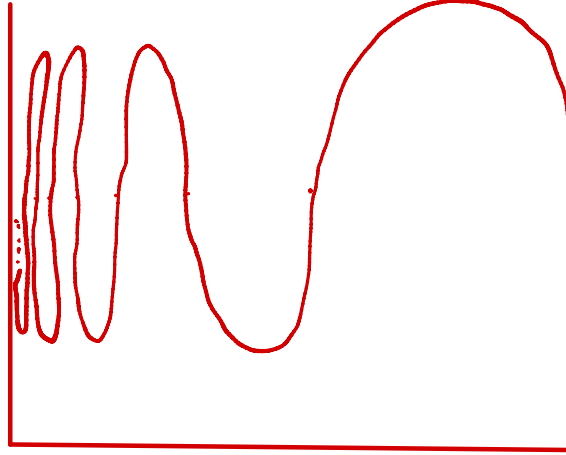
$$k: S^\infty \times I \rightarrow S^\infty, \quad x \mapsto \frac{(1-t) \cdot s(x) + t \cdot e_1}{\|(1-t) \cdot s(x) + t \cdot e_1\|}$$

for $e_1 = (1, 0, 0, \dots)$. Then k is a homotopy between s and the constant map $S^\infty \rightarrow S^\infty$ with value e_1 . Hence S^∞ with the subspace topology is contractible.

Assertion (v) follows from Theorem 2.20, Theorem 3.4, and Corollary 5.4 using assertion (ii). Alternatively, the proof for assertion (iv) does carry over to assertion (v).

Assertion (vi) is a direct consequence of the other assertions.

Example 5.6 (Warsaw circle). Let W be the *Warsaw circle*, i.e., the compact subsets of $\mathbb{R}^2$ given by the union of $\{(x, \sin(2\pi/x)) \mid x \in (0, 1]\}$, $\{(1, y) \mid y \in [-2, 0]\}$, $\{(x, -2) \mid x \in [0, 1]\}$ and $\{(0, y) \mid y \in [-2, 1]\}$.



Then the projection $p: W \rightarrow \{\bullet\}$ is a weak homotopy equivalence but not a homotopy equivalence. In particular W is a compact space which is not homotopy equivalent to a CW -complex.

Remark 5.7 (Whitehead Theorem for pairs). There is the following version of the Whitehead Theorem 5.1 (ii) for pairs. Let $(F, f): (Y, B) \rightarrow (Z, C)$ be a map of pairs. Then the following assertions are equivalent:

- (i) The maps $F: Y \rightarrow Z$ and $f: B \rightarrow C$ are weak homotopy equivalences;
- (ii) For every CW -pair (X, A) the maps of the homotopy classes of pairs induced by composition with (F, f)

$$(F, f)_*: [(X, A), (Y, B)] \rightarrow [(X, A), (Z, C)], \quad [(G, g)] \mapsto [(F \circ G, g \circ f)]$$

is bijective.

6. CW-APPROXIMATION

Definition 6.1 (n -coconnected maps). A map $f: X \rightarrow Y$ is called n -coconnected for $n \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$, if for any base point $x \in X$ the map $\pi_i(f, x): \pi_i(X, x) \rightarrow \pi_i(Y, f(x))$ is injective if $i = n$, and is bijective if $i > n$.

Consider a natural number n and a map $f: X \rightarrow Y$. Then f is a weak homotopy equivalence if and only if it is both n -connected and n -coconnected.

Definition 6.2 (n -CW-model for a pair). Consider a topological pair (Y, A) such that A is a CW -complex and $n \in \mathbb{Z}^{\geq 0}$. (The subcomplex A may be empty.) An n -CW-model for (Y, A) consists of an n -connected pair of CW -complexes (Z, A) together with an n -coconnected map $f: Z \rightarrow Y$ satisfying $f|_A = \text{id}_A$.

Theorem 6.3 (n -CW-models). Consider a topological pair (Y, A) such that A is a CW -complex and $n \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$. Then there exists an n -CW-model

$$(f, \text{id}_A): (Z, A) \rightarrow (Y, A)$$

such that $Z \setminus A$ contains no cells of dimension $\leq n$.

Proof. We construct a sequence of nested spaces $Z_n \subseteq Z_{n+1} \subseteq Z_{n+2} \subseteq \dots$ and maps $f_i: Z_i \rightarrow Y$ for $i \geq n$ such that the following holds:

- $Z_n = A$ and $f_n = \text{id}_A$;
- $f_i|_{Z_{i-1}} = f_{i-1}$ for $i = (n+1), (n+2), \dots$;
- There exists for $i \geq n$ a pushout of the shape

$$\begin{array}{ccc} \coprod_{j \in J_i} S^i & \xrightarrow{\coprod_{j \in J_i} q_i^j} & Z_i \\ \downarrow & & \downarrow \\ \coprod_{j \in J_i} D^{i+1} & \xrightarrow{\coprod_{j \in J_i} Q_i^j} & Z_{i+1} \end{array}$$

such that the image of each map q_j does not meet any closed cell in A of dimension $> i$;

- For any base point $z \in Z_i$ and $i > n$, the map $\pi_j(f_i, z)$ is injective for $j = n$, bijective for $n < j < i$, and surjective for $j = i$.

Before we explain the construction of these data, we explain how we get the desired n -CW-model from it. Namely, we define $Z = \text{colim}_{i \rightarrow \infty} Z_i$ and $f = \text{colim}_{i \rightarrow \infty} f_i: Z \rightarrow X$. Then (Z, A) is a CW-pair and the i -skeleton Z_i of Z is the complement of the union of the open cells of dimension $> i$ of A in Z_i . In particular $Z \setminus Z_i$ contains no cells of dimension $\leq i$. Since $Z \setminus A$ contains no k -cells for $0 \leq k \leq n$, the pair (Z, A) is n -connected by Corollary 4.5. We conclude from Corollary 4.5 again that the map $\pi_m(Z_i, z_i) \rightarrow \pi_m(Z, z_i)$ induced by the inclusion $Z_i \rightarrow Z$ is bijective for $m < i$ and surjective for $m = i$ for any $i \geq n$ and $z_i \in Z_i$. Hence the map f is n -coconnected by Theorem 2.20.

Finally we carry out the construction of the sequence $Z_n \subseteq Z_{n+1} \subseteq Z_{n+2} \subseteq \dots$ and the sequence of maps $f_i: Z_i \rightarrow Y$. The induction beginning is obvious, take $Z_n = A$ and $f_n = \text{id}_A$. The induction step how to construct Z_{i+1} and f_{i+1} , when Z_i and f_i have already been established, is done as follows. For each path component C of A choose a zero-cell x_C in A which is contained in C . Then for every element u in the kernel of the map $\pi_i(f_i, x_C): \pi_i(Z_i, x_C) \rightarrow \pi_i(Y, x_C)$ choose a pointed map $q_{C,u}: (S^i, s) \rightarrow (Z_i, x_C)$ with $u = [q_{C,u}]$. The define Z'_{i+1} as the pushout

$$\begin{array}{ccc} \coprod_{\substack{C \in \pi_0(A) \\ u \in \ker(\pi_i(f_i, x_C))}} S^i & \xrightarrow{\coprod_{\substack{C \in \pi_0(A) \\ u \in \ker(\pi_i(f_i, x_C))}} q_{C,u}} & Z_i \\ \downarrow & & \downarrow \\ \coprod_{\substack{C \in \pi_0(A) \\ u \in \ker(\pi_i(f_i, x_C))}} D^{i+1} & \xrightarrow{\quad} & Z'_{i+1}. \end{array}$$

Since each $[q_{C,u}]$ lies in the kernel of $\pi_i(f_i, x_C)$, each map $f_i \circ q_{C,u}: S^i \rightarrow Y$ can be extended to a map $\overline{q_{C,u}}: D^{i+1} \rightarrow Y$. By the Cellular Approximation Theorem 4.1 we can additionally arrange that the image of each map $\overline{q_{C,u}}$ has trivial intersection with any open cell of A of dimension $i > i+1$.

The collection of these extensions yield a map $f'_{i+1}: Z'_{i+1} \rightarrow Y$ by the pushout property. We have for $j \leq (i+1)$ and $C \in \pi_0(A)$ the commutative diagram

$$\begin{array}{ccc} \pi_j(Z_i, x_C) & \xrightarrow{\pi_j(f_i, x_C)} & \pi_j(Y, x_C) \\ \downarrow & \nearrow \pi_j(f'_{i+1}, x_C) & \\ \pi_j(Z'_{i+1}, x_C) & & \end{array}$$

where the vertical arrow is induced by the inclusion $Z_i \rightarrow Z_{i+1}$. The vertical arrow is bijective for $j \leq (i-1)$ and surjective for $j = i$ by Corollary 4.5. Hence

$\pi_j(f'_{i+1}, x_C)$ is injective for $j = n$ and bijective for $n < j \leq (i-1)$, as $\pi_j(f_i, x_C)$ has these properties by the induction hypothesis. Consider an element v in the kernel of $\pi_i(f'_{i+1}, x_C)$. Choose $u \in \pi_i(Z_i, x_C)$ whose image under the vertical arrow is v . Then u lies in the kernel of $\pi_i(f_i, x_C)$. By construction u lies in the kernel of the vertical arrow. Hence v is trivial. Therefore $\pi_i(f'_{i+1}, x_C)$ is injective. As $\pi_i(f_i, x_C)$ is surjective by the induction hypothesis, $\pi_i(f'_{i+1}, x_C)$ is surjective. This implies that $\pi_j(f'_{i+1}, x_C)$ is injective for $i = n$ and bijective for $n < j \leq i$ for all $C \in \pi_0(A)$.

Now consider any $C \in \pi_0(A)$ and any element $[w_C] \in \pi_{i+1}(Y, x_C)$. Choose a map $w_C: (S^{i+1}, s) \rightarrow (Y, x_C)$ representing $[w_C]$. Define the desired space Z_{i+1} and the desired map $f_{i+1}: Z_{i+1} \rightarrow Y$ by

$$\begin{aligned} Z_{i+1} &= Z'_{i+1} \vee \bigvee_{\substack{C \in \pi_0(A) \\ [w_C] \in \pi_{i+1}(Y, x_C)}} S^{i+1}, \\ f_{i+1} &= f'_{i+1} \vee \bigvee_{\substack{C \in \pi_0(A) \\ [w_C] \in \pi_{i+1}(Y, x_C)}} w_C. \end{aligned}$$

We have for $j \leq (i+1)$ and $C \in \pi_0(A)$ the commutative diagram

$$\begin{array}{ccc} \pi_j(Z'_{i+1}, x_C) & \xrightarrow{\pi_j(f'_{i+1}, x_C)} & \pi_j(Y, x_C) \\ \downarrow & \nearrow \pi_j(f_{i+1}, x_C) & \\ \pi_j(Z_{i+1}, x_C) & & \end{array}$$

where the vertical arrow is induced by the inclusion $Z'_{i+1} \rightarrow Z_{i+1}$. The left vertical arrow is bijective for $j < i$ and surjective for $j = i$ by Corollary 4.5. It is also injective for $j = i$, since the inclusion $Z'_{i+1} \rightarrow Z_{i+1}$ has an obvious retraction $Z_{i+1} \rightarrow Z'_{i+1}$. Hence the left vertical arrow is bijective for $j \leq i$. This implies that $\pi_j(f_{i+1}, x_C)$ is injective for $i = n$ and bijective for $n < j \leq i$ for all $C \in \pi_0(A)$. Moreover, by construction any element $[w_C]$ is in the image of $\pi_j(f_{i+1}, x_C)$. Hence $\pi_j(f_{i+1}, x_C)$ is surjective for all $C \in \pi_0(A)$. Since $\pi_0(A) \rightarrow \pi_0(Z_{i+1})$ is surjective, we conclude from the diagrams (2.1) and (2.6) that for any base point $z \in Z_{i+1}$ the map $\pi_j(f_{i+1}, z)$ is injective for $i = n$, bijective for $n < j \leq i$, and surjective for $j = (i+1)$.

This finishes the proof of Theorem 6.3. $\square$

Remark 6.4. One can think of the n -CW-model $f: (Z, A) \rightarrow (Y, A)$ as a sort of homotopy theoretic hybrid of A and Y . If $n = 0$ and Y is path connected, then the hybrid looks like Y in the sense that f is a weak homotopy equivalence. As n increases, the hybrid looks more and more like A , and less and less like Y . If we take $n = \infty$, then the inclusion $A \rightarrow Z$ is a weak homotopy equivalence and can actually be realized by $Z = A$ and id_A .

More precisely, if $k: A \rightarrow Z$ and $l: A \rightarrow Y$ are the inclusions and $a \in A$ is a base point, we get a factorization

$$\pi_i(l, a): \pi_i(A, a) \xrightarrow{\pi_i(k, a)} \pi_i(Z, a) \xrightarrow{\pi_i(f, a)} \pi_i(Y, a)$$

such that the following holds:

- If $i < n$, then the first map $\pi_i(k, a)$ is bijective;
- If $i = n$, then the first map $\pi_i(k, a)$ is surjective and the second map $\pi_i(f, a)$ is injective;
- If $i > n$, then the second map $\pi_i(f, a)$ is bijective.

Corollary 6.5. Consider a CW-pair (X, A) and $n \in \mathbb{N}$. Then the following assertions are equivalent:

- (i) There is a CW -pair (Z, A) such that (X, A) and (Z, A) are homotopy equivalent relative A and $Z \setminus A$ contains no cells of dimension $\leq n$;
- (ii) The pair (X, A) is n -connected.

Proof. (i) $\implies$ (ii) This follows from Corollary 4.5.

(ii) $\implies$ (i) We obtain from Theorem 6.3 an n -model $(f, \text{id}_A): (Z, A) \rightarrow (X, A)$ such that $Z \setminus A$ contains no cells of dimension $\leq n$. Since (Z, A) and (X, A) are n -connected and f is n -coconnected, $f: Z \rightarrow X$ is a weak homotopy equivalence inducing the identity on A . A version of the Whitehead Theorem 5.1 (ii) relative A implies that (X, A) and (Z, A) are homotopy equivalent relative A . $\square$

In particular any path connected CW -complex is homotopy equivalent to a CW -complex Z having precisely one 0-cell.

Example 6.6. Let X be path connected CW -complex. We conclude from Theorem 2.7 that a 1-connected CW -model for $X = (X, \emptyset)$ is given by the universal covering $\tilde{X} \rightarrow X$.

For this section the case $n = 0$ is important which we treat next.

Definition 6.7. Consider a space Y . A CW -approximation (X, f) of Y is a CW -complex X together with a weak homotopy equivalence $f: X \rightarrow Y$.

Theorem 6.8 (Existence and uniqueness of CW -approximations). *Let Y be a topological space. Then:*

- (i) There exists a CW -approximation (X, f) of Y ;
- (ii) Let (X, f) and (X', f') be two CW -approximations of Y . Then there exists a homotopy equivalence $g: X \rightarrow X'$ for which the following diagram commutes up to homotopy

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ & \searrow f & \swarrow f' \\ & Y & \end{array}$$

The homotopy equivalence g is up to homotopy uniquely determined by the property $f' \circ g \simeq f$.

Proof. (i) Consider a path component C of Y . From Theorem 6.3 applied to the pair $(C, \emptyset)$ and $n = 0$ we obtain a CW -complex X_C and weak homotopy equivalence $f_C: X_C \rightarrow C$. Then we get from $X = \coprod_{C \in \pi_0(Y)} X_C$ and $f = \coprod_{C \in \pi_0(Y)} f_C$ a CW -approximation of Y .

(ii) We conclude from the Whitehead Theorem 5.1 (ii) that there exists a map $g: X \rightarrow X'$ which is uniquely determined up to homotopy by the property $f' \circ g \simeq f$. The map g is a weak homotopy equivalence and hence a homotopy equivalence by Corollary 5.4. $\square$

Remark 6.9. One may think of Theorem 6.8 as the topological analogue of the fact that any positive R -chain complex C_* possesses a *projective R -resolution* $f_*: P_* \rightarrow C_*$, i.e., a projective positive R -chain complex P_* together with an R -chain map $f_*: P_* \rightarrow C_*$ inducing an isomorphism on all homology modules, and that for two projective resolutions (P_*, f_*) and (P'_*, f'_*) of C_* there is a R -chain homotopy equivalence $g_*: P_* \rightarrow P'_*$ which is a up to R -chain homotopy uniquely determined by the property $f'_* \circ g_* \simeq f_*$.

Theorem 6.10. *Let $f: X \rightarrow Y$ be a weak homotopy equivalence of spaces. Then the induced map on singular homology $H_n(f): H_n(X) \rightarrow H_n(Y)$ is bijective for all $n \geq 0$.*

Proof. See [31, Theorem 9.5.3 on page 237]. $\square$

Remark 6.11 (*CW*-approximations for pairs). Consider a pair (Y, B) . Choose a *CW*-approximation $u: A \rightarrow B$ for B . Let $\text{cyl}(u)$ be the mapping cylinder of u . It contains the *CW*-complex A as subspace. Let $g: (X, A) \rightarrow (\text{cyl}(f), A)$ be a 0-*CW*-model which exists by Theorem 6.3. Thus we obtain a pair of *CW*-complexes (X, A) together with a weak homotopy equivalence $g: X \rightarrow Y$ satisfying $g|_A = \text{id}_A$. Let $p: \text{cyl}(f) \rightarrow Y$ be the projection which is a homotopy equivalence and satisfies $p|_A = u$. Let $f: X \rightarrow Y$ be the composite $p \circ g$. Then $f: X \rightarrow Y$ and $f|_A = u: A \rightarrow B$ are weak homotopy equivalences. So we obtain a *CW*-approximation $(f, u): (X, A) \rightarrow (Y, B)$ for pairs.

A relative version of the Whitehead Theorem 5.1 (ii), see Remark 5.7, shows that for two such *CW*-approximations $f: (X, A) \rightarrow (Y, B)$ and $f': (X', A') \rightarrow (Y, B)$ there is a homotopy equivalence of pairs $g: (X, A) \rightarrow (X', A')$ which is up to homotopy uniquely determined by the property that f and $f' \circ g$ are homotopic as maps of pairs $(X, A) \rightarrow (Y, B)$.

7. THE CATEGORY OF COMPACTLY GENERATED SPACES

We briefly recall some basics about compactly generated spaces. More information and proofs can be found in [25]. A topological space X is *compactly generated* if it is a Hausdorff space and a set $A \subseteq X$ is closed if and only if for any compact subset $C \subset X$ the intersection $C \cap A$ is a closed subspace of C .

Every locally compact space, and every space satisfying the first axiom of countability, e.g., a metrizable space, is compactly generated. If $p: X \rightarrow Y$ is an identification of topological spaces and X is compactly generated and Y is Hausdorff, then Y is compactly generated. A closed subset of a compactly generated space is again compactly generated. For open subsets one has to be careful as it is explained in Subsection 7.1.

7.1. Open subsets. Recall that a topological space B is called *regular* if for any point $x \in X$ and closed set $A \subseteq X$ there exists open subsets U and V with $x \in U$, $A \subseteq V$ and $U \cap V = \emptyset$. A Hausdorff space is called *locally compact* if every $x \in X$ possesses a compact neighborhood. Equivalently, for every $x \in X$ and open neighborhood U there exists an open neighborhood V of x such that the closure of V in X is compact and contained in U , see [23, Lemma 8.2 in Section 3-8 on page 185].

Definition 7.1 (Quasi-regular open set and regular space). An open subset $U \subseteq B$ is called *quasi-regular* if for any $x \in X$ there exists an open neighborhood V_x whose closure in B is contained in U .

Lemma 7.2. (i) Let B be a compactly generated Hausdorff space. A quasi-regular open subset $U \subseteq B$ equipped with the subspace topology is compactly generated;

(ii) Let $f: X \rightarrow Y$ be a (continuous) map between (not necessarily compactly generated) spaces. If $V \subseteq Y$ is a quasi-regular open subset, then $f^{-1}(V) \subseteq X$ is a quasi-regular open subset;

(iii) The intersection of finitely many quasi-regular open subsets is again a quasi-regular open subset;

(iv) A space is regular if and only if every open subset is quasi-regular;

(v) Any locally compact Hausdorff space, any metrizable space, and every *CW*-complex are regular;

(vi) Every open subset of a *CW*-complex is quasi-regular and, equipped with the subspace topology, compactly generated.

Proof. (i) See [25, page 135].

(ii) Consider a point $x \in f^{-1}(V)$. Choose an open set W of Y such that $f(x) \in W$ and the closure of W in B is contained in V . Then $f^{-1}(W)$ is an open subset of X which contains x and whose closure in X is contained in $f^{-1}(V)$.

(iii) Let $U_1, U_2, \dots, U_r$ be quasi-regular open subsets. Consider $x \in U := \bigcap_{i=1}^r U_i$. Choose for every $i = 1, 2, \dots, r$ an open subset V_i with $x \in V_i$ such that the closure $\overline{V_i}$ of V_i in B is contained in U_i . Put $V := \bigcap_{i=1}^r V_i$. Then $x \in V$ and $\overline{V} \subseteq \bigcap_{i=1}^r \overline{V_i} \subseteq U$. Hence U is a quasi-regular open subset.

(iv) See [23, Lemma 2.1 in Section 4-2 on page 196].

(v) This is obvious for locally compact spaces. Metrizable spaces are treated in [23, Theorem 2.3 in Section 4-2 on page 198]. Every CW -complex is paracompact, see [22], and hence in particular regular, see [23, Theorem 4.1 in Section 6-4 on page 255].

(vi) This follows from assertions (i), (iv), and (v). $\square$

7.2. The retraction functor k . There is a construction which assigns to a topological Hausdorff space X a new topological space $k(X)$ such that X and $k(X)$ have the same underlying sets, $k(X)$ is compactly generated, X and $k(X)$ have the same compact subsets, the identity $k(X) \rightarrow X$ is continuous and is a homeomorphism if and only if X is compactly generated. Namely, define the new topology on $k(X)$ by declaring a subset $A \subseteq X$ to be closed if and only if for every compact subset of X the intersection $A \cap C$ is a closed subset of C .

7.3. Mapping spaces, product spaces, and subspaces. Given two compactly generated spaces X and Y , denote by $\text{map}(X, Y)_{k.o.}$ the set of maps $X \rightarrow Y$ with the compact-open-topology, i.e., a subbasis for the compact-open-topology is given by the sets $W(C, U) = \{f: X \rightarrow Y \mid f(C) \subseteq U\}$, where C runs through the compact subsets of X and U runs through the open subsets of Y . Note that $\text{map}(X, Y)_{k.o.}$ is not compactly generated in general. We denote by $\text{map}(X, Y)$ the topological space given by $k(\text{map}(X, Y)_{k.o.})$. Sometimes we abbreviate $\text{map}(X, Y)$ by Y^X and denote for a map $f: Y \rightarrow Z$ the induced map $\text{map}(\text{id}_X, f): \text{map}(X, Y) \rightarrow \text{map}(X, Z)$, $g \mapsto f \circ g$ by $f^X: Y^X \rightarrow Z^X$. If X and Y are compactly generated spaces, then $X \times Y$ stands for $k(X \times_p Y)$, where $X \times_p Y$ is the topological space with respect to the “classical” product topology.

If $A \subseteq X$ is a subset of a compactly generated space, the subspace topology means that we take $k(A_{st})$ for A_{st} the topology space given by the “classical” subspace topology on A .

Roughly speaking, all the usual constructions of topologies are made compactly generated by passing from Y to $k(Y)$ in order to stay within the category of compactly generated spaces.

7.4. Basic features of the category of compactly generated spaces. The category of compactly generated spaces has the following convenient features:

- A map $f: X \rightarrow Y$ of compactly generated spaces is continuous if and only if its restriction $f|_C: C \rightarrow Y$ to any compact subset $C \subseteq X$ is continuous;
- If X, Y , and Z are compactly generated spaces, then the obvious maps

$$\text{map}(X, \text{map}(Y, Z)) \xrightarrow{\cong} \text{map}(X \times Y, Z);$$

$$\text{map}(X, Y \times Z) \xrightarrow{\cong} \text{map}(X, Y) \times \text{map}(X, Z),$$

are homeomorphisms and the map given by composition

$$\text{map}(X, Y) \times \text{map}(Y, Z) \rightarrow \text{map}(X, Z)$$

is continuous;

- The product of two identifications is again an identification;
- If X is locally compact and Y compactly generated, then $X \times Y$ and $X \times_p Y$ are the same topological spaces;
- Let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ be a sequence of inclusions of compactly generated spaces such that X_i is a closed subspace of X_{i+1} for $i = 0, 1, 2, \dots$

Then the colimit $\text{colim}_{i \rightarrow \infty} X_i$ exists in the category of compactly generated Hausdorff spaces. Moreover, if Y is a compactly generated space, then $\text{colim}_{i \rightarrow \infty} (X_i \times Y)$ exists in the category of compactly generated spaces and the canonical map

$$\text{colim}_{i \rightarrow \infty} (X_i \times Y) \xrightarrow{\cong} (\text{colim}_{i \rightarrow \infty} X_i) \times Y$$

is a homeomorphism;

- In the category of compactly generated spaces the pushout of a diagram $X_1 \xleftarrow{f_1} X_0 \xrightarrow{f_2} X_2$ exists if f_1 or f_2 is the inclusion of a closed subspace;
- Given a Hausdorff space Y , the canonical map $k(Y) \rightarrow Y$ is a weak homotopy equivalence and induces an isomorphism on singular homology.
- Given a pushout in the category of compactly generated spaces, its product with a compactly generated space is again a pushout in the category of compactly generated spaces.
- The product of two CW -complexes is again a CW -complex;

Remark 7.3 (Compactly generated weak Hausdorff spaces). There is also the category of compactly generated weak Hausdorff spaces, see [26]. The main advantage in contrast to the category of compactly generated Hausdorff spaces, see [25], is that in the category of compactly generated weak Hausdorff spaces colimits for small diagrams, for instance pushouts or filtered colimits, always exist, see [26, Corollary 2.23]. In the category of compactly generated spaces one can define the pushout of a diagram $X_1 \xleftarrow{f_1} X_0 \xrightarrow{f_2} X_2$ only if for the pushout in the classical setting

$$\begin{array}{ccc} X_0 & \xrightarrow{f_1} & X_1 \\ f_2 \downarrow & & \downarrow \overline{f_2} \\ X_2 & \xrightarrow{\overline{f_1}} & X \end{array}$$

the space X is Hausdorff, since the retraction functor k digests only Hausdorff spaces. Note that X is Hausdorff if f_1 or f_2 is an inclusion of a closed subspace. Therefore in the case treated in the manuscript this condition is always satisfied and the pushout exists in the category of compactly generated Hausdorff spaces.

The same discussion applies to the colimit $\text{colim}_{i \rightarrow \infty} X_i$ of a sequence of inclusions of compactly generated spaces of $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$.

For simplicity we will discuss these issues not anymore and will work in the category of compactly generated Hausdorff spaces throughout this manuscript.

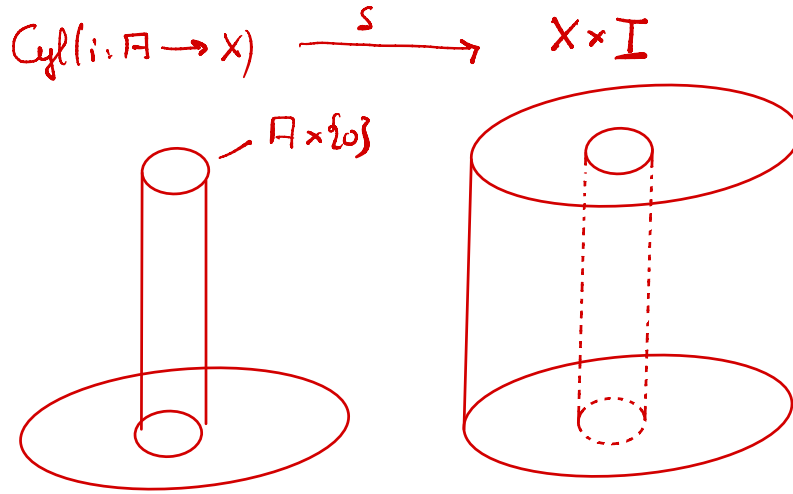
8. COFIBRATIONS

8.1. Basics about cofibrations.

Definition 8.1 (Homotopy extension property). A map $i: A \rightarrow X$ has the *homotopy extension property* (HEP) for the space Y , if for any map $f: X \rightarrow Y$ and any homotopy $h: A \times I \rightarrow Y$ with $h_0 = f \circ i$, there exists a homotopy $H: X \times I \rightarrow Y$ with $H_0 = f$ and $H \circ (i \times \text{id}_I) = h$.

(8.2)

and there is a canonical map $s: \text{cyl}(i) \rightarrow X \times I$ defined as the pushout of the inclusion $i_0^X: X \rightarrow X \times I$ and the map $i \times \text{id}_I: A \times I \rightarrow X \times I$.



Proposition 8.6. *The following assertions are equivalent for a map $i: A \rightarrow X$:*

- (i) *The map $i: A \rightarrow X$ is a cofibration;*
- (ii) *The map $i: A \rightarrow X$ has the homotopy extension property for the mapping cylinder $\text{cyl}(i)$;*
- (iii) *The canonical map $s: \text{cyl}(i) \rightarrow X \times I$ has a retraction $r: X \times I \rightarrow \text{cyl}(i)$.*

Proof. (i) $\implies$ (ii) This is obvious.

(ii) $\implies$ (iii) If we apply the homotopy lifting property to the map $k: X \rightarrow \text{cyl}(i)$ and the homotopy $l: A \times I \rightarrow \text{cyl}(f)$, we obtain a map $r: X \times I \rightarrow \text{cyl}(i)$ such that $r \circ i_0^X = k$ and $r \circ (i \times \text{id}_I) = l$ hold. Since we have $r \circ s \circ k = r \circ i_0^X = k$ and $r \circ s \circ l = r \circ (i \times \text{id}_I) = l$, we conclude from the pushout property that $r \circ s = \text{id}_{\text{cyl}(i)}$ holds.

(iii) $\implies$ (i) Consider any map $f: X \rightarrow Y$ and any homotopy $h: A \times I \rightarrow Y$ satisfying $h_0 = f|_A$. We obtain from the pushout property a map $a: \text{cyl}(i) \rightarrow Y$ such that $a \circ k = f$ and $a \circ l = h$ hold. Now define $H: X \times I \rightarrow Y$ to be $a \circ r$. Then $H \circ (i \times \text{id}_I) = a \circ r \circ (i \times \text{id}_I) = a \circ r \circ s \circ l = a \circ l = h$ and $H \circ i_0^X = a \circ r \circ i_0^X = a \circ r \circ s \circ k = a \circ k = f$ hold. Therefore i has the homotopy lifting property for every space Y and hence is a cofibration. $\square$

Remark 8.7 (Cofibrations are closed embeddings). Note that Proposition 8.6 implies that a cofibration $i: A \rightarrow X$ is a *closed embedding*, i.e., its image $i(A)$ is a closed subspace of X and that i induces a homeomorphism $A \xrightarrow{\cong} i(A)$. Namely, the composite $j: A \xrightarrow{i_1^A} A \times I \xrightarrow{l} \text{cyl}(i)$ is a closed embedding and j can be written as the composite $A \xrightarrow{i} X \xrightarrow{i_1^X} X \times I \xrightarrow{r} \text{cyl}(i)$ because of $r \circ i_1^X \circ i = r \circ (i \times \text{id}_I) \circ i_1^A = r \circ s \circ l \circ i_1^A = l \circ i_1^A = j$. Now use the fact that a map u is a closed embedding if the composite $v \circ u$ of it with some other map v is a closed embedding.

Lemma 8.8.

- (i) *If $i: A \rightarrow X$ is a cofibration, then there exists a retraction $r: X \times I \rightarrow X \times \{0\} \cup_{A \times \{0\}} A \times I$;*

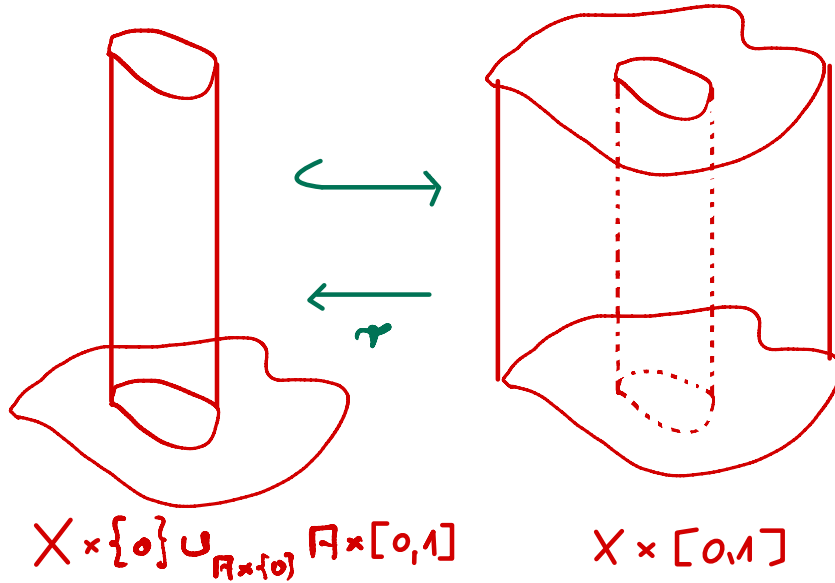
(ii) If there exists a retraction $r: X \times I \rightarrow X \times \{0\} \cup_{A \times \{0\}} A \times I$ and the inclusion $A \rightarrow X$ is a closed embedding, then the inclusion $i: A \rightarrow X$ is a cofibration.

Proof. (i) Put $Y = X \times \{0\} \cup_{A \times \{0\}} A \times I$. Define $f: X \rightarrow Y$ by $f(x) = (x, 0)$ and $h: A \times I \rightarrow Y$ by $h(a, t) = (a, t)$. From the homotopy extension property applied to Y , f , and h we obtain the desired retraction $r = H: X \times I \rightarrow X \times \{0\} \cup_{A \times \{0\}} A \times I$.
(ii) Note that $X \times \{0\} \cup_{A \times \{0\}} A \times I \subseteq X \times I$ is to be understood to be equipped with the subspace topology. Since the inclusion $A \rightarrow X$ is a closed embedding, we get with this topology a pushout

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & X \times \{0\} \\ \downarrow & & \downarrow \\ A \times I & \longrightarrow & X \times \{0\} \cup_{A \times \{0\}} A \times I \end{array}$$

where all maps are inclusions.

Consider $f: X \rightarrow Y$ and $h: A \times I \rightarrow Y$ with $f \circ i_0^A = h_0$. Consider the map $g := f \cup h: X \times \{0\} \cup_{A \times \{0\}} A \times I \rightarrow Y$. The desired homotopy $H: X \times I \rightarrow Y$ is then given by $g \circ r$. $\square$



Lemma 8.9. Consider a pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow i & & \downarrow \bar{i} \\ X & \xrightarrow{\bar{f}} & Y \end{array}$$

such that $i: A \rightarrow X$ is a cofibration.

Then $\bar{i}: B \rightarrow Y$ is a cofibration.

Proof. Suppose that $h: B \times I \rightarrow Z$ and $\varphi: Y \rightarrow Z$ with $h_0 = \varphi \circ \bar{i}$ are given. Then we get a homotopy $h' = h \circ (f \times \text{id}_I)$ and a map $\varphi' = \varphi \circ \bar{f}: X \rightarrow Z$ satisfying

$h'_0 = \varphi' \circ i$. Since i is a cofibration, we get a homotopy $H': X \times I \rightarrow Z$ satisfying $H'_0 = \varphi'$ and $H' \circ (i \times \text{id}_I) = h'$. We have the pushout

$$\begin{array}{ccc} A \times I & \xrightarrow{f \times \text{id}_I} & B \times I \\ i \times \text{id}_I \downarrow & & \downarrow \bar{i} \times \text{id}_I \\ X \times I & \xrightarrow{\bar{f} \times \text{id}_I} & Y \times I. \end{array}$$

Hence H' and h define a map $H: Y \times \text{id}_I \rightarrow Z$ which is uniquely determined by $H \circ (\bar{f} \times \text{id}_I) = H'$ and $H \circ (\bar{i} \times \text{id}_I) = h$. We get $H_0 = \varphi$, since H_0 and φ have the same composite with $\bar{f}$ and $\bar{i}$. $\square$

Lemma 8.10. *Let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ be a sequence of cofibrations. Let X be its colimit $\text{colim}_{i \rightarrow \infty} X_i$.*

Then the canonical map $X_0 \rightarrow X$ is a cofibration.

Proof. Because of Remark 8.7 we can assume without loss of generality that X_i is a closed subspace of both X_{i+1} and X for $i = 0, 1, 2, \dots$ and $X = \bigcup_{i=0}^{\infty} X_i$. Because of Lemma 8.8 (ii) it suffices to construct a retraction $R: X \times I \rightarrow X \times \{0\} \cup_{X_0 \times \{0\}} X_0 \times I$. Since $X \times I = \text{colim}_{i \rightarrow \infty} X \times \{0\} \cup_{X_i \times \{0\}} (X_i \times I)$ holds, it suffices to construct a sequence of maps

$$r_i: X \times \{0\} \cup_{X_i \times \{0\}} X_i \times I \rightarrow X \times \{0\} \cup_{X_0 \times \{0\}} X_0 \times I$$

for $i = 0, 1, 2, \dots$ such that $r_i|_{X \times \{0\} \cup_{X_{i-1} \times \{0\}} X_{i-1} \times I} = r_{i-1}$ holds for $i = 1, 2, 3, \dots$ and $r_0 = \text{id}_{X \times \{0\} \cup_{X_0 \times \{0\}} X_0 \times I}$ holds.

We construct the desired retractions r_i by induction over $i = 0, 1, 2, \dots$. The induction beginning is obvious. The induction step from $(i-1)$ to $i \geq 1$ is done as follows. Since $X_{i-1} \rightarrow X_i$ is a cofibration, there exists a retraction $r'_i: X_i \times I \rightarrow X_i \times \{0\} \cup_{X_{i-1} \times \{0\}} X_{i-1} \times I$ by Lemma 8.8 (ii). It extends to a retraction

$$r'_i: X \times \{0\} \cup_{X_i \times \{0\}} X_i \times I \rightarrow X \times \{0\} \cup_{X_{i-1} \times \{0\}} X_{i-1} \times I$$

by $\text{id}_{X \times \{0\}} \cup r'_i$. Now define r_i to be the composite $r_{i-1} \circ r''_i$. $\square$

8.2. Cofibrations and NDR-pairs.

Definition 8.11 (NDR-pair). We call a pair (X, A) an NDR-pair or neighborhood deformation retract, if there are maps $h: X \times I \rightarrow X$ and $v: X \rightarrow I$ satisfying:

- $h(a, t) = a$ for $a \in A$ and $t \in I$;
- $h(x, 0) = x$ for $x \in X$;
- $v^{-1}(0) = A$;
- $h(x, t) \in A$ for $x \in X$ and $t \in I$ with $v(x) < t$.

Lemma 8.12. *Let (X, A) be a pair. Let $i: A \rightarrow X$ be an inclusion. Then the following assertions are equivalent:*

- (i) *The map $i: A \rightarrow X$ is a cofibration;*
- (ii) *The pair (X, A) is an NDR-pair.*

Proof. (i) $\implies$ (ii) We get from Remark 8.7 that $A \subseteq X$ is closed and from Lemma 8.8 (i) a retraction $r: X \times I \rightarrow X \times \{0\} \cup_{A \times \{0\}} A \times I$. Define

$$h: X \times I \rightarrow X, \quad x \mapsto \text{pr}_X \circ r(x),$$

and

$$v: X \rightarrow I, \quad x \mapsto \sup\{|t - \text{pr}_I \circ r(x, t)| \mid t \in I\},$$

where $\text{pr}_X: X \times I \rightarrow X$ and $\text{pr}_I: X \times I \rightarrow I$ are the canonical projections. One easily checks that h and v satisfy the conditions appearing in Definition 8.11.

(ii) $\implies$ (i) Given the maps h and v , we can define a retraction $r: X \times I \rightarrow X \times \{0\} \cup_{A \times \{0\}} A \times I$ by

$$r(x, t) = \begin{cases} (h(x, t), 0) & \text{if } t \leq v(x); \\ (h(x, t), t - v(x)) & \text{if } t \geq v(x). \end{cases}$$

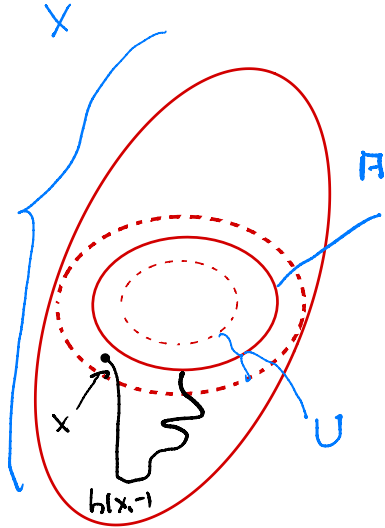
Since $A = v^{-1}(0)$ holds, $A \subseteq X$ is closed. Moreover, A is a G_δ -subset of X , i.e., A is the intersection of countably many open subsets of X . Lemma 8.8 (ii) implies that $i: A \rightarrow X$ is a cofibration. $\square$

Definition 8.13 (Strong neighborhood deformation retraction). Consider a pair (X, A) . We call A a *strong neighborhood deformation retraction* of X , if $A \subseteq X$ is closed, there is an open neighborhood U of A in X such that the inclusion $i: A \rightarrow U$ has retraction $r: U \rightarrow A$, and there exists a homotopy relative A between id_U and $i \circ r$, or, equivalently, $A \subseteq X$ is closed, there is an open neighborhood U of A in X and a homotopy $h: U \times I \rightarrow U$ such that $h(u, 0) = u$ and $h(u, 1) \in A$ holds for $u \in U$ and we have $h(a, t) = a$ for $a \in A$ and $t \in I$.

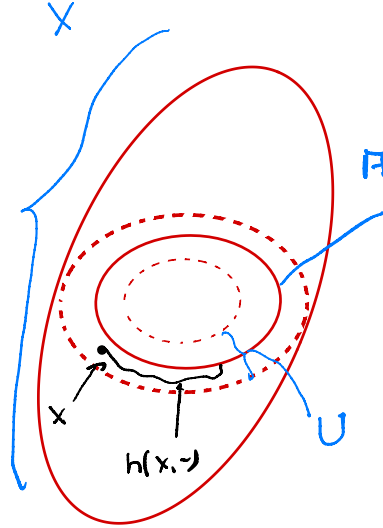
Definition 8.14 (Neighborhood deformation retraction). Consider a pair (X, A) . We call A a *neighborhood deformation retraction* of X , if $A \subseteq X$ is closed, there is an open neighborhood U of A in X and a homotopy $h: U \times I \rightarrow X$ such that $h(u, 0) = u$ and $h(u, 1) \in A$ hold for $u \in U$ and we have $h(a, t) = a$ for $a \in A$ and $t \in I$.

Remark 8.15 (Strong neighborhood deformation retraction versus neighborhood deformation retraction). The difference between Definition 8.13 and Definition 8.14 is that in Definition 8.13 the target of h is U , whereas in Definition 8.14 the target of h is X . Hence a strong neighborhood deformation retraction is a neighborhood deformation retraction. The converse is not true in general.

NEIGH. DEF. RETRACTION



STRONG NEIGH. DEF. RETRACTION



Remark 8.16 (NDR-pairs versus neighborhood deformation retractions). Let (X, A) be an NDR-pair in the sense of Definition 8.11 which is equivalent to $i: A \rightarrow X$ being a cofibration by Lemma 8.12. Then it is a neighborhood deformation retraction in the sense of Definition 8.14. Namely, given $h: X \times I \rightarrow X$ and $v: X \rightarrow I$

as in Definition 8.11, we get by $U = v^{-1}([0, 1])$ and $h|_{U \times I}: U \times I \rightarrow X$ the data required in Definition 8.14. The converse is not true in general.

Now suppose that (X, A) is neighborhood deformation retraction in the sense of Definition 8.14 and assume additionally that there is a map $w: X \rightarrow I$ satisfying $w^{-1}(0) = A$ and $U = w^{-1}([0, 1])$. The latter additional condition is known to be automatically satisfied if X is a perfectly normal space, i.e., a metric space or a CW -complex, and $A \subseteq X$ is closed, or if X is a normal space and $A \subseteq X$ is a closed G_δ -subset of X . Then we obtain a retraction $r: X \times I \rightarrow X \times \{0\} \cup_{A \times \{0\}} A \times I$ by

$$r(x, t) = \begin{cases} (x, t) & \text{if } x \in w^{-1}(0); \\ (h(x, t/2w(x)), 0) & \text{if } x \in w^{-1}((0, 1/2]), t \leq 2w(x); \\ (h(x, 1), t - 2w(x)) & \text{if } x \in w^{-1}((0, 1/2]), 2w(x) \leq t \leq 1; \\ (h(x, 2t(1 - w(x))), 0) & \text{if } x \in w^{-1}([1/2, 1)); \\ (x, 0) & x \in w^{-1}(1). \end{cases}$$

Hence $i: A \rightarrow X$ is a cofibration by Lemma 8.8 (ii) which is equivalent to (X, A) being an NDR-pair by Lemma 8.12.

8.3. Relative CW -complexes are cofibrations.

Theorem 8.17 (Relative CW -complexes are cofibrations). *Let (X, A) be a relative CW -complex. Then the inclusion $i: A \rightarrow X$ is a cofibration.*

Proof. Because of Lemma 8.10 it suffices to prove that the inclusion $X_i \rightarrow X_{i+1}$ is a cofibration for $i = 0, 1, 2, \dots$. Choose a pushout

$$\begin{array}{ccc} \coprod_{j \in J_i} S^i & \longrightarrow & X_i \\ \downarrow & & \downarrow \\ \coprod_{j \in J_i} D^{i+1} & \longrightarrow & X_{i+1}. \end{array}$$

By Lemma 8.9 it suffices to show that the left vertical arrow in the diagram above is a cofibration. This follows from the fact that the inclusion $S^i \rightarrow D^{i+1}$ is a cofibration which is a consequence of Lemma 8.8 (ii). $\square$

One can actually show the following stronger result which we state without giving the proof which follows essentially from the fact that (D^{i+1}, S^i) is a strong neighborhood deformation retraction and is similar to the one of Theorem 8.17.

Theorem 8.18 (CW -complexes and strong neighborhood deformation retraction). *Let (B, A) be a strong neighborhood deformation retraction. Let (X, B) be a relative CW -complex. Then the pair (X, A) is a strong neighborhood deformation retraction.*

We omit the proof of the following result whose proof is similar to the one of Theorem 8.18.

Theorem 8.19 (CW -complexes are locally contractible). *Every CW -complex X is locally contractible, i.e., for every point $x \in X$ and every open neighborhood V of x in X there exists an open neighborhood U of x in X such that $\bar{U} \subseteq V$ holds and U and $\bar{U}$ are contractible.*

8.4. Well-pointed spaces.

Definition 8.20 (Well-pointed space). A well-pointed space (X, x) is a pointed space such that the inclusion of the base point $\{x\} \rightarrow X$ is a cofibration.

Lemma 8.21. *Let $\{(X_i, x_i) \mid i \in I\}$ be a collection of well-pointed spaces. Then $\bigvee_{i \in I} (X_i, x_i)$ with the canonical base point is well-pointed.*

Proof. We have the pushout

$$\begin{array}{ccc} \coprod_{i \in I} \{x_i\} & \longrightarrow & \{\bullet\} \\ \downarrow & & \downarrow \\ \coprod_{i \in I} X_i & \longrightarrow & \bigvee_{i \in I} (X_i, x_i) \end{array}$$

where the left vertical arrow is the obvious inclusion and a cofibration. Now apply Lemma 8.9. $\square$

8.5. Comparing pointed homotopy and homotopy. Consider a well-pointed space (X, x) and space Y . Next we define a covariant functor

$$(8.22) \quad \gamma = \gamma_{(X, x), Y} : \Pi(Y) \rightarrow \mathbf{Sets}$$

from the fundamental groupoid $\Pi(Y)$ to the category $\mathbf{Sets}$ of sets. It sends the element $y \in Y$ to the set $[(X, x), (Y, y)]^0$ of pointed homotopy classes of pointed maps $(X, x) \rightarrow (Y, y)$. Consider a morphism $[w] : y_0 \rightarrow y_1$ in $\Pi(Y)$ represented by path $w : I \rightarrow Y$ with $w(0) = y_0$ and $w(1) = y_1$ and an element $[f] \in [(X, x), (Y, y_0)]^0$ represented by a pointed map $f : (X, x) \rightarrow (Y, y_0)$. Since the inclusion $\{x\} \rightarrow X$ is a cofibration, we can find a homotopy $h : X \times I \rightarrow Y$ such that $h_0 = f_0$ and $h(x, t) = w(t)$ holds. Now we define $\gamma([w])([f]) = [h_1]$. We omit the proof, which is essentially based on the fact that the inclusion $\{x\} \times I \rightarrow X \times I$ is a cofibration, that this definitions makes sense and yields the functor γ announced in (8.22).

If we fix a point $y \in Y$, we get using the identification of $\pi_1(Y, y)$ with $\text{aut}_{\Pi(Y)}(y)$ an operation

$$(8.23) \quad \pi_1(Y, y) \times [(X, x), (Y, y)]^0 \rightarrow [(X, x), (Y, y)]^0.$$

One easily checks that the forgetful map $[(X, x), (Y, y)]^0 \rightarrow [X, Y]$ induces a bijection

$$(8.24) \quad \pi_1(Y, y) \backslash [(X, x), (Y, y)]^0 \xrightarrow{\cong} [X, Y].$$

Note that for a simply connected pointed space (Y, y) , the bijection (8.24) reduces to a bijection $[(X, x), (Y, y)]^0 \xrightarrow{\cong} [X, Y]$.

If we take $(X, x) = (S^n, s)$, the operation (8.23) yields an operation of $\pi_1(Y, y)$ on $\pi_n(Y, y)$. If $n = 1$, this is the conjugation action, where $[w]$ acts on $[u] \in \pi_1(Y, y)$ by $[u] \mapsto [w] \cdot [u] \cdot [w]^{-1}$. If $n \geq 2$, then $\pi_n(Y, y)$ is abelian and the $\pi_1(Y, y)$ -action is by automorphism of abelian groups. Hence we get a left $\mathbb{Z}[\pi_1(Y, y)]$ -module structure on $\pi_n(Y, y)$ for $n \geq 2$.

Suppose that Y is path connected and has a universal covering $p : \tilde{Y} \rightarrow Y$. Choose $\tilde{y} \in \tilde{Y}$ with $p(\tilde{y}) = y$. Recall that $\tilde{Y}$ comes with a $\pi_1(Y, y)$ -action. Fix a natural number $n \geq 2$. We get for $[u] \in \pi_1(Y, y)$ a homeomorphism $l_{[u]} : \tilde{Y} \xrightarrow{\cong} \tilde{Y}$ by left multiplication with $[u]$. Choose a path $v : I \rightarrow \tilde{Y}$ from $[u] \cdot \tilde{y}$ to $\tilde{y}$. Then we get an isomorphism $T_n([v]) : \pi_n(\tilde{Y}, [u] \cdot \tilde{y}) \xrightarrow{\cong} \pi_n(\tilde{Y}, \tilde{y})$ from (2.6), which is independent of the choice of v as $\tilde{Y}$ is simply connected. Now we define a left $\pi_1(Y, y)$ -action on $\pi_n(\tilde{Y}, \tilde{y})$ by letting $[u] \in \pi_1(Y, y)$ act on $\pi_n(\tilde{Y}, \tilde{y})$ by the composite $\pi_n(\tilde{Y}, \tilde{y}) \xrightarrow{\pi_n(l_{[u], \tilde{y}})} \pi_n(\tilde{Y}, [u] \cdot \tilde{y}) \xrightarrow{T_n([v])} \pi_n(\tilde{Y}, \tilde{y})$. One easily checks that this defines a left $\mathbb{Z}[\pi_1(Y, y)]$ -module structure on the abelian group $\pi_n(\tilde{Y}, \tilde{y})$. Recall the isomorphism $\pi_n(p, \tilde{y}) : \pi_n(\tilde{Y}, \tilde{y}) \xrightarrow{\cong} \pi_n(Y, y)$ from Theorem 2.7. One easily checks that it is compatible with the left $\mathbb{Z}[\pi_1(Y, y)]$ -module structures on $\pi_n(\tilde{Y}, \tilde{y})$ and $\pi_n(Y, y)$ constructed above.

Lemma 8.25. *Let $f : (X, x) \rightarrow (Y, y)$ be a pointed map of well-pointed spaces. Suppose that f is a homotopy equivalence (after forgetting the base points).*

Then f is a pointed homotopy equivalence.

Proof. Choose a homotopy inverse $g': Y \rightarrow X$ of f . Because of (8.24) we can change g' up to homotopy such that $g'(y) = x$ holds. The map $f_*: [(Y, y), (X, x)]^0 \rightarrow [(Y, y), (Y, y)]^0$ sends $[g']$ to the element $[f \circ g']$ which is mapped under the projection $[(Y, y), (Y, y)]^0 \rightarrow [Y, Y]$ to $[\text{id}_Y]$. Because of (8.24) there is an element $v \in \pi_1(Y, y)$ satisfying $[\text{id}_Y] = v \cdot [f \circ g']$ in $[(Y, y), (Y, y)]^0$. Since $\pi_1(f, x): \pi_1(X, x) \rightarrow \pi_1(Y, y)$ is bijective, we can find $u \in \pi_1(X, x)$ with $\pi_1(f, x)(u) = v$. Let $g: (Y, y) \rightarrow (X, x)$ be a pointed map satisfying $[g] = u \cdot [g']$ in $[(Y, y), (X, x)]^0$. Then we get in $[(Y, y), (Y, y)]^0$

$$[f \circ g] = f_*([g]) = f_*(u \cdot [g']) = v \cdot f_*([g']) = v \cdot [f \circ g'] = [\text{id}_Y].$$

Hence g is a pointed homotopy right inverse of f . The same argument applied to g shows that $[g]$ has a pointed homotopy right inverse. This implies that f is a pointed homotopy equivalence. $\square$

8.6. The Homotopy Theorem for pushouts and cofibrations. Given a space B , let Top^B be the category of topological spaces under B . Objects are maps $u: B \rightarrow X$. A morphism from $u: B \rightarrow X$ to $v: B \rightarrow Y$ is a map $f: X \rightarrow Y$ satisfying $f \circ u = v$. We call two such morphisms $f_0, f_1: u \rightarrow v$ homotopic if they are homotopic through morphisms in Top^B , i.e., there exists a homotopy $h: X \times I \rightarrow Y$ such that $h_0 = f_0$ and $h_1 = f_1$ holds and we have $h_t \circ u = v$ for every $t \in I$. Let h-Top^B be the associated homotopy category, i.e., the set of objects of h-Top^B and Top^B agree and a morphism from u to v in h-Top^B is a homotopy class of morphisms from u to v in Top^B .

Let Cof^B and h-Cof^B respectively be the full subcategory of Top^B and h-Top^B respectively consisting of those objects $i: B \rightarrow X$ for which i is a cofibration.

Given two spaces A and B , define $\Pi(A, B)$ to be the following category. Objects are maps $f: A \rightarrow B$. A morphism from f_0 to f_1 is a homotopy class $[h]$ relative $A \times \{0, 1\}$ of maps $h: A \times I \rightarrow B$ with $h_0 = f_0$ and $h_1 = f_1$. Note that h itself is a homotopy between f_0 and f_1 and $[h]$ is the homotopy class of such homotopies represented by h . If $A = \{\bullet\}$, then $\Pi(\{\bullet\}, B)$ is the fundamental groupoid $\Pi(B)$ of B . Note that $\Pi(A, B)$ is a groupoid. Given a cofibration $i: A \rightarrow X$, we next sketch the construction of a contravariant functor

$$(8.26) \quad \beta_i: \Pi(A, B) \rightarrow \text{h-Cof}^B.$$

An object $f: A \rightarrow B$ is sent to the cofibration $\gamma_i(f): B \rightarrow Y_f$ given by the following pushout and Lemma 8.9

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ i \downarrow & & \downarrow \gamma_i(f) \\ X & \xrightarrow{\beta_i(f)} & Y_f. \end{array}$$

Now consider two maps $f_0, f_1: A \rightarrow B$ together with a homotopy $h: A \times I \rightarrow B$ with $h_0 = f_0$ and $h_1 = f_1$. As i is a cofibration, there exists a homotopy $H: X \times I \rightarrow Y_{f_0}$ with $H_0 = \beta_i(f_0)$ and $H \circ (i \times \text{id}_I) = \gamma_i(f_0) \circ h$. Since we have the pushout

$$\begin{array}{ccc} A & \xrightarrow{f_1} & B \\ i \downarrow & & \downarrow \gamma_i(f_1) \\ X & \xrightarrow{\beta_i(f_1)} & Y_{f_1} \end{array}$$

there is precisely one map $u: Y_{f_1} \rightarrow Y_{f_0}$ uniquely determined by the property that $u \circ \beta_i(f_1) = \beta_i(f_0)$ and $u \circ \gamma_i(f_1) = \gamma_i(f_0)$ hold. Obviously u is a morphism from

$\gamma_i(f_1): B \rightarrow Y_{f_1}$ to $\gamma_i(f_0): B \rightarrow Y_{f_0}$ in Cof^B . Thanks to H , we have

$$(8.27) \quad u \circ \beta_i(f_1) \simeq \beta_i(f_0).$$

We omit the proof that $[u]$ depends only on $[h]$ which can be found in [31, Proposition 5.2.1 on page 107]. So we can define $\beta_i([h]) = [u]$. We also omit the proof that β_i is a contravariant functor. Note that $\beta_i([u])$ is represented by a homotopy equivalence as $\pi(A, B)$ is a groupoid.

Theorem 8.28 (Homotopy Theorem for pushouts and cofibrations). *Consider a pushout*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ i \downarrow & & \downarrow \bar{i} \\ X & \xrightarrow{\bar{f}} & Y \end{array}$$

such that $i: A \rightarrow X$ is a cofibration and f is a homotopy equivalence.

Then $\bar{i}: B \rightarrow Y$ is a cofibration and $\bar{f}$ is a homotopy equivalence.

Proof. The map $\bar{i}$ is a cofibration by Lemma 8.9. Let $g: B \rightarrow A$ be a homotopy inverse of f . Consider the pushout

$$\begin{array}{ccc} B & \xrightarrow{g} & A \\ \bar{i} \downarrow & & \downarrow \bar{\bar{i}} \\ Y & \xrightarrow{\bar{g}} & Z. \end{array}$$

The map $\bar{\bar{i}}$ is a cofibration by Lemma 8.9. Since $g \circ f \simeq \text{id}_A$, we get from the contravariant functor $\beta_i: \Pi(A, A) \rightarrow \text{h-Cof}^A$ of (8.26) and from (8.27) a homotopy equivalence $u: Z \rightarrow X$ such that $u \circ \bar{g} \circ \bar{f} \simeq \text{id}_X$ holds. Hence $\bar{f}$ has a left homotopy inverse. Interchanging the role of f and g shows that $\bar{f}$ has a right homotopy inverse. Hence $\bar{f}$ is a homotopy equivalence. $\square$

Theorem 8.28 can easily be extended to the following theorem.

Theorem 8.29 (Homotopy Theorem for maps between pushouts). *Let the following two diagrams be pushouts*

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ i_2 \downarrow & & \downarrow j_1 \\ X_2 & \xrightarrow{j_2} & X \end{array} \quad \begin{array}{ccc} Y_0 & \xrightarrow{k_1} & Y_1 \\ k_2 \downarrow & & \downarrow l_1 \\ Y_2 & \xrightarrow{l_2} & Y \end{array}$$

where the left vertical arrows i_2 and k_2 are cofibrations. Let $f_i: X_i \rightarrow Y_i$ be homotopy equivalences for $i = 0, 1, 2$ satisfying $f_1 \circ i_1 = k_1 \circ f_0$ and $f_2 \circ i_2 = k_2 \circ f_0$. Denote by $f: X \rightarrow Y$ the map induced by f_0, f_1 , and f_2 and the pushout property.

Then f is a homotopy equivalence.

Remark 8.30. The condition that the maps i_2 and k_2 are cofibrations appearing in Theorem 8.29 is necessary as the following examples shows.

We take as pushouts

$$\begin{array}{ccc} S^n & \xrightarrow{i_1} & D^{n+1} \\ i_2 \downarrow & & \downarrow j_1 \\ D^{n+1} & \xrightarrow{j_2} & S^{n+1} \end{array} \quad \begin{array}{ccc} S^n & \xrightarrow{k_1} & \{\bullet\} \\ k_2 \downarrow & & \downarrow l_1 \\ \{\bullet\} & \xrightarrow{l_2} & \{\bullet\} \end{array}$$

and define $f_0 = \text{id}_{S^n}$ and f_1, f_2 , and f to be the the projections.

Example 8.31. Let B be the compact subset of $\mathbb{R}^2$ given by

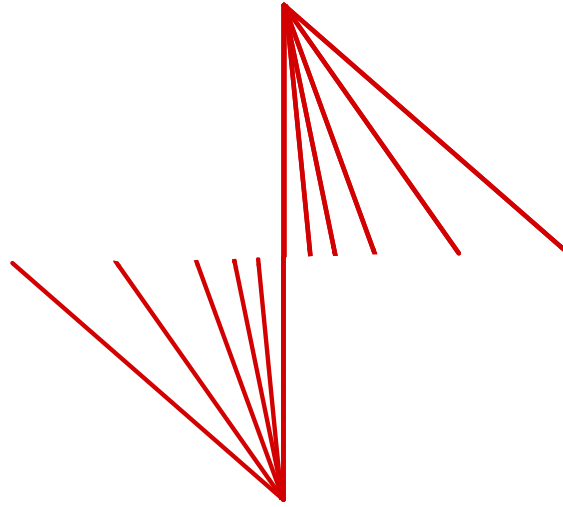
$$B = \{(1/n, 0) \in \mathbb{R}^2 \mid n \in \mathbb{Z}^{\geq 1}\} \amalg \{(0, 0)\}.$$

Let C be the cone over B with cone point $(0, 1)$ in $\mathbb{R}^2$, i.e.,

$$C = \{x \in \mathbb{R}^2 \mid \exists t \in I \text{ and } b \in B \text{ satisfying } x = t \cdot b + (1 - t) \cdot (0, 1)\}.$$

Define

$$A = \{x \in \mathbb{R}^2 \mid x \in C \text{ or } -x \in C\}.$$



Then we have:

- (i) The inclusion $\{(0, 1)\} \rightarrow C$ is a cofibration;
- (ii) C is contractible;
- (iii) A is not contractible;
- (iv) The inclusion $\{(0, 0)\} \rightarrow A$ is not a cofibration;
- (v) The inclusion $\{(0, 0)\} \rightarrow C$ is not a cofibration;
- (vi) C is a not CW -complex;
- (vii) A is not a CW -complex.

Since we have the pushout whose left vertical arrow is the obvious inclusion and a cofibration

$$\begin{array}{ccc} B \times \{1\} & \longrightarrow & \{\bullet\} \\ \downarrow & & \downarrow \\ B \times I & \longrightarrow & C \end{array}$$

assertions (i) and (ii) follow from Theorem 8.28.

Next we show assertion (iii). Suppose that A is contractible. Since A is in particular path connected, we can find a map $h: A \times I \rightarrow A$ with $h(a, 0) = (0, 0)$ and $h(a, 1) = a$ for all $a \in A$. Since any path from $(-1/n, 0)$ to $(0, 0)$ in A must go through $(0, -1)$, we can find elements $t_n^- \in I$ with $h(-1/n, t_n^-) = (0, -1)$ for $n \geq 1$. Since I is compact, we can find a strictly monotone increasing function $N: \mathbb{N} \rightarrow \mathbb{N}$ and an element $t^- \in I$ with $\lim_{n \rightarrow \infty} t_{N(n)}^- = t^-$. As $\lim_{n \rightarrow \infty} 1/N(n) = 0$ holds, we conclude $h((0, 0), t^-) = \lim_{n \rightarrow \infty} h(-1/N(n), t_{N(n)}^-) = (0, -1)$. Since h

is continuous, we can choose t^- such that $h((0,0),t) = (0,-1)$ for $t \in I$ implies $t^- \leq t$. Analogously we construct $t^+ \in I$ such that $h((0,0),t^+) = (0,1)$ holds and $h((0,0),t) = (0,1)$ for $t \in I$ implies $t^+ \leq t$. Next we consider only the case $t^+ \leq t^-$, the other case is completely analogous. Obviously $t^+ \neq t^-$ holds and hence $t^+ < t^-$. By continuity $\lim_{n \rightarrow 0} h(-1/n, t^+) = h(0, t^+) = (0,1)$. Hence there is a natural number n_0 such that $h(-1/n, t^+) = (0, u_n)$ with $u_n \geq 0$ holds for $n \geq n_0$. Since any path from $(-1/n, 0)$ to $(0, u)$ for $u \geq 0$ in A must go through $(0, -1)$ we can for every $n \geq n_0$ elements $s_n^- \in I$ with $h(-1/n, s_n^-) = (0, -1)$ and $s_n^- \leq t^+$. Since I is compact, we can find a strictly monotone increasing function $N': \mathbb{N} \rightarrow \mathbb{N}$ and an element $s^- \in I$ with $\lim_{n \rightarrow \infty} s_{N'(n)}^- = s^-$. Obviously $s^- \leq t^+$. As $\lim_{n \rightarrow \infty} 1/N'(n) = 0$ holds, we conclude $(0, -1) = \lim_{n \rightarrow \infty} h((1/n, 0), s_n^-) = h((0, 0), s^-)$. This implies $t^- \leq s^-$. Hence we get $t^- \leq t^+$, a contradiction.

Suppose that assertions (iv) is not true. As C and hence also $\{-x \in \mathbb{R}^n \mid x \in C\}$ are contractible by assertion (ii), Theorem 8.29 implies that A is contractible. Since we have already proved that A is not contractible, assertion (iv) follows.

Suppose that the inclusion $\{(0,0)\} \rightarrow C$ is a cofibration. Then also the inclusion $\{(0,0)\} \rightarrow \{-x \in \mathbb{R}^n \mid x \in C\}$ is a cofibration. This implies by Lemma 8.21 that the inclusion $\{(0,0)\} \rightarrow A$ is a cofibration. Hence assertion (iv) implies assertion (v).

Since the point $(0,0)$ in C has the property that any neighborhood of it in C which does not contain $(1,0)$ is not contractible, assertion (vi) follows from Theorem 8.19. The proof of assertion (vii) is analogous.

8.7. (Pointed) cylinders, cones and suspensions. Consider a space X . Recall that its *cylinder* is defined by $X \times I$, its *cone* $\text{cone}(X)$ by the pushout

$$\begin{array}{ccc} X \times \{1\} & \longrightarrow & \{\bullet\} \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & \text{cone}(X), \end{array}$$

and its *suspension* by the pushout

$$\begin{array}{ccc} X & \longrightarrow & \text{cone}(X) \\ \downarrow & & \downarrow \\ \text{cone}(X) & \longrightarrow & \Sigma X. \end{array}$$

Equivalently, one can define ΣX to be the quotient of $X \times [-1, 1]$ under the equivalence relation generated by $(x_0, 1) \sim (x_1, 1)$ for $x_0, x_1 \in X$ and $(x_0, -1) \sim (x_1, -1)$ for $x_0, x_1 \in X$. There is an obvious pushout

$$\begin{array}{ccc} X & \longrightarrow & \text{cone}(X) \\ \downarrow & & \downarrow \\ \{\bullet\} & \longrightarrow & \Sigma X. \end{array}$$

Consider a pointed space (X, x) . Its *reduced mapping cylinder* $\text{cyl}(X, x)$, its *reduced mapping cone* $\text{cone}(X, x)$, and its *reduced suspension* $\Sigma(X, x)$ are defined by the pushouts

$$(8.32) \quad \begin{array}{ccc} \{x\} \times I & \longrightarrow & \{\bullet\} \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & \text{cyl}(X, x), \end{array}$$

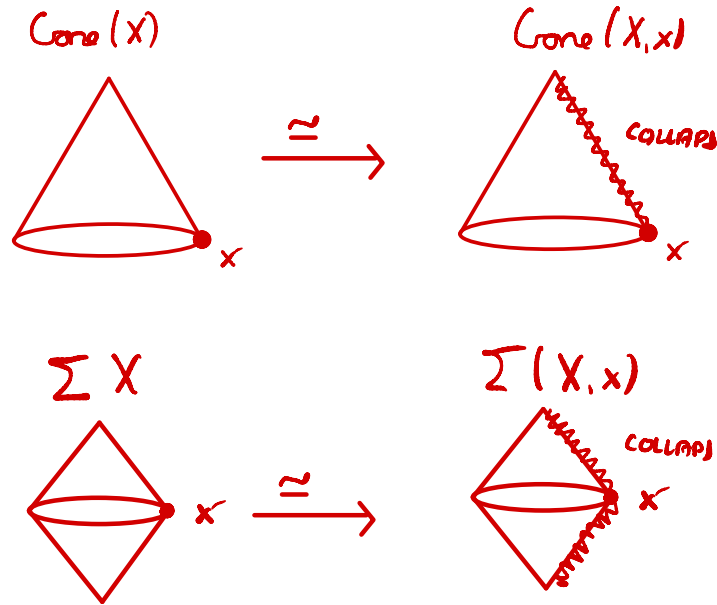
$$(8.33) \quad \begin{array}{ccc} \{x\} \times I \cup X \times \{1\} & \longrightarrow & \{\bullet\} \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & \text{cone}(X, x), \end{array}$$

and

$$(8.34) \quad \begin{array}{ccc} \{x\} \times I \cup X \times \{0, 1\} & \longrightarrow & \{\bullet\} \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & \Sigma(X, x). \end{array}$$

and come with a preferred base point. Note that $\Sigma(X, x)$ can be identified with the smash product $(S^1, s) \wedge (X, x) = (S^1 \times X) / (S^1 \times x \cup \{s\} \times X)$.

Given a well-pointed pointed space (X, x) , the canonical projections $\text{cyl}(X) \rightarrow \text{cyl}(X, x)$, $\text{cone}(X) \rightarrow \text{cone}(X, x)$, and $\Sigma X \rightarrow \Sigma(X, x)$ are pointed homotopy equivalences by Lemma 8.25, Theorem 8.28, and Theorem 8.29.



There are obvious pushouts

$$\begin{array}{ccc} \{x\} \times I & \longrightarrow & \{\bullet\} \\ \downarrow & & \downarrow \\ \text{cyl}(X, x) & \longrightarrow & \text{cone}(X, x), \end{array}$$

and

$$\begin{array}{ccc} X & \longrightarrow & \text{cone}(X, x) \\ \downarrow & & \downarrow \\ \text{cone}(X, x) & \longrightarrow & \Sigma(X, x). \end{array}$$

Let $f: (X, x) \rightarrow (Y, y)$ be a pointed map. Its *reduced mapping cone* $\text{cone}(f, x)$ is defined by the pushout

$$(8.35) \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \text{cone}(X, x) & \longrightarrow & \text{cone}(f, x) \end{array}$$

or, equivalently by the pushout

$$(8.36) \quad \begin{array}{ccc} (\{x\} \times I) \cup (X \times \{0\}) \cup (X \times \{1\}) & \xrightarrow{u} & Y \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & \text{cone}(f, x) \end{array}$$

where u sends (x, t) for $t \in I$ to y , $(z, 1)$ to $f(z)$ for $z \in X$ and $(z, 0)$ to y for $z \in X$.

Note that $\text{cone}(f, x)$ comes with a preferred base point for which the pushout (8.35) is a diagram of pointed maps of pointed spaces.

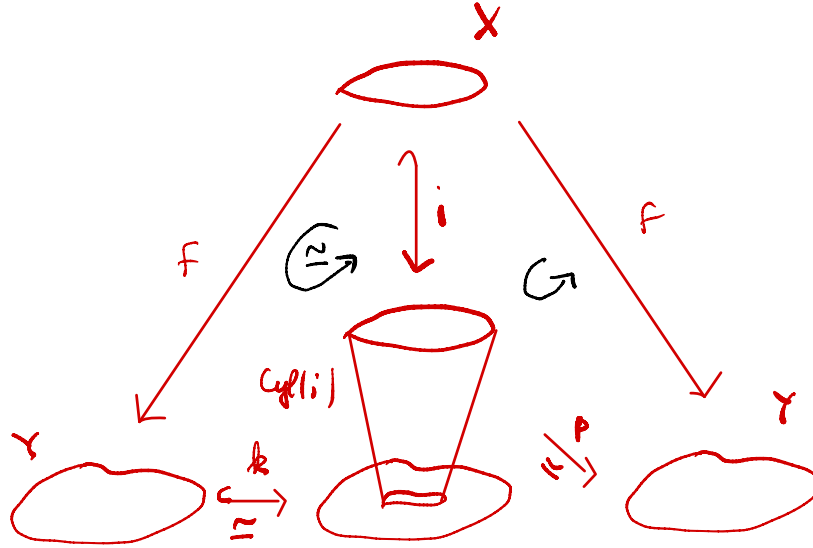
8.8. Turning a map into a cofibration. Consider a map $f: X \rightarrow Y$. Then it can be replaced up to homotopy equivalence by a cofibration. Namely let $i: X \rightarrow \text{cyl}(f)$ be the canonical inclusion and $p: \text{cyl}(f) \rightarrow Y$ be the projection. Then we get the factorization

$$f: X \xrightarrow{i} \text{cyl}(f) \xrightarrow{p} Y$$

where i is a cofibration and p is a homotopy equivalence. Actually we get a diagram

$$(8.37) \quad \begin{array}{ccccc} & & X & & \\ & f \swarrow & \downarrow i & \searrow f & \\ Y & \xrightarrow[k]{\simeq} & \text{cyl}(f) & \xrightarrow[p]{\simeq} & Y \end{array}$$

where the left triangle commutes up to homotopy, the right triangle commutes, the two horizontal maps k and p are homotopy equivalences which are homotopy inverse to one another, and the vertical arrow i is a cofibration.



8.9. The Cofiber Sequence. A pointed map $f: (X, x) \rightarrow (Y, y)$ induces by composition for every pointed space (B, b) a map

$$f^*: [(Y, y), (B, b)]^0 \rightarrow [(X, x), (B, b)]^0, \quad [u] \mapsto [u \circ f]$$

which depends only on the pointed homotopy class of f . A sequence $(X, x) \xrightarrow{f} (Y, y) \xrightarrow{g} (Z, z)$ of maps of pointed spaces is called *homotopy coexact* if for each pointed space (B, b) the induced sequence of pointed sets

$$[(Z, z), (B, b)]^0 \xrightarrow{g^*} [(Y, y), (B, b)]^0 \xrightarrow{f^*} [(X, x), (B, b)]^0$$

is exact at $[(Y, y), (B, b)]^0$ in the sense that the image of g^* is the preimage of f^* of the base point in $[(X, x), (B, b)]^0$ given by $[c_b]$ for the constant pointed map $c_b: (X, x) \rightarrow (B, b)$. Note that this implies that $g \circ f$ is pointed homotopy equivalent to the constant map $c_z: (X, x) \rightarrow (Z, z)$.

Lemma 8.38. *Let $f: (X, x) \rightarrow (Y, y)$ be a pointed map. Let $j: (Y, y) \rightarrow \text{cone}(f, x)$ be the canonical inclusion which is a map of pointed spaces.*

Then the sequence

$$(X, x) \xrightarrow{f} (Y, y) \xrightarrow{j} (\text{cone}(f, x), *)$$

is homotopy coexact.

Proof. This is a direct consequence of the pushout (8.36) which says that a pointed map $(\text{cone}(f, x), *) \rightarrow (B, b)$ is the same as a pointed map $v: (Y, y) \rightarrow (B, b)$ together with a pointed homotopy $h: X \times I \rightarrow B$ between the constant map c_b and $v \circ f$. $\square$

One can iterate this construction and obtains a homotopy coexact sequence of pointed sets, infinite to the right,

$$X \xrightarrow{f} Y \xrightarrow{j} \text{cone}(f) \xrightarrow{f_2} \text{cone}(j) \xrightarrow{f_3} \text{cone}(f_2) \xrightarrow{f_4} \dots$$

where we omit the base points from the notation and homotopy coexact means that it is exact as a sequence of pointed sets at $Y, \text{cone}(f), \text{cone}(j), \dots$

The further investigation replace the iterated mapping cones with homotopy equivalent spaces which are more appealing, namely iterated suspensions.

Suppose additionally, that the pointed map $f: (X, x) \rightarrow (Y, y)$ is a cofibration (after forgetting the base points). Note that then we can think of X as a closed subspace of Y and f as the inclusion of X into Y , see Remark 8.7. Then we obtain a pointed map $p: (\text{cone}(f, x), *) \rightarrow (Y/X, *)$ which is homotopy equivalence by Theorem 8.29. We conclude from Lemma 8.25 that p is a pointed homotopy equivalence. Hence the following diagram of pointed sets commutes

$$(8.39) \quad \begin{array}{ccccc} (X, x) & \xrightarrow{f} & (Y, y) & \xrightarrow{j} & (\text{cone}(f, x), *) \\ \downarrow \text{id}_X & & \downarrow \text{id}_Y & & \downarrow p \\ (X, x) & \xrightarrow{f} & (Y, y) & \xrightarrow{q} & (Y/X, *) \end{array}$$

where $q: X \rightarrow X/Y$ is the canonical projection and all vertical arrows are pointed homotopy equivalences. Hence the sequence $(X, x) \xrightarrow{f} (Y, y) \xrightarrow{q} (Y/X, *)$ is homotopy coexact.

Note that $j: (Y, y) \rightarrow \text{cone}(f, x)$ is a cofibration and $\text{cone}(f, x)/Y$ is homeomorphic to $\Sigma(X, x)$ regardless whether f is a cofibration or not. Hence we obtain from (8.39) a commutative diagram of pointed sets

$$\begin{array}{ccccc} (Y, y) & \xrightarrow{j} & (\text{cone}(f, x), *) & \xrightarrow{k} & (\text{cone}(j, *), *) \\ \downarrow \text{id}_Y & & \downarrow \text{id}_{\text{cone}(f, x)} & & \downarrow p' \\ (Y, y) & \xrightarrow{j} & (\text{cone}(f, x), *) & \xrightarrow{g} & (\Sigma(X, x), *) \end{array}$$

where all vertical arrows are pointed homotopy equivalences and $g = p' \circ k$. Hence the sequence

$$(X, x) \xrightarrow{f} (Y, y) \xrightarrow{j} (\text{cone}(f, x), *) \xrightarrow{g} (\Sigma(X, x), *)$$

is homotopy coexact. Iterating this process leads to the following result. Denote by Σ^n the n -fold suspension.

Theorem 8.40 (Cofiber sequence). *Consider a pointed map $f: (X, x) \rightarrow (Y, y)$. Then we obtain a homotopy coexact sequence, infinite to the right*

$$(8.41) \quad \begin{aligned} (X, x) &\xrightarrow{f} (Y, y) \xrightarrow{j} (\text{cone}(f, x), *) \xrightarrow{g} (\Sigma(X, x), *) \xrightarrow{\Sigma f} (\Sigma(Y, y), *) \\ &\xrightarrow{\Sigma j} \Sigma(\text{cone}(f, x), *) \xrightarrow{\Sigma g} (\Sigma^2(X, x), *) \xrightarrow{\Sigma^2 f} (\Sigma^2(Y, y), *) \\ &\xrightarrow{\Sigma^2 j} \Sigma^2(\text{cone}(f, x), *) \xrightarrow{\Sigma^2 g} (\Sigma^3(X, x), *) \xrightarrow{\Sigma^3 f} \dots \end{aligned}$$

Note that this sequence (8.41) is natural in f . Moreover, it yields for every pointed space (B, b) the following exact sequence of pointed sets, which is infinite to the left, natural in both f and (B, b) , and sometimes called the *Puppe sequence*:

$$(8.42) \quad \begin{aligned} \dots &\xrightarrow{\Sigma^2 g^*} [\Sigma^2 \text{cone}(f), B]^0 \xrightarrow{\Sigma^2 j^*} [\Sigma^2 Y, B]^0 \xrightarrow{\Sigma^2 f^*} [\Sigma^2 X, B]^0 \xrightarrow{\Sigma g^*} [\Sigma \text{cone}(f), B]^0 \\ &\xrightarrow{\Sigma j^*} [\Sigma Y, B]^0 \xrightarrow{\Sigma f^*} [\Sigma X, B]^0 \xrightarrow{g^*} [\text{cone}(f), B]^0 \xrightarrow{j^*} [Y, B]^0 \xrightarrow{f^*} [X, B]^0. \end{aligned}$$

Here and also sometimes in the sequel we omit the base points from the notation. Note the obvious fact that the map $f^*: [Y, B]^0 \rightarrow [X, B]^0$ is not necessarily surjective.

8.10. Group structures on the Puppe Sequence. Let (X, x) be a well-pointed space and (Y, y) be a pointed space. We have the pinching map $\nabla_1: S^1 \rightarrow S^1 \vee S^1$ of (2.5). It induces a pinching map

$$\begin{aligned} \nabla_X: \Sigma(X, x) &= S^1 \wedge X \\ &\xrightarrow{\nabla_1 \wedge \text{id}_X} (S^1 \vee S^1) \wedge X = (S^1 \wedge X) \vee (S^1 \wedge X) = \Sigma(X, x) \vee \Sigma(X, x). \end{aligned}$$

Now we can define a group structure on $[\Sigma X, Y]^0$ by

$$(8.43) \quad [\Sigma X, Y]^0 \times [\Sigma X, Y]^0 \rightarrow [\Sigma X, Y]^0, \quad [f] \cdot [g] \mapsto [(f \vee g) \circ \nabla_X].$$

Analogously to the proof of Lemma 2.4, one can show that this group structure is abelian on $[\Sigma^n X, Y]^0$ for $n \geq 2$. If we take (X, x) to be (S^0, s) , then the groups $[\Sigma^n S^0, (Y, y)]^0 = [S^n, Y]^0$ and $\pi_n(Y, y)$ agree.

The exact Puppe sequence (8.42) appearing in Theorem 8.40 is an exact sequence of groups or abelian groups in the ranges where the group structures are defined on the sets of pointed homotopy classes.

9. FIBRATIONS

9.1. Basics about fibrations.

Definition 9.1 (Homotopy lifting property). A map $p: E \rightarrow B$ has the *homotopy lifting property* (HLP) for the space X , if for each homotopy $h: X \times I \rightarrow B$ and each map $f: X \rightarrow E$ satisfying $p \circ f = h_0$, there is a homotopy $H: X \times I \rightarrow E$ with $p \circ H = h$ and $H_0 = f$.

In other words, the HLP for a space X means that the extension problem indicated by the following diagram has a solution $\bar{H}$ for every map $f: X \rightarrow E$ and homotopy $h: X \times I \rightarrow B$ satisfying $p \circ f = h_0$

$$(9.2) \quad \begin{array}{ccccc} & & E & & \\ & \swarrow p & & \searrow f & \\ B & & & & X \\ & \nwarrow e_B^0 & \swarrow \text{map}(I, E) & \nwarrow \bar{H} & \\ & & \text{map}(I, B) & & \end{array}$$

(Note: The diagram in the image is a commutative diagram with nodes B , E , X , $\text{map}(I, E)$, and $\text{map}(I, B)$. Arrows are: $B \xleftarrow{p} E$, $E \xleftarrow{e_E^0} \text{map}(I, E)$, $\text{map}(I, E) \xleftarrow{f} X$, $X \xleftarrow{\bar{h}} \text{map}(I, B)$, $\text{map}(I, B) \xleftarrow{e_B^0} B$, $\text{map}(I, E) \xleftarrow{\text{map}(\text{id}_I, p)} \text{map}(I, B)$, and $\text{map}(I, E) \xleftarrow{\bar{H}} X$ (dashed arrow).)

where e_B^0 and e_E^0 are given by evaluation at 0 and $\bar{h}$ is the adjoint of h under the canonical adjunction homeomorphism $\text{map}(X \times I, B) \xrightarrow{\cong} \text{map}(X, \text{map}(I, B))$, and analogously for H and $\bar{H}$.

Equivalently, one may describe the HLP with by the following diagram

$$(9.3) \quad \begin{array}{ccc} X & \xrightarrow{f} & E \\ \downarrow i_0^X & \nearrow H & \downarrow p \\ X \times I & \xrightarrow{h} & B \end{array}$$

Definition 9.4 (Fibration). A map $p: E \rightarrow B$ is called a *fibration* or *Hurewicz fibration* if it has the homotopy lifting property for every space X .

A map $p: E \rightarrow B$ is called a *Serre fibration* if it has the homotopy lifting property for the cube I^n for all $n \geq 1$.

Define for a map $p: E \rightarrow B$ the space $W(p)$ by the pullback

$$(9.5) \quad \begin{array}{ccc} W(p) & \xrightarrow{e_p} & E \\ \bar{p} \downarrow & & \downarrow p \\ \text{map}(I, B) & \xrightarrow{e_B^0} & B. \end{array}$$

Explicitly $W(p) = \{(e, w) \in E \times \text{map}(I, B) \mid p(e) = w(0)\} \subseteq E \times \text{map}(I, B)$ and e_p sends (e, w) to e and $\bar{p}$ sends (e, w) to w . Note that we obtain from (9.5) a map

$$r: \text{map}(I, E) \rightarrow W(p)$$

uniquely determined by the property that $e_p \circ r = e_E^0$ and $\bar{p} \circ r = \text{map}(\text{id}_I, p)$ holds.

If we have base points $e \in E$ and $b \in B$ with $f(e) = b$, then $W(p)$ inherits the base point $* = (e, c_b)$ for the constant map $c_b: I \rightarrow B$ with image $\{b\}$ and the diagram (9.5) is a diagram of pointed spaces.

Proposition 9.6. *The following assertions are equivalent for a map $p: E \rightarrow B$:*

- (i) p is a fibration;
- (ii) p has the HLP for $W(p)$;
- (iii) The map $r: \text{map}(I, E) \rightarrow W(p)$ has a section s .

Proof. (i) $\implies$ (ii) This is obvious.

(ii) $\implies$ (iii) If we apply the HLP to the map $e_p: W(p) \rightarrow E$ and the homotopy $h: W(p) \times I \rightarrow B$ which corresponds under the adjunction homeomorphism $\text{map}(W(p) \times I, B) \xrightarrow{\cong} \text{map}(W(p), \text{map}(I, B))$ to $\bar{p}$, we get a map $\bar{s}: W(p) \times I \rightarrow E$. Let $s: W(p) \rightarrow \text{map}(I, E)$ be the map corresponding to $\bar{s}$ under the adjunction homeomorphism $\text{map}(W(p) \times I, E) \xrightarrow{\cong} \text{map}(W(p), \text{map}(I, E))$. Since the composite of $r \circ s$ and of $\text{id}_{W(p)}$ with both e_p and $\bar{p}$ agree, we get $r \circ s = \text{id}_{W(p)}$.

(iii) $\implies$ (i) Consider a homotopy $h: X \times I \rightarrow B$ and a map $f: X \rightarrow E$ satisfying $p \circ f = h_0$. Because of the pullback (9.5) we get from (f, p) a map $u: X \rightarrow W(p)$. Let $\bar{H}: X \rightarrow \text{map}(I, E)$ be the composite $s \circ u$. Let $H: X \times I \rightarrow E$ be the homotopy corresponding to $\bar{H}$ under the adjunction homeomorphism $\text{map}(X \times I, E) \xrightarrow{\cong} \text{map}(X, \text{map}(I, E))$. Then H is a solution to the HLP given by (h, f) . This shows that p is a fibration. $\square$

Proposition 9.7. *Consider the pullback*

$$\begin{array}{ccc} X & \xrightarrow{\bar{f}} & Y \\ \bar{q} \downarrow & & \downarrow q \\ B & \xrightarrow{f} & C \end{array}$$

If q is a fibration, then its pullback $\bar{q}$ along f is a fibration.

Proof. Consider a map $u: A \rightarrow X$ and a homotopy $h: A \times I \rightarrow B$ such that $h_0 = \bar{q} \circ u$ holds. As q is a fibration, we get from the HLP applied to the map $\bar{f} \circ u: A \rightarrow Y$ and the homotopy $f \circ h: A \times [0, 1] \rightarrow Y$ a homotopy $\bar{H}: A \times I \rightarrow Y$ satisfying $q \circ \bar{H} = f \circ h$ and $\bar{H}_0 = \bar{f} \circ u$. Since the diagram above is a pullback, we get a map $H: A \times [0, 1] \rightarrow X$ uniquely determined by $\bar{f} \circ H = \bar{H}$ and $\bar{q} \circ H = h$. Since H_0 and u have the the same composite with both $\bar{f}$ and $\bar{q}$, we get $H_0 = f$. Hence $\bar{q}$ has the HLP and therefore is a fibration. $\square$

The elementary proof of the next result can be found in [31, Proposition 5.5.4 on page 116 and Proposition 5.5.5 and 5.5.6 on page 117].

Proposition 9.8. *Let Z be a (compactly generated) space. Let $i: A \rightarrow B$ be a cofibration and $p: E \rightarrow B$ be a fibration. Then:*

(i) *The induced map*

$$\text{map}(i, \text{id}_Z): \text{map}(B, Z) \rightarrow \text{map}(A, Z)$$

is a fibration;

(ii) *The induced map*

$$\text{map}(\text{id}_Z, p): \text{map}(Z, E) \rightarrow \text{map}(Z, B)$$

is a fibration;

(iii) *The canonical map $\text{map}(I, E) \rightarrow W(p)$ sending v to $(v(0), p \circ v)$ is a fibration;*

(iv) *Consider the pullback*

$$\begin{array}{ccc} E_i & \xrightarrow{\bar{i}} & E \\ \bar{p} \downarrow & & \downarrow p \\ A & \xrightarrow{i} & B. \end{array}$$

Then the upper horizontal arrow $\bar{i}: E_i \rightarrow E$ is a cofibration.

The elementary proof the next result can be found in [31, Corollary 5.5.3 on page 116].

Proposition 9.9 (Improved HLP). *Let $p: E \rightarrow B$ be a fibration and $i: A \rightarrow B$ be a cofibration which is the inclusion of a closed subspace A of B . Consider a homotopy $h: X \times I \rightarrow B$ and a map $f: A \times I \cup X \times \{0\} \rightarrow E$. Let $j: A \times I \cup X \times \{0\} \rightarrow X \times I$ be the obvious inclusion. Suppose $p \circ f = h \circ j$.*

Then there exists a homotopy $H: X \times I \rightarrow E$ satisfying $p \circ H = h$ and $H \circ j = f$, in other words, we can solve the following extension problem

$$\begin{array}{ccc} A \times I \cup X \times \{0\} & \xrightarrow{f} & E \\ \downarrow j & \nearrow H & \downarrow p \\ X \times I & \xrightarrow{h} & B. \end{array}$$

9.2. Turning a map into a fibration. Let $f: X \rightarrow Y$ be a map. Consider the space $W(f)$ defined in 9.5. Then the composite $q_f: W(f) \xrightarrow{\bar{f}} \text{map}(I, Y) \xrightarrow{e_Y^1} Y$ is a fibration by the following argument.

Consider a homotopy $h: A \times I \rightarrow Y$ and a map $u: A \rightarrow X$ satisfying $f \circ u = h_0$. Since (9.5) is a pullback, there is a homotopy $H: A \times I \rightarrow W(f)$ which is uniquely determined by the properties that $e_f \circ H$ is the composite $A \times I \xrightarrow{\text{pr}_A} A \xrightarrow{u} X$ for pr_A the canonical projection and that $\bar{f} \circ H: A \times I \rightarrow \text{map}(I, Y)$ agrees under the adjunction $\text{map}(A \times I, \text{map}(I, Y)) \xrightarrow{\cong} \text{map}(A \times I \times I \rightarrow Y)$ with the composite $A \times I \times I \xrightarrow{\text{id}_A \times v} A \times I \xrightarrow{h} Y$ for the map $v: I \times I \rightarrow I$ sending $(s, t) \rightarrow s \cdot t$. Explicitly H sends (a, t) to the pair $(u(a), w)$ where $w: I \rightarrow Y$ sends s to $h(a, st)$. One easily checks that $H_0 = u$ and $q_f \circ H = h$ holds.

We have the inclusion $i: X \rightarrow W(f)$ sending x to $(x, c_{f(x)})$. Its composite with the map $e_f: W(f) \rightarrow X$ appearing in (9.5) is the identity on X . Define a homotopy $k: W(f) \times [0, 1] \rightarrow W(f)$ by sending $((x, w), t)$ to (x, w_t) for the path $w_t: I \rightarrow Y$ sending s to $w(st)$. Then $k_0 = i \circ e_f$ and $k_1 = \text{id}_{W(f)}$. Hence e_f is a homotopy equivalence with homotopy inverse i . Obviously $q_f \circ i = f$ holds. So we get a

factorization $f: X \xrightarrow{i} W(f) \xrightarrow{q_f} Y$ into a homotopy equivalence i followed by a fibration q_f . Actually we obtain a diagram

$$(9.10) \quad \begin{array}{ccccc} X & \xrightarrow[\simeq]{i} & W(f) & \xrightarrow[\simeq]{e_f} & X \\ & \searrow f & \downarrow q_f & \swarrow f & \\ & & Y & & \end{array}$$

such that the left triangle commutes, the right triangle commutes up to homotopy, the two horizontal arrows are homotopy equivalences and homotopy inverse to one another, and the middle vertical arrow q_f is a fibration. Recall that we have

$$\begin{aligned} W(f) &= \{(x, w) \in X \times \text{map}(I, Y) \mid p(x) = w(0)\}; \\ i(x) &= (x, c_{f(x)}); \\ e_f(x, w) &= x; \\ q_f(x, w) &= w(1). \end{aligned}$$

9.3. Homotopy Theorem for pullbacks and fibrations.

Theorem 9.11 (Homotopy Theorem for pullbacks and fibrations). *Consider the pullback*

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{f}} & E \\ \bar{p} \downarrow & & \downarrow p \\ A & \xrightarrow{f} & B. \end{array}$$

Suppose that p is a fibration and f is a homotopy equivalence. Then $\bar{p}$ is a fibration and $\bar{f}$ is a homotopy equivalence.

Proof. We have already shown in Proposition 9.7 that $\bar{p}$ is a fibration. The proof that $\bar{f}$ is a homotopy equivalence is omitted and can be found in [31, Proposition 5.5.10 on page 118]. $\square$

Theorem 9.11 can easily be extended to the following theorem.

Theorem 9.12 (Homotopy Theorem for maps between pullbacks). *Let the following two diagrams be pullbacks*

$$\begin{array}{ccc} X & \xrightarrow{i_1} & X_1 \\ i_2 \downarrow & & \downarrow j_1 \\ X_2 & \xrightarrow{j_2} & X_0 \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{k_1} & Y_1 \\ k_2 \downarrow & & \downarrow l_1 \\ Y_2 & \xrightarrow{l_2} & Y_0 \end{array}$$

where the right vertical arrows j_1 and l_1 are fibrations. Let $f_i: X_i \rightarrow Y_i$ be homotopy equivalences for $i = 0, 1, 2$ satisfying $l_1 \circ f_1 = f_0 \circ j_1$ and $l_2 \circ f_2 = f_0 \circ j_2$. Denote by $f: X \rightarrow Y$ the map induced by f_0, f_1 , and f_2 and the pullback property.

Then f is a homotopy equivalence.

Remark 9.13. The condition that j_1 and l_1 are fibrations appearing in Theorem 9.12 is necessary as the following examples shows.

Given a pointed space (X, x) , let $P(X, x)$ be the subspace of $\text{map}(I, X)$ consisting of path w with $w(0) = x$ and $\Omega(X, x)$ be the subspace of $\text{map}(I, X)$ consisting of path w with $w(0) = w(1) = x$. Often $\Omega(X, x)$ is called the *loop space* of X . One easily checks that $P(X, x)$ is contractible. We take as pullbacks

$$\begin{array}{ccc}
\Omega(X, x) & \xrightarrow{i_1} & P(X, x) \\
i_2 \downarrow & & \downarrow j_1 \\
\{\bullet\} & \xrightarrow{j_2} & X
\end{array}
\qquad
\begin{array}{ccc}
\{\bullet\} & \xrightarrow{k_1} & \{\bullet\} \\
k_2 \downarrow & & \downarrow l_1 \\
\{\bullet\} & \xrightarrow{l_2} & X
\end{array}$$

where j_1 is by evaluation at 1 and j_2 , l_1 , and l_2 have as image the base point x . Take $f_1: P(X, x) \rightarrow \{\bullet\}$ to be the projection, $f_2 = \text{id}_{\{\bullet\}}$, and $f_0 = \text{id}_X$. Note that $\Omega(X, x)$ is in general not contractible.

9.4. The fiber transport. Let $p: E \rightarrow B$ be a fibration. Next we construct a functor

$$(9.14) \quad \tau: \Pi(B) \rightarrow \mathbf{h}\text{-Top}.$$

It sends an object x in the fundamental groupoid to the fiber $F_x := p^{-1}(x)$ of p over x . Consider a morphism $[w]: x \rightarrow y$. Choose a path $w: I \rightarrow B$ with $w(0) = x$ and $w(1) = y$ representing w . Apply HLP to the inclusion $i_x: F_x \rightarrow E$ and the homotopy $h: F_x \times I \xrightarrow{\text{pr}_I} I \xrightarrow{w} B$ for the projection pr_I . This yields a homotopy $H: F_x \times I \rightarrow E$ with $H_0 = i_x$ and $p \circ H = h$. Then H_1 is a map $F_x \rightarrow F_y$ and we define $\tau([w]) = [H_1]$. We leave it to the reader to check that $[H_1]$ depends only on $[w]$ and is independent of the choices of w and H and yields a covariant functor.

Proposition 9.15. *Let $p: E \rightarrow B$ be a fibration over a path connected space B . Then for any two points x and y the fibers F_x and F_y are homotopy equivalent.*

Proof. This follows from the functor τ of (9.14) and the fact that $\Pi(B)$ is a groupoid. $\square$

9.5. Homotopy equivalences and fibrations.

Definition 9.16 (Fiber homotopy equivalence). Let $p_0: E_0 \rightarrow B$ and $p_1: E_1 \rightarrow B$ be fibrations over B .

A *fiber preserving map* $f: p_0 \rightarrow p_1$ is a map $f: E_0 \rightarrow E_1$ satisfying $p_1 \circ f = p_0$.

Two such fiber preserving maps $f_0, f_1: p_0 \rightarrow p_1$ are called *fiber homotopy equivalent* if there is a homotopy $h: E_0 \times I \rightarrow E_1$ such that $h_0 = f_0$ and $h_1 = f_1$ hold and $h_t: E_0 \rightarrow E_1$ is a fiber preserving map $h_t: p_0 \rightarrow p_1$ for each $t \in I$.

A fiber preserving map $f: E_0 \rightarrow E_1$ is a *fiber homotopy equivalence* if there is a fiber preserving map $g: E_1 \rightarrow E_0$ such that $g \circ f$ is fiber homotopy equivalent to id_{E_0} and $f \circ g$ is fiber homotopy equivalent to id_{E_1} .

Theorem 9.17 (Characterization of fiber homotopy equivalences). *Let $p_0: E_0 \rightarrow B$ and $p_1: E_1 \rightarrow B$ be fibrations over B .*

Then a fiber preserving map $f: p_0 \rightarrow p_1$ is a fiber homotopy equivalence if and only if the underlying map $f: E_0 \rightarrow E_1$ is a homotopy equivalence.

Proof. The proof is indicated for instance in [20, Proposition in Section 5 of Chapter 7 on page 52]. $\square$

Theorem 9.18 (Homotopy Covering Theorem). *Let $p: E \rightarrow B$ be a fibration. Consider two maps $f_0, f_1: X \rightarrow B$ which are homotopic. Let $p_i: E_i \rightarrow X$ be the fibration obtained by the pulling back of p to f_i for $i = 0, 1$.*

Then p_0 and p_1 are fiber homotopy equivalent.

Proof. See [27, Proposition 15.16 on page 344]. $\square$

Corollary 9.19. *Let $p: E \rightarrow B$ be a fibration over a contractible space B . Then p is fiber homotopy equivalent to a trivial fibration $B \times F \rightarrow B$.*

9.6. The Fiber Sequence. A pointed map $f: (X, x) \rightarrow (Y, y)$ induces by composition for every pointed space (A, a) a map

$$f_*: [(A, a), (X, x)]^0 \rightarrow [(A, a), (Y, y)]^0, \quad [u] \mapsto [f \circ u]$$

which depends only on the pointed homotopy class of f . A sequence $(X, x) \xrightarrow{f} (Y, y) \xrightarrow{g} (Z, z)$ of maps of pointed space is called *homotopy exact* if for each pointed space (A, a) the induced sequence of pointed sets

$$[(A, a), (X, x)]^0 \xrightarrow{f_*} [(A, a), (Y, y)]^0 \xrightarrow{g_*} [(A, a), (Z, z)]^0$$

is exact at $[(A, a), (Y, y)]^0$ in the sense that the image of f_* is the preimage of g_* of the base point in $[(A, a), (Z, z)]$ given by $[c_z]$ for the constant pointed map $c_z: (B, b) \rightarrow (Z, z)$. Note that this implies that $g \circ f$ is pointed homotopy equivalent to the constant map $c_z: (X, x) \rightarrow (Z, z)$.

Recall that $P(Y, y)$ is the subspace of $\text{map}(I, Y)$ consisting of path w with $w(0) = y$. It has the constant path c_y as base point. Equivalently, one define $P(Y, y)$ by the pullback

$$(9.20) \quad \begin{array}{ccc} P(Y, y) & \xrightarrow{\overline{c_y}} & \text{map}(I, Y) \\ \downarrow & & \downarrow e_Y^0 \\ \{\bullet\} & \xrightarrow{c_y} & Y. \end{array}$$

Define the space $P(f, x)$ by the pullback

$$(9.21) \quad \begin{array}{ccc} P(f, x) & \xrightarrow{\bar{f}} & P(Y, y) \\ p_f \downarrow & & \downarrow e_Y^1 \\ X & \xrightarrow{f} & Y. \end{array}$$

The space $P(f, x)$ inherits from the base points $x \in X$ and $c_y \in P(Y, y)$ a base point $*$ for which the diagram (9.21) becomes a diagram of pointed spaces. Explicitly $P(f, x)$ is the subspace of $X \times \text{map}(I, Y)$ consisting of those pairs (z, w) for which $w(0) = y$ and $w(1) = f(z)$ holds. The map $\bar{f}$ sends (z, w) to w and p_f sends (z, w) to z .

Lemma 9.22. *Let $f: (X, x) \rightarrow (Y, y)$ be a pointed map of pointed spaces. Let $p_f: P(f, x) \rightarrow X$ be the map defined in (9.21). Then*

$$(P(f, x), *) \xrightarrow{p_f} (X, x) \xrightarrow{f} (Y, y)$$

is homotopy exact.

Proof. This is a direct consequence of the pullback (9.21) and the adjunction $\text{map}(A, \text{map}(I, X)) \xrightarrow{\cong} \text{map}(A \times I, X)$. Namely, they imply that a pointed map $(A, a) \rightarrow (P(f, x), *)$ is the same as a pointed map $u: (A, a) \rightarrow (X, x)$ together with a pointed homotopy $h: (A, a) \times I \rightarrow (Y, y)$ between the constant map $c_y: B \rightarrow Y$ and $f \circ u: (A, a) \rightarrow (Y, y)$. $\square$

One can iterate this process and obtains a homotopy exact sequence, infinite to the left

$$\dots \xrightarrow{f_5} P(f_3) \xrightarrow{f_4} P(f_2) \xrightarrow{f_3} P(f_1) \xrightarrow{f_2} P(p_f) \xrightarrow{f_1} P(f) \xrightarrow{p_f} X \xrightarrow{f} Y.$$

Here and also sometimes in the sequel we omit the base points from the notation.

The further investigations replace the spaces $P(f_1), P(f_2), \dots$ by more appealing spaces, namely by iterated loop spaces $\Omega^n(X, x)$. Recall that the loop space $\Omega(X, x)$

is the subspace of $\text{map}(I, X)$ consisting of maps $w: I \rightarrow X$ with $w(0) = w(1) = x$. It can also be described by the pullback

$$(9.23) \quad \begin{array}{ccc} \Omega(X, x) & \xrightarrow{i} & P(X, x) \\ \downarrow & & \downarrow e_X^1 \\ \{\bullet\} & \xrightarrow{c_x} & X \end{array}$$

or, equivalently,

$$(9.24) \quad \begin{array}{ccc} \Omega(X, x) & \xrightarrow{i} & \text{map}(I, X) \\ \downarrow & & \downarrow \text{map}(i, \text{id}_X) \\ \{\bullet\} & \longrightarrow & \text{map}(\partial I, X) \end{array}$$

where $i: \partial I \rightarrow I$ is the inclusion and the lower horizontal arrow has the constant map c_x with value x as image.

Suppose additionally that the pointed map $f: (X, x) \rightarrow (Y, y)$ is a fibration (after forgetting the base points) and that (X, x) and (Y, y) are well-pointed. We have the commutative diagram

$$\begin{array}{ccccc} \{\bullet\} & \xrightarrow{c_y} & Y & \xleftarrow{f} & X \\ \simeq \downarrow j & & \downarrow \text{id}_Y & & \downarrow \text{id}_X \\ P(Y, y) & \xrightarrow{e_Y^1} & Y & \xleftarrow{f} & X \end{array}$$

where j is the map onto the base point $*$ on $P(Y, y)$. The pullback of the upper row is $f^{-1}(y)$, whereas the pullback of the lower row is $P(f, x)$ because of (9.21). All vertical maps are homotopy equivalences. Hence the diagram induces by the Homotopy Theorem 9.11 a homotopy equivalence

$$g: f^{-1}(y) \xrightarrow{\simeq} P(f, x).$$

It is a pointed homotopy equivalence by Lemma 8.25, since its domain and codomain are well-pointed. The following diagram of well-pointed spaces commutes

$$(9.25) \quad \begin{array}{ccccc} (f^{-1}(y), x) & \xrightarrow{i_y} & (X, x) & \xrightarrow{f} & (Y, y) \\ g \downarrow \simeq & & \downarrow \text{id}_X & & \downarrow \text{id}_Y \\ (P(f, y), *) & \xrightarrow{p_f} & (X, x) & \xrightarrow{f} & (Y, y) \end{array}$$

for i_x the inclusion and all vertical arrows are pointed homotopy equivalences. Since the lower row is homotopy exact, the upper row is homotopy exact.

The map $p_f: P(f, x) \rightarrow X$ is a fibration by Proposition 9.7 applied to the pullback (9.7) since the inclusion $\{y\} \rightarrow Y$ is a cofibration and hence $e_Y^1: P(Y, y) \rightarrow Y$ is a fibration by Proposition 9.8 (i). Then we obtain a pullback

$$\begin{array}{ccc} (\Omega(Y, y), *) & \xrightarrow{i_f} & P(f, x) \\ \downarrow & & \downarrow p_f \\ \{\bullet\} & \xrightarrow{c_x} & X \end{array}$$

from the pullbacks (9.21) and (9.23). Explicitly $i_f: \Omega(Y, y) \rightarrow P(f, x)$ sends w to (x, w) and induces a homeomorphism $i_f: \Omega(Y, y) \rightarrow p_f^{-1}(x)$. Hence the sequence of pointed spaces

$$\Omega(Y, y) \xrightarrow{i_f} (P(f, x), *) \xrightarrow{p_f} (X, x) \xrightarrow{f} (Y, y)$$

is homotopy exact. Iterating this process yields the following result.

Theorem 9.26 (Fiber sequence). *Let $f: (X, x) \rightarrow (Y, y)$ be a map of well-pointed spaces. Then we obtain a homotopy exact sequence, infinite to the left,*

$$(9.27) \quad \dots \xrightarrow{\Omega^2 p_f} (\Omega^2(X, x), *) \xrightarrow{\Omega^2 f} (\Omega^2(Y, y), *) \xrightarrow{\Omega i_f} (\Omega(P(f, x), *), *) \xrightarrow{\Omega p_f} (\Omega(X, x), *) \xrightarrow{\Omega f} (\Omega(Y, y), *) \xrightarrow{i_f} (P(f, x), *) \xrightarrow{p_f} (X, x) \xrightarrow{f} (Y, y).$$

Note this sequence (9.27) is natural in f and yields for any pointed space (B, b) the long exact sequence of pointed sets, infinite to the left,

$$(9.28) \quad \dots \xrightarrow{(\Omega^2 p_f)_*} [B, \Omega^2(X, x)]^0 \xrightarrow{(\Omega^2 f)_*} [B, \Omega^2(Y, y)]^0 \xrightarrow{(\Omega i_f)_*} [B, \Omega P(f, x)]^0 \xrightarrow{(\Omega p_f)_*} [B, \Omega(X, x)]^0 \xrightarrow{(\Omega f)_*} [B, \Omega(Y, y)]^0 \xrightarrow{(i_f)_*} [B, P(f, x)]^0 \xrightarrow{(p_f)_*} [B, X]^0 \xrightarrow{f_*} [B, Y]^0$$

where we have omitted the base points of the pointed spaces involved. Note the obvious fact that the map $f_*: [B, X]^0 \rightarrow [B, Y]^0$ is not surjective in general.

9.7. Group structures on pointed sets associated to the Fiber Sequence.

Definition 9.29 (Group object in $\mathbf{h}\text{-Top}^0$). A *group object in $\mathbf{h}\text{-Top}^0$* is a pointed space (X, x) together with pointed maps

$$\begin{aligned} m: (X \times X, (x, x)) &\rightarrow (X, x); \\ i: (X, x) &\rightarrow (X, x), \end{aligned}$$

satisfying the following conditions:

- (i) The two pointed maps $(X, x) \rightarrow (X, x)$ sending y to $m(x, y)$ and $m(y, x)$ respectively are pointed homotopic to the identity;
- (ii) The two pointed maps $m \circ (\text{id}_X \times m)$ and $m \circ (m \times \text{id}_X)$ from $(X \times X \times X, (x, x, x))$ to (X, x) are pointed homotopic;
- (iii) The two pointed maps $m \circ (\text{id}_X \times i)$ and $m \circ (i \times \text{id}_X)$ from $(X \times X, (x, x))$ to (X, x) are pointed homotopic to the constant map c_x .

Sometimes group objects in $\mathbf{h}\text{-Top}^0$ are called *associative H -spaces with inverse*.

Example 9.30 (Examples for group object in $\mathbf{h}\text{-Top}^0$). A topological group is obviously an example of a group object in $\mathbf{h}\text{-Top}^0$. Our main example is the loop space $\Omega(X, x)$ of a well-pointed space (X, x) where $m: \Omega(X, x) \times \Omega(X, x) \rightarrow \Omega(X, x)$ sends (v, w) to the concatenation $v * w$ and $i: \Omega(X, x) \rightarrow \Omega(X, x)$ sends w to the inverse path w^- .

Remark 9.31. Let (B, b) be a pointed space and (X, x) be a group object in $\mathbf{h}\text{-Top}^0$. Then $[(B, b), (X, x)]$ inherits a group structure by the multiplication given by

$$[(B, b), (X, x)]^0 \times [(B, b), (X, x)]^0 \rightarrow [(B, b), (X, x)]^0, \quad ([f], [g]) \mapsto [m \circ (f \times g)].$$

The unit is given by the class $[c_x]$ of the constant map. The inverse of $[f] \in [(B, b), (X, x)]^0$ is given by $[i \circ f]$.

In particular we obtain for a well-pointed space (X, x) a group structure on $[(B, b), (\Omega(X, x), *)]^0$. This group structure on $[(B, b), (\Omega^n(X, x), *)]^0$ is abelian for

$n \geq 2$. The sequence (9.28) is compatible with the group structures as long as they exist.

9.8. The adjunction between suspension and loop spaces. Let (X, x) and (Y, y) be well-pointed spaces. Then there is a natural adjunction homeomorphism

$$(9.32) \quad \text{ad}: \text{map}((\Sigma(X, x), *), (Y, y))^0 \xrightarrow{\cong} \text{map}((X, x), (\Omega(Y, y), *))^0$$

between mapping spaces of pointed spaces. It is uniquely determined by the property that it makes the following diagram commutative,

$$\begin{array}{ccc} \text{map}((\Sigma(X, x), *), (Y, y))^0 & \xrightarrow{\text{ad}} & \text{map}((X, x), (\Omega(Y, y), *))^0 \\ \downarrow & & \downarrow \\ \text{map}(X \times I, Y) & \xrightarrow{\cong} & \text{map}(X, \text{map}(I, Y)) \end{array}$$

where the lower horizontal arrow is the natural adjunction homeomorphism, the left vertical is the closed embedding coming from the projection $X \times I \rightarrow \Sigma(X, x)$ and the right vertical arrow is the closed embedding coming from the canonical inclusion $\Omega(Y, y) \rightarrow \text{map}(I, Y)$. By passing to π_0 , we obtain from (9.32) natural adjunction bijection

$$(9.33) \quad [(\Sigma(X, x), *), (Y, y)]^0 \xrightarrow{\cong} [(X, x), (\Omega(Y, y), *)]^0.$$

It is compatible with the group structure on the domain introduced in (8.43) and on the codomain introduced in Remark 9.31.

If we take $(X, x) = (S^n, s)$, we obtain for $n = 0, 1, 2, \dots$ a natural bijection of groups

$$(9.34) \quad \pi_{n+1}(Y, y) \xrightarrow{\cong} \pi_n(\Omega(Y, y), *)$$

for $n \geq 0$. Iterating this, we get a bijection of groups

$$(9.35) \quad [(S^0, s), (\Omega^n(X, x), *)]^0 = \pi_0(\Omega^n(X, x)) \xrightarrow{\cong} \pi_n(X, x).$$

9.9. Locally trivial bundles are fibrations. The proof of the following result can be found in [31, Theorem 13.4.1 on page 32].

Theorem 9.36 (Being a fibration is a local property). *Let $p: E \rightarrow B$ be a continuous map and let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open covering of B . Suppose that $\mathcal{U}$ is numerable, i.e., admits a subordinate partition of unity, and that $p|_{p^{-1}(U_i)}: p^{-1}(U_i) \rightarrow U_i$ is a fibration for every $i \in I$.*

Then p is a fibration.

Recall that a *partition of unity subordinate to $\mathcal{U}$* is a family $\{t_i: U_i \rightarrow [0, 1] \mid i \in I\}$ of functions t_i satisfying:

- The support $\text{supp}(t_i) := \overline{\{b \in B \mid t_i(b) \neq 0\}} \subseteq B$ of t_i is contained in U_i for $i \in I$;
- The family $\{t_i: U_i \rightarrow [0, 1] \mid i \in I\}$ is *locally finite*, i.e., for every $b \in B$ there is an open neighborhood V together with a finite subset $I_0 \subseteq I$ such that $t_i(v) = 0$ holds for all $v \in V$ and $i \in I \setminus I_0$;
- The (finite) sum $\sum_{i \in I} t_i(b)$ is 1 for every $b \in B$.

A space B is called *paracompact* if every open covering $\mathcal{U} = \{U_i \mid i \in I\}$ has a refinement $\mathcal{V} = \{V_j \mid j \in J\}$ which is *locally finite*, i.e., for every $b \in B$ there exists an open neighborhood W of b in B and a finite subset $I_0 \subseteq I$ satisfying $W \cap U_i \implies i \in I_0$. Note that such $\mathcal{V}$ is automatically numerable. Every metric space is paracompact, see [23, Theorem 4.3 on page 256]. Every *CW-complex* is

paracompact, see [22] or [10, Theorem 1.3.5]. Theorem 9.36 and the discussion above imply the following result.

Theorem 9.37 (Locally trivial bundles are fibrations). *Let $p: E \rightarrow B$ be a locally trivial bundle over a paracompact space, e.g., a principal G -bundle for a topological group G , a vector bundle, or a covering over a space B which is a CW-complex or a metric space.*

Then p is a fibration.

9.10. Duality between cofibrations and fibrations. There is a kind of duality between cofibrations and fibrations which we want to discuss next. One has to interchange $X \times I$ and $\text{map}(I, X)$, interchange pushouts with pullbacks, Σ with Ω and invert all arrows. Here is a list of some examples.

- (HEP) and (HLP)

Consider the diagrams (8.3) and (9.3)

$$\begin{array}{ccc} Y & \xleftarrow{f} & X \\ e_Y^0 \uparrow & \swarrow H & \uparrow i \\ \text{map}(I, Y) & \xleftarrow{\bar{h}} & A \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{f} & E \\ i_0^X \downarrow & \swarrow H & \downarrow p \\ X \times I & \xrightarrow{h} & B; \end{array}$$

- mapping cylinder and $W(p)$

Consider the diagrams (8.5) and (9.5)

$$\begin{array}{ccc} \text{cyl}(i) & \xleftarrow{k} & X \\ l \uparrow & & \uparrow i \\ A \times I & \xleftarrow{i_0^A} & A \end{array} \quad \text{and} \quad \begin{array}{ccc} W(p) & \xrightarrow{e_p} & E \\ \bar{p} \downarrow & & \downarrow p \\ \text{map}(I, B) & \xrightarrow{e_B^0} & B; \end{array}$$

- Turning a maps into cofibration or fibration

Consider the diagrams (8.37) and (9.10)

$$\begin{array}{ccc} Y & \xleftarrow{p} \text{cyl}(f) & \xleftarrow{k} Y \\ \swarrow f & \uparrow i & \searrow f \\ & X & \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{i} W(f) & \xrightarrow{e_p} X \\ \swarrow f & \downarrow q_f & \searrow f \\ & Y & \end{array}$$

- The Homotopy Theorems 8.28 and 9.11 where the relevant diagrams are

$$\begin{array}{ccc} Y & \xleftarrow{\bar{f}} & X \\ \bar{i} \uparrow & & \uparrow i \\ B & \xleftarrow{f} & A \end{array} \quad \text{the pushout} \quad \text{and the pullback} \quad \begin{array}{ccc} \bar{X} & \xrightarrow{\bar{f}} & E \\ \bar{p} \downarrow & & \downarrow p \\ A & \xrightarrow{f} & B. \end{array}$$

- The Cofiber Sequence appearing in Theorem 8.40 and the Fiber Sequence appearing in Theorem 9.26 as well as the long exact sequences of pointed homotopy classes associated to them, see (8.42) and (9.28).

10. THE LONG EXACT HOMOTOPY SEQUENCE ASSOCIATED TO A FIBRATION

10.1. The homotopy sequence.

Theorem 10.1 (The long exact homotopy sequence associated to a fibration). *Let (E, e) and (B, b) be well-pointed spaces. Let $p: E \rightarrow B$ be a fibration with $p(e) = b$. Put $F_b = p^{-1}(b)$. Denote by $i: F_b \rightarrow E$ the inclusion.*

Then we obtain a long exact sequence, infinite to the left

$$(10.2) \quad \begin{aligned} \cdots \xrightarrow{\partial_{n+1}} \pi_n(F_b, e) &\xrightarrow{\pi_n(i, e)} \pi_n(E, e) \xrightarrow{\pi_n(p, e)} \pi_n(B, b) \xrightarrow{\partial_n} \\ &\cdots \xrightarrow{\partial_2} \pi_1(F_b, e) \xrightarrow{\pi_1(i, e)} \pi_1(E, e) \xrightarrow{\pi_1(p, e)} \pi_1(B, b) \\ &\xrightarrow{\partial_1} \pi_0(F_b) \xrightarrow{\pi_0(i)} \pi_0(E) \xrightarrow{\pi_0(p)} \pi_0(B) \end{aligned}$$

with the following properties:

- It is an exact sequence of groups in the range until $\pi_1(B, b)$;
- It is exact at $\pi_1(B, b)$ in the sense that the image of $\pi_1(p, e)$ is the preimage of the component in F_b containing e under ∂_1 ;
- It is exact at $\pi_0(F_b)$ in the sense that the image of ∂_1 is the preimage of the component in E containing e under $\pi_0(i)$;
- It is exact at $\pi_0(E)$ in the sense that the image of $\pi_0(i)$ is the preimage of the component in B containing b under $\pi_0(p)$;
- The boundary operator $\partial_{n+1}: \pi_{n+1}(B, b) \rightarrow \pi_n(F_b, e)$ is defined as follows. Consider $u \in \pi_{n+1}(B, b)$. Choose a map $h: S^n \times [0, 1] \rightarrow B$ which sends $S^n \times \{0, 1\} \cup \{s\} \times I$ to b such that for the pointed standard homeomorphism

$$q: (S^n \times [0, 1] / (S^n \times \{0, 1\} \cup \{s\} \times I), *) \xrightarrow{\cong} (S^{n+1}, b)$$

the composite $\bar{h} = h \circ q: (S^{n+1}, s) \rightarrow (B, b)$ represents u . Choose a solution H to the lifting problem

$$\begin{array}{ccc} S^n \times \{0\} \cup \{s\} \times I & \xrightarrow{c_e} & E \\ \downarrow i & \nearrow H & \downarrow p \\ S^n \times [0, 1] & \xrightarrow{h} & B \end{array}$$

which exists by Proposition 9.9. Then $\partial_{n+1}(x)$ is represented by the pointed map $H_1: (S^n, s) \rightarrow (F_b, e)$.

Proof. This follows from the exact sequence (9.28) applied in the case $(B, b) = (S^0, s)$,

$$\begin{aligned} \cdots &\xrightarrow{(\Omega^2 p_p)_*} [S^0, \Omega^2 E]^0 \xrightarrow{(\Omega^2 p)_*} [S^0, \Omega^2 B]^0 \xrightarrow{(\Omega i_p)_*} [S^0, \Omega P(p, e)]^0 \\ &\xrightarrow{(\Omega p_p)_*} [S^0, \Omega E]^0 \xrightarrow{(\Omega p)_*} [S^0, \Omega B]^0 \\ &\xrightarrow{(i_p)_*} [S^0, P(p, e)]^0 \xrightarrow{(p_p)_*} [S^0, E]^0 \xrightarrow{p_*} [S^0, B]^0, \end{aligned}$$

the isomorphism (9.35)

$$[(S^0, s), (\Omega^n(X, x), *)]^0 = \pi_0(\Omega^n(X, x)) \xrightarrow{\cong} \pi_n(X, x)$$

and the diagram (9.25) which becomes in the situation considered here

$$\begin{array}{ccccc} (F_b, e) & \xrightarrow{i_b} & (X, x) & \xrightarrow{f} & (Y, y) \\ g \downarrow \simeq & & \downarrow \text{id}_X & & \downarrow \text{id}_Y \\ (P(p, e), *) & \xrightarrow{p_p} & (X, x) & \xrightarrow{f} & (Y, y). \end{array}$$

□

Remark 10.3 (Serre fibrations and the homotopy sequence). In order to have the long exact homotopy sequence of Theorem 10.1 available, one needs only to know that $p: E \rightarrow B$ is a Serre fibration, see [31, Theorem 6.3.2 on page 130]. The obvious

version of Theorem 9.36 holds also for Serre fibrations, see [31, Theorem 6.3.3 on page 130].

10.2. The Hopf fibration. Fix $d \in \mathbb{Z}^{\geq 1}$. We can consider S^1 as a subgroup of $\mathbb{C} \setminus \{0\}$ with respect to multiplication of complex numbers. In particular S^1 acts diagonally on $\mathbb{C}^{d+1}$. Then $S^{2d+1} \subseteq \mathbb{C}^{d+1}$ inherits an S^1 -action, which is free. Recall that $\mathbb{CP}^d$ is the set of 1-dimensional complex vector spaces of $\mathbb{C}^{d+1}$ and is equipped with the quotient topology with respect to the map $f: \mathbb{C}^{d+1} \setminus \{0\} \rightarrow \mathbb{CP}^d$ sending $z \in \mathbb{C}^{d+1} \setminus \{0\}$ to the 1-dimensional complex vector space generated by z . Consider the map $p: S^{2d+1} \rightarrow \mathbb{CP}^d$ obtained by restricting f to S^{2d+1} . Then $p: S^{2d+1} \rightarrow \mathbb{CP}^d$ is an identification. Obviously it factorizes over the projection $\text{pr}: S^{2d+1} \rightarrow S^{2d+1}/S^1$ into a bijective map $u: S^{2d+1}/S^1 \xrightarrow{\cong} \mathbb{CP}^d$. Since p and pr are identifications, u is a bijective identification and hence a homeomorphism. Now one easily checks that $p: S^{2d+1} \rightarrow \mathbb{CP}^d$ is a principal S^1 -bundle. Theorem 9.36 implies that p is a fibration. From Theorem 10.1 we obtain a long exact sequence of groups

$$\begin{aligned} \cdots \xrightarrow{\partial_{n+1}} \pi_n(S^1) \rightarrow \pi_n(S^{2d+1}) \xrightarrow{\pi_n(p)} \pi_n(\mathbb{CP}^d) \xrightarrow{\partial_n} \\ \cdots \xrightarrow{\partial_2} \pi_1(S^1) \rightarrow \pi_1(S^{2d+1}) \xrightarrow{\pi_1(p)} \pi_1(\mathbb{CP}^d) \rightarrow \{1\}. \end{aligned}$$

Since $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_n(S^1) = \{0\}$ for $n \geq 2$ by (2.8), we obtain an isomorphism

$$(10.4) \quad \pi_n(p): \pi_n(S^{2d+1}) \rightarrow \pi_n(\mathbb{CP}^d) \quad \text{for } n \geq 3$$

and an exact sequence of abelian groups

$$\{0\} \rightarrow \pi_2(S^{2d+1}) \rightarrow \pi_2(\mathbb{CP}^d) \rightarrow \pi_1(S^1) \rightarrow \pi_1(S^{2d+1}) \rightarrow \pi_1(\mathbb{CP}^d) \rightarrow \{0\}.$$

Recall that $\pi_m(S^n) \cong \{0\}$ for $m < n$ and $\pi_n(S^n) \cong \mathbb{Z}$ by Theorem 3.4. Hence $\mathbb{CP}^d$ is simply connected and we get an isomorphism

$$\pi_2(\mathbb{CP}^d) \cong \mathbb{Z}.$$

Since $\mathbb{CP}^1$ is homeomorphic to S^2 , we get from (10.4) the following theorem, which we have already briefly discussed in Example 3.5.

Theorem 10.5 ($\pi_3(S^2)$ is infinite cyclic). *The abelian group $\pi_3(S^2)$ is an infinite group with $[p]$ for the so called Hopf map $p: S^3 \rightarrow \mathbb{CP}^1 = S^2$ as generator.*

10.3. Homotopy groups of loop spaces.

Proposition 10.6. *Let (E, e) and (B, b) be well-pointed spaces. Let $p: E \rightarrow B$ be a fibration with $p(e) = b$. Put $F_b = p^{-1}(b)$. Suppose that E is weakly contractible. Then we get isomorphisms*

$$\pi_{n+1}(B, b) \xrightarrow{\cong} \pi_n(F_b, e)$$

for $n \geq 1$.

Proof. This follows directly from Theorem 10.1. □

Suppose that (X, x) is a path connected well-pointed space. Then we have the fibration $p: P(X, x) \rightarrow X$ whose fiber over $x \in X$ is the loop space $\Omega(X, x)$. As $P(X, x)$ is contractible, we get from Proposition 10.6 for $n \geq 0$ a preferred isomorphism of groups

$$(10.7) \quad \partial_{n+1}(X, x): \pi_{n+1}(X, x) \xrightarrow{\cong} \pi_n(\Omega(X, x)).$$

Note that $\pi_0(\Omega(X, x)) = [(S^0, s), (\Omega(X, x), *)]^0$ has a group structure by Example 9.30 and Remark 9.31. Iterating this, we get for every $n \geq 1$ a group isomorphism $\pi_n(X, x) \cong \pi_0(\Omega^n(X, x))$, as already mentioned in (9.35).

10.4. Homotopy groups of classifying spaces BG . Let G be a topological group and $p: EG \rightarrow BG$ be the universal principal G -bundle. Recall that it has the property that the pullback construction defines for every CW -complex X a bijection

$$(10.8) \quad [X, BG] \xrightarrow{\cong} \{\text{isomorphism classes of principal } G\text{-bundles over } X\}$$

and is up to isomorphism of G -bundles uniquely characterized by the property that EG is weakly contractible. Proposition 10.6 implies that BG is path connected and satisfies for $n \geq 1$

$$(10.9) \quad \pi_n(BG) \cong \pi_{n-1}(G)$$

for $e \in G$ the unit element.

10.5. On the homotopy groups of some classical Lie groups. Denote by $\mathbb{F}$ one of the (skew)fields $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ given by the reals numbers, the complex numbers, or the quaternions. We have the associated orthogonal, unitary, or symplectic groups which are Lie groups:

$$\begin{aligned} \mathrm{O}(n) &= \mathrm{O}(n, \mathbb{R}) = \{A \in \mathrm{M}(n, n, \mathbb{R}) \mid A^t A = I_n\}; \\ \mathrm{SO}(n) &= \mathrm{SO}(n, \mathbb{R}) = \{A \in \mathrm{O}(n) \mid \det(A) = 1\}; \\ \mathrm{U}(n) &= \mathrm{O}(n, \mathbb{C}) = \{A \in \mathrm{M}(n, n, \mathbb{C}) \mid A^* A = I_n\}; \\ \mathrm{SU}(n) &= \mathrm{SO}(n, \mathbb{C}) = \{A \in \mathrm{U}(n) \mid \det(A) = 1\}; \\ \mathrm{Sp}(n) &= \mathrm{O}(n, \mathbb{H}). \end{aligned}$$

The action of these groups on the unit spheres yield locally trivial fiber bundles and hence by Theorem 9.36 fibrations for $d = \dim_{\mathbb{R}}(\mathbb{F})$

$$\begin{aligned} \mathrm{O}(n, \mathbb{F}) &\xrightarrow{i} \mathrm{O}(n+1, \mathbb{F}) \rightarrow S^{d(n+1)-1}; \\ \mathrm{SO}(n, \mathbb{F}) &\xrightarrow{j} \mathrm{SO}(n+1, \mathbb{F}) \rightarrow S^{d(n+1)-1}. \end{aligned}$$

The inclusions i and j come from $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.

By passing to colimits we get the topological groups

$$\begin{aligned} \mathrm{O}(\infty; \mathbb{F}) &= \operatorname{colim}_{n \rightarrow \infty} \mathrm{O}(n, \mathbb{F}); \\ \mathrm{SO}(\infty; \mathbb{F}) &= \operatorname{colim}_{n \rightarrow \infty} \mathrm{SO}(n, \mathbb{F}). \end{aligned}$$

Since $S^{d(n+1)-1}$ is $(d(n+1)-2)$ -connected by Theorem 3.4, we conclude from Theorem 2.20 and Theorem 10.1

Proposition 10.10.

- (i) For $1 \leq n < m$ the inclusions $\mathrm{O}(n, \mathbb{F}) \rightarrow \mathrm{O}(m, \mathbb{F})$ and $\mathrm{SO}(n, \mathbb{F}) \rightarrow \mathrm{SO}(m, \mathbb{F})$ are $(d(n+1)-2)$ -connected;
- (ii) For $1 \leq n$ the inclusions $\mathrm{O}(n, \mathbb{F}) \rightarrow \mathrm{O}(\infty, \mathbb{F})$ and $\mathrm{SO}(n, \mathbb{F}) \rightarrow \mathrm{SO}(\infty, \mathbb{F})$ are $(d(n+1)-2)$ -connected.

The associated *Stiefel manifold* of orthogonal k -frames in $\mathbb{F}^n$ are defined by

$$\begin{aligned} V_k(\mathbb{R}^n) &= \mathrm{O}(n, \mathbb{R}) / \mathrm{O}(n-k, \mathbb{R}) \cong \mathrm{SO}(n, \mathbb{R}) / \mathrm{SO}(n-k, \mathbb{R}); \\ V_k(\mathbb{C}^n) &= \mathrm{U}(n, \mathbb{R}) / \mathrm{U}(n-k, \mathbb{C}) \cong \mathrm{SU}(n, \mathbb{R}) / \mathrm{SU}(n-k, \mathbb{C}); \\ V_k(\mathbb{H}^n) &= \mathrm{Sp}(n) / \mathrm{Sp}(n-k). \end{aligned}$$

We have the fibration $\mathrm{O}(n-k, \mathbb{R}) \rightarrow \mathrm{O}(n, \mathbb{R}) \rightarrow V_k(\mathbb{R}^n)$ and analogous fibrations for $V_k(\mathbb{C}^n)$ and $V_k(\mathbb{H}^n)$. The next proposition is a direct consequence of Theorem 10.1 and Proposition 10.10.

Proposition 10.11. *The space $V_k(\mathbb{F}^n)$ is $(d(n-k+1)-2)$ -connected.*

There is a fibration $V_k(\mathbb{F}^n) \rightarrow V_{k+1}(\mathbb{F}^{n+1}) \xrightarrow{p} V_1(\mathbb{F}^{n+1})$, where p sends a frame $\{v_1, v_2, \dots, v_{k+1}\}$ to the frame $\{v_{k+1}\}$. The next proposition follows from Theorem 10.1 and Proposition 10.11.

Proposition 10.12. *The inclusion $V_k(\mathbb{F}^n) \rightarrow V_{k+1}(\mathbb{F}^{n+1})$ is $(d(n+1)-2)$ -connected.*

Proposition 10.13. *We have*

$$\begin{aligned} \pi_{2(n-k)+1}(V_k(\mathbb{C}^n)) &\cong \mathbb{Z}; \\ \pi_{4(n-k)+3}(V_k(\mathbb{H}^n)) &\cong \mathbb{Z}; \\ \pi_{n-k}(V_k(\mathbb{R}^n)) &\cong \begin{cases} \mathbb{Z} & \text{if } k = 1 \text{ or } (n-k) \text{ even;} \\ \mathbb{Z}/2 & \text{if } k \geq 2 \text{ and } (n-k) \text{ odd.} \end{cases} \end{aligned}$$

Proof. The cases $\mathbb{F} = \mathbb{C}$ and $\mathbb{H}$ follows by induction using $V_1(\mathbb{F}^n) \cong S^{dn-1}$ and $\pi_n(S^n) \cong \mathbb{Z}$ and Proposition 10.12. The case $\mathbb{F} = \mathbb{R}$ needs more than we have accumulated so far and can be found in [31, Proposition 6.8.5 on page 148]. $\square$

Example 10.14. There are homeomorphisms $\mathrm{SO}(2) \xrightarrow{\cong} S^1$ and $\mathrm{SO}(3) \xrightarrow{\cong} \mathbb{RP}^3$. Since the universal covering of S^1 is the principal $\mathbb{Z}$ -bundle $\mathbb{R} \rightarrow S^1$ and the universal covering of $\mathbb{RP}^3$ is the principal $\mathbb{Z}/2$ -bundle $S^3 \rightarrow \mathbb{RP}^3$, we conclude from Proposition 10.10

$$\pi_1(\mathrm{SO}(n)) \cong \begin{cases} \mathbb{Z} & n = 2; \\ \mathbb{Z}/2 & n \geq 3, \end{cases}$$

and $\pi_1(\mathrm{SO}) \cong \mathbb{Z}/2$.

We conclude $\pi_2(\mathrm{SO}(3)) \cong \pi_2(\mathbb{RP}^3) \cong \pi_2(S^3) \cong \{0\}$ from Theorem 2.7 and Theorem 3.4. Proposition 10.10 (ii) implies $\pi_2(\mathrm{SO}) = \{0\}$. Actually for every compact Lie group G we have $\pi_2(G, g) = \{0\}$ for any base point $g \in G$, see [5, Proposition 7.5 on page 225].

11. THE EXCISION THEOREM OF BLAKERS-MASSEY

11.1. The statement of the Excision Theorem of Blakers-Massey. One basic feature of a homology theory is excision. Consider any (generalized) homology theory $\mathcal{H}_*$ with values in R -modules for a commutative ring R . Consider a CW -complex Y with CW -subcomplexes Y_0 , Y_1 , and Y_2 satisfying $Y = Y_1 \cup Y_2$ and $Y_0 = Y_1 \cap Y_2$, or a topological space Y with open subspaces Y_0 , Y_1 , and Y_2 satisfying $Y = Y_1 \cup Y_2$ and $Y_0 = Y_1 \cap Y_2$. Then the map induced by the inclusion $(Y_2, Y_0) \rightarrow (Y, Y_1)$ induces for all $n \in \mathbb{Z}$ an R -isomorphism

$$\mathcal{H}_n(Y_2, Y_0) \xrightarrow{\cong} \mathcal{H}_n(Y, Y_1).$$

This yields a long exact Mayer-Vietoris sequence of R -modules, infinite to both sides,

$$\begin{aligned} \dots \xrightarrow{\mathcal{H}_{n+1}(j_1) - \mathcal{H}_{n+1}(j_2)} \mathcal{H}_{n+1}(X) \xrightarrow{\partial_{n+1}} \mathcal{H}_n(X_0) \xrightarrow{\mathcal{H}_n(i_1) \oplus \mathcal{H}_n(i_2)} \mathcal{H}_n(X_1) \oplus \mathcal{H}_n(X_2) \\ \xrightarrow{\mathcal{H}_n(j_1) - \mathcal{H}_n(j_2)} \mathcal{H}_n(X) \xrightarrow{\partial_n} \mathcal{H}_{n-1}(X_0) \xrightarrow{\mathcal{H}_{n-1}(i_1) \oplus \mathcal{H}_{n-1}(i_2)} \dots \end{aligned}$$

where $i_k: X_0 \rightarrow X_k$ and $j_k: X_k \rightarrow X$ for $k = 1, 2$ are the inclusions. The corresponding statement is not true for homotopy groups as the following example shows.

Example 11.1. Consider the CW -complex $Y = S^1 \vee S^n$ with the CW -subcomplexes $Y_1 = S^1$, $Y_2 = S^n$, and $Y_0 = \{\bullet\}$ for $n \in \mathbb{Z}^{\geq 2}$. Suppose that $\pi_n(Y_2, X_0) \xrightarrow{\cong} \pi_n(Y, Y_1)$ is an isomorphism. Since $\pi_n(S^n, \{\bullet\})$ is isomorphic to $\pi_n(S^n)$ by the long exact homotopy sequence of the pair $(S^n, \{\bullet\})$, we conclude from Theorem 3.4

that $\pi_n(S^1 \vee S^n, S^1)$ is infinite cyclic. Theorem 3.4 and the exact sequence of abelian groups $\pi_n(S^1) \rightarrow \pi_n(S^1 \vee S^n) \rightarrow \pi_n(S^1 \vee S^n, S^1)$ coming from the long exact homotopy sequence of the pair $(S^1 \vee S^n, S^1)$, see Theorem 2.11 imply that $\pi_n(S^1 \vee S^n)$ is a subgroup of an infinite cyclic group and hence a finitely generated abelian group. This contradicts Example 3.6. Hence $\pi_n(S^n, \{\bullet\}) \rightarrow \pi_n(S^1 \vee S^n, S^1)$ is not bijective.

One of the main results of this course is the next theorem due to Blakers and Massey which shows excision in a very special case for homotopy groups.

Theorem 11.2 (The Excision Theorem of Blakers-Massey). *Consider $p, q \in \mathbb{Z}^{\geq 2}$. Let Y be a topological space with open subspaces Y_0, Y_1 , and Y_0 satisfying $Y = Y_1 \cup Y_2$ and $Y_0 = Y_1 \cap Y_2$. Suppose that for any base point $y_0 \in Y_0$ we have*

$$\begin{aligned}\pi_i(Y_1, Y_0, y_0) &= \{0\} \quad \text{for } 0 < i \leq p; \\ \pi_i(Y_2, Y_0, y_0) &= \{0\} \quad \text{for } 0 < i \leq q.\end{aligned}$$

Then, for every base point $y_0 \in Y_0$, the map induced by the inclusion $i: (Y_2, Y_0) \rightarrow (Y_1, Y_0)$

$$\pi_n(i, y_0): \pi_n(Y_2, Y_0, y_0) \rightarrow \pi_n(Y, Y_1, y_0)$$

is surjective for $1 \leq n = p + q$ and bijective for $1 \leq n \leq p + q - 1$.

If $p = 1$, then there is no condition on (Y_1, Y_0) in Theorem 11.2. Note that in Theorem 11.2 only the case $n \geq 1$ is treated, we will say something for $n = 0$ in Subsection 11.3.

11.2. The proof of the Excision Theorem of Blakers-Massey. The following rather elementary proof of the Excision Theorem 11.2 of Blakers-Massey is due to Dieter Puppe. The proof needs some preparation.

We begin with introducing some notation.

Notation 11.3 (Cubes and faces in $\mathbb{R}^n$). A cube in $\mathbb{R}^n$ for $n \geq 1$ is a subset of the form

$$W = W(a, \delta, L) := \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid a_i \leq x_i \leq a_i + \delta \text{ for } i \in L, x_i = a_i \text{ for } i \notin L\}$$

for $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, $\delta > 0$, and a (possibly empty) subset $L \subseteq \{1, 2, \dots, n\}$. The dimension $\dim(W)$ of W is defined to be $|L|$.

A face of W of W is a subset of W of the form

$$W' = \{x \in W \mid x_i = a_i \text{ for } i \in L_0, x_j = a_j + \delta \text{ for } j \in L_1\}$$

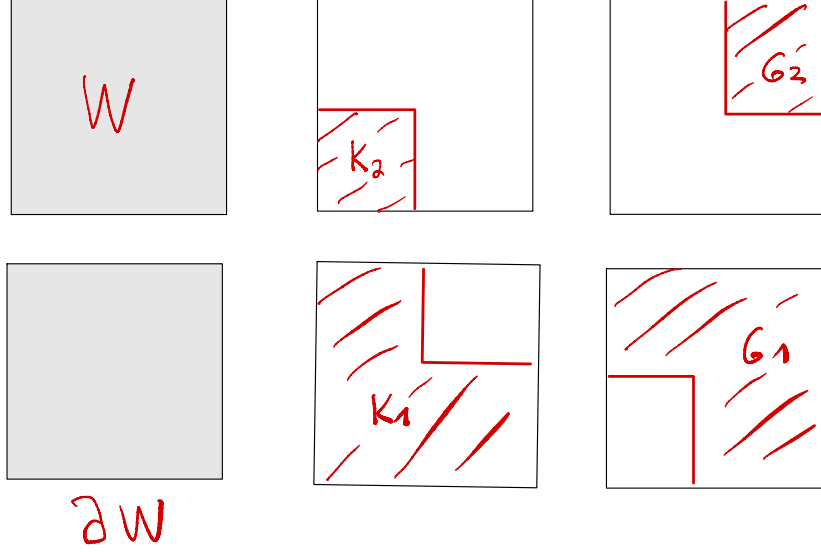
for some (possibly empty) subsets $L_0 \subseteq L$ and $L_1 \subseteq L$. (The subset W' may be empty or equal to W .)

Let ∂W be the union of all faces W' of W which are not equal to W .

For $1 \leq p \leq n$ we define the following subsets of W :

$$\begin{aligned}K_p(W) &= \{x \in W \mid x_i < a_i + \delta/2 \text{ for at least } p \text{ values } i \in L\}; \\ G_p(W) &= \{x \in W \mid x_i > a_i + \delta/2 \text{ for at least } p \text{ values } i \in L\}.\end{aligned}$$

For $p > \dim(W)$ we define $K_p(W)$ and $G_p(W)$ to be the empty sets. Note that $K_p(W)$ and $G_p(W)$ become smaller and smaller as p becomes bigger and $G_p(W) \cap K_q(W) = \emptyset$ if $p + q > \dim(W)$ hold.



Next we prove a technical lemma which will enter in the important Proposition 11.5. It essentially says that a map $W \rightarrow Y$, which satisfies a certain condition on the boundary ∂W , can be changed up to homotopy relative ∂W such that the resulting map satisfies the analog of this condition on W and not only on ∂W .

Lemma 11.4. *Consider a pair (Y, A) , a cube $W \subseteq \mathbb{R}^n$, and a map $f: W \rightarrow Y$. Suppose that for $p \leq \dim(W)$ we have $f^{-1}(A) \cap W' \subseteq K_p(W')$ for all faces $W' \subseteq \partial W$.*

Then there exists a map $g: W \rightarrow Y$ with the following properties:

- (i) *g is homotopic relative ∂W to f ;*
- (ii) *We have $g^{-1}(A) \subseteq K_p(W)$.*

The same conclusion holds if we replace $K_p(W)$ by $G_p(W)$ in assertion (ii).

Proof. Obviously we can assume without loss of generality that W is the standard cube $I^n = \prod_{i=1}^n [0, 1] = W((0, 0, \dots, 0), 1, \{1, 2, \dots, n\})$. Let I_2^n be the subcube of I^n given by $\prod_{i=1}^n [0, 1/2] = W((0, 0, \dots, 0), 1/2, \{1, 2, \dots, n\})$. Put $x_4 = (1/4, 1/4, \dots, 1/4)$.

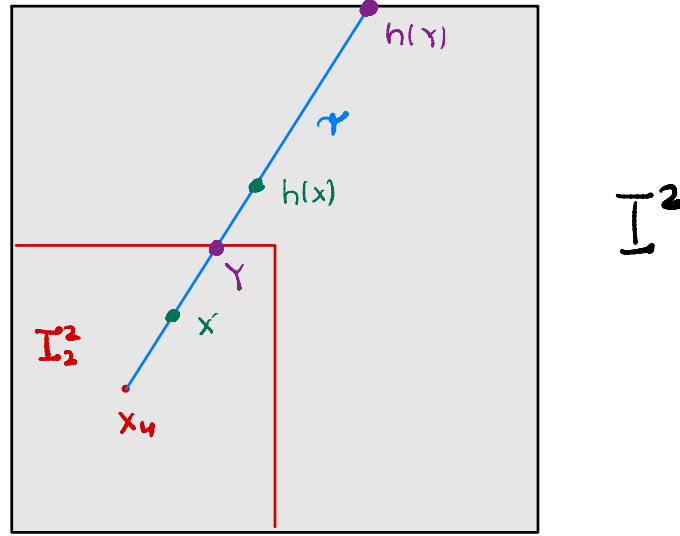
Define a map

$$h: I^n \rightarrow I^n$$

by expanding I_2^n to I^n by radial projection with center x_4 . Here is the precise definition of h . Let $x \in I^n$ be any point. If $x = x_4$, we define $h(x) = x_4$. Suppose that $x \neq x_4$. Consider the ray

$$r: \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^n, \quad t \mapsto x_4 + t \cdot (x - x_4)$$

starting at x_4 through x . Let $P(x)$ be its intersection point with ∂I_2^n and $Q(x)$ be its intersection point with ∂I^n . If x lies in the segment $[P(x), Q(x)]$ of the ray r , it is sent to $Q(x)$. Suppose that x lies on the segment $[x_4, P(x)]$. If we write $x = x_4 + t \cdot (P(x) - x_4)$ for some $t \in [0, 1]$ then $h(x)$ is defined to be $x_4 + t \cdot (Q(x) - x_4)$. In other words, h sends the segment $[x_4, P(x)]$ affinely to the segment $[x_4, Q(x)]$ and the segment $[P(x), Q(x)]$ to the point $Q(x)$. Obviously h is homotopic relative ∂W to id_W .

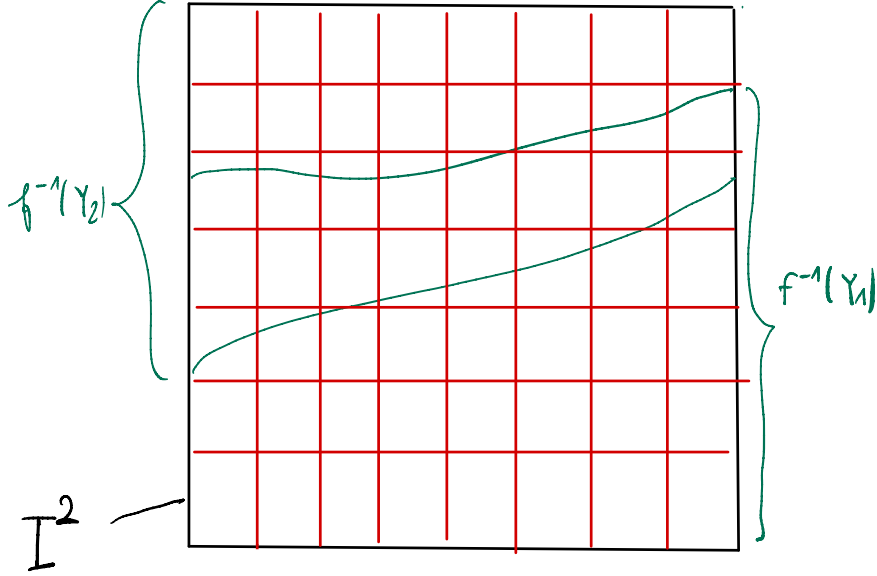


Now we set $g = f \circ h$. Then g is homotopic relative ∂W to f . It remains to show $g^{-1}(A) \subseteq K_p(W)$.

Consider $z \in I^n$ with $g(z) \in A$. If $z_i < 1/2$ holds for $i = 1, 2, \dots, n$, then $z \in K_n(I^n) \subseteq K_p(I^n)$. So it suffices to treat the case, where $z_i \geq 1/2$ holds for at least one $i \in \{1, 2, \dots, n\}$. Then $h(z) \in \partial I^n$ holds by definition. Choose a face $W' \subseteq \partial I^n$ with $h(z) \in W'$. Since $h(z) \in f^{-1}(A)$ holds, we get $h(z) \in W' \cap f^{-1}(A)$ and hence $h(z) \in K_p(W')$. Hence we have $h(z)_i < 1/2$ for at least p many elements $i \in \{1, 2, \dots, n\}$. For $i \in \{1, 2, \dots\}$ with $h(z)_i < 1/2$ we get $h(z)_i = 1/4 + t \cdot (z_i - 1/4)$ with $t \geq 1$ and hence $z_i < 1/2$. This shows $z \in K_p(W)$. This finishes the proof of Lemma 11.4 for $K_p(W)$, the version for $G_p(W)$ is proven analogously. $\square$

The next proposition contains the main technical result needed for the proof of Theorem 11.2.

For the remainder of this subsection let Y be a topological space with open subspaces Y_0 , Y_1 , and Y_2 satisfying $Y = Y_1 \cup Y_2$ and $Y_0 = Y_1 \cap Y_2$ and we consider $p, q \in \mathbb{Z}^{\geq 0}$.



Proposition 11.5. *Suppose that (Y_1, Y_0) is p -connected and (Y_2, Y_0) is q -connected. Let $f: I^n \rightarrow Y$ be a map. Let $\mathcal{W} = \{W\}$ be a subdivision of I^n into cubes W such that either $f(W) \subseteq Y_1$ or $f(W) \subseteq Y_2$ holds. (It exists as I^n is compact.)*

Then there exists a homotopy $h: I^n \times I \rightarrow Y$ with $h_0 = f$ satisfying for every $W \in \mathcal{W}$:

- (i) *If $f(W) \subseteq Y_j$ holds, then we have $h_t(W) \subseteq Y_j$ for every $t \in I$, where $j \in \{0, 1, 2\}$;*
- (ii) *If $f(W) \subseteq Y_0$ holds, then we have $h_t|_W = f|_W$ for every $t \in I$;*
- (iii) *If $f(W) \subseteq Y_1$ holds, then we have $h_1^{-1}(Y_1 \setminus Y_0) \cap W \subseteq K_{p+1}(W)$;*
- (iv) *If $f(W) \subseteq Y_2$ holds, then we have $h_1^{-1}(Y_2 \setminus Y_0) \cap W \subseteq G_{q+1}(W)$.*

Proof. We enlarge the collection of cubes $\mathcal{W}$ such that for every $W \in \mathcal{W}$ all of its faces belong to $\mathcal{W}$. Let $C_k \subseteq I^n$ be the union of all cubes $W \in \mathcal{W}$ with $\dim(W) \leq k$. We construct for $k = 0, 1, 2, \dots, n$ a homotopy $h[k]: C_k \times I \rightarrow Y$ satisfying for each cube $W \in \mathcal{W}$ of dimension $\leq k$ the conditions (i), (ii), (iii), and (iv) such that $h[k]|_{C_{k-1} \times I} = h[k-1]$ holds for $k = 1, 2, \dots, n$. Then the desired homotopy is $h = h[n]$.

Note in the sequel that for a cube $W \in \mathcal{W}$, for which we have $f(W) \subseteq Y_0$ and condition (ii) holds, conditions (iii) and (iv) are automatically satisfied, since then $h_1^{-1}(Y_1 \setminus Y_0)$ and $h_1^{-1}(Y_2 \setminus Y_0)$ are empty. Moreover, if a cube $W \in \mathcal{W}$ satisfies both $f(W) \subseteq Y_1$ and $f(W) \subseteq Y_2$, then we have $f(W) \subseteq Y_0$, and for each cube $W \in \mathcal{W}$ we have $f(W) \subseteq Y_1$ or $f(W) \subseteq Y_2$. So every cube $W \in \mathcal{W}$ satisfies precisely one of the following conditions:

- $f(W) \subseteq Y_0$;
- $f(W) \subseteq Y_2$ and $f(W) \not\subseteq Y_1$;
- $f(W) \subseteq Y_1$ and $f(W) \not\subseteq Y_2$.

We begin with $k = 0$. Consider a cube W in $\mathcal{W}$ of dimension 0. If $W_0 \subseteq Y_0$, define $h[0]_t(W_0) = W_0$ for $t \in I$. This is forced upon us by condition (ii). Suppose $f(W) \subseteq Y_1$ and $f(W) \not\subseteq Y_2$ hold. As (Y_1, Y_0) is 0-connected, we can choose a path $w: I \rightarrow Y_1$ from $f(W)$ to a point $y \in Y_0$. We define $h[0](W, t) = w(t)$ for $t \in I$. Then conditions (i) and (iii) are satisfied for trivial reasons. Analogously one defines $h[0]$ in the case, where $f(W) \subseteq Y_2$ and $f(W) \not\subseteq Y_0$ hold. This finishes the

construction of $h[0]$. One easily checks that all the conditions (i), (ii), (iii), and (iv) are satisfied for every 0-dimensional cube W by $h[0]$.

Next we deal with the induction step from $(k-1)$ to k . Consider a cube of dimension k . Then $\partial W = W \cap C_{k-1}$. Since $\partial W \rightarrow W$ is a cofibration, we can extend $h[k-1]_{\partial W \times I}$ to $W \times I$ such that conditions (i) and (ii) are satisfied. So we get a homotopy $h[k]': C_k \times I \rightarrow Y$ such that conditions (i) and (ii) hold for $h[k]'$ and any $W \in \mathcal{W}$ with $\dim(W) \leq k$ and the restriction of $h[k]'$ to $C_{k-1} \times I$ satisfies conditions (i), (ii), (iii), and (iv) for any $W \in \mathcal{W}$ with $\dim(W) = (k-1)$.

The homotopy $h[k]'$ is not yet the desired homotopy $h[k]$. It remains to explain why we can change $h[k]'$ further such that all the conditions (i), (ii), (iii), and (iv) are satisfied for each cube W with $\dim(k) \leq k$. For this purpose we consider the map $h[k]_1': C_k \rightarrow Y$ and construct an appropriate homotopy $h[k]'' : C_k \times I \rightarrow Y$ with $h[k]_0'' = h[k]_1'$ and will get the desired homotopy $h[k] : C_k \times I \rightarrow Y$ by $h[k]' * h[k]''$.

Consider a cube W . We explain how to define $h[k]''|_{W \times [0,1]}$ with $\dim(W) = k$. If $h[k]_1'(W) \subseteq Y_0$, then we define $h[k]_t''|_W = h[k]_1'|_W$ for $t \in [0,1]$. Suppose that $h[k]_1'(W) \subseteq Y_1$ and $h[k]_1'(W) \not\subseteq Y_2$ holds. If $\dim(W) \leq p$, there exists a homotopy l relative ∂W with $l_0 = h[k]_1'$ and $l_1(W) \subseteq Y_0$, since the pair (Y_1, Y_0) is p -connected. Define $h[k]''|_{W \times I}$ by l . If $\dim(W) > p$, we use Lemma 11.4 with $f = h[k]_1'|_W$ to define $h[k]''|_{W \times I}$. We treat the case $h[k]_1'(W) \subseteq Y_2$ and $h[k]_1'(W) \not\subseteq Y_1$ analogously. This finishes the construction of $h[k]''$ and hence of the desired homotopy $h[k]$. Note that $h[k]''$ is stationary on C_{k-1} . One easily checks that $h(k)$ satisfies conditions (i), (ii), (iii), and (iv) for any $W \in \mathcal{W}$ with $\dim(W) \leq k$. Hence the proof of Proposition 11.5 is finished. $\square$

Denote by $F(Y_1, Y, Y_2)$ the subspace of $\text{map}(I, Y)$ given by

$$F(Y_1, Y, Y_2) := \{w: I \rightarrow Y \mid w(0) \in Y_1, w(1) \in Y_2\}.$$

So we are looking at paths in Y starting somewhere in Y_1 and ending somewhere in Y_2 . Define $F(Y_1, Y_1, Y_0)$ to be the subspace of $\text{map}(I, Y_1)$ given by

$$F(Y_1, Y_1, Y_0) := \{w: I \rightarrow Y_1 \mid w(1) \in Y_0\}.$$

So here we are looking at paths in Y_1 ending somewhere in Y_0 . Since we can think of $\text{map}(I, Y_1)$ as a subspace of $\text{map}(I, Y)$, we can also think of $F(Y_1, Y_1, Y_0)$ as a subspace of $F(Y_1, Y, Y_2)$.

Proposition 11.6. *Suppose that (Y_1, Y_0) is p -connected and (Y_2, Y_0) is q -connected. Then the inclusion*

$$F(Y_1, Y_1, Y_0) \rightarrow F(Y_1, Y, Y_2)$$

is $(p+q-1)$ -connected.

Proof. Consider a map of pairs

$$\varphi: (I^n, \partial I^n) \rightarrow (F(Y_1, Y, Y_2), F(Y_1, Y_1, Y_0))$$

for any $n \leq (p+q-1)$. We have to find a homotopy h with $h_0 = \varphi$ such that the image of h_1 is contained in $F(Y_1, Y_1, Y_0)$.

By the adjunction $\text{map}(I^n \times I, Z) \xrightarrow{\cong} \text{map}(I^n, \text{map}(I; Z))$ the map φ is the same as a map $\Phi: I^n \times I \rightarrow Y$ satisfying:

- (i) $\Phi(x, 0) \in Y_1$ for $x \in I^n$;
- (ii) $\Phi(x, 1) \in Y_2$ for $x \in I^n$;
- (iii) $\Phi(x, 1) \in Y_0$ for $x \in \partial I^n, t \in I$.

In the sequel we call a map $\Phi: I^n \times I \rightarrow Y$ satisfying the three conditions above admissible. We have to show that any such admissible map Φ can be homotoped through admissible maps to an admissible map $\Phi': I^n \times I \rightarrow Y$ with the property $\Phi'(I^n \times I) \subseteq Y_1$.

Starting with an admissible map $\Phi: I^{n+1} = I^n \times I \rightarrow Y$, we apply Proposition 11.5 and obtain a new admissible map Ψ . One easily checks that the homotopy coming from Proposition 11.5 is a homotopy through admissible maps.

Consider the projection $\text{pr}: I^n \times I \rightarrow I^n$. Next we show that the images of $\Psi^{-1}(Y \setminus Y_1)$ and $\Psi^{-1}(Y \setminus Y_2)$ under pr are disjoint. Suppose the contrary. So there are $y \in I^n$, $z_1 \in \Psi^{-1}(Y \setminus Y_1)$ and $z_2 \in \Psi^{-1}(Y \setminus Y_1)$ with $\text{pr}(z_1) = y = \text{pr}(z_2)$. Choose a cube $W \subseteq I^{n+1}$ with $z_1 \in W$. Since $z_1 \in \Psi^{-1}(Y \setminus Y_1)$ holds, we conclude $z_1 \in K_{p+1}(W)$ from condition (iii) appearing in Proposition 11.5. This implies that $y \in K_p(I^n)$ holds. Analogously one shows $y \in G_q(I^n)$, now using condition (iv) appearing in Proposition 11.5. This is a contradiction since $K_p(I^n) \cap G_q(I^n)$ is empty if $n < p + q$ holds.

The intersection of $\text{pr}(\Psi^{-1}(Y \setminus Y_1))$ and ∂I^n is empty since Ψ is admissible and hence $\Psi(\partial I^n) \times I \subseteq Y_1$ holds. Hence the closed subsets $\text{pr}(\Psi^{-1}(Y \setminus Y_1))$ and $\partial I^n \cup \text{pr}(\Psi^{-1}(Y \setminus Y_2))$ of I^n are disjoint. Choose a continuous function $\tau: I^n \rightarrow I$ which assumes the value 0 on $\Psi^{-1}(Y \setminus Y_1)$ and the value 1 on $\partial I^n \cup \text{pr}(\Psi^{-1}(Y \setminus Y_2))$. Then we obtain a homotopy through admissible maps

$$h: (I^n \times I) \times I \rightarrow Y, \quad ((x, t), s) \mapsto \Psi(x, (1-s)t + s\tau(x))$$

such that $h_0 = \Psi$ and $h_1(I^n \times I) \subseteq Y_1$ holds. This finishes the proof of Proposition 11.6. $\square$

Now we are ready to give the proof of the Excision Theorem 11.2.

Proof of Theorem 11.2. We have the path fibration map $\text{map}(I, Y) \rightarrow Y$ sending w to $w(0)$, see Proposition 9.8 (i). The induced map $p: F(Y_1, Y, Y_2) \rightarrow Y_1$ sending w to $w(0)$ is a fibration by Proposition 9.7. The fiber over a point $y_1 \in Y_1$ is $F(\{y_0\}, Y, Y_2)$. We obtain a commutative diagram of fibrations

$$\begin{array}{ccc} F(\{y_0\}, Y_1, Y_0) & \xrightarrow{i} & F(\{y_0\}, Y, Y_2) \\ \downarrow & & \downarrow \\ F(Y_1, Y_1, Y_0) & \xrightarrow{j} & F(Y_1, Y, Y_2) \\ \downarrow & & \downarrow \\ Y_1 & \xrightarrow{\text{id}} & Y_1 \end{array}$$

where i and j are the inclusions. We have already shown that j is $(p + q - 1)$ -connected by Proposition 11.6. Using the long exact homotopy sequences of the two fibrations above and a kind of Five-Lemma argument shows that i is also $(p + q - 1)$ -connected. There is a commutative diagram for $n \geq 1$

$$\begin{array}{ccc} \pi_{n-1}(F(\{\bullet\}, Y_1, Y_0), *) & \xrightarrow{i} & \pi_{n-1}(F(\{\bullet\}, Y, Y_2), *) \\ \downarrow \cong & & \downarrow \cong \\ \pi_n(Y_1, Y_0, y_0) & \longrightarrow & \pi_n(Y, Y_2, y_0) \end{array}$$

whose lower horizontal arrow is induced by the inclusion and vertical arrows are bijections by a version of (9.34) for pairs. Hence the lower vertical arrow is surjective for $1 \leq n = p + q$ and bijective for $1 \leq n \leq p + q - 1$. This finishes the proof of Theorem 11.2. $\square$

11.3. The Excision Theorem for $n = 0$. Note that in Theorem 11.2 only $n \geq 1$ is considered. We also want to treat the case $n = 0$.

Proposition 11.7. *Let Y be a topological space Y with open subspaces Y_0 , Y_1 , and Y_2 satisfying $Y = Y_1 \cup Y_2$, $Y_0 = Y_1 \cap Y_2$ and $Y_0 \neq \emptyset$.*

Then the canonical map $\iota: \pi_0(Y_2, Y_0) \rightarrow \pi_0(Y, Y_1)$ is bijective.

Proof. Since every element in Y belongs to Y_1 or Y_2 , the map ι is obviously surjective. Injectivity is proved as follows.

Consider elements $C_1, C_2 \in \pi_0(Y_2)$. Let $\overline{C_1}$ and $\overline{C_2}$ be the classes represented by them in $\pi_0(Y_2, Y_0)$. Suppose that they have the same image under ι . Then we have to show $\overline{C_1} = \overline{C_2}$.

We first treat the case, where $\iota(\overline{C_1}) = \iota(\overline{C_2})$ is different from the base point in $\pi_0(Y, Y_1)$. Then the images of C_1 and C_2 under the map $\pi_0(Y_2) \rightarrow \pi_0(Y)$ agree. Hence we can find a path $w: I \rightarrow Y$ with $w(0) \in C_1$ and $w(1) \in C_2$. Since $\iota(\overline{C_1}) = \iota(\overline{C_2})$ is different from the base point in $\pi_0(Y, Y_1)$, this path cannot meet $w^{-1}(Y_1)$. Hence it is a path $w: I \rightarrow Y_2$. This implies $C_1 = C_2 \in \pi_0(Y_2)$ and hence $\overline{C_1} = \overline{C_2}$.

Next we treat the case, where $\iota(\overline{C_1}) = \iota(\overline{C_2})$ is the base point in $\pi_0(Y, Y_1)$. It suffices to show that then $\overline{C_1}$ is the base point $*$ in $\pi_0(Y_2, Y_0)$. As $\iota(\overline{C_1})$ is the base point in $\pi_0(Y, Y_1)$, there is a path component D in $\pi_0(Y_1)$ such that the image of C_1 under $\pi_0(Y_2) \rightarrow \pi_0(Y)$ and the image of D under $\pi_0(Y_1) \rightarrow \pi_0(Y)$ agree. Hence we can find a path $w: I \rightarrow Y$ with $w(0) \in C_1$ and $w(1) \in Y_1$. If $w(0) \in Y_0$ holds, $\overline{C_1}$ is obviously the base point in $\pi_0(Y_2, Y_0)$. Hence we can assume without loss of generality that $y_1 \notin Y_0$ holds. Since $y_1 \notin Y_0$ and $y_1 \in Y_1$ hold, we have $y_1 \notin Y_2$. If $w^{-1}(Y_2)$ is empty, w is a path in Y_0 and hence $\overline{C_1}$ is the base point in $\pi_0(Y_2, Y_0)$. Hence we can assume without loss of generality that $w^{-1}(Y_2)$ is not empty and $y_1 \notin Y_2$ holds.

Let t_0 be the infimum of $w^{-1}(Y_2) \subseteq I$. As $w^{-1}(Y_2)$ is open, we have $0 \leq t_0 < 1$. Since $0 \notin w^{-1}(Y_2)$ holds, we get $t_0 \notin w^{-1}(Y_2)$. Hence $[0, t_0] \subseteq W_1$ holds. There exists $t_1 \in I$ with $t_0 < t_1$ such that $[0, t_1] \subseteq w^{-1}(Y_1)$ holds. Now choose $t_2 \in [0, 1]$ satisfying $t_0 < t_2 < t_1$ and $t_2 \in w^{-1}(Y_2)$. Note that then $t_2 \in w^{-1}(Y_1) \cap w^{-1}(Y_2) = w^{-1}(Y_0)$ holds. Consider the path $v: I \rightarrow Y$ sending s to $w(st_2)$. Then v is a path in Y_2 from $v(0) = x_2$ to $v(1) \in Y_0$. This implies that $\overline{C_1}$ is the base point in $\pi_0(Y_2, Y_0)$. This finishes the proof of Proposition 11.7. $\square$

Another shorter proof of Proposition 11.7 comes from the following observation. The map on singular homology $\mu: H_0(Y_2, Y_0; \mathbb{Z}) \xrightarrow{\cong} H_0(Y, Y_1; \mathbb{Z})$ induced by the inclusion is an isomorphism by excision. The abelian group $H_0(Y_2, Y_0; \mathbb{Z})$ is a free $\mathbb{Z}$ -modules with basis B_1 which is the complement of the image of $\pi_0(Y_0) \rightarrow \pi_0(Y_2)$ in $\pi_0(Y_2)$. The abelian group $H_0(Y, Y_1; \mathbb{Z})$ is a free $\mathbb{Z}$ -modules with basis B_2 which is the complement of the image of $\pi_0(Y_1) \rightarrow \pi_0(Y)$ in $\pi_0(Y)$. The map μ sends an element in $\pi_0(Y_2) \setminus \pi_0(Y_0)$ to an element in $\pi_0(Y) \setminus \pi_0(Y_1)$ or to 0.

11.4. Some applications of the Excision Theorem of Blakers-Massey.

Proposition 11.8. *Let Y be a topological space Y with subspaces Y_0 , Y_1 , and Y_2 satisfying $Y = Y_1 \cup Y_2$, $Y_0 = Y_1 \cap Y_2$, and $Y_0 \neq \emptyset$. Consider $m, n \in \mathbb{Z}^{\geq 0}$.*

- (i) *Suppose that (Y_2, Y_0) is n -connected. Then (Y, Y_1) is n -connected;*
- (ii) *Suppose that (Y_1, Y_0) is m -connected and (Y_2, Y_0) is n -connected. Then:*
 - (a) *The map $\pi_0(Y_2, Y_0) \rightarrow \pi_0(Y, Y_1)$ is bijective;*
 - (b) *For every base point y_0 the following holds: The map $\pi_i(Y_2, Y_0, y_0) \rightarrow \pi_i(Y, Y_1, y_0)$ induced by the inclusion is bijective for $1 \leq i \leq m+n-1$ and surjective for $i = m+n$.*

Proof. We only give the proof for assertion (i), the one for assertion (ii) is analogous, if one takes Theorem 11.2 into account.

Recall that (Y, Y_1) is n -connected if and only if $\pi_0(Y, Y_1)$ is trivial, i.e., consist of one element, and for every element $i \in \{1, 2, \dots, n\}$ and every base point $y_1 \in Y_1$ the set $\pi_i(Y, Y_1, y_1)$ is trivial. The analogous statement holds for (Y_2, Y_0) .

Proposition 11.7 implies that $\pi_0(Y, Y_1)$ is trivial.

Consider i with $1 \leq i$. We conclude from Theorem 11.2 that $\pi_i(Y, Y_1, y_0)$ is trivial for every base point $y_0 \in Y_0$. We need to check that $\pi_i(Y, Y_1, y_1)$ is trivial for every element $i \in \{1, 2, \dots, n\}$ and every base point $y_1 \in Y_1$. Since the map $\pi_0(Y_0) \rightarrow \pi_0(Y_1)$ is surjective by assumption, we can connect y_1 by a path in Y_1 to a point y_0 in Y_0 . The obvious version of (2.6) for pointed pairs implies $\pi_i(Y, Y_1, y_1) \cong \pi_i(Y, Y_1, y_0)$ and hence $\pi_i(Y, Y_1, y_1)$ is trivial. $\square$

Proposition 11.9. *Consider $m, n \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$ and a pushout*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ i \downarrow & & \downarrow \bar{i} \\ X & \xrightarrow{\bar{f}} & Y \end{array}$$

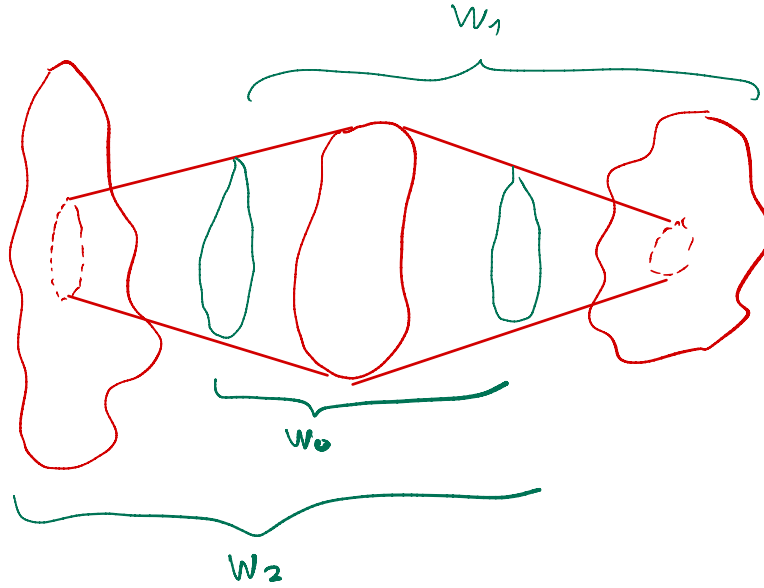
such that $i: A \rightarrow X$ is a cofibration.

- (i) *If f is n -connected, then $\bar{f}$ is n -connected;*
- (ii) *If f is n -connected and i is m -connected, then*
 - *The map $\pi_0(\bar{f}, f): \pi_0(X, A) \rightarrow \pi_0(Y, B)$ is bijective;*
 - *For every $a \in A$ the map $\pi_k(\bar{f}, f, a): \pi_k(X, A, a) \rightarrow \pi_k(Y, B, f(a))$ is bijective for $1 \leq k \leq m + n - 1$ and surjective for $1 \leq k = m + n$,**where we use the convention that $m + n - 1$ and $m + n$ mean ∞ if $m = \infty$ or $n = \infty$ holds.*

Proof. Because of Subsection 8.8 and Theorem 8.29 we can replace f and i by the inclusions into their mapping cylinders. Hence it suffices to consider in Proposition 11.9 the diagram of closed subspaces

$$\begin{array}{ccc} A & \longrightarrow & \text{cyl}(f) \\ \downarrow & & \downarrow \\ \text{cyl}(i) & \longrightarrow & \text{cyl}(i) \cup_A \text{cyl}(f). \end{array}$$

Now one easily constructs open subsets W_0 , W_1 , and W_2 of $\text{cyl}(i) \cup_A \text{cyl}(f)$ such that $A \subseteq W_0$, $\text{cyl}(i) \subseteq W_1$, and $\text{cyl}(f) \subseteq W_2$ hold, the corresponding inclusions are homotopy equivalences, and we have $\text{cyl}(i) \cup_A \text{cyl}(f) = W_1 \cup W_2$ and $W_0 = W_1 \cap W_2$.



Hence it suffices to show if we put $W = \text{cyl}(i) \cup_A \text{cyl}(f)$ and (W_1, W_0) is m -connected and (W_2, W_0) is n -connected for $m, n \in \mathbb{Z}^{\geq 0}$:

- The pair (W, W_2) is n -connected;
- The map induced by the inclusion $\pi_0(W_1, W_0) \rightarrow \pi_0(W, W_2)$ is bijective;
- The map induced by the inclusion $\pi_k(W_1, W_0, a) \rightarrow \pi_k(W, W_2, a)$ is bijective for $1 \leq k \leq m + n - 1$ and surjective for $1 \leq k = m + n$.

This has already been done in Proposition 11.8. $\square$

We leave it to the reader to prove the following generalization of Proposition 11.9 (i).

Proposition 11.10. *Let the following two diagrams be pushouts*

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ i_2 \downarrow & & \downarrow j_1 \\ X_2 & \xrightarrow{j_2} & X \end{array} \quad \begin{array}{ccc} Y_0 & \xrightarrow{k_1} & Y_1 \\ k_2 \downarrow & & \downarrow l_1 \\ Y_2 & \xrightarrow{l_2} & Y \end{array}$$

where the left vertical arrows i_2 and k_2 are cofibrations. Let $f_i: X_i \rightarrow Y_i$ be maps for $i = 0, 1, 2$ satisfying $f_1 \circ i_1 = k_1 \circ f_0$ and $f_2 \circ i_2 = k_2 \circ f_0$. Denote by $f: X \rightarrow Y$ the map induced by f_0, f_1 , and f_2 and the pushout property.

Consider $n \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$. Suppose that f_1 and f_2 are n -connected and f_0 is $(n - 1)$ -connected with the convention $\infty - 1 = \infty$.

Then f is n -connected.

Proposition 11.11. *Consider $m, n \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$ and a cofibration $i: A \rightarrow X$. Suppose that i is m -connected and A is n -connected. Let $\text{pr}: X \rightarrow X/A$ be the canonical projection. Consider any $a \in A$*

Then the map

$$\pi_k(\text{pr}, a): \pi_k(X, A, a) \rightarrow \pi_k(X/A, \{*\}, *) = \pi_k(X/A, *)$$

is bijective for $0 \leq k \leq m + n$ and surjective for $k = m + n + 1$, where we use the convention that $m + n$ and $m + n + 1$ mean ∞ if $m = \infty$ or $n = \infty$ holds.

Proof. Consider the pushout

$$\begin{array}{ccc} A & \xrightarrow{j} & \text{cone}(A) \\ \downarrow i & & \downarrow \\ X & \longrightarrow & X \cup_A \text{cone}(A) \end{array}$$

for j the inclusion. The map j is $(n+1)$ -connected as A is n -connected. Proposition 11.9 implies that $\pi_k(X, A, a) \rightarrow \pi_k(X \cup_A \text{cone}(A), \text{cone}(A), *)$ induced by the inclusion is bijective, if $0 \leq k \leq m+n$ hold, and is surjective for $k = m+n+1$. The projection $\text{pr}: X \cup_A \text{cone}(A) \rightarrow X/A$ is a homotopy equivalence by Theorem 8.28 and hence induces an isomorphism $\pi_k(\text{pr}): \pi_k(X \cup_A \text{cone}(A), *) \xrightarrow{\cong} \pi_k(X/A, *)$ for every $k \geq 0$ by Lemma 8.25. $\square$

The next two results are consequence of Proposition 11.9. Their rather elementary proof is left to the reader and can be found in [31, Theorem 6.10.5 on page 154 and Proposition 6.10.9 on page 156].

Proposition 11.12. *Let X and Y be well pointed spaces. Consider $m, n \in \mathbb{Z}^{\geq 1}$. Suppose that X is m -connected and Y is n -connected.*

- (i) *The inclusion $X \vee Y \rightarrow X \times Y$ induces an isomorphism $\pi_k(X \vee Y) \xrightarrow{\cong} \pi_k(X \times Y)$ for $0 \leq k \leq m+n$;*
- (ii) *$\pi_k(X \times Y, X \vee Y)$ and $\pi_k(X \wedge Y)$ are trivial for $0 \leq k \leq m+n+1$;*
- (iii) *The canonical map $\pi_k(X \vee Y) \rightarrow \pi_k(X) \times \pi_k(Y) \rightarrow$ is bijective for $0 \leq k \leq m+n$.*

Note that in Proposition 11.12 we assume that $m, n \geq 1$ holds which implies that X and Y are simply connected. This assumption is need as the Example 3.6 shows.

The join $X * Y$ of X and Y is defined by the pushout

$$(11.13) \quad \begin{array}{ccc} X \times Y & \longrightarrow & X \times \text{cone}(Y) \\ \downarrow & & \downarrow \\ \text{cone}(X) \times Y & \longrightarrow & X * Y. \end{array}$$

One can describe $X * Y$ also as the quotient space of $X \times I \times Y$ under the equivalence relation generated by $(x, 0, y_0) \sim (x, 0, y_1)$ and $(x_0, 1, y) \sim (x_1, 1, y)$ for $x, x_0, x_1 \in X$ and $y, y_0, y_1 \in Y$. Intuitively it says that each point in X is connected to each point in Y by a unit interval. One easily checks that $S^m * S^n$ is homeomorphic to S^{m+n+1} and $S^0 * X$ is homeomorphic to ΣX .

The proof of the next proposition is left to the reader and can be found in [31, Proposition 6.10.9 on page 156].

Proposition 11.14. *Consider $m, n \in \mathbb{Z}^{\geq -1}$. Let X and Y be spaces such that X is m -connected and Y is n -connected, where (-1) -connected means that there is no condition.*

Then their join is $(m+n+2)$ -connected.

11.5. The Freudenthal Suspension Theorem. Let (X, x) be a pointed space. The pointed suspension is a functor and hence yields a map

$$[(S^n, s), (X, x)]^0 \rightarrow [(\Sigma(S^n, s), *), (\Sigma(X, x), *)]^0.$$

Using the standard identification $(\Sigma(S^n, s), *) = (S^{n+1}, s)$ we obtain a group homomorphism called *suspension homomorphism* for $k \in \mathbb{Z}^{\geq 1}$

$$(11.15) \quad \sigma_k(X, x): \pi_k(X, x) \rightarrow \pi_{k+1}(\Sigma(X, x), *)$$

Note that σ_k is also defined for $k = 0$ but not a group homomorphism in this case.

Theorem 11.16 (Freudenthal Suspension Theorem). *Let (X, x) be a well pointed space. Consider $n \in \mathbb{Z}^{\geq 0}$. Suppose that X is n -connected. Then the suspension homomorphism*

$$\sigma_k(X, x): \pi_k(X, x) \rightarrow \pi_{k+1}(\Sigma(X, x), *)$$

is bijective for $0 \leq k \leq 2n$ and surjective for $k = 2n + 1$.

Proof. If X is path connected, then $\Sigma(X, x)$ is simply connected and hence the map $\sigma_0(X, x): \pi_0(X, x) \rightarrow \pi_1(\Sigma(X, x), *)$ is obviously bijective. Hence we can assume $k \geq 1$ in the sequel.

There is a homeomorphism $u: \text{cone}(X, x)/X \xrightarrow{\cong} \Sigma(X, x)$. The following diagram commutes

$$\begin{array}{ccc} \pi_k(X, x) & \xrightarrow{\sigma_k(X, x)} & \pi_{k+1}(\Sigma(X, x), *) \\ \partial_{k+1} \uparrow & & \uparrow \pi_{k+1}(u, *) \\ \pi_{k+1}(\text{cone}(X, x), X, x) & \xrightarrow{\pi_{k+1}(\text{pr}, x)} & \pi_{k+1}(\text{cone}(X, x)/X, \{*\}, *) = \pi_{k+1}(\text{cone}(X, x)/X, *) \end{array}$$

where $\text{pr}: \text{cone}(X, x) \rightarrow \text{cone}(X, x)/X$ is the projection and where the left vertical arrow ∂_{n+1} is the boundary operator of the long exact homotopy sequence of the pair $(\text{cone}(X, x), X)$, see Remark 2.13, and is bijective, since $\text{cone}(X, x)$ is contractible. The right vertical arrow is bijective, as u is a pointed homeomorphism. Hence it remains to show that the map $\pi_l(\text{pr}, x): \pi_l(\text{cone}(X, x), X, x) \rightarrow \pi_l(\text{cone}(X, x)/X, \{*\}, *)$ is bijective for $2 \leq l \leq 2n + 1$ and surjective for $l = 2n + 2$. This follows from Proposition 11.11. This finishes the proof of the Freudenthal Suspension Theorem 11.16. $\square$

Remark 11.17. We have the degree homomorphism $\deg_n: \pi_n(S^n, s) \rightarrow \mathbb{Z}$, which is known to be bijective for $n = 1$ by elementary covering theory and compatible with the suspension homomorphisms. We conclude that S^n is simply connected for $n \geq 2$ from the Seifert-van Kampen Theorem. Note that the Freudenthal Suspension Theorem 11.16 implies for $n \geq 1$ that the suspension homomorphism $\sigma_k(S^n, s): \pi_k(S^n, s) \rightarrow \pi_{k+1}(S^{n+1}, s)$ is bijective for $0 \leq k \leq 2n - 2$ and surjective for $k = 2n - 1$. This gives another proof of Theorem 3.4, which does not use differential topology.

Remark 11.18. Let $\mathcal{H}_*$ be a (generalized) homology theory. Then we have for every $n \in \mathbb{Z}$ the suspension isomorphism

$$\sigma_n(X, x): \mathcal{H}_n(X, \{x\}) \xrightarrow{\cong} \mathcal{H}_{n+1}(\Sigma(X, x), \{*\})$$

whereas in Freudenthal Suspension Theorem 11.16 the suspension homomorphism is only bijective in a range depending on the connectivity of X . The connectivity assumptions appearing in Theorem 11.16 are necessary and actually sharp. For instance, we know $\pi_3(S^2) \cong \mathbb{Z}$ from Theorem 10.5, and one can show that $\pi_4(S^3)$ is cyclic of order two and that the suspension homomorphism $\sigma_3(S^2, s): \pi_3(S^2, s) \rightarrow \pi_4(S^3)$ is surjective and obviously not injective. Moreover, the suspension homomorphism $\sigma_2(S^1): \pi_2(S^1) \rightarrow \pi_3(S^2)$ is not surjective as its domain is trivial and its codomain is not trivial.

11.6. Stable homotopy groups. Let (X, x) be a pointed space and $n \in \mathbb{Z}^{\geq 0}$. Consider the sequence given by the suspension homomorphisms of (11.15)

$$(11.19) \quad \pi_n(X) \xrightarrow{\sigma_n(X)} \pi_{n+1}(\Sigma X) \xrightarrow{\sigma_{n+1}(\Sigma X)} \pi_{n+2}(\Sigma^2 X) \\ \xrightarrow{\sigma_{n+2}(\Sigma^2 X)} \pi_{n+3}(\Sigma^3 X) \xrightarrow{\sigma_{n+3}(\Sigma^3 X)} \dots$$

where we omit the base points. Recall that $\Sigma^m X$ is $(m-1)$ -connected for $m \in \mathbb{Z}^{\geq 1}$ by Proposition 11.11 and hence by the Freudenthal Suspension Theorem 11.16 the map $\sigma_{n+m}(\Sigma^m X): \pi_{n+m}(\Sigma^m X) \rightarrow \pi_{m+n+1}(\Sigma^{m+1} X)$ is surjective for $m = (n+1)$ and bijective for $m \geq (n+2)$. So after finitely many steps all these suspension homomorphism are isomorphism of abelian groups.

Definition 11.20 (Stable homotopy groups). Define the abelian group $\pi_n^s(X, x)$, called *nth stable homotopy group* of (X, x) to be the direct limit of the sequence (11.19).

Given a (unpointed) space Y , define

$$\pi_n^s(Y) := \pi_n^s(Y_+)$$

where Y_+ is the pointed space $(Y \amalg \{*\}, *)$ obtained from Y by adjoining an extra base point.

Obviously $\pi_n^s(X, x)$ is a functor from the category of pointed spaces to the category of abelian groups. Moreover, if the two pointed maps $f_0, f_1: (X, x) \rightarrow (Y, y)$ are pointed homotopic, then the induced homomorphisms $\pi_n^s(f_0)$ and $\pi_n^s(f_1)$ from $\pi_n^s(X, x)$ to $\pi_n^s(Y, y)$ agree. The stable homotopy groups come with a natural map

$$(11.21) \quad \iota_n(X, x): \pi_n(X, x) \rightarrow \pi_n^s(X, x)$$

and with a natural suspension homomorphism

$$(11.22) \quad \sigma_n^s(X, x): \pi_n^s(X, x) \rightarrow \pi_{n+1}^s(\Sigma(X, x), *).$$

The map $\iota_n(X, x)$ is in general neither injective nor surjective. If X is m -connected for $m \in \mathbb{Z}^{\geq 0}$, then $\iota_n(X, x)$ is surjective if $n = 2m + 1$ and is bijective if $n \leq 2m$ by the Freudenthal Suspension Theorem 11.16. The construction of the stable homotopy groups is designed so that $\sigma_n^s(X, x)$ of (11.22) is bijective for every pointed space (X, x) and $n \geq 0$.

Given a (unpointed) topological pair (X, A) , we define

$$(11.23) \quad \pi_n^s(X, A) = \pi_n^s(X_+ \cup_{A_+} \text{cone}(A_+, *), *).$$

Thus we obtain a functor from the category of pairs to the category of abelian groups which is homotopy invariant, i.e., for two maps of pairs $f_0, f_1: (X, A) \rightarrow (Y, B)$ the induced homomorphisms $\pi_n^s(f_0)$ and $\pi_n^s(f_1)$ from $\pi_n^s(X, A)$ to $\pi_n^s(Y, B)$ agree if f_0 and f_1 are homotopic as maps of pairs.

We record the following theorem whose proof we will give later when we are dealing more generally with spectra.

Theorem 11.24 (Stable homotopy groups form a (generalized) homology theory). *There exist natural transformation $\partial_{n+1}(X, A): \pi_{n+1}^s(X, A) \rightarrow \pi_n^s(A)$ for $n \in \mathbb{Z}^{\geq 0}$ such that stable homotopy π_*^s defines a homology theory on the category of pairs satisfying the disjoint union axiom.*

Obviously π_*^s also satisfies the weak homotopy equivalence axiom saying that a weak homotopy equivalence induces isomorphisms on the stable homotopy groups. It does not satisfy the dimension axiom.

Definition 11.25 (Stable stems). Define the n -th stable stem π_n^s to be $\pi_n^s(\{\bullet\}) = \pi_n^s(S^0, *)$ for $n \geq 0$.

Note that π_n^s is the direct limit of the directed system

$$(11.26) \quad \pi_n(S^0, *) \xrightarrow{\sigma_n(S^0, *)} \pi_{n+1}(S^1, *) \xrightarrow{\sigma_{n+1}(S^1, *)} \pi_{n+2}(S^2, *) \\ \xrightarrow{\sigma_{n+2}(S^2, *)} \pi_{n+3}(S^3, *) \xrightarrow{\sigma_{n+3}(S^3, *)} \dots$$

where we have used the standard identification $(S^{n+1}, *) = (\Sigma(S^n, *), *)$. Recall that the map $\sigma_{n+m}(S^m, *): \pi_{n+m}(S^m, *) \rightarrow \pi_{n+m+1}(S^{m+1}, *)$ is surjective for $m = (n+1)$ and bijective for $m \geq (n+2)$.

Remark 11.27 (Outlook about $\pi_k^s(S^n)$). Obviously it is easier to compute π_n^s instead of $\pi_n(S^m)$ for $m > n$. Nevertheless it is an open (and extremely hard) problem to compute π_n^s general. At the time of writing it is fair to say that we do not know π_n^s in the range $n \geq 100$. Only some asymptotic results are known in that range. There is not even a formula known which might give the answer. There is no obvious pattern in the computations, one has carried out so far. At least one knows that π_n^s is finite for $n \geq 1$, see [24] or Corollary 32.4 and one knows its values for $n \leq 61$ and also for some other values for $n \leq 99$. For instance we have

$$(11.28) \quad \begin{array}{c|cccccccccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \hline \pi_n^s & \mathbb{Z} & \mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/24 & 0 & 0 & \mathbb{Z}/2 & \mathbb{Z}/240 & \mathbb{Z}/2^2 & \mathbb{Z}/2^3 & \mathbb{Z}/6 & \mathbb{Z}/504 \\ \hline n & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ \hline \pi_n^s & 0 & \mathbb{Z}/3 & \mathbb{Z}/2^2 & \mathbb{Z}/480 \times \mathbb{Z}/2 & \mathbb{Z}/2^2 & \mathbb{Z}/2^4 & \mathbb{Z}/8 \times \mathbb{Z}/2 & \mathbb{Z}/264 \times \mathbb{Z}/2 \end{array}$$

where A^m means $\bigoplus_{i=1}^m A$. The table above is taken from Toda [30]. More information about the stable stems can be found for instance in [13, 14].

12. THE HUREWICZ THEOREM

12.1. The Hurewicz homomorphism. Let (X, x) be a pointed space. Next we define for $n \in \mathbb{Z}^{\geq 1}$ a homomorphism of groups, which is natural in X and called *n-th Hurewicz map* or *n-th Hurewicz homomorphism*.

$$(12.1) \quad \text{hur}_n(X, x): \pi_n(X, x) \rightarrow H_n(X),$$

where $H_n(X)$ denotes singular homology (with coefficients in $\mathbb{Z}$).

Given an element $[f]$ in $\pi_n(X, x)$ represented by a pointed map $f: (S^n, s) \rightarrow (X, x)$, define $\text{hur}_n(X, x)([f])$ to be the image of the standard fundamental class $[S^n] \in H_n(S^n)$ under the map $H_n(f): H_n(S^n) \rightarrow H_n(X)$ induced by f . Obviously this definition is independent of the choice of representative f of $[f]$. Let $\nabla_n: S^n \rightarrow S^n \vee S^n$ be the pinching map, see (2.5). Let $\text{pr}_k: S^1 \vee S^1 \rightarrow S^1$ be the projection onto the k -th functor for $k = 1, 2$. Then the following diagram commutes

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{H_n(\nabla_n)} & H_n(S^n \vee S^n) \\ & \searrow \Delta_n & \downarrow \cong \\ & & H_n(\text{pr}_1) \times H_n(\text{pr}_2) \\ & & \downarrow \\ & & H_n(S^n) \times H_n(S^n) \end{array}$$

where Δ_n is the diagonal map sending z to (z, z) and the right vertical arrow is an isomorphism. Note that in $\pi_n(X, x)$ the inverse of $[f]$ is given by $[f \circ u]$ for any map $u: (S^n, s) \rightarrow (S^n, s)$ of degree -1 and $H_n(u): H_n(S^n) \rightarrow H_n(S^n)$ sends $[S^n]$ to $-[S^n]$. Now one easily checks that $\text{hur}_n(X, x)$ is a group homomorphism. Obviously it is natural in (X, x) .

The elementary proof of the following lemma is left to the reader.

Lemma 12.2. *Let $w: I \rightarrow X$ be a path from x to y . Then the following diagram commutes*

$$\begin{array}{ccc}
 \pi_n(X, x) & & \\
 \downarrow T_n([w]) \cong & \searrow \text{hur}_n(X, x) & \\
 & & H_n(X) \\
 & \nearrow \text{hur}_n(X, y) & \\
 \pi_n(X, y) & &
 \end{array}$$

where $T_n([w])$ is the isomorphism introduced in Subsection 2.3.

12.2. The Hurewicz Theorem. Before we investigate the Hurewicz homomorphism further, we consider the following two special cases. The first one is the case $n = 1$ and has already been dealt with in a previous lecture course.

Proposition 12.3. *If X is a path connected space, then for any base point x the map induced by the Hurewicz homomorphism*

$$\pi_1(X, x)_{\text{ab}} \rightarrow H_1(X)$$

for $\pi_1(X, x)_{\text{ab}} = \pi_1(X)/[\pi_1(X), \pi_1(X)]$ the abelianization of $\pi_1(X, x)$ is an isomorphism.

Lemma 12.4. *For $n \in \mathbb{Z}^{\geq 1}$ the Hurewicz homomorphism $\text{hur}_n(S^n, s): \pi_n(S^n, s) \rightarrow H_n(S^n)$ of (12.1) is bijective.*

Proof. The map $f: \mathbb{Z} \rightarrow \pi_n(S^n)$ sending 1 to the class of $[\text{id}_{S^n}]$ is bijective by Theorem 3.4. The composite of $\text{hur}_n(S^n, s)$ and f is the homomorphism $\mathbb{Z} \rightarrow H_n(S^n)$ sending 1 to $[S^n]$ and hence bijective. This implies that $\text{hur}_n(S^n, s)$ is bijective. $\square$

Next we prove one of the main results of the course.

Theorem 12.5 (Hurewicz Theorem). *Consider $n \in \mathbb{Z}^{\geq 2}$. Let X be an $(n-1)$ -connected space. Then the Hurewicz homomorphism*

$$\text{hur}_n(X, x): \pi_n(X, x) \rightarrow H_n(X)$$

of (12.1) is bijective for any base point $x \in X$.

Proof. Since X is n -connected, it is weakly homotopy equivalent to a CW-complex Y which has precisely one 0-cell and no cells of dimension d for $1 \leq d \leq (n-1)$ by Corollary 6.5. The inclusion $i: Y_{n+1} \rightarrow Y$ induces for the base point $y_0 \in Y_0$ bijections

$$\begin{aligned}
 \pi_n(i, y_0): \pi_n(Y_{n+1}, y_0) &\xrightarrow{\cong} \pi_n(Y, y_0); \\
 H_n(i): H_n(Y_{n+1}) &\xrightarrow{\cong} H_n(Y),
 \end{aligned}$$

by Corollary 4.5 and the long exact homotopy sequence of the pair (Y, Y_{n+1}) . Hence we can assume without loss of generality that X has precisely one 0-cell $\{x_0\}$ and the dimension $\dim(e)$ of every cell e satisfies $\dim(e) \in \{0, n, (n+1)\}$.

This implies that $X_n = \bigvee_{i \in I} S^n$ and there exists a pushout

$$\begin{array}{ccc}
 \prod_{j \in J} S^n & \xrightarrow{\prod_{j \in J} q_j} & X_n = \bigvee_{i \in I} S^n \\
 \downarrow & & \downarrow \\
 \prod_{j \in J} D^{n+1} & \longrightarrow & X.
 \end{array}$$

The Cellular Approximation Theorem 4.1 implies that each map $q_j: S^n \rightarrow X_n$ is homotopic to a map $q'_j: S^n \rightarrow X_n$ sending s to x_0 . Choose a homotopy $h_j: \coprod_{i \in I} S^n \times [0, 1] \rightarrow X_n$ with $(h_j)_0 = \coprod_{j \in J} q_j$ and $(h_j)_1 = \coprod_{j \in J} q'_j$. Consider the following commutative diagram

$$\begin{array}{ccccc}
 \coprod_{j \in J} D^{n+1} & \xleftarrow{\quad} & \coprod_{j \in J} S^n & \xrightarrow{\coprod_{j \in J} q_j} & X_n \\
 \simeq \downarrow l_0 & & \simeq \downarrow k_0 & & \simeq \downarrow \text{id}_{X_n} \\
 \coprod_{j \in J} D^{n+1} \times I & \xleftarrow{\quad} & \coprod_{j \in J} S^n \times I & \xrightarrow{\coprod_{j \in J} h_j} & X_n \\
 \simeq \uparrow l_1 & & \simeq \uparrow k_1 & & \simeq \uparrow \text{id}_{X_n} \\
 \coprod_{j \in J} D^{n+1} & \xleftarrow{\quad} & \coprod_{j \in J} S^n & \xrightarrow{\coprod_{j \in J} q'_j} & X_n.
 \end{array}$$

where k_0 and l_0 are the obvious inclusions coming from $0 \in I$ and k_1 and l_1 are the obvious inclusions coming from $1 \in I$. All vertical arrows are homotopy equivalences and all left horizontal arrows are cofibrations. Hence the induced maps from the pushout of the upper row to the pushout of the middle row as well as the arrow from the pushout of the lower row to the pushout of middle row are homotopy equivalences by Theorem 8.28. Therefore we can assume without loss of generality that q_j sends the base point $s \in S^n$ to x_0 and we can write X as a pushout

$$\begin{array}{ccc}
 \bigvee_{j \in J} S^n & \xrightarrow{f} & \bigvee_{i \in I} S^n \\
 \downarrow & & \downarrow k \\
 \bigvee_{j \in J} D^{n+1} & \longrightarrow & X
 \end{array}$$

where f respects the base points and k is the inclusion $X_n = \bigvee_{i \in I} S^n \rightarrow X_{n+1} = X$. We obtain a commutative diagram of abelian groups with exact rows

$$\begin{array}{ccccccc}
 \pi_n \left(\bigvee_{j \in J} S^n, * \right) & \xrightarrow{\pi_n(f)} & \pi_n \left(\bigvee_{i \in I} S^n, * \right) & \xrightarrow{\pi_n(k)} & \pi_n(X, x_0) & \longrightarrow & \{0\} \\
 \downarrow \text{hur}_n(\bigvee_{j \in J} S^n, *) & & \downarrow \text{hur}_n(\bigvee_{i \in I} S^n, *) & & \downarrow \text{hur}_n(X, x_0) & & \\
 H_n \left(\bigvee_{j \in J} S^n \right) & \xrightarrow{H_n(f)} & H_n \left(\bigvee_{i \in I} S^n \right) & \xrightarrow{H_n(k)} & H_n(X) & \longrightarrow & \{0\}
 \end{array}$$

The lower row is exact by excision, the long exact homology sequence for pairs, and the fact that $H_{n-1} \left(\bigvee_{j \in J} S^n \right) = \{0\}$ holds. The upper row is exact by Theorem 11.9, the long exact homotopy sequence for pairs, and the conclusion from Theorem 3.4 and Proposition 11.12 (iii) that $\pi_{n-1} \left(\bigvee_{j \in J} S^n \right) = \{0\}$ holds. Hence by the Five Lemma it suffices to prove that the left vertical arrow and the middle vertical arrow are bijective. The following diagram commutes

$$\begin{array}{ccc}
 \bigoplus_{i \in I} \pi_n(S^n, s) & \xrightarrow[\cong]{\bigoplus_{i \in I} \pi_n(k_i, s)} & \pi_n \left(\bigvee_{i \in I} S^n \right) \\
 \bigoplus_{i \in I} \text{hur}_n(S^n, s) \downarrow & & \downarrow \text{hur}_n \left(\bigvee_{i \in I} S^n, * \right) \\
 \bigoplus_{i \in I} H_n(S^n, s) & \xrightarrow[\cong]{\bigoplus_{i \in I} H_n(k_i)} & H_n \left(\bigvee_{i \in I} S^n \right).
 \end{array}$$

and has bijections as horizontal arrows by Proposition 11.12 (iii), where k_i is the inclusion of the summand belonging to $i \in I$. Since the left vertical arrow is bijective by Lemma 12.4, the right vertical arrow is bijective. Lemma 12.2 implies that the

Hurewicz homomorphism $\text{hur}_n(X, x): \pi_n(X, x) \rightarrow H_n(X)$ is bijective for all base points $x \in X$. This finishes the proof of the Hurewicz Theorem 12.5. $\square$

Remark 12.6. The condition that X simply connected in Theorem 12.5 is necessary. Consider a non-trivial group G for which G_{ab} is trivial, e.g., the simple finite group A_n for $n \geq 5$. Choose a path connected CW-complex X with $\pi_1(X) \cong G$. Then X is 0-connected and Lemma 12.3 implies that the Hurewicz homomorphism $\text{hur}_1(X, x): \pi_1(X, x) \rightarrow H_1(X)$ is not injective.

The condition that X is simply connected is also necessary in the following Proposition 12.7.

Proposition 12.7. *Let X be a simply connected space and $n \in \mathbb{Z}^{\geq 1}$. Then*

- (i) *The following assertions are equivalent:*
 - X is n -connected;
 - $H_i(X) = 0$ holds for $1 \leq i \leq n$;
 - $H_i(X) = 0$ holds for $2 \leq i \leq n$;
- (ii) *The following assertions are equivalent:*
 - X is weakly contractible;
 - $H_i(X) = 0$ holds for $1 \leq i$;
 - $H_i(X) = 0$ holds for $2 \leq i$.

Proof. (i) This follows by induction over $n = 1, 2, \dots$. The induction beginning $n = 1$ follows from the conclusion of Proposition 12.3 that $H_1(X) = 0$ vanishes for a simply connected space X . The induction step from $(n-1) \geq 1$ to n follows from Theorem 12.5.

(ii) This follows from assertion (i). $\square$

We record the following stronger version of the Hurewicz Theorem whose proof can be found in [27, Theorem 10.25 on page 185].

Theorem 12.8 (Improved Hurewicz Theorem). *Consider $n \in \mathbb{Z}^{\geq 2}$. Let X be an $(n-1)$ -connected space. Then for any base point $x \in X$ the Hurewicz homomorphism*

$$\text{hur}_m(X, x): \pi_m(X, x) \rightarrow H_m(X)$$

of (12.1) is bijective for $m = n$ and surjective for $m = n + 1$.

12.3. The relative Hurewicz Theorem. There is also a relative version of the Hurewicz map for a pointed pair (X, A, a) for $n \geq 1$

$$(12.9) \quad \text{hur}_n(X, A, a): \pi_n(X, A, a) \rightarrow H_n(X, A),$$

which sends $[f] \in \pi_n(X, A, a)$ represented by a map of triples $f: (D^n, S^{n-1}, \{s\}) \rightarrow (X, A, \{a\})$ to the image of the standard fundamental class $[D^n, S^{n-1}]$ under the homomorphism $H_n(f): H_n(D^n, S^{n-1}) \rightarrow H_n(X, A)$. It is a group homomorphism for $n \geq 2$ and the following diagram commutes for $n \geq 2$

$$(12.10) \quad \begin{array}{ccccc} \pi_n(X, a) & \longrightarrow & \pi_n(X, A, a) & \longrightarrow & \pi_{n-1}(A, a) \\ \downarrow \text{hur}_n(X, a) & & \downarrow \text{hur}_n(X, A, a) & & \downarrow \text{hur}_{n-1}(X, a) \\ H_n(X) & \longrightarrow & H_n(X, A) & \longrightarrow & H_{n-1}(A, a), \end{array}$$

where the exact upper row and the exact lower row are parts of the long exact sequences associated to the pair (X, A) .

Theorem 12.11 (The relative Hurewicz Theorem). *Consider $n \in \mathbb{Z}^{\geq 2}$. Let (X, A) be a pair. Suppose that A and X are simply connected and (X, A) is $(n-1)$ -connected. Then:*

(i) *The Hurewicz homomorphism*

$$\text{hur}_n(X, A, a): \pi_n(X, A, a) \rightarrow H_n(X, A)$$

of (12.10) is bijective for any base point $a \in A$;

(ii) *The homology group $H_i(X, A)$ vanishes for $2 \leq i \leq (n-1)$.*

Proof. We can arrange that the inclusion $A \rightarrow X$ is a cofibration by Subsection 8.8. Let $\text{pr}: (X, A, \{a\}) \rightarrow (X/A, \{*\})$ be the projection. We obtain a commutative diagram

$$\begin{array}{ccc} \pi_n(X, A, a) & \xrightarrow[\cong]{\pi_n(\text{pr}, a)} & \pi_n(X/A, \{*\}, *) = \pi_n(X/A, *) \\ \text{hur}_n(X, A, a) \downarrow & & \downarrow \text{hur}_n(X/A, *) \\ H_n(X, A) & \xrightarrow[\cong]{H_n(\text{pr})} & H_n(X/A, \{*\}) = H_n(X/A). \end{array}$$

The upper row is bijective by Proposition 11.11. The lower row is bijective by excision. If X/A is $(n-1)$ -connected, then Theorem 12.5 implies that right vertical arrow is also bijective. Hence it suffices to show for $n \in \mathbb{Z}^{\geq 2}$:

- The space X/A is $(n-1)$ -connected;
- The homology group $H_i(X/A)$ vanishes for $2 \leq i \leq (n-1)$.

This is done by induction over $n = 2, 3, 4, \dots$

Since A and X are simply connected, X/A is simply connected by the Seifert-von-Kampen Theorem. Now the induction beginning $n = 2$ follows. The induction step from $(n-1) \geq 2$ to n is done as follows.

By induction hypothesis applied to the $(n-2)$ -connected pair (X, A) , we know that $\text{hur}_{n-1}(X, A, a): \pi_{n-1}(X, A, a) \rightarrow H_{n-1}(X, A)$ is bijective and the homology group $H_i(X, A)$ vanishes for $2 \leq i \leq (n-2)$. As $\pi_{n-1}(X, A, a)$ vanishes, the homology group $H_i(X, A)$ vanishes for $2 \leq i \leq (n-1)$. As the projection pr induces an isomorphism $H_i(X, A) \xrightarrow{\cong} H_i(X/A)$ for $i \geq 1$, the homology group $H_i(X/A)$ vanishes for $2 \leq i \leq (n-1)$. This finishes the proof of Theorem 12.11. $\square$

12.4. Applications of the Hurewicz Theorem. Next we generalize Proposition 12.7 to maps.

Proposition 12.12. *Let $f: X \rightarrow Y$ be a map of simply connected spaces.*

- (i) *The following assertions are equivalent for $n \in \mathbb{Z}^{\geq 1}$:*
- *f is n -connected;*
 - *$H_i(f): H_i(X) \rightarrow H_i(Y)$ is bijective for $2 \leq i \leq (n-1)$ and surjective for $i = n$;*
- (ii) *The following assertions are equivalent:*
- *f is a weak homotopy equivalence;*
 - *$H_i(f): H_i(X) \rightarrow H_i(Y)$ is bijective for $i \geq 2$.*

Proof. Since we can replace f by the inclusion into its mapping cylinder, this follows from the relative Hurewicz Theorem 12.11. $\square$

The next theorem is called also sometimes the Whitehead Theorem, see also Theorem 5.1.

Theorem 12.13 (Whitehead Theorem). *Let $f: X \rightarrow Y$ be a map of simply connected CW-complexes. Then the following assertions are equivalent:*

- *f is a homotopy equivalence;*
- *f is a weak homotopy equivalence;*
- *$H_i(f): H_i(X) \rightarrow H_i(Y)$ is bijective for $i \geq 2$.*

Proof. This is a direct consequence of Theorem 5.1 and Proposition 12.12. $\square$

The condition that X and Y are simply-connected is necessary in Theorem 12.13. Here is a more general version of Theorem 12.13 which does not need the assumption that X and Y are simply connected.

Theorem 12.14. *Let $f: X \rightarrow Y$ be a map of path connected CW-complexes. Suppose that for one (and hence all) base point $x \in X$ the map $\pi_1(X, x) \rightarrow \pi_1(Y, f(x))$ is bijective. We can lift f to a map between the universal coverings $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$, i.e., we have the commutative diagram:*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ p_X \downarrow & & \downarrow p_Y \\ X & \xrightarrow{f} & Y \end{array}$$

Then the following assertions are equivalent:

- f is a homotopy equivalence;
- $H_i(\tilde{f}): H_i(\tilde{X}) \rightarrow H_i(\tilde{Y})$ is bijective for $i \geq 2$.

Proof. We conclude from Theorem 2.7 that f is a weak homotopy equivalence if and only if $\tilde{f}$ is a weak homotopy equivalence. By Theorem 12.13 $\tilde{f}$ is a weak homotopy equivalence if and only if $H_i(\tilde{f}): H_i(\tilde{X}) \rightarrow H_i(\tilde{Y})$ is bijective for $i \geq 2$. Now Theorem 12.14 follows from Theorem 5.1. $\square$

Example 12.15. Consider the spaces $X = S^n \vee S^n \vee S^{2n}$ and $Y = S^n \times S^n$ for $n \in \mathbb{Z}^{\geq 2}$. Then X and Y are simply connected and $H_i(X) \cong H_i(Y)$ holds for $i \geq 0$. But the cohomology rings of X and Y are not isomorphic and hence there is no homotopy equivalence from X to Y .

Note that this does not contradict Theorem 12.13, since there the existence of a map $f: X \rightarrow Y$ is required which implements the isomorphism $H_i(X) \xrightarrow{\cong} H_i(Y)$.

Theorem 12.16 (Recognizing the sphere up to homotopy). *Let X be a path connected CW-complex. Then the following assertions are equivalent for $n \geq 2$:*

- The space X is homotopy equivalent to S^n ;
- The space X is simply connected, $H_i(X)$ vanishes for all $i \geq 2$ with $i \neq n$ and $H_n(X)$ is isomorphic to $\mathbb{Z}$.

Proof. Suppose that the space X is simply connected, $H_i(X)$ vanishes for all $i \geq 2$ with $i \neq n$ and $H_n(X)$ is isomorphic to $\mathbb{Z}$. Proposition 12.7 (i) implies that X is $(n-1)$ -connected. We conclude from the Hurewicz Theorem 12.5 that the Hurewicz homomorphism $\text{hur}_n(X, x): \pi_n(X, x) \rightarrow H_n(X, x)$ is bijective. Therefore we can find a map $f: S^n \rightarrow X$ such that $H_n(f): H_n(S^n) \rightarrow H_n(X)$ is an isomorphism. Hence $H_i(f): H_i(S^n) \rightarrow H_i(X)$ is an isomorphism for $2 \leq i$. Proposition 12.12 (i) implies that f is a homotopy equivalence.

The other implication is obviously true. $\square$

13. MOORE SPACES

Definition 13.1 (Moore space). Consider $n \geq 1$ and an abelian group G . A Moore space (X, φ) of type (G, n) consists of a path connected CW-complex X and an isomorphism $\varphi: H_n(X) \xrightarrow{\cong} G$ such that $H_i(X) = \{0\}$ for $i \notin \{0, n\}$ holds and that X is simply connected if $n \geq 2$.

Sometimes (X, x, φ) is denoted by $M(G, n)$

Lemma 13.2. Consider $n \geq 1$ and a group G which is assumed to be abelian if $n \geq 2$ holds. Then:

(i) There exists an $(n + 1)$ -dimensional CW-complex X with the following properties:

- The space X is the reduced mapping cone of some map $f: \bigvee_{i \in I} S^n \rightarrow \bigvee_{j \in J} S^n$;
- There is an exact sequence of groups

$$0 \rightarrow \pi_n\left(\bigvee_{j \in J} S^n\right) \xrightarrow{\pi_n(f, x)} \pi_n\left(\bigvee_{i \in I} S^n\right) \xrightarrow{\pi_n(k)} \pi_n(X, x) \rightarrow 0$$

for $k: X_n = \bigvee_{j \in J} S^n \rightarrow X$ the inclusion and $n \geq 2$;

- We have $\{\bullet\} = X_0 = X_{n-1}$;
- The space X is $(n - 1)$ -connected;
- We have $\pi_n(X, x) \cong G$ for any base point $x \in X$;
- The homology group $H_i(X)$ vanishes for $i \geq (n + 1)$ if G abelian;
- The homology group $H_i(X)$ vanishes for $1 \leq i \leq (n - 1)$;
- If G is finitely presented or if G is finitely generated abelian, then X can be chosen to be a finite CW-complex.

(ii) Suppose that $n \geq 2$ holds. Let X be the space constructed in the proof of assertion (i). Let $x \in X$ be any base point. Let (Y, y) be any pointed CW-complex. Let $\psi: \pi_n(X, x) \xrightarrow{\cong} \pi_n(Y, y)$ be any group homomorphism.

Then there is a pointed map $u: (X, x) \rightarrow (Y, y)$ satisfying $\pi_n(u, x) = \psi$.

Proof. (i) For $n = 1$ one can choose a presentation of the group and consider the associated presentation CW-complex, which is path connected and satisfies $\pi_1(X, x) \cong G$ for any base point x . In general $H_2(X)$ is not trivial. This can be arranged if G is abelian. Choose an exact sequence of abelian groups $0 \rightarrow \bigoplus_{i \in I} \mathbb{Z} \xrightarrow{\alpha} \bigoplus_{j \in J} \mathbb{Z} \rightarrow G \rightarrow 0$. If G is finitely generated abelian, one can choose I and J to be finite. Then one can find a map $f: \bigvee_{i \in I} S^1 \rightarrow \bigvee_{j \in J} S^1$ such that the following diagram commutes

$$\begin{array}{ccc} \bigoplus_{i \in I} \mathbb{Z} & \xrightarrow{\alpha} & \bigoplus_{j \in J} \mathbb{Z} \\ \cong \downarrow & & \downarrow \cong \\ H_1\left(\bigvee_{i \in I} S^1\right) & \xrightarrow{H_1(f)} & H_1\left(\bigvee_{j \in J} S^1\right) \end{array}$$

where the vertical maps are the obvious isomorphisms. We have the short exact sequence

$$\begin{aligned} H_2\left(\bigvee_{j \in J} S^1\right) &\rightarrow H_2(X) \rightarrow H_1\left(\bigvee_{i \in I} S^1\right) \xrightarrow{H_1(f)} H_1\left(\bigvee_{j \in J} S^1\right) \rightarrow H_1(X) \\ &\rightarrow H_0\left(\bigvee_{i \in I} S^1\right) \xrightarrow{H_0(f)} H_0\left(\bigvee_{j \in J} S^1\right) \end{aligned}$$

This implies $H_2(X) = \{0\}$ and $H_1(X) \cong G$. Since X is 2-dimensional, we get $H_i(X) = \{0\}$ for $i \geq 2$.

Suppose $n \geq 2$. Then G is an abelian group and we can choose an exact sequence of abelian groups $0 \rightarrow \bigoplus_{i \in I} \mathbb{Z} \xrightarrow{\alpha} \bigoplus_{j \in J} \mathbb{Z} \rightarrow G \rightarrow 0$. If G is finitely generated, one can choose I and J to be finite. Let $\{e_i \mid i \in I\}$ and $\{e'_j \mid j \in J\}$ be the standard basis of $\bigoplus_{i \in I} \mathbb{Z}$ and $\bigoplus_{j \in J} \mathbb{Z}$. Then there is a set of integers $\{d_{i,j} \mid i \in I, j \in J\}$ such that the set $\{j \in J \mid d_{i,j} \neq 0\}$ is finite for every $i \in I$ and $\alpha(e_i) = \sum_{j \in J} d_{i,j} \cdot e'_j$.

holds. The canonical maps

$$(13.3) \quad \bigoplus_{i \in I} \pi_n(S^n, s) \xrightarrow{\cong} \pi_n\left(\bigvee_{i \in I} S^n, s\right);$$

$$(13.4) \quad \bigoplus_{j \in J} \pi_n(S^n, s) \xrightarrow{\cong} \pi_n\left(\bigvee_{j \in J} S^n, s\right),$$

are bijective by Proposition 11.12 (iii). Because of Theorem 3.4 there is a pointed map $f_i: (S^n, s) \rightarrow (\bigvee_{j \in J} S^n, s)$ such that for every $j \in J$ its composite with the projection $\text{pr}_k: \bigvee_{j \in J} S^n \rightarrow S^n$ to the factor k belonging to $k \in J$ has degree $d_{i,k}$. Define the pointed map

$$f := \bigvee_{i \in I} f_i: \bigvee_{i \in I} S^n \rightarrow \bigvee_{j \in J} S^n.$$

Under the obvious identifications coming from the isomorphisms (13.3) and (13.4) the homomorphism α can be identified with

$$\pi_n(f, s): \pi_n\left(\bigvee_{i \in I} S^n, s\right) \rightarrow \pi_n\left(\bigvee_{j \in J} S^n, s\right).$$

Hence the cokernel of the latter map is isomorphic to G and its kernel is trivial. Let X be the reduced mapping cone of f with the preferred base point $x \in X_0$. The sequence

$$0 \rightarrow \pi_n\left(\bigvee_{i \in I} S^n, s\right) \xrightarrow{\pi_n(f, s)} \pi_n\left(\bigvee_{j \in J} S^n, s\right) \rightarrow \pi_n(X, x) \rightarrow 0$$

is exact by Theorem 2.11, Theorem 3.4, and Theorem 11.11. Hence X is a path connected CW -complex which has precisely one zero cell, no cells e of dimension $1 \leq \dim(e) \leq n-1$, is $(n-1)$ -connected by Corollary 4.5, satisfies $\pi_n(X, x) \cong G$ for the base point $x \in X_0$ and hence for all base points in X by the diagram (2.6), and the homology groups $H_i(X)$ vanish for $1 \leq i \leq (n-1)$. It remains to show that $H_i(X)$ vanishes for $i \geq (n+1)$. As X is $(n+1)$ -dimensional, it suffices to do this for $i = (n+1)$. The following diagram commutes

$$\begin{array}{ccc} & & H_{n+1}(\bigvee_{j \in J} S^n) = \{0\} \\ & & \downarrow H_{n+1}(k) \\ & & H_{n+1}(X) \\ & & \downarrow \partial_{n+1} \\ \{0\} & & H_n(\bigvee_{i \in I} S^n) \\ \downarrow & \xrightarrow[\cong]{\text{hur}_n(\bigvee_{i \in I} S^n, s)} & \downarrow \\ \pi_n(\bigvee_{i \in I} S^n, s) & & H_n(\bigvee_{i \in I} S^n) \\ \downarrow \pi_n(f, s) & & \downarrow H_n(f) \\ \pi_n(\bigvee_{j \in J} S^n, s) & \xrightarrow[\cong]{\text{hur}_n(\bigvee_{j \in J} S^n, s)} & H_n(\bigvee_{j \in J} S^n) \\ \downarrow \pi_n(k, x) & & \downarrow H_n(k) \\ \pi_n(X, x) & \xrightarrow[\cong]{\text{hur}_n(X, x)} & H_n(X) \\ \downarrow & & \downarrow \\ \{0\} & & \{0\} \end{array}$$

where the column are exact and the horizontal arrows are bijective by Hurewicz Theorem 12.5 since X , $\bigvee_{i \in I} S^n$, and $\bigvee_{j \in J} S^n$ are $(n-1)$ -connected by Corollary 4.5. Hence $H_{n+1}(X)$ vanishes.

(ii). We start with the case $n \geq 2$. Recall that X is the reduced mapping cone of a specific map

$$f := \bigvee_{i \in I} f_i: \bigvee_{i \in I} S^n \rightarrow \bigvee_{j \in J} S^n$$

such that we have an exact sequence of abelian groups

$$0 \rightarrow \pi_n\left(\bigvee_{i \in I} S^n\right) \xrightarrow{\pi_n(f, x)} \pi_n\left(\bigvee_{j \in J} S^n\right) \xrightarrow{\pi_n(k)} \pi_n(X, x) \rightarrow 0$$

for $k: X_n = \bigvee_{j \in J} S^n \rightarrow X$ the inclusion and that we have isomorphisms (13.3) and (13.4). Because of (2.6) we can assume without loss of generality that the base point of X is the standard base point s in $X_n = \bigvee_{j \in J} S^n \subseteq X$. For every $j \in J$ let $a_j: (S^n, s) \rightarrow (Y, y)$ be the pointed map whose class in $\pi_n(Y, y)$ is the image of the standard generator of $\pi_n(S^n, s)$ under the composite

$$\pi_n(S^n, s) \xrightarrow{l_j} \bigoplus_{j \in J} \pi_n(S^n, s) \xrightarrow{\cong} \pi_n\left(\bigvee_{j \in J} S^n, s\right) \xrightarrow{\pi_n(k)} \pi_n(X, x) \xrightarrow{\psi} \pi_n(Y, y),$$

where l_j is the inclusion of the j -th summand. Define the pointed map

$$u_n := \bigvee_{j \in J} a_j: (X_n, x) = \left(\bigvee_{j \in J} S^n, s\right) \rightarrow (Y, y).$$

Then the composite of the homomorphism $\pi_n(u_n, s): \pi_n\left(\bigvee_{j \in J} S^n, s\right) \rightarrow \pi_n(Y, y)$ with the homomorphism $\pi_n(f, s): \pi_n\left(\bigvee_{i \in I} S^n, s\right) \rightarrow \pi_n\left(\bigvee_{j \in J} S^n, s\right)$ is trivial. This implies that the composite $u_n \circ f$ is pointed nullhomotopic. Hence u_n extends to a pointed map $u: (X, x) \rightarrow (Y, y)$. We get by construction $\pi_n(u, x) = \psi$. $\square$

Theorem 13.5 (Existence and uniqueness of Moore spaces). *Consider $n \in \mathbb{Z}^{\geq 1}$ and two abelian groups G and G' . Then:*

- (i) *There exists a Moore space (X, ϕ) of type (G, n) such that the $X_{n-1} = X_0 = \{x\}$ holds;*
- (ii) *If $n \geq 2$ and (X, ϕ) and (X', ϕ') are Moore spaces of type (G, n) , then there is a homotopy equivalence $f: X \rightarrow X'$ satisfying $\phi' \circ H_n(f) = \phi$.*

Proof. (i) This follows from Lemma 13.2 (i).

(ii) We can suppose without loss of generality that X is a CW -complex as it occurs in Lemma 13.2 (i). Then we obtain from Lemma 13.2 (ii) and Hurewicz Theorem 12.5 a map $f: X \rightarrow X'$ such that the following diagram with isomorphisms as vertical maps commutes

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{\pi_n(f)} & \pi_n(X') \\ \text{hur}_n \downarrow \cong & & \cong \downarrow \text{hur}_n \\ H_n(X) & \xrightarrow{H_n(f)} & H_n(X') \\ \phi \downarrow \cong & & \cong \downarrow \phi' \\ G & \xrightarrow{\alpha} & G'. \end{array}$$

This implies that $H_i(f): H_i(X) \rightarrow H_i(X')$ is bijective for $i \geq 1$. Since X and X' are simply connected, f is a homotopy equivalence by Whitehead Theorem 12.13. $\square$

14. EILENBERG-MACLANE SPACES

Definition 14.1 (Eilenberg-MacLane space). Consider $n \geq 1$ and a group G which is assumed to be abelian if $n \geq 2$ holds. An *Eilenberg MacLane space* (X, x, ψ) of type (G, n) consists of a path connected pointed CW -complex (X, x) and a group isomorphism $\psi: \pi_n(X, x) \xrightarrow{\cong} G$ such that $\pi_i(X, x) = \{0\}$ holds for $1 \leq i$ with $i \neq n$.

Sometimes (X, x, φ) is denoted by $K(G, n)$.

Lemma 14.2. *Let (A, a) be a path connected pointed CW -complex and $n \in \mathbb{Z}^{\geq 0}$. Then there is a CW -pair (X, A) such that X is path connected and obtained from A by attaching cells of dimension $\geq (n+2)$, the inclusion $j: A \rightarrow X$ induces an isomorphism $\pi_i(A, a) \rightarrow \pi_i(X, a)$ for $1 \leq i \leq n$, and $\pi_i(X, a) = \{1\}$ holds for $i \geq (n+1)$.*

Proof. Consider the pair $(\text{cone}(A), A)$. Apply Theorem 6.3 to it for the natural number $(n+1)$. Then we obtain a CW -pair (X, A) and a map of pairs $(f, \text{id}_A) \rightarrow (\text{cone}(A), A)$ such that the pair (X, A) is $(n+1)$ -connected and the homomorphism $\pi_i(f, a): \pi_i(X, a) \rightarrow \pi_i(\text{cone}(f), a)$ is injective for $i = (n+1)$ and bijective for $i \geq (n+2)$. Since $\text{cone}(A)$ is contractible, this implies $\pi_i(X, a) = \{1\}$ for $i \geq (n+1)$. As (X, A) is $(n+1)$ -connected, the map $\pi_i(A, a) \rightarrow \pi_i(X, a)$ is bijective for $1 \leq i \leq n$. $\square$

Theorem 14.3 (Existence and uniqueness of Eilenberg-MacLane spaces). *Consider $n \in \mathbb{Z}^{\geq 1}$ and two groups G and G' which are assumed to be abelian if $n \geq 2$ holds. Then:*

- (i) *There exists an Eilenberg-MacLane space (X, x, φ) of type (G, n) such that the $X_{n-1} = X_0 = \{x\}$ holds;*
- (ii) *Let (X, x) be a pointed n -connected CW -complex and (X', x', φ') be an Eilenberg-MacLane space of type (G', n) .*

We obtain a bijection

$$[(X, x), (X', x')]^0 \xrightarrow{\cong} \text{hom}(\pi_n(X, x), G'), \quad [f] \mapsto \varphi' \circ \pi_n(f, x).$$

Moreover, the forgetful map $[(X, x), (X', x')]^0 \rightarrow [X, X']$ is bijective for $n \geq 2$;

- (iii) *Let (X, x, φ) be an Eilenberg-MacLane space of type (G, n) and (X', x', φ') be an Eilenberg-MacLane space of type (G', n) .*

We obtain a bijection

$$[(X, x), (X', x')]^0 \xrightarrow{\cong} \text{hom}(G, G'), \quad [f] \mapsto \varphi' \circ \pi_n(f, x) \circ \varphi^{-1}.$$

Moreover, the forgetful map $[(X, x), (X', x')]^0 \rightarrow [X, X']$ is bijective for $n \geq 2$;

- (iv) *Let (X, x, φ) and (X', x', φ') be two Eilenberg-MacLane spaces of type (G, n) . Then there exists a pointed homotopy equivalence $f: (X, x) \rightarrow (X', x')$ which is up to pointed homotopy equivalence uniquely determined by the property that $\varphi' \circ \pi_1(f, x) = \varphi$ holds.*

Proof. (i) This follows from Lemma 13.2 (i) and Lemma 14.2.

(ii) The forgetful map $[(X, x), (X', x')]^0 \rightarrow [X, X']$ is bijective for $n \geq 2$ because of (8.24), since X' is simply connected.

We can find by Corollary 6.5 a CW -pair (Y, y) together with a pointed homotopy equivalence $(Y, y) \rightarrow (X, x)$ such that $Y_{n-1} = Y_0 = \{y\}$ holds. Hence we can assume without generality that $X_{n-1} = X_0 = \{x\}$ holds. Then X_n looks like $\bigvee_{i \in I} S^n$ and

we have canonical isomorphisms

$$\bigoplus_{i \in I} \pi_n(S^n, s) \xrightarrow{\cong} \pi_n\left(\bigvee_{i \in I} S^n, x\right);$$

$$\mathbb{Z} \xrightarrow{\cong} \pi_n(S^n, s).$$

Consider a homomorphism $v: \pi_n(X, x) \rightarrow G'$. For every $i \in I$ choose a pointed map $u_i: (S^n, s) \rightarrow (X', x')$ such that the composite

$$\pi_n(S^n, s) \xrightarrow{j_i} \bigoplus_{i \in I} \pi_n(S^n, s) \xrightarrow{\cong} \pi_n(X_n, x) \xrightarrow{\pi_n(k)} \pi_n(X, x) \xrightarrow{v} G' \xrightarrow{\phi'^{-1}} \pi_n(X', x')$$

sends $[\text{id}_{S^n}]$ to $[u_i]$, where j_i is the inclusion of the i -th summand and $k: X_n \rightarrow X$ is the inclusion. Define the map

$$f_n = \bigvee_{i \in I} u_i: X_n = \bigvee_{i \in I} S^n \rightarrow X'.$$

It sends the basepoint x of X_n to the base points x' of X' . The map $\pi_n(f_n, x): \pi_n(X_n, x) \rightarrow \pi_n(X', x')$ agrees with the composite

$$\pi_n(X_n, x) \xrightarrow{\pi_n(k)} \pi_n(X, x) \xrightarrow{v} G' \xrightarrow{(\phi')^{-1}} \pi_n(X', x').$$

We can define inductively maps $f_j: (X_j, x) \rightarrow (X', x')$ for $j = n, (n+1), (n+2), \dots$ satisfying $f_{j+1}|_{X_j} = f_j$ for $j = n, (n+1), (n+2), \dots$, since the attaching map $q: S^n \rightarrow X_n$ of any $(n+1)$ -cell of X lies in the kernel of $\pi_n(k): \pi_n(X_n, x) \rightarrow \pi_n(X)$ and $\pi_j(X', x') = 0$ holds for $j = (n+1), (n+2), \dots$. Define the map

$$f := \text{colim}_{j \rightarrow \infty} f_j: X = \text{colim}_{j \rightarrow \infty} X_j \rightarrow X'$$

Then $f(x) = x'$ holds and v agrees with the composite $\pi_n(X, x) \xrightarrow{\pi_n(f, x)} \pi_n(X', x') \xrightarrow{\phi'} G'$. This proves surjectivity.

Injectivity is proved as follows. Consider two pointed maps $f_0, f_1: (X, x) \rightarrow (X', x')$ such that $\pi_n(f_0, x) = \pi_n(f_1, x)$ holds. We have to construct a pointed homotopy equivalence $h: (X, x) \times I \rightarrow (X', x')$ between f_0 and f_1 . We construct inductive maps $h_j: X_j \times I \cup X \times \{0, 1\} \rightarrow X'$ for $j = 0, 1, 2, \dots$ such that

$$h_0: X_0 \times I \cup X \times \{0, 1\} = \{x\} \times I \cup X \times \{0, 1\} \rightarrow X'$$

sends every element in $\{x\} \times I$ to x and is given on $X \times \{k\}$ by f_k for $k = 0, 1$ and we have for $j = 0, 1, 2, \dots$

$$h_{j+1}|_{X_j \times I \cup X \times \{0, 1\}} = h_j.$$

Since $X \times I$ is $\text{colim}_{j \rightarrow \infty} X_j \times I \cup X \times \{0, 1\}$, we can define the desired pointed homotopy h by $\text{colim}_{j \rightarrow \infty} h_j$.

It remains to construct the map h_j for $j = 0, 1, 2, \dots$. We have constructed h_0 already. Since $X_{n-1} = X_0$ holds, we have $X_{n-1} \times I \cup X \times \{0, 1\} = X_0 \times I \cup X \times \{0, 1\}$ and can define $h_j = h_0$ for $1 \leq j \leq (n-1)$. Next we construct h_n . We have $X_n = \bigvee_{i \in I} S^n$. We have to specify for each $i \in I$ a map $h_{n,i}: S_i^{n-1} \times I \rightarrow X'$ such that $h_{n,i}$ sends an element in $\{s\} \times I$ to x' and its restriction to $S_i^{n-1} \times \{k\}$ is $f_k|_{S_i^{n-1}}$ for $k = 0, 1$, where S_i^n is the i -th summand in $\bigvee_{i \in I} S^n$, since then the collection of the maps $h_{n,i}$ -s yields the desired map h_n by $h_n|_{X_0 \times I \cup X \times \{0, 1\}} = h_0$ and $h_n|_{S_i^n \times I} = h_{n,i}$. The existence of $h_{n,i}$ follows from $\pi_n(f_0, x) = \pi_n(f_1, x)$ since this implies that the pointed maps $f_0|_{S_i^{n-1}}$ and $f_1|_{S_i^{n-1}}$ from $(S_i^{n-1}, s) \rightarrow (X', x')$ are pointed homotopic. This finishes the construction of h_n .

Since $X_{i+1} \times I \cup X \times \{0, 1\}$ is obtained from $X_i \times I \cup X \times \{0, 1\}$ by attaching cells of dimension $(i+2)$ and $\pi_{i+1}(X', x')$ vanishes, we can extend h_i to h_{i+1} for $i = n, (n+1), (n+2), \dots$. This finishes the proof of assertion (ii).

(iii) This follows from assertion (ii).

(iv) This follows from assertion (iii). $\square$

Remark 14.4 (Eilenberg MacLane space of type $(G, 1)$ and unpointed homotopy classes). In Theorem 14.3 we have treated unpointed homotopy classes only for $n \geq 2$. We briefly discuss what happens in the case $n = 1$.

Consider the situation of assertion (ii) of Theorem 14.3. Then one obtains a bijection from $[X, X']$ to the set $[\Pi(X), \Pi(X')]$ of natural equivalence classes of functors from $\Pi(X)$ to $\Pi(X')$ by sending $[f]$ to $[\Pi(f)]$. In terms of fundamental groups one obtains a bijection for $\text{Inn}(G')$ the group of inner automorphisms of G'

$$[X, X'] \xrightarrow{\cong} \text{Inn}(G') \setminus \text{hom}(\pi_1(X, x), G')$$

defined as follows. For $[f]$ we can choose a representative f with $f(x) = x'$ and associate to $[f]$ the class of $\varphi' \circ \pi_1(f, x)$. These claims follow from Theorem 14.3 (ii) using the bijection (8.24).

In the situation of assertion (iii) of Theorem 14.3 we obtain a bijection

$$[X, X'] \xrightarrow{\cong} \text{Inn}(G') \setminus \text{hom}(G, G').$$

Note that $\text{Inn}(G')$ is trivial if and only if G' is abelian. So for abelian G' we get also for $n = 1$ that the forgetful map $[(X, x), (X', x')] \rightarrow [X, X']$ is bijective in assertions (ii) and (iii) of Theorem 14.3.

Consider an abelian group G and $n \in \mathbb{Z}^{\geq 1}$. Let (X, x, φ) be an Eilenberg-MacLane space of type (G, n) . Then the Hurewicz homomorphism $\text{hur}_n(X, x, \varphi): \pi_n(X, x) \xrightarrow{\cong} H_n(X)$ is bijective by Theorem 12.5. Moreover $H_i(X)$ is trivial for $1 \leq i < n$ by Proposition 12.7 (i) and $H_0(X) \cong \mathbb{Z}$. By the Universal Coefficient Theorem we obtain an isomorphism $\alpha_n: H^n(X; G) \xrightarrow{\cong} \text{hom}_{\mathbb{Z}}(H_n(X), G)$. Hence the following composite is an isomorphism

$$\begin{aligned} \beta: \text{hom}_{\mathbb{Z}}(G, G) &\xrightarrow{\text{hom}_{\mathbb{Z}}(\varphi, \text{id}_G)} \text{hom}_{\mathbb{Z}}(\pi_n(X, x), G) \\ &\xrightarrow{\text{hom}_{\mathbb{Z}}(\text{hur}_n(X, x)^{-1}, \text{id}_G)} \text{hom}_{\mathbb{Z}}(H_n(X), G) \xrightarrow{\alpha_n^{-1}} H^n(X; G). \end{aligned}$$

Let

$$(14.5) \quad \iota_n(X, x, \varphi) \in H^n(X; G)$$

be the element which is uniquely determined by $\beta_n(\text{id}_G) = \iota_n(X, x, \varphi)$. Let Y be a CW-complex. Consider the map

$$(14.6) \quad \gamma_n(Y): [Y, X] \rightarrow H^n(Y; G), \quad [f] \mapsto H^n(f; G)(\iota_n(X, x, \varphi)).$$

We will later give the proof of the following theorem.

Theorem 14.7. *Consider $n \in \mathbb{Z}^{\geq 1}$. Let G be an abelian group. Let (X, x, φ) be an Eilenberg-MacLane space of type (G, n) . Let Y be a CW-complex.*

Then the map

$$\gamma_n(Y): [Y, X] \rightarrow H^n(Y; G)$$

defined in (14.6) is bijective.

Example 14.8 (Homotopy classes of maps to S^1). We conclude from Remark 14.4 or from Theorem 14.7 that we obtain for a CW-complex Y a bijection of groups

$$[Y, S^1] \xrightarrow{\cong} H^1(Y; \mathbb{Z})$$

by sending $[f]$ to the image of a fixed generator of the infinite cyclic group $H^1(S^1; \mathbb{Z})$ under the homomorphism $H^1(f; \mathbb{Z}): H^1(S^1; \mathbb{Z}) \rightarrow H^1(Y; \mathbb{Z})$.

15. POSTNIKOV TOWERS

Let X be a connected CW -complex X . A *Postnikov tower* for X consists of a sequence of spaces $\tau_{\leq k}X$ for $k \in \mathbb{Z}^{\geq 1}$, a sequence of maps $\varphi_k: X \rightarrow \tau_{\leq k}X$ for $k \in \mathbb{Z}^{\geq 1}$, and a sequence of fibrations $p_k: \tau_{\leq k}X \rightarrow \tau_{\leq k-1}X$ for $k \in \mathbb{Z}^{\geq 2}$ with the following properties:

- $\pi_i(\tau_{\leq k}) = \{0\}$ for $i \geq k + 1$;
- The map φ_k induces isomorphisms $\pi_i(\varphi_k): \pi_i(X) \xrightarrow{\cong} \pi_i(\tau_{\leq k}X)$ for $1 \leq i \leq k$;
- We have $p_{k+1} \circ \varphi_{k+1} = \varphi_k$ for $k \in \mathbb{Z}^{\geq 1}$,
- Each space X_n has the homotopy type of a CW -complex.

The following diagram commutes

$$\begin{array}{c}
 \vdots \\
 \downarrow p_5 \\
 \tau_{\leq 4}X \\
 \downarrow p_4 \\
 \tau_{\leq 3}X \\
 \downarrow p_3 \\
 \tau_{\leq 2}X \\
 \downarrow p_2 \\
 \tau_{\leq 1}X.
 \end{array}
 \quad
 \begin{array}{c}
 \nearrow \varphi_4 \\
 \nearrow \varphi_3 \\
 \nearrow \varphi_2 \\
 \nearrow \varphi_1 \\
 \xrightarrow{\quad}
 \end{array}
 \begin{array}{c}
 X \\
 \\
 \\
 \\
 \\
 \end{array}$$

Note that $\tau_{\leq 1}X$ is necessarily a model for $K(\pi_1(X), 1)$ and each map φ_k is k -connected and has $K(\pi_k(X), k)$ as fiber.

Remark 15.1 (The construction of a Postnikov tower). The details of the construction of a Postnikov tower can be found for instance in [33, Chapter IX]. The basic idea is the following.

For $k \in \mathbb{Z}^{\geq 0}$ we can construct a CW -complex X'_k obtained from X by attaching cells of dimension $\geq (k+2)$ such that inclusion $\varphi'_k: X \rightarrow X'_k$ induces isomorphisms $\pi_i(\varphi'_k): \pi_i(X) \rightarrow \pi_i(X'_k)$ for $i = 0, 1, 2, \dots, k$ and $\pi_i(X'_k) = \{0\}$ holds for $i \geq (k+1)$, see Lemma 14.2. The inclusion $\varphi'_i: X \rightarrow X'_{i-1}$ extends to a map $p'_k: X'_k \rightarrow X'_{k-1}$ since X'_k is obtained from X by attaching cells of dimension $\geq (k+2)$ and $\pi_i(X'_{k-1})$ vanishes for $i \geq k$.

So we get a commutative diagram

$$\begin{array}{c}
 \vdots \\
 \downarrow p_5 \\
 X'_4 \\
 \downarrow p'_4 \\
 X'_3 \\
 \downarrow p'_3 \\
 X'_2 \\
 \downarrow p'_2 \\
 X'_1
 \end{array}
 \quad
 \begin{array}{c}
 \nearrow \varphi'_4 \\
 \nearrow \varphi'_3 \\
 \nearrow \varphi'_2 \\
 \nearrow \varphi'_1 \\
 \longrightarrow
 \end{array}
 \begin{array}{c}
 X \\
 \\
 \\
 \\
 \\
 \longrightarrow X
 \end{array}$$

such that

- $\pi_i(X'_k) = \{0\}$ for $i \geq k+1$;
- The map φ'_k induces isomorphisms $\pi_i(\varphi'_k): \pi_i(X) \xrightarrow{\cong} \pi_i(X'_k)$ for $1 \leq i \leq k$.

Then by turning a map into a fibration starting with p'_2 and working inductively upwards, we obtain the desired Postnikov tower.

There is a canonical map from X to the inverse limit $\text{invlim}_{k \rightarrow \infty} \{\tau_{\leq k} X, p_k\}$ which is a weak homotopy equivalence.

Recall that $p_k: \tau_{\leq k} X \rightarrow \tau_{\leq k-1} X$ has a fiber $K(\pi_k(X), k)$. Suppose that X is a simple space, i.e., the action of the fundamental group $\pi_1(X)$ on the homotopy groups $\pi_n(X)$ is trivial for $n \in \mathbb{Z}^{\geq 1}$. (Note that this implies that $\pi_1(X)$ is abelian.) Then one can actually extend p_k to a fiber sequence

$$K(\pi_k(X), k) \rightarrow \tau_{\leq k} X \xrightarrow{p_k} \tau_{\leq k-1} X \rightarrow K(\pi_k(X), k+1).$$

It determines a class

$$(15.2) \quad [p_k] \in [\tau_{\leq k} X; K(\pi_k(X), k+1)] = H^{k+1}(\tau_{\leq k-1} X; \pi_k(X)),$$

called *k-invariant of the Postnikov tower* which determines p_k up to strong fiber homotopy equivalence.

Example 15.3. The first few terms of the Postnikov tower for the sphere S^2 can be understood explicitly. The first homotopy groups of the sphere are given by

$$\pi_n(S^2) \cong \begin{cases} \{0\} & n = 0, 1; \\ \mathbb{Z} & n = 2, 3; \\ \mathbb{Z}/2 & n = 4. \end{cases}$$

Hence $\tau_{\leq 2} S^2$ is $K(\mathbb{Z}, 2)$ for which $\mathbb{CP}^\infty$ is a model. The fibration $p_3: \tau_{\leq 3} S^2 \rightarrow \tau_{\leq 2} S^2$ is classified by the 3-invariant which is an element in $H^4(\tau_{\leq 2} S^2; \pi_3(S^2)) \cong H^4(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}$. This invariant is trivial if and only if $\tau_{\leq 3} S^2 \simeq K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}, 3)$. Actually, it is known that the 3-invariant is non-trivial, see [33, Example 1 in IX.5 on page 437].

16. SPECTRA

16.1. Basics about spectra. Note that in the sequel we often omit the base points from the notation. Moreover, pointed space means always well pointed space. Recall that we are working in the category of compactly generated spaces.

Definition 16.1 (Spectrum). A *spectrum* $\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$ is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps*

$$\sigma(n): E(n) \wedge S^1 \longrightarrow E(n+1)$$

for $n \in \mathbb{Z}$. A *map of spectra* $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{E}'$ is a sequence of maps $f(n): E(n) \rightarrow E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e., we have $f(n+1) \circ \sigma(n) = \sigma'(n) \circ (f(n) \wedge \text{id}_{S^1})$ for all $n \in \mathbb{Z}$.

Maps of spectra are sometimes called functions in the literature, they should not be confused with the notion of a map of spectra in the stable category, see [1, III.2.].

Note that we are not requiring that $E(n) = \{\bullet\}$ holds for $n \leq -1$.

Example 16.2 (Suspension spectrum of a pointed space). Given a pointed space X , define its *suspension spectrum* $\Sigma^\infty X$ by $\Sigma^\infty X(n) = \{\bullet\}$ for $n \leq -1$ and $\Sigma^\infty X(n) = X \wedge S^n$ for $n \geq 0$. Note that $\Sigma^\infty X(0) = X \wedge S^0$ can be identified with X itself. Since S^{n+1} can be identified with $S^n \wedge S^1$, we can define the n -structure map to be

$$\begin{aligned} \text{id}_{(X \wedge S^n) \wedge S^1}: \Sigma^\infty X(n) \wedge S^1 &= (X \wedge S^n) \wedge S^1 \\ &\rightarrow (X \wedge S^n) \wedge S^1 = X \wedge (S^n \wedge S^1) = X \wedge S^{n+1} = \Sigma^\infty X(n+1). \end{aligned}$$

Example 16.3 (Sphere spectrum). If we take $X = S^0$ in Example 16.2, we obtain the *sphere spectrum* $\mathbf{S}$. Note that $S(n) = \{\bullet\}$ for $n \leq -1$ and $S(n) = S^n$ for $n \geq 0$ hold and that the n -th structure map comes from the identification $S^n \wedge S^1 = S^{n+1}$.

Example 16.4 (Eilenberg-MacLane spectrum). Given an abelian group G , we define the associated *Eilenberg Mac-Lane spectrum* $\mathbf{K}(G)$ as follows. We put $\mathbf{K}(G)(n) = \{\bullet\}$ for $n \leq 0$ and put $\mathbf{K}(G)(n) = K(G, n)$ for some model $K(G, n)$ of the Eilenberg MacLane space of type (G, n) for $n \geq 1$. In order to define the n -th structure map for $n \geq 0$, it suffices to specify a map $\sigma(n): K(G, n) \rightarrow \Omega K(G, (n+1))$ because of the adjunction (9.32). Recall that we have a preferred isomorphism $\partial_{n+1}(K(G, n+1)): \pi_{n+1}(K(G, n+1)) \xrightarrow{\cong} \pi_n(\Omega K(G, n+1))$, see (10.7). We conclude from Theorem 14.3 (iii) that there is a homotopy equivalence $\overline{\sigma(n)}: K(G, n) \rightarrow \Omega K(G, n+1)$ which is uniquely determined by the property that under the identifications $\pi_n(K(G, n)) = G$ and $\pi_{n+1}(K(G, n+1)) = G$ the map $\pi_n(\overline{\sigma(n)}): \pi_n(K(G, n)) \rightarrow \pi_n(\Omega K(G, (n+1)))$ and the preferred isomorphism ∂_{n+1} are inverse to one another.

Definition 16.5 (Homotopy groups of a spectrum). For $n \in \mathbb{Z}$ the n th *homotopy groups of a spectrum* $\mathbf{E}$ is defined by

$$\pi_n(\mathbf{E}) := \text{colim}_{k \rightarrow \infty} \pi_{n+k}(E(k))$$

where the k th structure map of the system $\pi_{n+k}(E(k))$ is given by the composite

$$\begin{aligned} a_{n,k}(\mathbf{E}): \pi_{n+k}(E(k)) &\xrightarrow{\sigma_{n+k}(E(k))} \pi_{n+k+1}(S^1 \wedge E(k)) \\ &\xrightarrow{\pi_{n+k+1}(\text{flip})} \pi_{n+k+1}(E(k) \wedge S^1) \\ &\xrightarrow{\pi_{n+k+1}(\sigma(k))} \pi_{n+k+1}(E(k+1)) \end{aligned}$$

of the suspension homomorphism $\sigma_{n+k}(E(k))$ of (11.15), the map induced by the flip map $\text{flip}: S^1 \wedge E(k) \xrightarrow{\cong} E(k) \wedge S^1$, and the homomorphism induced by the structure map $\sigma(k)$.

A *weak equivalence of spectra* is a map $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{F}$ of spectra inducing an isomorphism on all homotopy groups.

A spectrum $\mathbf{E}$ is called an Ω -spectrum if the adjoint $\overline{\sigma(n)}: E_n \rightarrow \Omega E_{n+1}$ of $\sigma(n)$ under the adjunction (9.32) induces for every $n \geq 1$ and every $n \in \mathbb{Z}$ a bijection $\pi_n(\overline{\sigma(n)}): \pi_n(E(n)) \rightarrow \pi_n(\Omega E(n+1))$. The Eilenberg MacLane spectrum $\mathbf{K}(G)$ is an Ω -spectrum.

Note that it may happen that $\pi_n(\mathbf{E})$ is non-trivial for some $n \in \mathbb{Z}$ with $n \leq -1$. Each of the groups $\pi_n(\mathbf{E})$ is abelian.

Note that for $k, n \in \mathbb{Z}$ with $k+n \geq 1$ there is a natural map

$$(16.6) \quad \psi_{k,n}: \pi_{k+n}(E(k)) \rightarrow \pi_n(\mathbf{E}).$$

In general this map is not bijective. It is bijective if $\mathbf{E}$ is an Ω -spectrum since in view of the adjunction (9.32) one can compute $\pi_n(\mathbf{E})$ of a spectrum as the colimit of the sequence

$$(16.7) \quad \pi_n(E(0)) \xrightarrow{\pi_n(\overline{\sigma(0)})} \pi_n(\Omega E(1)) \xrightarrow{\pi_n(\overline{\Omega\sigma(1)})} \pi_n(\Omega^2 E(2)) \xrightarrow{\pi_n(\overline{\Omega^2\sigma(1)})} \dots$$

Note that our construction of the Eilenberg-MacLane spectrum $\mathbf{K}(G)$ of Example 16.4 depends on choices. In any case we get a preferred isomorphism $\pi_0(\mathbf{K}(G)) \xrightarrow{\cong} G$ and $\pi_n(\mathbf{K}(G)) = \{0\}$ for $n \neq 0$. Moreover, for any two such constructions with different choices of $\mathbf{K}(G)$, there is a weak homotopy equivalence between the resulting spectra inducing the identity on π_0 under the identification of π_0 with G above.

Note that for a pointed space X the n -stable homotopy group $\pi_n^s(X_+)$ of Definition 11.20 agrees with $\pi_n^s(\Sigma^\infty X)$ of Definition 16.5.

Given a spectrum $\mathbf{E}$ and a pointed space X , we can define their *smash product* to be the spectrum $X \wedge \mathbf{E}$ whose n -th spaces is $(X \wedge \mathbf{E})(n) := X \wedge E(n)$ and whose n -th structure map is $\text{id}_X \wedge \sigma(n): X \wedge E(n) \rightarrow X \wedge E(n+1)$. Next introduce the associated *mapping spectrum* $\text{map}(X; \mathbf{E})^0$. Its n -th space is given by $\text{map}(X; \mathbf{E})^0(n) = \text{map}(X, E(n))^0$ for $n \in \mathbb{Z}$. Its n -th structure map for $n \in \mathbb{Z}$ is defined to be the composite

$$\text{map}(X, E(n))^0 \xrightarrow{\text{map}(\text{id}_X, \overline{\sigma(n)})^0} \text{map}(X; \Omega E(n+1))^0 \xrightarrow{i_n} \Omega \text{map}(X, E(n+1))^0$$

having the adjunction 9.32 in mind. Here i_n is the homeomorphism which assigns to the pointed map $f: X \rightarrow \Omega E(n) = \text{map}(S^1, E(n))^0$ the pointed map $g: S^1 \rightarrow \text{map}(X, E(n))$ sending $s \in S^1$ to the map $X \rightarrow E(n)$, $x \mapsto f(x)(s)$. It can be also written as the composite

$$\begin{aligned} \text{map}(X, \Omega E(n))^0 &= \text{map}(X, \text{map}(S^1, E(n))^0)^0 \xrightarrow{\cong} \text{map}(X \wedge S^1, E(n))^0 \\ &\xrightarrow{\cong} \text{map}(S^1 \wedge X, E(n))^0 \xrightarrow{\cong} \text{map}(S^1, \text{map}(X, E(n))^0)^0 = \Omega \text{map}(X, E(n))^0. \end{aligned}$$

16.2. Homology and cohomology theories for pointed spaces and pairs.

Fix a commutative ring R .

Definition 16.8 (Homology theory for pointed spaces). A *homology theory for pointed spaces with values in R -modules* $\tilde{\mathcal{H}}_* = (\tilde{\mathcal{H}}_*, s_*)$ consists of a covariant functor $\tilde{\mathcal{H}}_*$ from the category Top^0 of pointed spaces to the category of $\mathbb{Z}$ -graded R -modules together with a natural transformation $s_*: \tilde{\mathcal{H}}_*(-) \rightarrow \tilde{\mathcal{H}}_{*+1}(S^1 \wedge -)$ such that the following conditions are satisfied:

- *Pointed homotopy invariance*

Let f and g be pointed maps $(X, x) \rightarrow (Y, y)$ which are pointed homotopic. Then for every $n \in \mathbb{Z}$ the R -homomorphisms $\tilde{\mathcal{H}}_n(f)$ and $\tilde{\mathcal{H}}_n(g)$ from $\tilde{\mathcal{H}}_n(X, x)$ to $\tilde{\mathcal{H}}_n(Y, y)$ agree:

- *Exactness*

Let $f: (X, x) \rightarrow (Y, y)$ be a pointed map. Let $i: Y \rightarrow \text{cone}(f, x)$ be the inclusion into the pointed mapping cone of f . Then for every $n \in \mathbb{Z}$ the sequence of R -modules

$$\tilde{\mathcal{H}}_n(X, x) \xrightarrow{\tilde{\mathcal{H}}_n(f)} \tilde{\mathcal{H}}_n(Y, y) \xrightarrow{\tilde{\mathcal{H}}_n(i)} \tilde{\mathcal{H}}_n(\text{cone}(f), *)$$

is exact;

- *Suspension isomorphism*

For every pointed space (X, x) and every $n \in \mathbb{Z}$ the map

$$s_n(X, x): \tilde{\mathcal{H}}_n(X, x) \xrightarrow{\cong} \tilde{\mathcal{H}}_{n+1}(S^1 \wedge X, *)$$

is bijective.

We say that $\tilde{\mathcal{H}}_*$ satisfies the *one point union axiom* if for any collection of pointed spaces $\{(X_i, x_i) \mid i \in I\}$ and every $n \in \mathbb{Z}$ the map

$$\bigoplus_{i \in I} \tilde{\mathcal{H}}_n(j_i): \bigoplus_{i \in I} \tilde{\mathcal{H}}_n(X_i, x_i) \xrightarrow{\cong} \tilde{\mathcal{H}}_n\left(\bigvee_{i \in I} (X_i, x_i)\right)$$

is bijective for $j_i: (X_i, x_i) \rightarrow \bigvee_{i \in I} (X_i, x_i)$ the inclusion of the i -th summand.

We say that $\tilde{\mathcal{H}}_*$ satisfies the *dimension axiom* if we have $\tilde{\mathcal{H}}_n(S^0, s) = 0$ for $n \neq 0$.

Definition 16.9 (Cohomology theory for pointed spaces). A *cohomology theory for pointed spaces with values in R -modules* $\tilde{\mathcal{H}}^* = (\tilde{\mathcal{H}}^*, s^*)$ consists of a contravariant functor $\tilde{\mathcal{H}}^*$ from the category $\mathbf{Top}^0$ of pointed spaces to the category of $\mathbb{Z}$ -graded R -modules together with a natural transformation $s^*: \tilde{\mathcal{H}}^*(-) \rightarrow \tilde{\mathcal{H}}^{*+1}(S^1 \wedge -)$ such that the following conditions are satisfied:

- *Pointed homotopy invariance*

Let f and g be pointed maps $(X, x) \rightarrow (Y, y)$ which are pointed homotopic. Then for every $n \in \mathbb{Z}$ the R -homomorphisms $\tilde{\mathcal{H}}^n(f)$ and $\tilde{\mathcal{H}}^n(g)$ from $\tilde{\mathcal{H}}^n(Y, y)$ to $\tilde{\mathcal{H}}^n(X, x)$ agree;

- *Exactness*

Let $f: (X, x) \rightarrow (Y, y)$ be a pointed map. Let $i: Y \rightarrow \text{cone}(f, x)$ be the inclusion into the pointed mapping cone of f . Then for every $n \in \mathbb{Z}$ the sequence of R -modules

$$\tilde{\mathcal{H}}^n(\text{cone}(f, x), *) \xrightarrow{\tilde{\mathcal{H}}^n(i)} \tilde{\mathcal{H}}^n(Y, y) \xrightarrow{\tilde{\mathcal{H}}^n(f)} \tilde{\mathcal{H}}^n(X, x)$$

is exact;

- *Suspension isomorphism*

For every pointed space (X, x) and every $n \in \mathbb{Z}$ the map

$$s^n(X, x): \tilde{\mathcal{H}}^n(X, x) \xrightarrow{\cong} \tilde{\mathcal{H}}^{n+1}(S^1 \wedge X)$$

is bijective.

We say that $\tilde{\mathcal{H}}^*$ satisfies the *one point union axiom* if for any collection of pointed spaces $\{(X_i, x_i) \mid i \in I\}$ and every $n \in \mathbb{Z}$ the map

$$\prod_{i \in I} \tilde{\mathcal{H}}^n(j_i): \tilde{\mathcal{H}}^n\left(\bigvee_{i \in I} (X_i, x_i)\right) \rightarrow \prod_{i \in I} \tilde{\mathcal{H}}^n(X_i, x_i)$$

is bijective for $j_i: (X_i, x_i) \rightarrow \bigvee_{i \in I} (X_i, x_i)$ the inclusion of the i -th summand.

We say that $\tilde{\mathcal{H}}^*$ satisfies the *dimension axiom* if we have $\tilde{\mathcal{H}}^n(S^0, s) = 0$ for $n \neq 0$.

A cohomology theory for pointed spaces is to be understood to be cohomology theory for pointed spaces with values in $\mathbb{Z}$ -modules, and analogously for pairs and homology theories.

Remark 16.10 (Correspondence between (co-)homology theories for pointed space and pairs). There is a one-to-one correspondence between homology theories for pointed spaces with values in R -modules and homology theories for pairs with values in R -modules. Let $\tilde{\mathcal{H}}_*$ be a homology theory for pointed spaces with values in R -modules. Then we can define a homology theory $\mathcal{H}_*$ for pairs with values in R -modules as follows. For a pair (X, A) define

$$\mathcal{H}_n(X, A) := \tilde{\mathcal{H}}_n(X_+ \cup_{A_+} \text{cone}(A_+, *)),$$

where $X_+ = X \amalg \{*\}$ is the pointed space obtained from X by adjoining an extra base point. If A is empty, we get $\mathcal{H}_n(X) = \tilde{\mathcal{H}}_n(X_+, *)$. Recall that we also have to specify for a pair (X, A) a boundary operator $\partial_n(X, A): \mathcal{H}_n(X, A) \rightarrow \mathcal{H}_{n-1}(A)$ for $n \in \mathbb{Z}$. It is defined by the composite

$$\begin{aligned} \partial_n(X, A): \mathcal{H}_n(X, A) = \tilde{\mathcal{H}}_n(X_+ \cup_{X_+} \text{cone}(A_+, *), *) &\xrightarrow{\mathcal{H}_n(\text{pr})} \mathcal{H}_n(S^1 \wedge A_+, *) \\ &\xrightarrow{s_{n-1}(A_+, *)^{-1}} \tilde{\mathcal{H}}_{n-1}(A_+, *) = \mathcal{H}_{n-1}(A) \end{aligned}$$

for the projection $\text{pr}: X_+ \cup_{X_+} \text{cone}(A_+, *) \rightarrow S^1 \wedge A_+$. We leave the elementary proof that $(\mathcal{H}_*, \partial_*)$ is a homology theory to the reader.

Given a homology theory for pairs with values in R -modules $(\mathcal{H}_*, \partial_*)$, we can define a homology theory for pointed spaces with values in R -modules $\tilde{\mathcal{H}}_*$ by $\tilde{\mathcal{H}}_*(X, x) = \mathcal{H}_*(X, \{x\})$. We leave the construction of the natural transformation $s_*(X, x): \tilde{\mathcal{H}}_n(X, x) \rightarrow \tilde{\mathcal{H}}_{n+1}(S^1 \wedge X, *)$ and the proof that $(\tilde{\mathcal{H}}_*, s_*)$ is a homology theory for pointed spaces with values in R -modules to the reader.

The analogous statements and construction yields a one-to-one-correspondence between cohomology theories for pointed spaces with values in R -modules and cohomology theories for pairs with values in R -modules.

More details can be found for instance in [31, Section 7.6 on page 176-177].

Proposition 16.11. *Let X be a (compactly generated) topological Hausdorff space with a sequence of closed subspaces $X_0 \subset X_1 \subseteq X_2 \subseteq \dots \subseteq X$ such that X is the union of the X_i -s and carries the colimit topology. Then:*

- (i) *Suppose that the homology theory with values in R -modules $\mathcal{H}_*$ satisfies the disjoint union axiom for countable index sets.*

Then there is for every $n \in \mathbb{Z}$ a natural R -isomorphism

$$\text{colim}_{k \rightarrow \infty} \mathcal{H}_n(X_k) \xrightarrow{\cong} \mathcal{H}_n(X);$$

- (ii) *Suppose that the cohomology theory $\mathcal{H}^*$ with values in R -modules satisfies the disjoint union axiom for countable index sets. Then there is for every $n \in \mathbb{Z}$ a natural short exact sequence*

$$0 \rightarrow \text{invlim}_{k \rightarrow \infty}^1 \mathcal{H}^{n-1}(X_k) \rightarrow \mathcal{H}^n(X) \rightarrow \text{invlim}_{k \rightarrow \infty} \mathcal{H}_n(X_k) \rightarrow 0.$$

Proof. The proof can be found in [27, Proposition 7.53 on page 121 and Proposition 7.66 on page 127] in the special case that X is a CW -complex and X_k is its k -skeleton. The proof carries directly over to our more general setting. $\square$

Proposition 16.12.

- (i) *Let $\mathbf{t}_*: \mathcal{H}_* \rightarrow \mathcal{K}_*$ be a transformation of homology theories with values in R -modules satisfying the disjoint union axiom. Suppose that the homomorphism $\mathbf{t}_n(\{\bullet\}): \mathcal{H}_n(\{\bullet\}) \rightarrow \mathcal{K}_n(\{\bullet\})$ is bijective for all $n \in \mathbb{Z}$.*

Then $\mathbf{t}_n(X, A): \mathcal{H}_n(X, A) \rightarrow \mathcal{K}_n(X, A)$ is bijective for every CW -pair (X, A) and $n \in \mathbb{Z}$;

(ii) Let $\mathbf{t}^*: \mathcal{H}^* \rightarrow \mathcal{K}^*$ be a transformation of cohomology theories with values in R -modules satisfying the disjoint union axiom. Suppose that the homomorphism $\mathbf{t}^n(\{\bullet\}): \mathcal{H}^n(\{\bullet\}) \rightarrow \mathcal{K}^n(\{\bullet\})$ is bijective for all $n \in \mathbb{Z}$.

Then $\mathbf{t}^n(X, A): \mathcal{H}^n(X, A) \rightarrow \mathcal{K}^n(X, A)$ is bijective for every CW -pair (X, A) and $n \in \mathbb{Z}$.

Proof. By the long exact sequence of a pair and the Five-Lemma one can reduce the claim to the case $A = \emptyset$. The claim follows for zero-dimensional CW -complexes X from the disjoint union axiom. Inductively over the dimension one proves the claim for finite-dimensional CW -complexes, where in the induction step the Mayer-Vietoris sequence and homotopy invariance comes in. Using Proposition 16.11 one obtains the general case from the finite-dimensional case. $\square$

16.3. The homology and cohomology theory assigned to a spectrum.

Lemma 16.13. For a spectrum $\mathbf{E}$ and $n \in \mathbb{Z}$ there are equivalences

$$(16.14) \quad \pi_n(\mathbf{E}) \simeq \pi_{n+1}(S^1 \wedge \mathbf{E}) \quad \text{and} \quad \pi_n(\mathbf{E}) \simeq \pi_{n-1}(\text{map}(S^1, \mathbf{E})^0)$$

which are natural in $\mathbf{E}$.

Proof. We claim that the maps

$$(16.15) \quad (-1)^k \sigma_{n+k}(E(k)): \pi_{n+k}(E(k)) \rightarrow \pi_{n+k+1}(S^1 \wedge E(k))$$

assemble to an equivalence

$$(16.16) \quad \begin{aligned} \pi_n(\mathbf{E}) &= \text{colim}_{k \rightarrow \infty} \pi_{n+k}(E(k)) \\ &\rightarrow \text{colim}_{k \rightarrow \infty} \pi_{n+k+1}(S^1 \wedge E(k)) = \pi_{n+1}(S^1 \wedge \mathbf{E}). \end{aligned}$$

The following commutative diagram shows that the maps $(-1)^k \sigma_{n+k}(E(k))$ commute with the structure maps of the respective colimits and induce the map (16.16):

$$(16.17) \quad \begin{array}{ccc} \pi_{n+k}(E(k)) & \xrightarrow{-\sigma_{n+k}(E(k))} & \pi_{n+k+1}(S^1 \wedge E(k)) \\ \sigma_{n+k}(E(k)) \downarrow & \nearrow -\text{id} & \downarrow \sigma_{n+k+1}(S^1 \wedge E(k)) \\ \pi_{n+k+1}(S^1 \wedge E(k)) & & \pi_{n+k+2}(S^1 \wedge (S^1 \wedge E(k))) \\ \pi_{n+k+1}(\text{flip}) \downarrow & & \downarrow \pi_{n+k+2}(\text{flip}) \\ \pi_{n+k+1}(E(k) \wedge S^1) & & \pi_{n+k+2}((S^1 \wedge E(k)) \wedge S^1) \\ \pi_{n+k+1}(\sigma(k)) \downarrow & & \downarrow \pi_{n+k+2}(\sigma(k)) \\ \pi_{n+k+1}(E(k+1)) & \xrightarrow{\sigma_{n+k+1}(E(k+1))} & \pi_{n+k+2}((S^1 \wedge E(k)) \wedge S^1). \end{array}$$

To prove commutativity of diagram 16.17, one starts with an element $[f] \in \pi_{n+k}(E(k))$ represented by a pointed map $f: S^{n+k} \rightarrow E(k)$. Its image under the composite of the left vertical arrows is $[g] \in \pi_{n+k+2}(S^1 \wedge E(k+1))$ for the pointed map

$$\begin{aligned} g: S^1 \wedge S^1 \wedge S^{n+k} &\xrightarrow{\text{id}_{S^1} \wedge \text{id}_{S^1} \wedge f} S^1 \wedge S^1 \wedge E(k) \\ &\xrightarrow{\text{id}_{S^1} \wedge \text{flip}} S^1 \wedge (E(k) \wedge S^1) \xrightarrow{\text{id}_{S^1} \wedge \sigma(k)} S^1 \wedge E(k+1). \end{aligned}$$

However, the image of $[f] \in \pi_{n+k}(E(k))$ under the composite of the right vertical arrows is $[g] \in \pi_{n+k+2}(S^1 \wedge E(k+1))$ is $g \circ (\text{flip} \wedge \text{id}_{S^{n+k}})$ (and not $[g]$). Since the homomorphism $\text{flip} \wedge \text{id}_{S^{n+k}}: S^1 \wedge S^1 \wedge S^{n+k} \rightarrow S^1 \wedge S^1 \wedge S^{n+k}$ has degree -1 , Theorem 3.4 implies that $[g \circ (\text{flip} \wedge \text{id}_{S^{n+k}})] = -[g]$ holds.

It remains to show that the map (16.16) is an equivalence. Recall the general fact that for a directed system of abelian groups $A_0 \xrightarrow{\phi_0} A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \dots$ every element in the colimit can be written as $\psi_m(a_m)$ for some $m \in \mathbb{Z}^{\geq 0}$ and some

$a_m \in A_m$ for the structure map $\psi_m: A_m \rightarrow \operatorname{colim}_{k \rightarrow \infty} A_k$ and the element $\psi_m(a_m)$ is zero in the colimit if and only if there exists $n \in \mathbb{Z}^{\geq 0}$ with $n \geq m$ such that the composite $\phi_{n-1} \circ \phi_{n-2} \circ \cdots \circ \phi_m: A_m \rightarrow A_n$ sends a_m to zero. Now note that the upper left triangle in diagram 16.17 with the dashed map commutes, which implies injectivity of (16.16). Surjectivity is more difficult. We only show that the twofold suspension $\pi_n(\mathbf{E}) \rightarrow \pi_{n+1}(S^1 \wedge \mathbf{E}) \rightarrow \pi_{n+2}(S^1 \wedge S^1 \wedge \mathbf{E})$ is surjective. As both maps are injective this proves that they are equivalences. Consider $f: S^{n+k} \rightarrow S^2 \wedge E(k)$. We will show that $[f] \in \pi_{n+2}(S^1 \wedge S^1 \wedge \mathbf{E})$ is the image of $[g] \in \pi_n(\mathbf{E})$ for the map

$$(16.18) \quad g: S^{n+k+2} \xrightarrow{f} S^2 \wedge E(k) \xrightarrow{\text{flip}} E(k) \wedge S^2 \xrightarrow{\sigma^2} E(k+2).$$

Consider the diagram

$$(16.19) \quad \begin{array}{ccc} S^2 \wedge S^{n+k+2} & \xrightarrow{\text{flip}} & S^{n+k+2} \wedge S^2 \\ \text{id} \wedge f \downarrow & & \downarrow f \wedge \text{id} \\ S^2 \wedge (S^2 \wedge E(k)) & \xrightarrow{\text{flip}} & (S^2 \wedge E(k)) \wedge S^2 \\ \text{id} \wedge \text{flip} \downarrow & \swarrow \alpha & \downarrow \text{id} \wedge \sigma^2 \\ S^2 \wedge E(k) \wedge S^2 & \xrightarrow{\text{id} \wedge \sigma^2} & S^2 \wedge E(k+2). \end{array}$$

The map $\alpha: S^2 \wedge E(k) \wedge S^2 \rightarrow S^2 \wedge E(k) \wedge S^2$ is the map swapping the first and last factor. As this swap map of $S^2 \wedge S^2$ has degree 1, we see that $\alpha \simeq \text{id}$. Similarly, the flip map $S^2 \wedge S^{n+k+2} \rightarrow S^{n+k+2} \wedge S^2$ is homotopic to the identity. This is the reason why we have to suspend twice! Now the upper square and middle triangle in diagram 16.19 obviously commute. The bottom right triangle commutes up to homotopy using $\text{id} \simeq \alpha$. The right vertical composite represents $[f] \in \pi_{n+2}(S^2 \wedge E)$. The bottom left composite is the image of $[g]$ under $\pi_n(\mathbf{E}) \rightarrow \pi_{n+2}(S^1 \wedge S^1 \wedge \mathbf{E})$.

The proof of the equivalence $\pi_n(\mathbf{E}) \simeq \pi_{n-1}(\operatorname{map}(S^1, \mathbf{E})^0)$ is easy using the equivalence $\pi_n(\Omega X) \simeq \pi_{n+1}(X)$ for a pointed space X . $\square$

Definition 16.20. Let $\mathbf{E}$ be a spectrum and X be a pointed space. Define the *reduced $\mathbf{E}$ -(co)homology of X* by

$$\tilde{H}_n(X; \mathbf{E}) = \pi_n(X \wedge \mathbf{E}) \quad \text{and} \quad \tilde{H}^n(X; \mathbf{E}) = \pi_{-n}(\operatorname{map}(X, \mathbf{E})^0).$$

Theorem 16.21 (The homology and cohomology theory assigned to a spectrum).

Let $\mathbf{E}$ be a spectrum. Then:

- (i) The reduced $\mathbf{E}$ -homology $\tilde{H}_*(-, \mathbf{E})$ is a homology theory of pointed spaces with values in $\mathbb{Z}$ -modules. Its associated homology theory on pairs of spaces satisfies the disjoint union axiom. For every $n \in \mathbb{Z}$ there is an isomorphism

$$\alpha_n(\mathbf{E}): H_n(\{\bullet\}; \mathbf{E}) \xrightarrow{\cong} \pi_n(\mathbf{E});$$

- (ii) The reduced $\mathbf{E}$ -cohomology $\tilde{H}^*(-, \mathbf{E})$ is a cohomology theory of pointed spaces with values in $\mathbb{Z}$ -modules. Its associated cohomology theory on pairs of spaces satisfies

$$\alpha^n(\mathbf{E}): H^n(\{\bullet\}; \mathbf{E}) \xrightarrow{\cong} \pi_{-n}(\mathbf{E}).$$

If $\mathbf{E}$ is an Ω -spectrum and we consider as input only pointed CW-complexes, then $H^*(-; \mathbf{E})$ satisfies the disjoint union axiom.

Proof. (i). Because of Remark 16.10 it suffices to construct a homology theory for pointed spaces $\tilde{\mathcal{H}}_*$.

We begin by showing how $\tilde{H}_n(X; \mathbf{E}) = \pi_n(X \wedge \mathbf{E})$ from definition 16.20 is a homology theory for pointed spaces. Is obviously a functor from pointed spaces to $\mathbb{Z}$ -graded abelian groups. Furthermore, it sends pointed homotopic maps to the same map on $\tilde{H}_n(-; \mathbf{E})$. As suspension isomorphism we use the isomorphism

$$s_n(x): \tilde{H}_n(X; \mathbf{E}) = \pi_n(X \wedge \mathbf{E}) \simeq \pi_{n+1}(S^1 \wedge X \wedge \mathbf{E}) = \tilde{H}_{n+1}(X; \mathbf{E})$$

from lemma 16.13 for the spectrum $X \wedge \mathbf{E}$, which is natural in X .

Next we prove exactness. Consider a pointed map $f: X \rightarrow Y$. We have to prove the exactness of the sequence

$$\pi_n(X \wedge \mathbf{E}) \xrightarrow{\pi_n(f \wedge \text{id}_{\mathbf{E}})} \pi_n(Y \wedge \mathbf{E}) \xrightarrow{\pi_n(i \wedge \text{id}_{\mathbf{E}})} \pi_n(\text{cone}(f) \wedge \mathbf{E})$$

for $i: Y \rightarrow \text{cone}(f)$ the inclusion into the pointed mapping cone of f . Since the composite $i \circ f$ is pointed nullhomotopic, we get $\text{im}(\pi_n(f \wedge \text{id}_{\mathbf{E}})) \subseteq \ker(\pi_n(i \wedge \text{id}_{\mathbf{E}}))$. It remains to show $\ker(\pi_n(i \wedge \text{id}_{\mathbf{E}})) \subseteq \text{im}(\pi_n(f \wedge \text{id}_{\mathbf{E}}))$.

Consider an element $z \in \ker(\pi_n(i \wedge \text{id}_{\mathbf{E}}))$. Then we can find $k \in \mathbb{Z}^{\geq 0}$ and a pointed map $g: S^{n+k} \rightarrow Y \wedge E(k)$ such that $[g] \in \pi_{n+k}(Y \wedge E(k))$ represents $z \in \ker(\pi_n(i \wedge \text{id}_{\mathbf{E}}))$ and $\pi_{n+k}(i \wedge \text{id}_{E(k)}): \pi_{n+k}(Y \wedge E(k)) \rightarrow \pi_{n+k}(\text{cone}(f) \wedge E(n))$ sends $[g]$ to zero. Let $h: \text{cone}(f) \wedge E(n) \wedge I \rightarrow \text{cone}(f) \wedge E(n)$ be a pointed homotopy with $h_0 = i \circ g$ and h_1 the constant map. Next we construct the following diagram

$$(16.22) \quad \begin{array}{ccccc} S^{n+k} & \xrightarrow{g} & Y \wedge E(n) & & \\ \downarrow j & & \downarrow i \wedge \text{id}_{E(n)} & \searrow i \wedge \text{id}_{E(n)} & \\ \text{cone}(\text{id}_{S^{n+k}}) & \xrightarrow{H} & \text{cone}(f \wedge \text{id}_{E(n)}) & \xrightarrow[\cong]{\phi} & \text{cone}(f) \wedge E(n) \\ \downarrow p(\text{id}_{S^n}) & & \downarrow p(f \wedge \text{id}_{E(n)}) & & \\ S^{n+k} \wedge S^1 & \xrightarrow{\beta} & X \wedge E(n) \wedge S^1 & \xrightarrow{\text{id}_X \wedge \sigma(n)} & X \wedge E(n+1) \\ \downarrow \text{id}_{S^{n+k} \wedge S^1} & & \downarrow f \wedge \text{id}_{E(n)} \wedge \text{id}_{S^1} & & \downarrow f \wedge \text{id}_{E(n+1)} \\ S^{n+k} \wedge S^1 & \xrightarrow{g \wedge \text{id}_{S^1}} & Y \wedge E(n) \wedge S^1 & \xrightarrow{\text{id}_Y \wedge \sigma(n)} & Y \wedge E(n+1) \end{array}$$

The left column is part of the cofibration sequence of the pointed map id_{S^n} , whereas the middle column is part of the cofibration sequence of the pointed map $f \wedge \text{id}_{E(n)}$, see Theorem 8.40. The map H is given by the map g and the homotopy h and makes the uppermost left square commutative. The map β is the map uniquely determined by the property that the left middle square commutes. The map ϕ is the canonical homeomorphism and makes the corresponding triangle commutative. The lowermost right square commutes. The left lowermost square does not commute but it does commute up to pointed homotopy. The elementary verification of this fact is left to the reader or can be extracted from [27, Lemma 8.31 on page 143].

Now the composite $(\text{id}_Y \wedge \sigma(n)) \circ (g \wedge \text{id}_{S^1}): S^{n+k} \wedge S^1 \rightarrow Y \wedge E(n+1)$ is another representative of $z \in \ker(\pi_n(i \wedge \text{id}_{\mathbf{E}}))$. We conclude from the diagram 16.22 that the composite $(\text{id}_X \wedge \sigma(n)) \circ \beta: S^{n+k} \wedge S^1 \rightarrow X \wedge E(n+1)$ represents an element in $\pi_n(X \wedge \mathbf{E})$ which sent by $\pi_n(f \wedge \text{id}_{\mathbf{E}})$ to z . This finishes the proof of exactness and hence of the assertion that $(\tilde{\mathcal{H}}_*(-; \mathbf{E}), s_*)$ defines homology theory for pointed spaces in the sense of Definition 16.9.

It remains to check that $\tilde{\mathcal{H}}_*(-; \mathbf{E})$ satisfies the one point union axiom, i.e., that the map

$$(16.23) \quad \bigoplus_{i \in I} \pi_n(X_i \wedge E) \rightarrow \pi_n\left(\bigvee_{i \in I} X_i \wedge E\right)$$

is an equivalence. Recall from exercise 47 on sheet 12 that for two spectra $\mathbf{E}_1$ and $\mathbf{E}_2$ the canonical map $\mathbf{E}_1 \vee \mathbf{E}_2 \rightarrow \mathbf{E}_1 \times \mathbf{E}_2$ is a weak homotopy equivalence. This shows that the map (16.23) is an equivalence if I is finite. For general I , use the equivalences

$$(16.24) \quad \pi_n(\bigvee_{i \in I} X_i \wedge E) \simeq \operatorname{colim}_{F \subseteq I \text{ finite}} \pi_n(\bigvee_{i \in F} X_i \wedge E) \quad \text{and}$$

$$(16.25) \quad \bigoplus_{i \in I} \pi_n(X_i \wedge E) \simeq \operatorname{colim}_{F \subseteq I \text{ finite}} \bigoplus_{i \in F} \pi_n(X_i \wedge E).$$

The first equivalence follows from the following argument: Consider a compact subset $C \subseteq \bigvee_{i \in I} (X_i, x_i)$. We want to show that there is a finite subset $J \subseteq I$ with $C \subseteq J$. Suppose that this is not the case. Then we can find a sequence of elements $j(1), j(2), j(3), \dots$ of pairwise distinct elements in I and a sequence of pairwise distinct points $s_1, s_2, s_3, \dots$ in C satisfying $s_i \in C \cap X_{j(i)} \setminus \{x_{j(i)}\}$. Consider the set $S = \{s_1, s_2, s_3, \dots\}$. Let $T \subseteq S$ be any subset of S . Then $T \cap X_i$ is either empty or consists of one point for $i \in I$. Since each X_i is Hausdorff, $T \cap X_i$ is closed in X_i for every $i \in I$. This implies that T is a closed subset of $\bigvee_{i \in I} X_i$. Hence S is a discrete subset of $\bigvee_{i \in I} X_i$ and contained in a compact subset C of X . This implies that S is finite, a contradiction.

The proof that the reduced $\mathbf{E}$ -cohomology

$$\tilde{\mathcal{H}}^n(X; \mathbf{E}) = \pi_{-n}(\operatorname{map}(X, \mathbf{E}))^0$$

is a cohomology theory for pointed spaces is analogous to the one for homology except that some care is necessary for the disjoint union axiom. The additional difficulty is that we have a homeomorphism

$$\operatorname{map}(\bigvee_{i \in I} X_i; \mathbf{E}(n))^0 = \prod_{i \in I} \operatorname{map}(X_i, \mathbf{E}(n))^0$$

and hence we get for k, n an isomorphism

$$\pi_{n+k}(\operatorname{map}(\bigvee_{i \in I} X_i; \mathbf{E}(n))^0) \xrightarrow{\cong} \prod_{i \in I} \pi_{n+k}(\operatorname{map}(X_i; \mathbf{E}(n))^0)$$

but colimits and products do not commute. Therefore we need the assumption that $\mathbf{E}$ is an Ω -spectrum, Namely, with this assumption, structure map of (16.6)

$$\psi_{n,k}: \pi_{n+k}(\operatorname{map}(Y, \mathbf{E}(k))^0) \rightarrow \pi_n(\operatorname{map}(Y, \mathbf{E}))^0$$

is an isomorphism for every pointed space (Y, y) and we do not have to take the colimit, since $\operatorname{map}(X; \mathbf{E})^0$ is an Ω -spectrum by Theorem 16.6 and Theorem 10.1 applied to the fibration $\operatorname{map}(X, E(n))^0 \rightarrow \operatorname{map}(X, E(n)) \rightarrow E(n)$, as $\mathbf{E}$ is an Ω -spectrum and X is a CW -complex. $\square$

Example 16.26 (Sphere spectrum and stable homotopy). Let $\mathbf{S}$ be the sphere spectrum of Example 16.3. Then the associated homology theory $\mathcal{H}_*(-; \mathbf{S})$ agrees with the stable homotopy theory $\pi_*^s(-)$ introduced in Definition 11.20 and Theorem 11.24 follows from Theorem 16.21 (i).

Example 16.27 (The Eilenberg-MacLane spectrum and singular homology). We have introduced for an abelian group G the Eilenberg-MacLane spectrum $\mathbf{K}(G)$ in Definition 16.4. Theorem 16.21 (ii) we obtain a cohomology theory which satisfies the disjoint union axiom and the dimension axiom and $\mathcal{H}^0(\{\bullet\}) \cong G$. Singular cohomology $H^*(-; G)$ with coefficients in G is also a cohomology theory which satisfies the disjoint union axiom and the dimension axiom and $H^0(\{\bullet\}; G) \cong G$. We obtain from the maps $\gamma^n(Y): [Y, K(G, n)] \rightarrow H^n(Y; G)$ of (14.6) a natural transformation of cohomology theories $\gamma^*: \mathcal{H}^*(-; \mathbf{K}(G)) \rightarrow H^*(-; G)$ which induces

an isomorphism $\gamma^0(\{\bullet\}): \mathcal{H}^*(\{\bullet\}; \mathbf{K}(G)) \rightarrow H^*(\{\bullet\}; G)$. Proposition 16.12 (ii) implies that we get a natural equivalence of cohomology theories

$$\gamma^*: \mathcal{H}^*(-; \mathbf{K}(G)) \xrightarrow{\cong} H^*(-; G).$$

In particular we see that for every $n \in \mathbb{Z}$ the map $\gamma^n(X): [X, K(G, n)] \rightarrow H^n(Y; G)$ of (14.6) is bijective for every CW -complex X , as predicted in Theorem 14.7.

We mention without proof that $\mathcal{H}_*(-; \mathbf{K}(A))$ can be identified with singular homology $H_*(-; A)$ with coefficients in A .

Example 16.28 (Hopf's Theorem revisited). Let M be a closed smooth manifold of dimension d . Let $K(\mathbb{Z}, d)$ be a model for the Eilenberg-MacLane space of type $(\mathbb{Z}, d)$. Choose a map $f: S^d \rightarrow K(\mathbb{Z}, d)$ inducing an isomorphism $\pi_d(f): \pi_d(S^d) \xrightarrow{\cong} \pi_d(K(\mathbb{Z}, d))$. Since f is $(d+1)$ -connected and any smooth d -dimensional manifold carries a d -dimensional CW -structure, we obtain a bijection $[M, S^d] \xrightarrow{\cong} [M, K(\mathbb{Z}, d)]$ by sending $[g]$ to $[f \circ g]$ from the Whitehead Theorem 5.1 (i). Composing it with the bijection $\gamma^n(Y): [Y, K(G, n)] \xrightarrow{\cong} H^n(Y; G)$ of (14.6) yields a bijection

$$\nu: [M, S^d] \xrightarrow{\cong} H^d(M), \quad [f] \mapsto H^d(f)([S^d])$$

for the fundamental class $[S^d] \in H^d(S^d)$.

Suppose that M is oriented. If we compose ν with the bijective homomorphism $H^d(M) \xrightarrow{\cong} \mathbb{Z}$ sending u to $\langle u, [M] \rangle$ for the fundamental class $[M] \in H_d(M)$, then we obtain a bijection

$$[M, S^d] \xrightarrow{\cong} \mathbb{Z}, \quad [f] \mapsto \deg(f).$$

Thus we rediscover Hopf's Degree Theorem 3.1.

Suppose that M is not orientable. We mention without giving the proof that $H^d(M; \mathbb{Z})$ is isomorphic to $\mathbb{Z}/2$ and $\nu: [M, S^d] \xrightarrow{\cong} \mathbb{Z}/2$ sends $f: M \rightarrow S^d$ to zero, if $H_d(f, \mathbb{Z}/2): H_d(M, \mathbb{Z}/2) \rightarrow H_d(S^d, \mathbb{Z}/2)$ is trivial, and to the generator if $H_d(f, \mathbb{Z}/2): H_d(M, \mathbb{Z}/2) \rightarrow H_d(S^d, \mathbb{Z}/2)$ is bijective.

16.4. Brown's Representation Theorem. Let $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{F}$ be a map of spectra. It induces in the obvious way a natural transformation of homology theories with values in $\mathbb{Z}$ -modules

$$(16.29) \quad t_*^{\mathbf{f}}: \mathcal{H}_*(-; \mathbf{E}) \rightarrow \mathcal{H}_*(-; \mathbf{F})$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{H}_n(\{\bullet\}; \mathbf{E}) & \xrightarrow{t_n^{\mathbf{f}}(\{\bullet\})} & \mathcal{H}_n(\{\bullet\}; \mathbf{F}) \\ \cong \downarrow & & \downarrow \cong \\ \pi_n(\mathbf{E}) & \xrightarrow{\pi_n(\mathbf{f})} & \pi_n(\mathbf{F}). \end{array}$$

It also induces in the obvious way a natural transformation of cohomology theories with values in $\mathbb{Z}$ -modules

$$(16.30) \quad t_{\mathbf{f}}^*: \mathcal{H}^*(-; \mathbf{E}) \rightarrow \mathcal{H}^*(-; \mathbf{F})$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{H}^n(\{\bullet\}; \mathbf{E}) & \xrightarrow{t_{\mathbf{f}}^n(\{\bullet\})} & \mathcal{H}^n(\{\bullet\}; \mathbf{F}) \\ \cong \downarrow & & \downarrow \cong \\ \pi_{-n}(\mathbf{E}) & \xrightarrow{\pi_{-n}(\mathbf{f})} & \pi_{-n}(\mathbf{F}). \end{array}$$

A proof of the next theorem can be found in [27, Theorem 9.27 on page 164 and Theorem 9.28 on page 165]

Theorem 16.31 (Brown's Representation Theorem).

- (i) Let $\mathcal{K}^*$ be a cohomology theory with values in $\mathbb{Z}$ -modules defined on the category of CW -pairs satisfying the disjoint union axiom. Then there is an Ω -spectrum $\mathbf{E}$ and a natural equivalence of cohomology theories

$$t^*: \mathcal{H}^*(-; \mathbf{E}) \xrightarrow{\cong} \mathcal{K}^*;$$

- (ii) Consider two Ω -spectra $\mathbf{E}$ and $\mathbf{F}$. Let $t^*: \mathcal{H}^*(-; \mathbf{E}) \rightarrow \mathcal{H}^*(-; \mathbf{F})$ be a natural transformation of cohomology theories.

Then there is a map of spectra $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{F}$ such that for every CW -pair (X, A) and $n \in \mathbb{Z}$ the maps $t^n(X, A)$ and $\mathcal{H}^n(X, A; \mathbf{f})$ from $\mathcal{H}^n(-; \mathbf{E})$ to $\mathcal{H}^n(-; \mathbf{F})$ agree.

If $t^*: \mathcal{H}^*(-; \mathbf{E}) \rightarrow \mathcal{H}^*(-; \mathbf{F})$ is a natural equivalence of cohomology theories with values in $\mathbb{Z}$ -modules, then $\mathbf{f}$ is a weak homotopy equivalence.

One knows for a homology theory $\mathcal{H}_*$ with values in $\mathbb{Z}$ -modules satisfying the disjoint union that it can be identified on CW -pairs with $\mathcal{H}_*(-; \mathbf{E})$ for some spectrum $\mathbf{E}$ but in contrast to cohomology with values in $\mathbb{Z}$ -modules $\mathbf{E}$ is not uniquely determined by this property up to weak homotopy equivalence.

16.5. Basics about vector bundles. Vector bundles are always to be understood to be finite dimensional real or complex vector bundles. For a vector bundle ξ we denote by $p_\xi: E \rightarrow B$ its bundle projection. For a finite dimensional real or complex vector space V and a CW -complex B we denote by $\underline{V}_B$ the trivial vector bundle over B whose bundle projection $B \times V \rightarrow B$ is the canonical projection onto B . If B is clear from the context, we simply write $\underline{V}$. If V is oriented, then $\underline{V}$ inherits an orientation. We will equip $\mathbb{R}^k$ always with the standard orientation and $\mathbb{C}^k$ considered as a real vector spaces with the preferred orientation coming from $\{v_1, iv_1, v_2, iv_2, \dots, v_n, iv_n\}$ for any complex basis $\{v_1, v_2, \dots, v_n\}$. Given two bundles ξ and η with projections $p_\xi: E_\xi \rightarrow B_\xi$ and $p_\eta: E_\eta \rightarrow B_\eta$, a bundle morphism $(f, \bar{f})$ consists of two maps for which the following diagram is commutative

$$\begin{array}{ccc} E_\xi & \xrightarrow{\bar{f}} & E_\eta \\ \downarrow p_\xi & & \downarrow p_\eta \\ B_\xi & \xrightarrow{f} & B_\eta \end{array}$$

and for each $b \in B_\xi$ the maps induced by $\bar{f}$ from the fiber $p_\xi^{-1}(b)$ of ξ over b to the fiber $p_\eta^{-1}(f(b))$ of η over $f(b)$ is a linear isomorphism. We call two bundles ξ and η over the same basis space B *isomorphic over B* , if there is a bundle map $(f, \bar{f})$ with $f = \text{id}_B$.

The proof of the next result can be found for instance in [12, Theorem 4.7 on page 30].

Proposition 16.32. *Let X and Y be CW -complexes. Let η be a vector bundle over the CW -complex Y . Let $f, g: X \rightarrow Y$ be maps which are homotopic.*

Then the vector bundles $f^*\eta$ and $g^*\eta$ over X obtained from the pull back construction applied to f and g are isomorphic over X . Moreover, if η is oriented, then $f^*\eta$ and $g^*\eta$ inherits orientations and are oriented isomorphic over X .

Let X be a CW -complex and let $\text{VB}_k(X)$ be the set of isomorphism classes $[\xi]$ of k -dimensional real vector bundles ξ over X . There is a universal k -dimensional bundle γ_k over a CW -complex $\text{BO}(k)$ such that for any k -dimensional real vector

bundle ξ there is a map $c_\xi: X \rightarrow \text{BO}(k)$ uniquely determined up to homotopy by the property that ξ is isomorphic over X to $c_\xi^* \gamma_k$. Moreover, the homotopy class $[c_\xi]$ depends only on the isomorphism class $[\xi]$. Sometimes c_ξ is called the *classifying map* of ξ . The space $\text{BO}(k)$ is uniquely up to homotopy determined by the property that there exists a principal $\text{O}(k)$ -bundle $\text{EO}(k) \rightarrow \text{BO}(k)$ with contractible total space $\text{EO}(k)$. The k -dimensional vector bundle γ_k is given by the canonical projection $\text{EO}(k) \times_{\text{O}(k)} \mathbb{R}^k \rightarrow \text{BO}(k)$.

All this has an analog for oriented k -dimensional real vector bundles. Namely, let X be a CW -complex and let $\overline{\text{VB}}_k(X)$ be the set of oriented isomorphism classes $[\xi]$ of oriented k -dimensional real vector bundles ξ over X . There is a universal oriented k -dimensional real bundle $\overline{\gamma}_k$ over a CW -complex $\text{BSO}(k)$ such that for any oriented k -dimensional real vector bundle ξ there is a map $\overline{c}_\xi: X \rightarrow \text{BSO}(k)$ uniquely determined up to homotopy by the property that ξ is oriented isomorphic over X to $\overline{c}_\xi^* \overline{\gamma}_k$. Moreover, the homotopy class $[\overline{c}_\xi]$ depends only on the oriented isomorphism class $[\xi]$. The space $\text{BSO}(k)$ is uniquely up to homotopy determined by the property that there exists a principal $\text{SO}(k)$ -bundle $\text{ESO}(k) \rightarrow \text{BSO}(k)$ with contractible total space $\text{ESO}(k)$. The oriented k -dimensional real vector bundle $\overline{\gamma}_k$ is given by the canonical projection $\text{ESO}(k) \times_{\text{SO}(k)} \mathbb{R}^k \rightarrow \text{BSO}(k)$ and inherits an orientation from the standard orientation of $\mathbb{R}^k$.

All this has an analog for k -dimensional complex vector bundles. Namely, let X be a CW -complex and let $\text{VB}_k^\mathbb{C}(X)$ be the set of isomorphism classes $[\xi]$ of k -dimensional complex vector bundles ξ over X . There is a universal k -dimensional complex bundle $\gamma_k^\mathbb{C}$ over a CW -complex $\text{BU}(k)$ such that for any k -dimensional complex k -vector bundle ξ there is a map $c_\xi^\mathbb{C}: X \rightarrow \text{BU}(k)$ uniquely determined up to homotopy by the property that ξ is isomorphic over X to $(c_\xi^\mathbb{C})^* \gamma_k^\mathbb{C}$. Moreover, the homotopy class $[c_\xi^\mathbb{C}]$ depends only on the isomorphism class $[\xi]$. The space $\text{BU}(k)$ is uniquely up to homotopy determined by the property that there exists a principal $\text{U}(k)$ -bundle $\text{EU}(k) \rightarrow \text{BU}(k)$ with contractible total space $\text{EU}(k)$. The k -dimensional vector bundle $\gamma_k^\mathbb{C}$ is given by the canonical projection $\text{EU}(k) \times_{\text{U}(k)} \mathbb{C}^k \rightarrow \text{BU}(k)$.

For the proof of the next theorem we refer, for instance, to [21, Chapter 5]. It is a prototype of a connection between a geometric classification problem to homotopy theory.

Theorem 16.33 (Classification of vector bundles). *If X is a CW -complex, then the maps*

$$\begin{aligned} \text{VB}_k(X) &\rightarrow [X, \text{BO}(k)], & [\xi] &\mapsto [c_\xi]; \\ \overline{\text{VB}}_k(X) &\rightarrow [X, \text{BSO}(k)], & [\xi] &\mapsto [\overline{c}_\xi]; \\ \text{VB}_k^\mathbb{C}(X) &\rightarrow [X, \text{BU}(k)], & [\xi] &\mapsto [c_\xi^\mathbb{C}], \end{aligned}$$

are bijective. Their inverses send $[f]$ to $[f^* \gamma_k]$, $[f^* \overline{\gamma}_k]$ and $[f^* \gamma_k^\mathbb{C}]$.

The spaces $\text{BO}(k)$, $\text{BSO}(k)$, and $\text{BU}(k)$ are path connected and unique up to homotopy.

For a real vector bundle $\xi: E \rightarrow X$ with Riemannian metric define its *disk bundle* $p_{DE}: DE \rightarrow X$ by $DE = \{v \in E \mid \|v\| \leq 1\}$ and its *sphere bundle* $p_{SE}: SE \rightarrow X$ by $SE = \{v \in E \mid \|v\| = 1\}$, where p_{DE} and p_{SE} are the restrictions of p . Its *Thom space* $\text{Th}(\xi)$ is defined by DE/SE . It has a preferred base point $\infty := SE/SE$. The Thom space can be defined without a choice of a Riemannian metric as follows. Put $\text{Th}(\xi) = E \cup \{\infty\}$ for some extra point ∞ . Equip $\text{Th}(\xi)$ with the smallest topology for which any open subset U of E is an open subset of $\text{Th}(\xi)$ and a basis of open neighbourhoods for ∞ is given by the complements of closed subsets $A \subset E$ for which $A \cap E_x$ is compact for each fiber E_x . If X is compact, E is locally compact

and $\text{Th}(\xi)$ is the one-point-compactification of E . The advantage of this definition is that any bundle map $(f, \bar{f}): \xi_0 \rightarrow \xi_1$ of vector bundles ξ_0 and ξ_1 canonically induces a pointed map $\text{Th}(\bar{f}, f): \text{Th}(\xi_0) \rightarrow \text{Th}(\xi_1)$. Denote by $\underline{\mathbb{R}}^k$ the trivial vector bundle with fiber $\mathbb{R}^k$. We mention that there are pointed homeomorphisms, see for instance [28, Proposition 12.28].

$$(16.34) \quad \text{Th}(\xi \times \eta) \cong \text{Th}(\xi) \wedge \text{Th}(\eta);$$

$$(16.35) \quad \text{Th}(\xi \oplus \underline{\mathbb{R}}^k) \cong \text{Th}(\xi) \wedge S^k.$$

16.6. Thom spaces and Thom spectra.

Definition 16.36 (Stable system of vector bundles). Given $l \in \mathbb{Z}^{\geq 0}$, an l -dimensional stable system of vector bundles $\mu = \{(\xi_k, (f_k, \bar{f}_k)) \mid k \in \mathbb{Z}^{\geq 0}\}$ is a sequence of vector bundles $\{\xi_k \mid k \in \mathbb{Z}^{\geq 0}\}$ such that ξ_k is a $(k+l)$ -dimensional vector bundle with projection $p_{\xi_k}: E_k \rightarrow B_k$ for a CW -complex B_k as basis together with a bundle maps

$$\begin{array}{ccc} E_k \oplus \underline{\mathbb{R}} & \xrightarrow{\bar{f}_k} & E_{k+1} \\ \downarrow p_k \oplus p_{\underline{\mathbb{R}}} & & \downarrow p_{\xi_{k+1}} \\ B_k & \xrightarrow{f_k} & B_{k+1} \end{array}$$

for $k \in \mathbb{Z}^{\geq 0}$.

We call the system *oriented* if each vector bundle ξ_k is oriented and each bundle map $(\bar{f}_k, f_k)$ respects the orientations.

Given an l -dimensional vector bundle ξ over B , we can associated to it an l -dimensional stable vector bundle system $\underline{\xi}$ by putting $\underline{\xi}_k = \xi \oplus \underline{\mathbb{R}}^k$ for $k \in \mathbb{Z}^{\geq 0}$ by defining the structure maps $(\text{id}_B, \bar{f}_k)$ to be the obvious bundle isomorphism over B from $\xi \oplus \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}} \xrightarrow{\cong} \xi \oplus \underline{\mathbb{R}}^{k+1}$.

Definition 16.37 (Thom spectrum of a stable system of vector bundles). Consider an l -dimensional stable system of vector bundles $\mu = \{(\xi_k, (f_k, \bar{f}_k)) \mid k \in \mathbb{Z}^{\geq 0}\}$. Define the associated Thom spectrum $\mathbf{Th}(\mu)$ as follows. Its k -th space $\text{Th}(\mu)_k$ is $\{\bullet\}$ for $k < 0$ and $\text{Th}(\mu)_k = \text{Th}(\xi_k)$ for $k \geq 0$. The k th-structure map is given by the composite

$$\text{Th}(\mu)_k \wedge S^1 = \text{Th}(\xi_k) \wedge S^1 \stackrel{(16.35)}{=} \text{Th}(\xi_k \oplus \underline{\mathbb{R}}) \xrightarrow{\text{Th}(f_k, \bar{f}_k)} \text{Th}(\xi_{k+1}) = \text{Th}(\mu)_{k+1}.$$

Example 16.38 (Suspension spectrum). Let X be a CW -complex. Consider the zero-dimensional vector bundle $\underline{\mathbb{R}}^0_X$ over X . Let $\underline{\mathbb{R}}^0_X$ be the associated stable system of bundles maps. We have introduced its Thom spectrum $\mathbf{Th}(\underline{\mathbb{R}}^0_X)$ in Definition 16.37. Note that its 0-th space is X_+ . Then $\mathbf{Th}(\underline{\mathbb{R}}^0_X)$ agrees with the suspension spectrum $\Sigma^\infty X_+$ of Example 16.2. If we take $X = \{\bullet\}$, then $\Sigma^\infty \{\bullet\}_+ = \Sigma^\infty(S^0, s)$ is the sphere spectrum $\mathbf{S}$ of Example 16.3.

Example 16.39 (The spectra $\mathbf{MO}$ and $\mathbf{MSO}$). Choose for any k a model for the universal k -dimensional bundle γ_k over $\text{BO}(k)$. By the universal property of γ_{k+1} we can choose a bundle map $(f_k, \bar{f}_k): \gamma_k \oplus \underline{\mathbb{R}}_{\text{BO}(k)} \rightarrow \gamma_{k+1}$. We can arrange by the construction of Subsection 8.8 that each map $f_k: \text{BO}(k) \rightarrow \text{BO}(k+1)$ is an inclusion of CW -complexes and in particular a cofibration. We obtain a stable system γ of bundle maps by the collection of the bundles γ_k and bundle maps $(f_k, \bar{f}_k)$. The associated Thom spectrum of Definition 16.37 is denoted by $\mathbf{MO} = \mathbf{Th}(\mu)$.

Note that $\mathbf{MO}$ depends on some choices. But one can show for the result $\mathbf{MO}'$ for any other choices that there are homotopy equivalences of spectra $\mathbf{MO} \rightarrow \mathbf{MO}'$ and $\mathbf{MO}' \rightarrow \mathbf{MO}$ which are homotopy inverse to one another. (Here it is crucial that the

maps f_k are cofibrations.) In particular $\mathbf{MO} \rightarrow \mathbf{MO}'$ and $\mathbf{MO}' \rightarrow \mathbf{MO}$ are weak homotopy equivalences and we get from Proposition 16.12 natural equivalences of (co)homology theories $\mathcal{H}_*(-\mathbf{MO}) \xrightarrow{\cong} \mathcal{H}_*(-\mathbf{MO}')$ and $\mathcal{H}*(-\mathbf{MO}) \xrightarrow{\cong} \mathcal{H}*(-\mathbf{MO}')$ which actually are inverse to one another. Moreover, there is actually a canonical construction $\mathbf{MO}$ for which no additional choices are made. Therefore we ignore this ambiguity about $\mathbf{MO}$ in the sequel.

There is an analog $\bar{\mu}$ and $\mathbf{MSO} = \mathbf{Th}(\bar{\mu})$, where one replaces γ_k by $\bar{\gamma}_k$ and $\mathbf{BO}(k)$ by $\mathbf{BSO}(k)$.

Note that we have constructed the sphere spectrum $\mathbf{S}$, the Eilenberg-MacLane spectrum $\mathbf{K}(A)$ for an abelian group A , and the spectra $\mathbf{MO}$ and $\mathbf{MSO}$ so far. Recall that associated to them are (co-)homology theories in Theorem 16.21. For $\mathbf{S}$ we have identified $\mathcal{H}_*(-; \mathbf{S})$ with the stable homotopy groups $\pi_*^s(-)$, see Example 16.26. These will be identified with more geometric terms, namely with framed bordism, in Theorem 17.15. For $\mathbf{K}(A)$ we have identified $\mathcal{H}*(-; \mathbf{K}(A))$ with the singular cohomology $H^*(-; A)$ with coefficients in A , see Example 16.27. We will identify $\mathcal{H}_*(-; \mathbf{MO})$ and $\mathcal{H}_*(-; \mathbf{MSO})$ with more geometric terms, namely with unoriented and oriented bordism theory, see Theorem 17.11 and Theorem 17.14.

16.7. Topological K -theory. One can define topological groups

$$\begin{aligned} \mathbf{O} &= \operatorname{colim}_{k \rightarrow \infty} \mathbf{O}(k); \\ \mathbf{SO} &= \operatorname{colim}_{k \rightarrow \infty} \mathbf{SO}(k); \\ \mathbf{U} &= \operatorname{colim}_{k \rightarrow \infty} \mathbf{U}(k), \end{aligned}$$

for the inclusions $\mathbf{O}(k) \rightarrow \mathbf{O}(k+1)$, $\mathbf{SO}(k) \rightarrow \mathbf{SO}(k+1)$, and $\mathbf{U}(k) \rightarrow \mathbf{U}(k+1)$ given by taking the block sum with the $(1, 1)$ matrix (1) .

There is a principal $\mathbf{O}$ -bundle $\mathbf{EO} \rightarrow \mathbf{BO}$ over a CW -complex $\mathbf{BO}$ for which $\mathbf{EO}$ is contractible. Up to homotopy one can obtain $\mathbf{BO}$ also as $\operatorname{colim}_{k \rightarrow \infty} \mathbf{BO}(k)$ if one chooses adequate models for $\mathbf{BO}(k)$ and arranges that each map $\mathbf{BO}(k) \rightarrow \mathbf{BO}(k+1)$ is an inclusion of CW -complexes and in particular a cofibration. Analogously one can construct spaces $\mathbf{BSO}$ and $\mathbf{BU}$. The spaces $\mathbf{BO}$, $\mathbf{BSO}$, and $\mathbf{BU}$ are path connected. We have $\pi_1(\mathbf{BO}) \cong \mathbb{Z}/2$ and the spaces $\mathbf{BSO}$ and $\mathbf{BU}$ are actually simply connected.

A deep theorem of Bott says that there are weak homotopy equivalences

$$\begin{aligned} \beta^{\mathbb{R}}: \mathbb{Z} \times \mathbf{BO} &\xrightarrow{\cong} \Omega^8(\mathbb{Z} \times \mathbf{BO}); \\ \beta^{\mathbb{C}}: \mathbb{Z} \times \mathbf{BU} &\xrightarrow{\cong} \Omega^2(\mathbb{Z} \times \mathbf{BU}), \end{aligned}$$

where $\mathbb{Z}$ is equipped with the discrete topology and the base point $0 \in \mathbb{Z}$ and we choose some base point in the path connected spaces $\mathbf{BO}$ and $\mathbf{BU}$.

For $n \in \mathbb{Z}$ define $k(n) \in \{0, 1, 2, 3, 4, 5, 6, 7\}$ to be the unique element satisfying $k(n) \equiv n \pmod{8}$. Define an Ω -spectrum $\mathbf{KO}$ by defining the n -th space $\mathbf{KO}_n^{\mathbb{R}}$ to be $\Omega^{8-k(n)}(\mathbb{Z} \times \mathbf{BO})$ if $k(n) \neq 0$ and to be $\mathbb{Z} \times \mathbf{BO}$ if $k(n) = 0$. The n -th structure map is

$$\operatorname{id}: \Omega^{8-k(n)}(\mathbb{Z} \times \mathbf{BO}) \rightarrow \Omega \Omega^{8-(k(n)+1)}(\mathbb{Z} \times \mathbf{BO}) = \Omega^{8-k(n)}(\mathbb{Z} \times \mathbf{BO})$$

if $k(n) \neq 0$ and $\beta^{\mathbb{R}}: (\mathbb{Z} \times \mathbf{BO}) \rightarrow \Omega \Omega^7(\mathbb{Z} \times \mathbf{BO}) = \Omega^8(\mathbb{Z} \times \mathbf{BO})$. So the spectrum $\mathbf{KO}$ is 8-periodic and looks in the range from 0 to 8 like

$$\mathbb{Z} \times \mathbf{BO}, \Omega^7(\mathbb{Z} \times \mathbf{BO}), \Omega^6(\mathbb{Z} \times \mathbf{BO}), \dots, \Omega^1(\mathbb{Z} \times \mathbf{BO}), \mathbb{Z} \times \mathbf{BO}.$$

Similarly we define the Ω -spectrum $\mathbf{K}$. Define $\mathbf{K}_n$ to be $\mathbb{Z} \times \mathbf{BU}$ if n is even, and to be $\Omega(\mathbb{Z} \times \mathbf{BU})$ if n is odd. The n -th structure map is the identity $\operatorname{id}_{\Omega(\mathbb{Z} \times \mathbf{BU})}$ if

n is odd, and is $\beta^{\mathbb{C}}$ if n is even. So the spectrum $\mathbf{K}$ is 2-periodic and looks in the range 0 to 2 like

$$\mathbb{Z} \times \mathrm{BU}, \Omega(\mathbb{Z} \times \mathrm{BU}), \mathbb{Z} \times \mathrm{BU}.$$

Associated to these Ω -spectra are cohomology theories satisfying the disjoint union axiom

$$\begin{aligned} KO^*(X, A) &:= H^*(X, A; \mathbf{KO}); \\ K^*(X, A) &= H^*(X, A; \mathbf{K}), \end{aligned}$$

called *real and complex topological K-theory*. Note that KO^* is 8-periodic, i.e., there are natural isomorphisms $KO^*(X, A) \xrightarrow{\cong} KO^{*+8}(X, A)$, whereas K^* is 2-periodic, i.e., there are natural isomorphisms $K^*(X, A) \xrightarrow{\cong} K^{*+2}(X, A)$.

Associated to these Ω -spectra are homology theories satisfying the disjoint union axiom

$$\begin{aligned} KO_*(X, A) &:= H_*(X, A; \mathbf{KO}); \\ K_*(X, A) &= H_*(X, A; \mathbf{K}), \end{aligned}$$

called *real and complex topological K-homology*. Note that KO_* is 8-periodic, i.e., there are natural isomorphisms $KO_*(X, A) \xrightarrow{\cong} KO_{*+8}(X, A)$, whereas K_* is 2-periodic, i.e., there are natural isomorphisms $K_*(X, A) \xrightarrow{\cong} K_{*+2}(X, A)$.

The coefficients are given for the real case by

$$(16.40) \quad KO_n(\{\bullet\}) = KO^{8-n}(\{\bullet\}) = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \{0\}, \mathbb{Z}, \{0\}, \{0\}, \{0\}, \mathbb{Z},$$

for $n = 0, 1, 2, \dots, 7$

and in the complex case by

$$(16.41) \quad K_n(\{\bullet\}) = K^n(\{\bullet\}) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even;} \\ \{0\} & \text{if } n \text{ is odd.} \end{cases}$$

If X is a finite CW -complex, the abelian group $KO^0(X)$ can be identified with the Grothendieck construction applied to the abelian semi-group of stable isomorphism classes of finite-dimensional real vector bundles over X , where two finite-dimensional real vector bundles ξ and η over X are called stably isomorphic if $\xi \oplus \mathbb{R}^k$ and $\eta \oplus \mathbb{R}^l$ are isomorphic for some natural numbers k and l and the addition comes from the Whitney sum. The analogous statement holds for the complex case.

Remark 16.42. Topological K -theory is a very valuable cohomology theory which had many applications to problems in topology. It was later extended to C^* -algebras and plays a prominent role in the classification and the theory of C^* -algebras and in index theory.

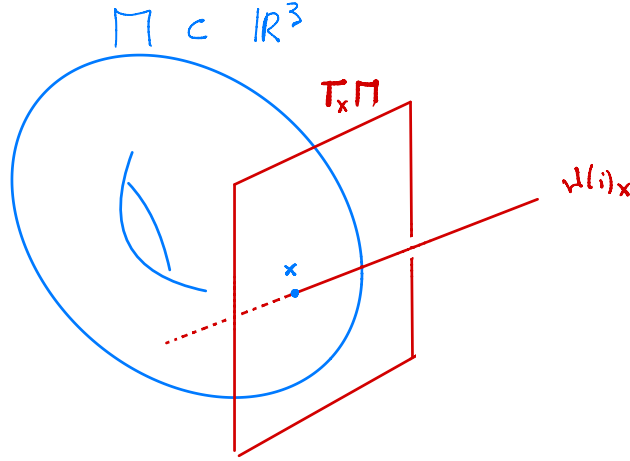
16.8. Outlook. The approach to spectra presented above can be called “classical” or “naive”. Moreover, we have not defined the notion of a smash product of two spectra and of a ring spectrum.

One can define the smash product $\mathbf{E} \wedge \mathbf{F}$ of two spectra $\mathbf{E}$ and $\mathbf{F}$ in the setting discussed in these notes but it depends on certain choices. Moreover associativity or commutativity of this smash product make only sense up to homotopy. This has led to the notions of highly structured spectra such as symmetric or orthogonal spectra, where the smash product is strictly defined and also associativity and commutativity of the smash product hold strictly.

Moreover, one works with spectra in the setting of higher category theory nowadays. An introduction to higher categories can be found for instance in [15].

17. THE PONTRJAGIN-THOM CONSTRUCTION

17.1. ξ -bordism. Let (M, i) be an embedding $i: M^n \rightarrow \mathbb{R}^{n+k}$ of a closed n -dimensional manifold M into $\mathbb{R}^{n+k}$. Note that $T\mathbb{R}^{n+k}$ comes with an explicit trivialisation $\mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \xrightarrow{\cong} T\mathbb{R}^{n+k}$ and the standard Euclidean inner product induces a Riemannian metric on $T\mathbb{R}^{n+k}$. Denote by $\nu(i)$ the *normal bundle*, which is the orthogonal complement of TM in $i^*T\mathbb{R}^{n+k}$ or can be thought of as the quotient bundle $i^*T\mathbb{R}^{n+k}/TM$. Sometimes we write $\nu(M)$ instead of $\nu(i)$ if i is clear from the context.



Next we apply this construction to bordism. Fix a space X together with a k -dimensional vector bundle ξ over X . We define the bordism set

$$(17.1) \quad \Omega_n(\xi)$$

of normal ξ -bordism classes of normal ξ -maps as follows.

Definition 17.2 (Normal ξ -map).

A *normal ξ -map* $(M, i, f, \bar{f})$ is a quadruple consisting of:

- A closed manifold M of dimension n ;
- An embedding $i: M \rightarrow \mathbb{R}^{n+k}$;
- A map $f: M \rightarrow X$;
- A bundle map $(f, \bar{f}): \nu(i) \rightarrow \xi$ covering f , where $\nu(i)$ is the normal bundle of the embedding i .

Definition 17.3 (Bordism of normal ξ -maps).

A *normal ξ -bordism* from the normal ξ -map $(M_0, i_0, f_0, \bar{f}_0)$ to the normal ξ -map $(M_1, i_1, f_1, \bar{f}_1)$ is a quadruple $(W, I, F, \bar{F})$ consisting of:

- A compact manifold W of dimension $(n+1)$ whose boundary ∂W is the disjoint union $\partial_0 W \amalg \partial_1 W$;
- An embedding of manifolds with boundary $I: W \rightarrow \mathbb{R}^{n+k} \times [0, 1]$ sending $\partial_m W$ to $\mathbb{R}^{n+k} \times \{m\}$ for $m = 0, 1$;
- Diffeomorphisms $u_m: M_m \rightarrow \partial_m W$ and $U_m: \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n+k} \times \{m\}$ for $m = 0, 1$ satisfying $I \circ u_m = U_m \circ i_m$;
- A map $F: W \rightarrow X \times [0, 1]$ satisfying $j_m \circ f_m = F \circ u_m$ for $m = 0, 1$ where $j_m: X \rightarrow X \times [0, 1]$ sends x to (x, m) ;

- A bundle map $(F, \bar{F}): \nu(I) \rightarrow \xi$ covering F such that $\bar{F} \circ \nu(u_m, U_m) = \bar{f}_m$ holds for $m = 0, 1$ where $(u_m, \nu(u_m, U_m)): \nu(i_m) \rightarrow \nu(I)$ is the obvious bundle map induced by Tu_m and TU_m .

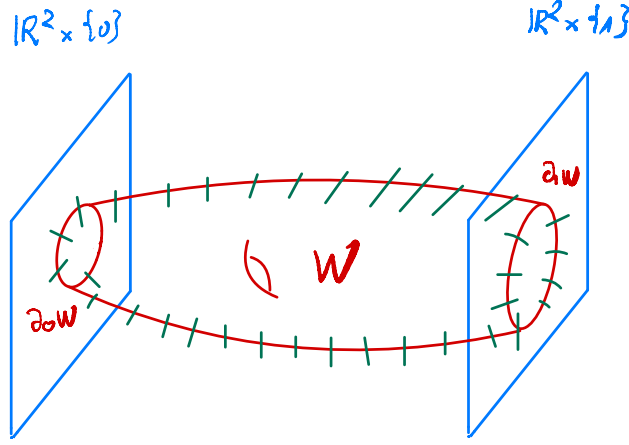
Remark 17.4. Note that in the definition above the following implicit identification

$$\nu(\partial W \subseteq \mathbb{R}^{n+k} \times \{0, 1\}) = \nu(W \subseteq \mathbb{R}^{n+1} \times [0, 1])|_{\partial W}$$

is used, which is based on the convention that at $\{0\}$ we take the inward normal field and at $\{1\}$ the outward normal vector field to get identifications

$$\begin{aligned} T\mathbb{R}^n \times [0, 1]|_{\mathbb{R}^n \times \{0, 1\}} &= T\mathbb{R}^n \times \{0, 1\} \oplus \mathbb{R}; \\ TW|_{\partial W} &= T\partial W \oplus \mathbb{R}. \end{aligned}$$

This convention guarantees that we can stack two cobordisms together to prove transitivity of the bordism relation.



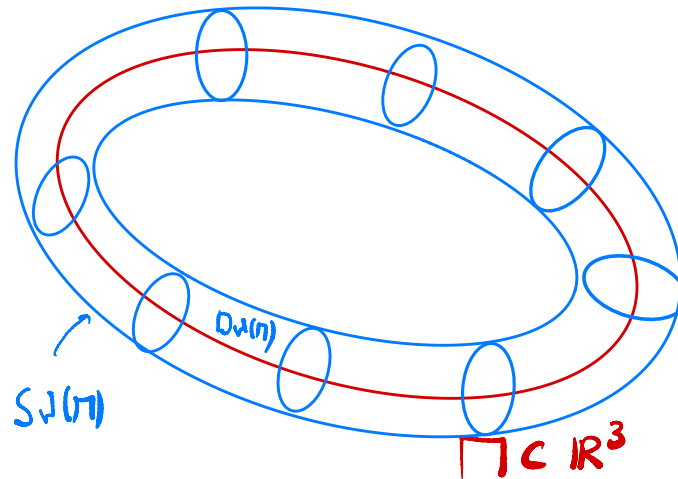
17.2. The Pontrjagin-Thom construction of ξ -bordism. Consider a normal ξ -map $(M, i, f, \bar{f})$, see Definition 17.2. Note that for any vector bundle η over a manifold B with total space E there exists a canonical bundle isomorphism $TB \oplus \eta \xrightarrow{\cong} s^*TE$ over B , where $s: B \rightarrow E$ is the zero-section. So we get an identification $TB \oplus \eta = TE|_B$. Let $(N(M), \partial N(M))$ be a tubular neighbourhood of M . Recall that there is a diffeomorphism

$$u: (D\nu(M), S\nu(M)) \rightarrow (N(M), \partial N(M))$$

with the property that its restriction to M is i and under the canonical identification $T(D\nu(M))|_M = TM \oplus \nu(i)$ the composite

$$\begin{aligned} \nu(i) = \{0\} \oplus \nu(i) &\rightarrow TM \oplus \nu(i) = T(D\nu(M))|_M \\ &\xrightarrow{Tu|_M} i^*T\mathbb{R}^{n+k} \rightarrow i^*T\mathbb{R}^{n+k}/TM = \nu(i) \end{aligned}$$

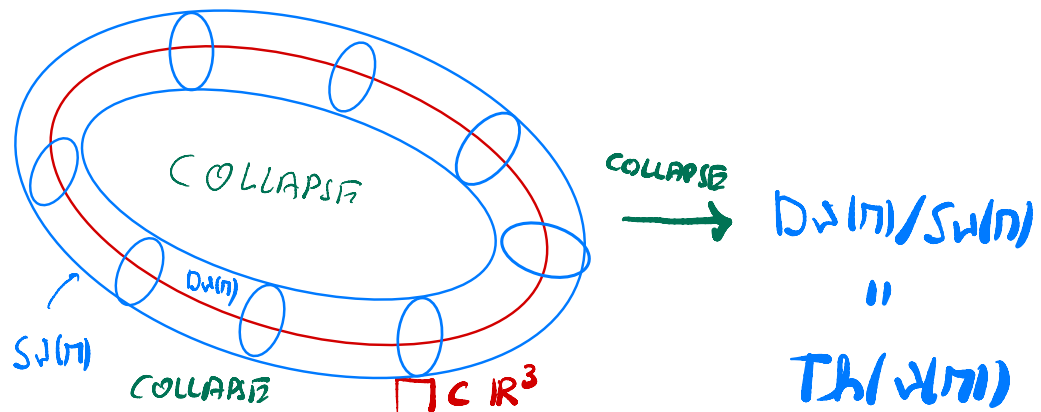
is the identity. Such a tubular neighborhood is unique up to diffeotopy on $\mathbb{R}^{n+k}$ relative M . See for instance [3, Theorem 21.11 on page 130 and Theorem 12.13 on page 131].



The *Thom collapse map*

$$(17.5) \quad c: S^{n+k} = \mathbb{R}^{n+k} \amalg \{\infty\} \rightarrow \text{Th}(\nu(M))$$

is the pointed map that is given by the diffeomorphism u^{-1} on the interior of $N(M)$ and sends the complement of the interior of $N(M)$ to the preferred base point ∞ .



The homology group $H_{n+k}(\text{Th}(\nu(M))) \cong H_{n+k}(N(M), \partial N(M))$ is infinite cyclic if M is connected, since $N(M)$ is a connected compact orientable $(n+k)$ -dimensional manifold with boundary $\partial N(M)$. The Hurewicz homomorphism

$$\text{hur}_{n+k}: \pi_{n+k}(\text{Th}(\nu(i))) \rightarrow H_{n+k}(\text{Th}(\nu(i)))$$

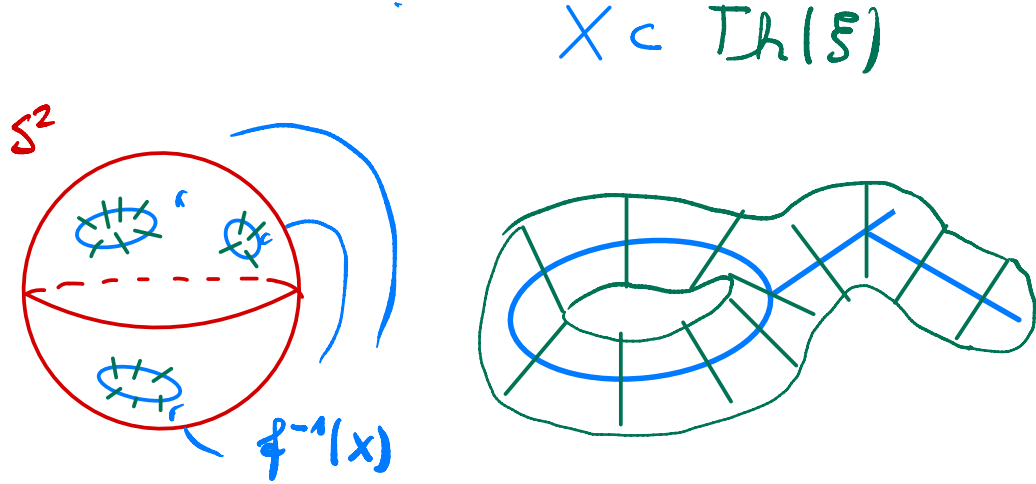
sends the class $[c]$ of c to a generator. This follows from the fact that any point in the interior of $N(M)$ is a regular value of c and has precisely one point in its preimage.

Theorem 17.6 (Pontrjagin-Thom Construction). *Let $\xi: E \rightarrow X$ be a k -dimensional vector bundle over a CW -complex X . Then the map*

$$P_n(\xi): \Omega_n(\xi) \rightarrow \pi_{n+k}(\text{Th}(\xi)),$$

which sends the bordism class of $(M, i, f, \overline{f})$ to the homotopy class of the composite $S^{n+k} \xrightarrow{c} \text{Th}(\nu(M)) \xrightarrow{\text{Th}(f, \overline{f})} \text{Th}(\xi)$, is a well-defined bijection, natural in ξ .

Proof. The details can be found in [4, Satz 3.1 on page 28, Satz 4.9 on page 35] or [11, Section 7.2 on page 172]. The basic idea becomes clear after we have explained the construction of the inverse for a finite CW -complex X . Consider a pointed map $(S^{n+k}, \infty) \rightarrow (\text{Th}(\xi), \infty)$. We can change f up to homotopy relative $\{\infty\}$ so that f becomes transverse to X . Note that transversality makes sense although X is not a manifold, one needs only the fact that X can be identified with the image of a zero-section of a vector bundle. Put $M = f^{-1}(X)$. The transversality construction yields a bundle map $(f|_M, \overline{f|_M}): \nu(M) \rightarrow \xi$ covering $f|_M$. Let $i: M \rightarrow \mathbb{R}^{n+k} = S^{n+k} - \{\infty\}$ be the inclusion. Then the inverse of $P_n(\xi)$ sends the class of f to the class of $(M, i, f|_M, \overline{f|_M})$.



□

17.3. The Pontrjagin-Thom construction and bordism for stable systems of bundles. Consider an n -dimensional system μ of vector bundles μ given by vector bundles ξ_k and bundle morphisms $(f_k, \bar{f}_k): \xi_k \oplus \underline{\mathbb{R}} \rightarrow \xi_{k+1}$. For $n, k \in \mathbb{Z}^{\geq 0}$ we next define a map

$$\Omega_n(f_k, \bar{f}_k): \Omega_n(\xi_k) \rightarrow \Omega_n(\xi_{k+1}).$$

Consider an element z in $\Omega_n(\xi_k)$ represented by normal ξ_k -map $(M, i, u, \bar{u})$. Let $j: \mathbb{R}^{n+k} = \mathbb{R}^{n+k} \times \{0\} \rightarrow \mathbb{R}^{n+k+1}$ be the standard inclusion. Then there is a natural identification of $\nu(j \circ i)$ with $\nu(i) \oplus \underline{\mathbb{R}}$. Consider the bundle map

$$(v, \bar{v}): \nu(j \circ i) = \nu(i) \oplus \underline{\mathbb{R}}_M \xrightarrow{(f, \bar{f}) \oplus (u, \underline{u})} \xi_k \oplus \underline{\mathbb{R}}_{B_k} \xrightarrow{(\bar{f}_k, f_k)} \xi_{k+1}$$

where $(u, \underline{u})$ is the obvious bundle map induced by u and $\text{id}_{\mathbb{R}}$. Then we obtain a normal ξ_{k+1} -map $(M, j \circ i, v, \bar{v})$. Its bordism class in $\Omega_n(\xi_{k+1})$ is the image of z under $\Omega_n(f_k, \bar{f}_k)$. We omit the proof that $\Omega_n(f_k, \bar{f}_k)$ is well-defined. We define the set

$$(17.7) \quad \Omega_n(\mu) = \text{colim}_{k \rightarrow \infty} \Omega_n(\xi_k)$$

with respect to the structure maps $\Omega_n(f_k, \bar{f}_k)$.

The set $\Omega_n(\mu)$ carries in contrast to each of the sets $\Omega_n(\xi_k)$ the structure of an abelian group. The unit is given by the class of the normal ξ_k -map $(M, i, v, \bar{v})$ with $M = \emptyset$ for any $k \in \mathbb{Z}^{\geq 0}$. Consider two elements z and z' in $\Omega_n(\mu)$. We can find representatives $(M, i, u, \bar{u})$ and $(M', i', u', \bar{u}')$ with $k = k'$ and $\text{im}(i) \cap \text{im}(i') = \emptyset$ and define $z_1 + z_2$ by the class of the disjoint union $(M \amalg M', i \amalg i', u \amalg u', \bar{u} \amalg \bar{u}')$. The inverse of a class represented by $(M, i, u, \bar{u})$ is the class represented by $(M, j \circ$

$i, f_k \circ u, \overline{f_k \circ u})$, where $j: \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n+k+1}$ is the standard inclusion, we identify $\nu(j \circ i)$ with $\nu(i) \oplus \mathbb{R}$ and $(f_k \circ, \overline{f_k \circ u})$ is the composite

$$(j \circ u, \overline{f_k \circ u}): \nu(j \circ i) = \nu(i) \oplus \mathbb{R}_M \xrightarrow{(u, \bar{u}) \oplus -\text{id}_{\mathbb{R}_u}} \xi_k \oplus \mathbb{R}_{B_k} \xrightarrow{(f_k, \bar{f}_k)} \xi_{k+1}$$

for $-\text{id}_{\mathbb{R}_u}$ the obvious bundle map induced by u and $-\text{id}_{\mathbb{R}}$.

Let $s_{n,k}: \pi_{n+k}(\text{Th}(\xi_k)) \rightarrow \pi_{n+k+1}(\text{Th}(\xi_{k+1}))$ be the composite of the suspension homomorphism $\pi_{n+k}(\text{Th}(\xi_k)) \rightarrow \pi_{n+k+1}(\text{Th}(\xi_k) \wedge S^1)$ and the homomorphism $\pi_{n+k+1}(\text{Th}(\xi_k) \wedge S^1) \rightarrow \pi_{n+k+1}(\text{Th}(\xi_{k+1}))$ induced by the k th structure map $\text{Th}(\xi_k) \wedge S^1 \rightarrow \text{Th}(\xi_{k+1})$ of the Thom spectrum $\mathbf{Th}(\mu)$ of Definition 16.37. Then we get from the definitions

$$\pi_n(\mathbf{Th}(\mu)) = \text{colim}_{k \rightarrow \infty} \pi_{n+k}(\text{Th}(\xi_k))$$

with respect to the structure maps $s_{n,k}$.

One easily checks that we obtain a commutative diagram

$$\begin{array}{ccc} \Omega_n(\xi_k) & \xrightarrow{\Omega_n(f_k, \tilde{f}_k)} & \Omega_n(\xi_{k+1}) \\ P_n(\xi_k) \downarrow \cong & & \cong \downarrow P_n(\xi_{k+1}) \\ \pi_{n+k}(\text{Th}(\xi_k)) & \xrightarrow{s_{n,k}} & \pi_{n+k+1}(\text{Th}(\xi_{k+1})). \end{array}$$

Therefore we obtain the following result.

Theorem 17.8 (Pontrjagin-Thom Construction for stable bundle systems).

Let μ be a stable bundle system. Then we obtain an isomorphism of abelian groups

$$P_n(\mu): \Omega_n(\mu) \rightarrow \pi_n(\mathbf{Th}(\mu))$$

by putting $P_n(\mu) = \text{colim}_{k \rightarrow \infty} P_n(\xi_k)$.

17.4. Unoriented bordism. Consider a pair (X, A) and $n \in \mathbb{Z}^{\geq 0}$. A *singular n -manifold over (X, A)* is a map $(u, \partial u): (M, \partial M) \rightarrow (X, A)$ with target (X, A) and a compact smooth manifold M with boundary ∂M of dimension n as source. Consider two singular n -maps $(u_k, \partial u_k): (M_k, \partial M_k) \rightarrow (X, A)$ for $k = 0, 1$. A singular $(n+1)$ -dimensional bordism between them consists of:

- A compact smooth manifold W of dimension $(n+1)$ with boundary ∂W ;
- A decomposition $\partial W = \partial_0 W \cup \partial_1 W \cup \partial_2 W$ for smooth submanifolds $\partial_0 W$, $\partial_1 W$, and $\partial_2 W$ of ∂W satisfying $\partial(\partial_0 W) \cap \partial(\partial_1 W) = \emptyset$ and $\partial(\partial_2 W) = \partial(\partial_0 W) \amalg \partial(\partial_1 W)$;
- A map $U: W \rightarrow X$;
- Diffeomorphisms $(v_k, \partial v_k): (M_k, \partial M_k) \xrightarrow{\cong} (\partial_k W, \partial(\partial_k W))$ for $k = 0, 1$ such that $\partial U \circ v_k = \partial u_k$ holds for $k = 0, 1$;
- We have $U(\partial_2 W) \subseteq A$.

If $(u_1, \partial u_1): (M_1, \partial M_1) \rightarrow (X, A)$ is given by $M_1 = \emptyset$, we call such a $(n+1)$ -dimensional bordism a *nullbordism* for $(u_0, \partial u_0): (M_0, \partial M_0) \rightarrow (X, A)$.

If there exists a bordism between two singular n -manifolds over (X, A) , we call them *bordant*. This turns out to be an equivalence relation, for transitivity one has to glue two bordisms together. So we can define the set $\mathcal{N}_n(X, A)$ to be the set of bordism classes of singular n -manifolds over (X, A) .

If A is empty, then for a singular bordism $(u, \partial u): (M, \partial M) \rightarrow (X, A)$ over $X = (X, \emptyset)$ we have $\partial M = \emptyset$ and hence M is just a closed manifold with a map $u: M \rightarrow X$. Also the notion of a bordism simplifies, since $\partial_2 W$ must be empty and hence ∂W is just the disjoint union $\partial_0 W \amalg \partial_1 W$.

The set $\mathcal{N}_n(X, A)$ inherits the structure of an abelian group as follows. The unit is given by the bordism class of the singular n -manifold $(u, \partial u): (M, \partial M) \rightarrow (X, A)$

for which $M = \emptyset$. Given two singular n -manifolds $(u_k, \partial u_k): (M_k, \partial M_k) \rightarrow (X, A)$ for $k = 0, 1$, define the sum of their bordism classes to be the bordism class of the disjoint union $(u_0, \partial u_0) \amalg (u_1, \partial u_1): (M_0, \partial M_0) \amalg (M_1, \partial M_1) \rightarrow (X, A)$. The inverse of the bordism class of $(u, \partial u): (M, \partial M) \rightarrow (X, A)$ is given by the bordism class of $(u, \partial u): (M, \partial M) \rightarrow (X, A)$ again, since a nullbordism for their disjoint union $(u, \partial u) \amalg (u, \partial u): (M, \partial M) \amalg (M, \partial M) \rightarrow (X, A)$ can be constructed from the cylinder $M \times I$ in the obvious way. Note that this implies that $\mathcal{N}_n(X, A)$ is actually an $\mathbb{F}_2$ -vector space.

A map of pairs $(F, f): (X, A) \rightarrow (Y, B)$ induces a homomorphism of $\mathbb{F}_2$ -vector spaces by sending the bordism class of $(u, \partial u): (M, \partial M) \rightarrow (X, A)$ to the bordism class of $(F \circ u, f \circ \partial u): (M, \partial M) \rightarrow (Y, B)$. We omit the proof that we obtain a covariant functor $\mathcal{N}_n(-)$ from the category of topological pairs to the category of $\mathbb{F}_2$ -vector spaces for $n \in \mathbb{Z}^{\geq 0}$. We define $\mathcal{N}_n(X, A)$ for $n \in \mathbb{Z}^{\leq -1}$ to be $\{0\}$. For a pair (X, A) define the homomorphism

$$(17.9) \quad \partial_n(X, A): \mathcal{N}_n(X, A) \rightarrow \mathcal{N}_{n-1}(A)$$

by sending the bordism class of $(u, \partial u): (M, \partial M) \rightarrow (X, A)$ to the bordism class of $\partial u: \partial M \rightarrow A$.

Theorem 17.10 (Singular bordism is a homology theory satisfying the disjoint union axiom).

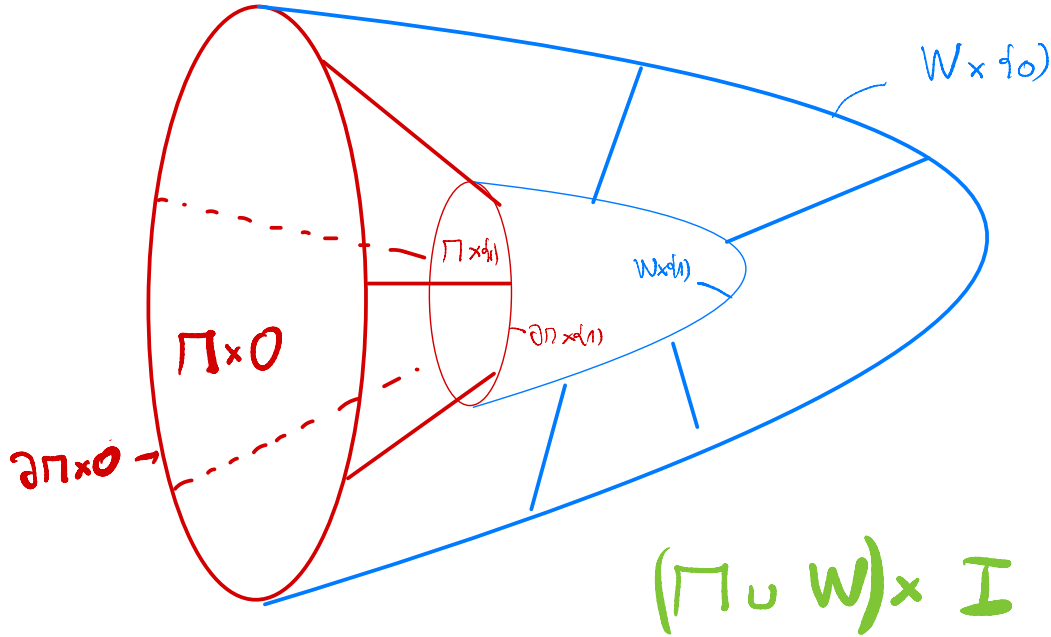
We obtain a homology theory with values in $\mathbb{F}_2$ -vector spaces satisfying the disjoint union axiom by $\mathcal{N}_*(-)$ and $\partial_*(-)$.

Sketch of the proof. We start with homotopy invariance. Consider for $k = 0, 1$ the maps $(F_k, f_k): (X, A) \rightarrow (Y, B)$ and a homotopy $h: (X, A) \times I \rightarrow (Y, B)$ between them. We have to show $\mathcal{N}(F_0, f_0) = \mathcal{N}(F_1, f_1)$. Consider a singular n -manifold $(u, \partial u): (M, \partial M) \rightarrow (X, A)$. We have to show that $(F_0, f_0) \circ (u, \partial u): (M, \partial M) \rightarrow (Y, B)$ and $(F_1, f_1) \circ (u, \partial u): (M, \partial M) \rightarrow (Y, B)$ are bordant. The desired bordism can easily be constructed from the composite $M \times I \xrightarrow{u \times \text{id}_I} X \times I \xrightarrow{h} Y$.

Consider a pair (X, A) . We have to show that we obtain a long exact sequence of $\mathbb{F}_2$ -vector spaces

$$\begin{aligned} \dots \xrightarrow{\partial_{n+2}} \mathcal{N}_{n+1}(A) &\xrightarrow{\mathcal{N}_{n+1}(i)} \mathcal{N}_{n+1}(X) \xrightarrow{\mathcal{N}_{n+1}(j)} \mathcal{N}_{n+1}(X, A) \\ &\xrightarrow{\partial_{n+1}} \mathcal{N}_n(A) \xrightarrow{\mathcal{N}_n(i)} \mathcal{N}_n(X) \xrightarrow{\mathcal{N}_n(j)} \mathcal{N}_n(X, A) \xrightarrow{\partial_n} \dots \end{aligned}$$

where $i: A \rightarrow X$ and $j: X = (X, \emptyset) \rightarrow (X, A)$ are the inclusions. We only explain exactness at $\mathcal{N}_{n+1}(X, A)$. Consider an element in $\mathcal{N}_{n+1}(X)$ given by the bordism class of $u: M \rightarrow X$. Its image under the composite $\partial_{n+1} \circ \mathcal{N}_{n+1}(j)$ is represented by the singular map with the empty set as domain and hence is zero. This shows $\text{im}(\mathcal{N}_{n+1}(j)) \subseteq \ker(\partial_{n+1})$. It remains to prove $\ker(\partial_{n+1}) \subseteq \text{im}(\mathcal{N}_{n+1}(j))$. Consider a singular $(n+1)$ -manifold $(u, \partial u): (M, \partial M) \rightarrow (X, A)$ over (X, A) such that its bordism class lies in $\ker(\partial_{n+1})$. Hence we can find a nullbordism for $\partial u_k: \partial M \rightarrow A$, i.e., a compact manifold W with boundary ∂W , a map $U: W \rightarrow A$ and a diffeomorphism $v: \partial M \rightarrow \partial W$ with $U \circ v = \partial u$. Then we obtain a singular n -manifold over X by $u \cup_v U: M \cup_v W \rightarrow X$. We claim that its bordism class is sent under $\mathcal{N}_{n+1}(j): \mathcal{N}_{n+1}(X) \rightarrow \mathcal{N}_{n+1}(X, A)$ to the bordism class of $(u, \partial u): (M, \partial M) \rightarrow (X, A)$ over (X, A) . This follows from the fact that one can construct from the composite $M \cup_v W \times I \xrightarrow{(u \cup_v U) \times \text{id}_I} X \times I \xrightarrow{\text{pr}_Y} Y$ for the pr_Y the canonical projection a bordism of singular $(n+1)$ -manifolds over (X, A) between $u \cup_v U: (M \cup_v W, \emptyset) \rightarrow (X, A)$ and $(u, \partial u): (M, \partial M) \rightarrow (X, A)$. This finishes the proof of exactness at $\mathcal{N}_{n+1}(X, A)$. The proof of exactness at the other places is similar.



The disjoint union axiom follows from the fact that for a compact subset C of the disjoint union $\coprod_{i \in I} X_i$ of the collection of spaces $\{X_i \mid i \in I\}$ there is a finite subset $J \subseteq I$ with $C \subseteq \coprod_{i \in J} X_i$.

We omit the proof that excision holds, i.e., if X is a space with subspaces $A \subseteq B \subseteq X$ satisfying $\overline{A} \subseteq B^\circ$, then the inclusion $i: (X \setminus A, B \setminus A) \rightarrow (X, A)$ induces for every $n \in \mathbb{Z}$ a bijection $\mathcal{N}_n(i): \mathcal{N}_n(X \setminus A, B \setminus A) \rightarrow \mathcal{N}_n(X, A)$. For a proof of the Mayer-Vietoris sequence for space X with open subspaces X_0 , X_1 , and X_2 satisfying $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$, we refer to [31, Proposition 21.1.7 on page 523]. The existence of such a Mayer-Vietoris sequence is essentially the same as excision. $\square$

Theorem 17.11 (Unoriented singular bordism and the spectrum $\mathbf{MO}$).

There is a natural equivalence of homology theories

$$\mathcal{N}_*(-) \xrightarrow{\cong} \mathcal{H}_*(-; \mathbf{MO})$$

where $\mathcal{H}_*(-; \mathbf{MO})$ is the homology theory associated in Theorem 16.21 (i), to the spectrum $\mathbf{MO}$ defined in Example 16.39.

Sketch of proof. We only construct for every space X and $n \in \mathbb{Z}$ an isomorphism of abelian groups $\mathcal{N}_n(X) \xrightarrow{\cong} \mathcal{H}_n(X; \mathbf{MO})$. We leave it to the reader to show that it can be extended to pairs (X, A) , is natural in (X, A) , and is compatible with the boundary operators of (X, A) and hence defines the desired natural equivalence of homology theories $\mathcal{N}_*(-) \xrightarrow{\cong} \mathcal{H}_*(-; \mathbf{MO})$.

Recall the stable system of vector bundles μ of Example 16.39 whose k -th vector bundle γ_k is the universal k -dimensional vector bundle over $\mathbf{BO}(k)$. For a space X , define the stable system of vector bundles $\text{pr}^* \mu$ as follows. The k -th vector

bundle is $\text{pr}^* \gamma_k$ over $X \times \text{BO}(k)$ which is obtained from the vector bundle γ_k by the pullback construction applied to the projection $\text{pr}: X \times \text{BO}(k) \rightarrow \text{BO}(k)$. The k -th bundle map is given by $\text{pr}^*(f_k, \overline{f_k})$ for the bundle map $(f_k, \overline{f_k}): \gamma_k \oplus \mathbb{R} \rightarrow \gamma_{k+1}$ using the obvious identification $\text{pr}^* \mathbb{R}_{\text{BO}(k)} = \mathbb{R}_{X \times \text{BO}(k)}$. Then we can identify the spectrum $\mathbf{Th}(\text{pr}^* \mu)$ with the spectrum $X_+ \wedge \mathbf{Th}(\mu)$ using (16.35). We get from Theorem 17.8 an isomorphism of abelian groups

$$\Omega_n(\text{pr}^* \mu) \xrightarrow{\cong} \pi_n(X_+ \wedge \mathbf{Th}(\mu)) = \mathcal{H}_n(X; \mathbf{MO}).$$

Hence it suffices to construct a bijection

$$(17.12) \quad \alpha_n: \Omega_n(\text{pr}^* \mu) \rightarrow \mathcal{N}_n(X).$$

One can define α_n as a forgetful map. More precisely, α_n sends the class of a normal μ_k -map $(M, i, f, \overline{f})$ to the bordism class of the singular n -manifold $\text{pr}_X \circ f: M \rightarrow X$ for the projection $\text{pr}: X \times \text{BO}(k) \rightarrow X$. Obviously α_n is a well-defined homomorphism of abelian groups. It remains to show that α_n is bijective.

We begin with surjectivity. Consider a singular n -manifold $f: M \rightarrow X$ representing an element $z \in \mathcal{N}_n(X)$. We can choose $k \in \mathbb{Z}^{\geq 0}$, actually $k = n + 1$ suffices, and an embedding $i: M \rightarrow \mathbb{R}^{n+k}$. Let $(u, \overline{u})$ be a bundle map from $\nu(i)$ to μ_k . We obtain a bundle map $(f \times u, f \times \overline{u})$ from $\nu(i)$ to $\text{pr}^* \gamma_k$. Then $(M, i, f \times u, f \times \overline{u})$ is a normal $\text{pr}^* \mu_k$ -map and hence defines an element $y \in \Omega_n(\text{pr}^* \mu)$. The image of y under α_n is z . Hence α_n is surjective.

Next we show injectivity of α_n . Consider an element $z \in \Omega_n(\text{pr}^* \mu)$ which is sent to zero under α_n . Choose $k \in \mathbb{Z}$ with $k \geq 2n + 3$ and a normal $\text{pr}^* \mu_k$ -map $(M, i, f \times u, f \times \overline{u})$ for $f: M \rightarrow X$ and $(u, \overline{u}): \nu(i) \rightarrow \gamma_k$, whose normal bordism class is z . Then $\alpha_n(z)$ is represented by the singular n -map $f: M \rightarrow X$. Hence we can find a compact $(n + 1)$ -dimensional manifold W with boundary ∂W , a diffeomorphism $t: M \rightarrow \partial W$, and a map $F: W \rightarrow X$ satisfying $F \circ t = f$. Now we have to find the right data to construct out of (W, F, w) a normal $\text{pr}^* \mu$ -nullbordism for $(M, i, f \times u, f \times \overline{u})$.

Since $k \geq 2n + 3$ we can construct an embedding of manifolds with boundary

$$(J, j): (W, \partial W) \rightarrow (\mathbb{R}^{n+k} \times [0, 1], \mathbb{R}^{n+k} \times \{0\}).$$

Then there is a natural identification $\nu(J)|_{\partial W} = \nu(j)$. The embedding $j \circ t: M \rightarrow \mathbb{R}^{n+k}$ and the given embedding $i: M \rightarrow \mathbb{R}^{n+k}$ are related by a diffeotopy $\Phi: \mathbb{R}^{n+k} \times \mathbb{R} \rightarrow \mathbb{R}^{n+k}$ because of $k \geq 2n + 3$. Hence we can find a diffeomorphism $T: \mathbb{R}^{n+k} \xrightarrow{\cong} \mathbb{R}^{n+k}$ such that $T \circ i = j \circ t$ holds. We get a bundle isomorphism $(t, \overline{t}): \nu(i) \xrightarrow{\cong} \nu(J)|_{\partial W} = \nu(j)$ coming from the differentials of t and T . Choose a bundle map $(v, \overline{v}): \nu(J) \rightarrow \mu_k$. Then the bundle maps $(v, \overline{v}) \circ (t, \overline{t})$ and $(u, \overline{u})$ from $\nu(i)$ to γ_k are homotopic. By a cofibration argument we can change $(v, \overline{v})$ up to homotopy of bundle maps $\nu(J) \rightarrow \gamma_k$ such that $(v, \overline{v}) \circ (t, \overline{t}) = (u, \overline{u})$ holds. These data yield a normal $\text{pr}^* \mu$ -nullbordism $(W, J, F \times v, F \times \overline{v})$ for $(M, i, f, \overline{f})$. Hence $z = 0$. This finishes the proof that the map α_n of (17.12) is bijective and therefore of Theorem 17.11. $\square$

17.5. The unoriented bordism ring. There is an external multiplicative structure on $\mathcal{N}_*(-)$ coming from taking the cartesian product. In particular we get for $m, n \in \mathbb{Z}^{\geq 0}$ and every two pairs (X, A) and (Y, B) a natural bilinear pairing

$$\mathcal{N}_m(X, A) \times \mathcal{N}_n(Y, B) \rightarrow \mathcal{N}_{m+n}((X, A) \times (Y, B)).$$

This induces on $\mathcal{N}_* = \mathcal{N}_*(\{\bullet\})$ the structure of a commutative $\mathbb{Z}$ -graded $\mathbb{F}_2$ -algebra whose unit is given by $\text{id}: \{\bullet\} \rightarrow \{\bullet\}$. Thom [29] has shown that $\mathcal{N}_*$, which is called the *unoriented bordism ring*, is a polynomial ring over $\mathbb{F}_2$ in variables x_i for $i \neq 2^k - 1$ and that for i even one can take the bordism class of $\mathbb{R}\mathbb{P}^i$ for x_i . Dold [7]

has constructed explicit closed manifolds representing x_i for i odd. In particular we get

$$(17.13) \quad \mathcal{N}_n = \mathbb{F}_2, \{0\}, \mathbb{F}_2, \{0\}, \mathbb{F}_2 \oplus \mathbb{F}_2, \mathbb{F}_2, \quad \text{for } n = 0, 1, 2, 4, 5.$$

Moreover, two closed manifolds are cobordant, or, equivalently, determine the same element in $\mathcal{N}_*$, if and only if they have the same Stiefel-Whitney numbers. For the definition of Stiefel-Whitney numbers we refer for instance to [21, Chapter 4].

17.6. Conventions about orientations. Let us discuss our orientation conventions for manifolds. For simplicity we will only consider a connected compact orientable n -dimensional manifold M with (possibly empty or non-connected) boundary ∂M , where orientable means that $H_n(M; \partial M)$ is infinite cyclic. Here is a list of desired properties or standard conventions.

- (i) On the vector space $\mathbb{R}^n$ for $n \geq 1$ we use the standard orientation given by the ordered standard basis $\{e_1, e_2, \dots, e_n\}$, where e_i is the vector

$$(0, 0, \dots, 0, 1, 0, \dots, 0)$$

whose only non-zero entry is at position i . If $n = 0$, an orientation on $\mathbb{R}^0$ is a choice of an element in $\{+, -\}$;

- (ii) For $n \geq 1$ an orientation on a TM is a choice of orientation on every $T_x M$ for $x \in M$ such that for every $x \in M$ there is an open neighbourhood U together with an isomorphism $TM|_U \xrightarrow{\cong} \underline{\mathbb{R}}^n$ of vector bundles over U with the property that for every $x \in U$ the isomorphism $T_x M \xrightarrow{\cong} \mathbb{R}^n$ respects the given orientation on $T_x M$ and the standard orientation of $\mathbb{R}^n$.

For $n = 0$ a choice of an orientation on TM is a choice of an element in $\{+, -\}$.

This makes actually sense for any vector bundle over M ;

- (iii) Since TD^n is $T\mathbb{R}^n|_{D^n}$ and we have the standard trivialisation $\underline{\mathbb{R}}^n \xrightarrow{\cong} T\mathbb{R}^n$, the standard orientation on the vector space $\mathbb{R}^n$ induces a standard orientation on TD^n . In particular on $D^1 = [-1, 1]$ we use the orientation on TD^1 coming from moving from -1 to 1 ;
- (iv) An orientation on M is a choice of a generator $[M, \partial M]$ of the infinite cyclic group $H_n(M, \partial M)$;
- (v) There is a preferred one-to-one-correspondence between the orientations on TM and the orientations on M which comes from the identification $H_n(T_x M, T_x M \setminus \{0\}) \xrightarrow{\cong} H_n(M, M \setminus \{x\})$ induced by the exponential map for $x \in M \setminus \partial M$;
- (vi) The boundary homomorphism $H_n(M, \partial M) \rightarrow H_{n-1}(\partial M)$ sends $[M, \partial M]$ to a class $[\partial M]$ which induces for every path component $C \in \partial M$ a generator $[C] \in H_{n-1}(C)$. Thus an orientation on M induces an orientation on C .
- (vii) We use the outward normal vector field and the canonical isomorphism $n_v \oplus Ti: \underline{\mathbb{R}} \oplus T\partial M \xrightarrow{\cong} TM|_{\partial M}$ in order to assign to an orientation on $T_x M$ an orientation on $T_x \partial M$ for $x \in X$. Thus an orientation on TM induces an orientation on TC for every path component C of ∂M ;
- (viii) On a product $M \times N$ of oriented connected closed smooth manifolds we use the orientation coming from the isomorphism induced by the cross product $H_{\dim(M)}(M; \mathbb{Z}) \otimes_{\mathbb{Z}} H_{\dim(N)}(N; \mathbb{Z}) \xrightarrow{\cong} H_{\dim(M \times N)}(M \times N; \mathbb{Z})$;
- (ix) On a direct sum $V \oplus W$ of oriented vector spaces we use the orientation coming from assigning to two ordered basis of V and W the obvious ordered basis of $V \oplus W$ by stacking the basis together.

This yields also a preferred procedure to define a preferred orientation on the Whitney sum $\xi \oplus \eta$ of two oriented vector bundles ξ and η ;

- (x) All the items above are compatible with one another;
- (xi) These conventions together with the standard orientation on the vector space $\mathbb{R}^n$ yield on S^1 respectively TS^1 the anticlockwise orientation and on $[-1, 1]$ and $T[-1, 1]$ respectively the orientation corresponding from moving from -1 to 1 ;
- (xii) With these conventions the standard orientation on $T[-1, 1]$ induces on $T\partial D^1 = T\partial[-1, 1] = T\{-1, 1\}$ the orientation which corresponds to $-$ on -1 and $+$ on 1 .

We leave it to the reader to check that this can be arranged if and only if we use the outward normal field and the convention that in the identification $n_v \oplus Ti: \mathbb{R} \oplus T\partial M \xrightarrow{\cong} TM|_{\partial M}$ we choose the order $\mathbb{R} \oplus T\partial M$ and not the order $T\partial M \oplus \mathbb{R}$. Namely (xii) forces us to use the outward normal field and the order is determined by (iii) and (xi).

17.7. Oriented bordism. Now we can modify the definition of the unoriented bordism group $\mathcal{N}_n(X, A)$ to the oriented bordism group $\Omega_n(X, A)$. We call a compact manifold M with (possibly empty) boundary ∂M *oriented* if for each path component C of M the homology group $H_n(C; \partial C)$ is infinite cyclic and we have chosen a generator $[C, \partial C] \in H_n(C; \partial C)$. Given an oriented compact manifold M , we denote by M^- the oriented compact manifold whose underlying manifold is M but where we use the orientation, where we replace $[C, \partial C]$ by $-[C, \partial C]$.

The difference in the new definition of $\Omega_n(X, A)$ and in the definition of $\mathcal{N}_n(X, A)$ appearing in Subsection 17.4 is that we additionally require for a singular n -manifold $(u, \partial u): (M, \partial M) \rightarrow (X, A)$ over (X, A) that M is an oriented manifold and in the definition of the bordism relation we additionally require that W is an oriented manifold and the diffeomorphism $(v_k, \partial v_k): (M_k, \partial M_k) \xrightarrow{\cong} (\partial_k W, \partial(\partial_k W))$ preserve the orientations for $k = 0, 1$. The addition and the unit is defined as before. However, the inverse of the bordism class of $(u, \partial u): (M, \partial M) \rightarrow (X, A)$ is given by the bordism class of $(u, \partial u): (M^-, \partial M^-) \rightarrow (X, A)$ and *not* by $(u, \partial u): (M, \partial M) \rightarrow (X, A)$; we have to reverse the orientations. This has the effect that $\Omega_n(X, A)$ is an abelian group but in general not a $\mathbb{F}_2$ -vector space which was the case for $\mathcal{N}_n(X, A)$. The proof that we get a homology theory $\Omega_*(-)$ satisfying the disjoint union axiom is analogous to the proof of Theorem 17.10.

The proof of the next theorem is analogous to the proof of Theorem 17.11.

Theorem 17.14 (Oriented singular bordism and the spectrum **MSO**).

There is a natural equivalence of homology theories

$$\Omega_*(-) \xrightarrow{\cong} \mathcal{H}_*(-; \mathbf{MSO})$$

where $\mathcal{H}_*(-; \mathbf{MSO})$ is the homology theory associated in Theorem 16.21 (i), to the spectrum **MSO** defined in Example 16.39.

17.8. The oriented bordism ring. There is an external multiplicative structure on $\Omega_*(-)$ coming from taking the cartesian product. In particular we get for $m, n \in \mathbb{Z}^{\geq 0}$ and every two pairs (X, A) and (Y, B) a natural bilinear pairing

$$\Omega_m(X, A) \times \Omega_n(Y, B) \rightarrow \Omega_{m+n}((X, A) \times (Y, B)).$$

This induces on $\Omega_* = \Omega_*(\{\bullet\})$ the structure of a commutative $\mathbb{Z}$ -graded ring whose unit is given by $\text{id}: \{\bullet\} \rightarrow \{\bullet\}$ with the standard orientation $+$ on the domain. Its structure was completely determined by Wall [32]. In particular $\Omega_* \otimes_{\mathbb{Z}} \mathbb{Q}$ is a polynomial $\mathbb{Q}$ -algebra whose generators as a polynomial $\mathbb{Q}$ -algebra can be taken to

be the oriented bordism classes of $\mathbb{CP}^{2n}$ for $n \geq 1$. Each Ω_n is a finitely generated abelian group in which the order of any nontrivial torsion element is 2.

Moreover, two oriented closed manifolds are oriented cobordant, or, equivalently, determine the same element in Ω_* , if and only if they have the same Pontrjagin and Stiefel-Whitney numbers. For the definition of Pontrjagin and Stiefel-Whitney numbers we refer for instance to [21, Chapter 4 and 16].

Here is some information about Ω_n in low degrees n :

- There is an isomorphism of abelian groups

$$\Omega_0 \xrightarrow{\cong} \mathbb{Z}$$

which sends the bordism class of a 0-dimensional oriented manifold which is just a finite collection of points equipped with a sign $+$ or $-$ to the sum of these signs;

- The signature defines an isomorphism of abelian groups

$$\text{sign}: \Omega_4 \xrightarrow{\cong} \mathbb{Z}$$

and the preimage of $1 \in \mathbb{Z}$ is the bordism class of $\mathbb{CP}^2$.

- We have $\Omega_n = \{0\}$ if and only if $n \in \{1, 2, 3, 6, 7\}$;
- We have

$$\Omega_n \cong \begin{cases} \mathbb{Z}/2 & n = 5; \\ \mathbb{Z} \oplus \mathbb{Z} & n = 8; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & n = 9; \\ \mathbb{Z}/2 & n = 10. \end{cases}$$

17.9. Framed bordism. Let ξ be an n -dimensional vector bundle over the space B . For $l \in \mathbb{Z}^{\geq 0}$ an l -framing of ξ is a bundle isomorphism $(\text{id}_B, \bar{u}): \underline{\mathbb{R}}^{n+l} \xrightarrow{\cong} \xi \oplus \underline{\mathbb{R}}^l$ over B . We call an l_0 -framing $(\text{id}_B, \bar{u}_0): \underline{\mathbb{R}}^{n+l_0} \xrightarrow{\cong} \xi \oplus \underline{\mathbb{R}}^{l_0}$ and an l_1 -framing $(\text{id}_B, \bar{u}_1): \underline{\mathbb{R}}^{n+l_1} \xrightarrow{\cong} \xi \oplus \underline{\mathbb{R}}^{l_1}$ equivalent if there exists $l \in \mathbb{Z}^{\geq 0}$ with $l \geq l_0, l_1$ such that for $i = 0, 1$ the two bundle isomorphisms over B

$$\underline{\mathbb{R}}^{n+l} = \underline{\mathbb{R}}^{n+l_i} \oplus \underline{\mathbb{R}}^{l-l_i} \xrightarrow{(\text{id}_B, \bar{u}_i) \oplus \text{id}_{\underline{\mathbb{R}}^{l-l_i}}} \xi \oplus \underline{\mathbb{R}}^{l_i} \oplus \underline{\mathbb{R}}^{l-l_i} = \xi \oplus \underline{\mathbb{R}}^l$$

are homotopic through bundle isomorphisms over B .

For a compact manifold M a *stable framing* is a stable framing of its tangent bundle TM . Of course not every compact manifold admits a framing. Every compact manifold with a stable framing is orientable and inherits from the stable framing an orientation. Let $i: (M, \partial M) \rightarrow (\mathbb{R}^m \times \mathbb{R}^{\geq 0}, \mathbb{R}^m \times \{0\})$ be an embedding of the compact manifold M (with possibly empty) boundary ∂M . Then we will tacitly use in the sequel the fact that there is a one-to-one correspondence between the stable framings of M and the stable framings of the normal bundle $\nu(i)$.

Now one can define for a pair (X, A) its framed bordism group $\Omega_n^{\text{fr}}(X, A)$ analogously to how we modified the definition of unoriented bordism $\mathcal{N}_*(-)$ to oriented bordism $\Omega_*(-)$. The difference in the new definition of $\Omega_n^{\text{fr}}(X, A)$ and in the definition of $\mathcal{N}_n(X, A)$ appearing in Subsection 17.4 is that we additionally require for a singular n -manifold $(u, \partial u): (M, \partial M) \rightarrow (X, A)$ over (X, A) that M comes with a stable framing and in the definition of the bordism relation we additionally require that W comes with a stable framing and the diffeomorphism $(v_k, \partial v_k): (M_k, \partial M_k) \xrightarrow{\cong} (\partial_k W, \partial(\partial_k W))$ is compatible with the stable framings. The addition and the unit is defined as before. However, the inverse of the class

represented is now defined by replacing a given stable framing by the new stable framing obtained by precomposition with the bundle automorphism for $l \in \mathbb{Z}^{\geq 1}$

$$\underline{\mathbb{R}^{n+l}} = \underline{\mathbb{R}^{n+l-1}} \oplus \underline{\mathbb{R}} \xrightarrow{\text{id}_{\underline{\mathbb{R}^{n+l-1}}} \oplus -\text{id}_{\underline{\mathbb{R}}}} \underline{\mathbb{R}^{n+l-1}} \oplus \underline{\mathbb{R}} = \underline{\mathbb{R}^{n+l}}.$$

The proof of the next theorem is a variation of the proof of Theorem 17.11 in view of Example 16.26 and Remark 16.37.

Theorem 17.15 (Framed bordism and the stable homotopy). *There is a natural equivalence of homology theories*

$$\Omega_*^{\text{fr}}(-) \xrightarrow{\cong} \pi_*^s(-).$$

Remark 17.16. One can give rather elementary geometric proofs of the formula

$$\Omega_n^{\text{fr}} \cong \begin{cases} \mathbb{Z} & n = 0; \\ \mathbb{Z}/2 & n = 1. \end{cases}$$

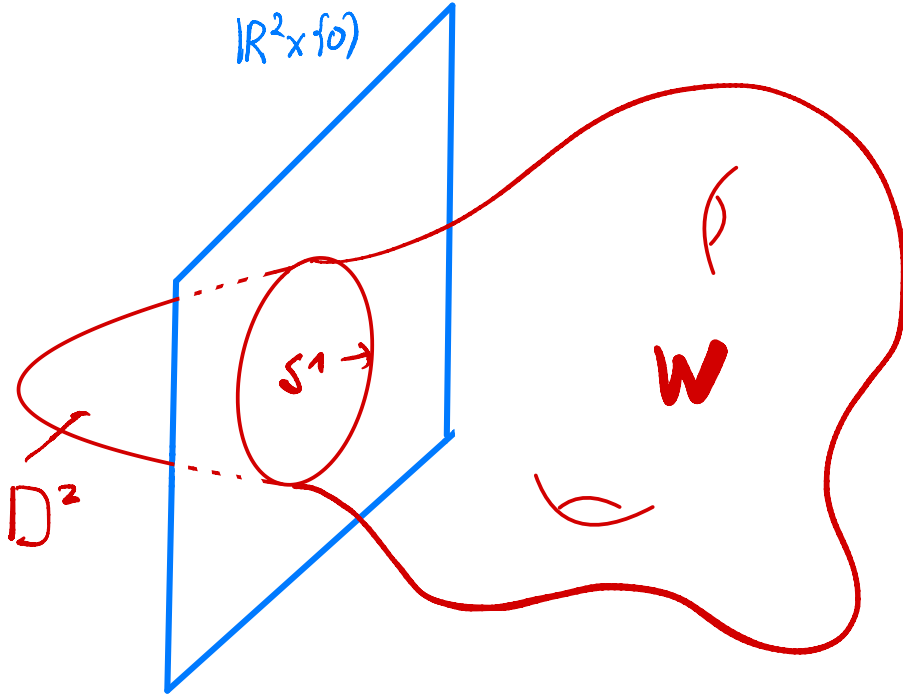
which agrees with the values of the n -stem π_n^s for $n = 0, 1$ by Theorem 17.15.

It is not hard to check that the forgetful map $\Omega_0^{\text{fr}} \rightarrow \Omega_0$ is bijective and we have computed Ω_0 already in Subsection 17.8.

Any connected closed 1-dimensional manifold M is diffeomorphic to S^1 . Because of Example 10.14 we get $[S^1, \text{SO}] \cong \mathbb{Z}/2$ and hence there are precisely two stable framings on S^1 . One of them extends to D^2 and the other does not. Then one can show that $\Omega_1^{\text{fr}} \cong \mathbb{Z}/2$ with the bordism class of S^1 equipped with the framing not extending to D^2 as generator. From $\pi_1^s \cong \mathbb{Z}/2$, Theorem 10.5, and the Freudenthal Suspension Theorem 11.20 we conclude

$$\pi_{n+1}(S^n) \cong \begin{cases} \{0\} & n = 1; \\ \mathbb{Z} & n = 2; \\ \mathbb{Z}/2 & n \geq 3. \end{cases}$$

Let us sketch the proof that $\Omega_1^{\text{fr}} \cong \mathbb{Z}/2$ with the bordism class of S^1 with the framing which does not extend to D^2 as generator. We first show that S^1 with the framing which does not extend to D^2 is not framed nullbordant. Suppose the contrary, i.e., that there is a framed nullbordism W for it. We can assume without loss of generality that W is path connected. In the sequel we identify $S^1 = \partial W$. Then $W \cup_{S^1} D^2$ is a closed 2-dimensional manifold. Its first Stiefel-Whitney class vanishes, since $H^1(W \cup_{S^1} D^2; \mathbb{F}_2) \rightarrow H^1(W; \mathbb{F}_2)$ is injective and sends it to the Stiefel-Whitney class of W which is trivial. Hence $W \cup_{S^1} D^2$ is orientable. We can choose an embedding $i: W \rightarrow \mathbb{R}^3$ such that i restricted to S^1 is given by the inclusion $S^1 \subset \mathbb{R}^2 = \mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^3$ and i maps D^2 to $(\mathbb{R}^3)^{\leq 0}$ and W to $(\mathbb{R}^3)^{\geq 0}$.



Since W and $\mathbb{R}^3$ are orientable, the normal bundle $\nu(i)$ is orientable and hence trivial. We choose an identification $\nu(i) = \mathbb{R}$. Note that this yields a specific stable framing on W . We get identifications $\nu(i|_{S^1}) = \mathbb{R}^2$, $\nu(i|_{D^2}) = \mathbb{R}$, and $\nu(i|_W) = \mathbb{R}$. So we get on S^1 the stable framing which extends to D^2 and comes from the specific stable framing on W . The problem is that there are different stable framing on W . They differ from the specific stable framing by an automorphism of $\underline{\mathbb{R}}^k_W$ for large $k \in \mathbb{Z}^{\geq 0}$. We conclude from Proposition 10.10 (ii) that the set of stable framing of W can be identified with $[W, \text{SO}]$. Hence it suffices to show for the inclusion $k: S^1 = \partial W \rightarrow W$ that the map $k^*: [W, \text{SO}] \rightarrow [S^1, \text{SO}]$ sending $[g]$ to $[g \circ k]$ is trivial, since then the induced stable framing on S^1 is the same for all possible stable framings on W and hence extends to D^2 . We have shown $\pi_2(\text{SO}) = \{0\}$ and $\pi_1(\text{SO}) \cong \mathbb{Z}/2$ in Example 10.14. We conclude from Theorem 14.3 (ii) that there is a map $f: \text{SO} \rightarrow K(\mathbb{Z}/2, 1)$ which is 3-connected. Since W and S^1 are CW -complexes of dimension ≤ 2 , we conclude from the Whitehead Theorem that it suffices to show that $k^*: [W, K(\mathbb{Z}/2, 1)] \rightarrow [S^1, K(\mathbb{Z}/2, 1)]$ is trivial. Because of Theorem 14.7 it suffices to show that $H^1(k; \mathbb{Z}/2): H^1(W, \mathbb{Z}/2) \rightarrow H^1(\partial W; \mathbb{Z}/2)$ is the trivial map. This follows from the part of the long exact cohomology sequence of $(W, \partial W)$

$$H^1(W, \mathbb{Z}/2) \xrightarrow{H^1(k; \mathbb{Z}/2)} H^1(\partial W; \mathbb{Z}/2) \rightarrow H^2(W, \partial W; \mathbb{Z}/2) \rightarrow H^2(W; \mathbb{Z}/2)$$

and from the computations $H^1(\partial W; \mathbb{Z}/2) \cong \mathbb{Z}/2$, $H^2(W, \partial W; \mathbb{Z}/2) \cong H_0(W; \mathbb{Z}/2) \cong \mathbb{Z}/2$, and $H^2(W; \mathbb{Z}/2) \cong H_0(W, \partial W; \mathbb{Z}/2) \cong \{0\}$. Thus we have shown that S^1 with the framing which does not extend to D^2 is not framed nullbordant and hence defines a non-trivial element in $\mathcal{N}_1$.

The framed bordism class of any framed 1-dimensional closed manifold is a $\mathbb{Z}$ -linear combination of the elements in $\mathcal{N}_1$ represented by S^1 with the framing which extends to D^2 and by S^1 with the framing that does not extend to D^2 . Obviously the class of S^1 with the stable framing which extends to D^2 represents zero in $\mathcal{N}_1$. One easily checks that the cylinder over S^1 gives a framed nullbordism for the disjoint union of two copies of S^1 equipped with the framing which does not extend to D^2 . Hence the element represented in Ω_1^{fr} by S^1 equipped with the framing which does not extend to D^2 has order precisely two and generates the abelian group $\mathcal{N}_1$.

18. THE ATIYAH-HIRZEBRUCH SPECTRAL SEQUENCE FOR HOMOLOGY FOR ARBITRARY FILTRATIONS

Fix a commutative ring R . Let $\mathcal{H}_*$ be a homology theory with values in R -modules satisfying the disjoint union axiom and the WHE-axiom, but not necessarily the dimension axiom. Recall that the WHE-axiom says that any weak homotopy equivalence $f: Y \rightarrow Z$ induces isomorphisms $\mathcal{H}_p(f): \mathcal{H}_p(Y) \rightarrow \mathcal{H}_p(Z)$ for every $p \in \mathbb{Z}$. Note that any homology theory which is defined only for pairs of CW -complexes can be extended to a homology theory for all pairs satisfying the WHE-axiom using cellular approximation. Moreover, stable homotopy and singular homology satisfy the WHE-axiom, see Theorem 6.10. Let X be a space coming with a filtration by subspaces

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X$$

such that $X = \text{colim}_{p \rightarrow \infty} X_p$.

Problem 18.1. How much can we say about $\mathcal{H}_n(X)$ for $n \in \mathbb{Z}$, if we know $\mathcal{H}_p(X_q, X_{q-1})$ for $p \in \mathbb{Z}^{\geq 0}$ and $q \in \mathbb{Z}$. More precisely, is there a machine which starts with the knowledge of $\mathcal{H}_p(X_q, X_{q-1})$ for $p \in \mathbb{Z}^{\geq 0}$ and $q \in \mathbb{Z}$ and gives some information about $\mathcal{H}_n(X)$, which in favourable situation leads to actual computations?

As a warmup we consider the following examples.

Example 18.2 (Cellular homology). In this example we appeal to some facts from the lecture course *Topology I*, see for instance [17, Chapter 3].

Let $\mathcal{H}_*$ be a homology theory with values in R -modules satisfying the disjoint union axiom and the dimension axiom. Let M be the R -module given by $\mathcal{H}_0(\{\bullet\})$. Let X be a CW -complex. Consider the R -chain complex $C_*^{\mathcal{H}}$ whose n -th chain module is $\mathcal{H}_n(X_n, X_{n-1})$ and whose n -th differential is the boundary operator of the triple (X_n, X_{n-1}, X_{n-2}) . Then there is a canonical R -isomorphism

$$H_n(C_*^{\mathcal{H}}(X)) \xrightarrow{\cong} \mathcal{H}_n(X).$$

So this is a recipe how to compute $\mathcal{H}_n(X)$ from the R -modules $\mathcal{H}_n(X_n, X_{n-1})$. Of course one needs to figure out the differentials of the R -chain complex $C_*^{\mathcal{H}}$. Recall that we have assigned to X its cellular $\mathbb{Z}$ -chain complex $C_*(X)$ and that there is an R -chain isomorphism

$$C_*^c(X) \otimes_{\mathbb{Z}} \mathcal{H}_0(\{\bullet\}) \xrightarrow{\cong} C_*^{\mathcal{H}}(X).$$

Hence we rediscover the fact that $\mathcal{H}_n(X)$ can be computed by the cellular homology $H_n(C_*(X) \otimes_{\mathbb{Z}} \mathcal{H}_0(\{\bullet\}))$ of X with coefficients in $\mathcal{H}_0(\{\bullet\})$.

One may say that in the sequel we want to drop the condition that $\mathcal{H}_*$ satisfies the dimension axiom. This is illustrated by the following example.

Example 18.3 (Long exact sequence of a pair). Let $\mathcal{H}_*$ be a homology theory with values in R -modules satisfying the disjoint union axiom. Suppose that the filtration of X has length 1, i.e., $X = X_1$. Then we can look at the long exact sequence of the pair (X, X_0)

$$\begin{aligned} \cdots \rightarrow \mathcal{H}_p(X_0, X_{-1}) \rightarrow \mathcal{H}_p(X, X_{-1}) \rightarrow \mathcal{H}_p(X, X_0) \rightarrow \mathcal{H}_{q-1}(X_0, X_{-1}) \\ \rightarrow \mathcal{H}_{p-1}(X, X_{-1}) \rightarrow \mathcal{H}_{p-1}(X, X_0) \rightarrow \cdots \end{aligned}$$

If we define

$$F_{p,q} = \text{im}(\mathcal{H}_{p+q}(X_p) \rightarrow \mathcal{H}_{p+q}(X)),$$

we get a filtration of $\mathcal{H}_q(X)$ by

$$\{0\} = F_{-1,q+1} \subseteq F_{0,q} \subseteq F_{1,q-1} = \mathcal{H}_q(X).$$

Put

$$E_{p,q}^\infty = F_{p,q} / F_{p-1,q+1}.$$

Then the computation of $E_{p,q}^\infty$ would give some information about $\mathcal{H}_{p+q}(X)$, namely we would obtain an extension

$$(18.4) \quad 0 \rightarrow E_{0,q}^\infty \rightarrow \mathcal{H}_q(X) \rightarrow E_{1,q-1}^\infty \rightarrow 0.$$

Then we are left with this *extension problem*. In general there are several solutions. In the favourable case that $E_{1,q-1}^\infty$ is free, we would even obtain an explicit answer, namely,

$$\mathcal{H}_q(X) \cong E_{1,q-1}^\infty \oplus E_{0,q}^\infty.$$

Sometimes one does not want to compute $\mathcal{H}_q(X)$ but only an invariant of it. For instance suppose that R is a principle ideal domain and we want to figure out $\text{rk}_R(\mathcal{H}_q(X))$. In this situation one gets a complete answer, namely

$$(18.5) \quad \text{rk}_R(\mathcal{H}_q(X)) = \text{rk}_R(E_{1,q-1}^\infty) + \text{rk}_R(E_{0,q}^\infty).$$

Consider the chain complex $C[q]_*$

$$\begin{aligned} \cdots \rightarrow C[q]_{p+1} = \mathcal{H}_{p+q+1}(X_{p+1}, X_p) \rightarrow C[q]_p = \mathcal{H}_{p+q}(X_p, X_{p-1}) \\ \rightarrow C[q]_{p-1} = \mathcal{H}_{p-q+q}(X_{p-1}, X_{p-2}) \rightarrow \cdots \end{aligned}$$

whose p -th differential is the composite

$$\mathcal{H}_{p+q}(X_p, X_{p-1}) \xrightarrow{\partial_{p+q}} \mathcal{H}_{p+q-1}(X_{p-1}) \rightarrow \mathcal{H}_{p+q-1}(X_{p-1}, X_{p-2}).$$

Next we show

$$(18.6) \quad E_{p,q}^\infty \cong H_p(C[q]_*).$$

For $p = 0$ we have $E_{0,q}^\infty = F_{0,q}^\infty = \text{im}(\mathcal{H}_q(X_0) \rightarrow \mathcal{H}_q(X))$. The chain complex $C[q]_*$ in dimensions looks like

$$\begin{aligned} \cdots \rightarrow \{0\} \rightarrow C[q]_2 = \mathcal{H}_{q+2}(X, X) = \{0\} \rightarrow C[q]_1 = \mathcal{H}_{q+1}(X, X_0) \rightarrow \\ C[q]_0 = \mathcal{H}_q(X_0) \rightarrow C[q]_{-1} = \mathcal{H}_q(\emptyset, \emptyset) = \{0\} \rightarrow \{0\} \rightarrow \cdots \end{aligned}$$

Hence $H_0(C[q]_*)$ is the cokernel of $\partial_{q+1}: \mathcal{H}_{q+1}(X, X_0) \rightarrow \mathcal{H}_q(X_0)$. This is by the long exact sequence isomorphic to $\text{im}(\mathcal{H}_q(X_0) \rightarrow \mathcal{H}_q(X))$. We have get from the definitions $E_{0,q}^\infty = F_{0,q}^\infty = \text{im}(\mathcal{H}_q(X_0) \rightarrow \mathcal{H}_q(X))$. Moreover, $H_1(C[q]_*)$ is the kernel of $\partial_{q+1}: \mathcal{H}_{q+1}(X, X_0) \rightarrow \mathcal{H}_q(X_0)$. This is by the long exact sequence isomorphic to $\text{cok}(\mathcal{H}_{q+1}(X_0) \rightarrow \mathcal{H}_{q+1}(X))$. We have get from the definitions $E_{1,q}^\infty = \text{cok}(\mathcal{H}_{q+1}(X_0) \rightarrow \mathcal{H}_{q+1}(X))$.

So we have a recipe how to compute $\mathcal{H}_{p+q}(X)$ from the modules $\mathcal{H}_{p+q}(X_p, X_{p-1})$ which is a two stage process in view of (18.4) and (18.6). But this requires that

we can figure out the differentials of $C[q]_*$ and then the homology of it. In the favourite case that $\mathcal{H}_{p+q}(X_p, X_{p-1}) = \{0\}$ holds for odd q and $p \in \mathbb{Z}$, we get

$$E_{p,q}^\infty = \begin{cases} 0 & \text{if } q \text{ is odd;} \\ \mathcal{H}_{p+q}(X_p, X_{p-1}) & \text{if } q \text{ is even.} \end{cases}$$

and hence

$$\mathcal{H}_n(X) = \begin{cases} \mathcal{H}_n(X, X_0) & \text{if } n \text{ is odd;} \\ \mathcal{H}_n(X_0) & \text{if } n \text{ is even.} \end{cases}$$

If we assume that R is a principle ideal domain, each R -module $\mathcal{H}_{p+q}(X_p, X_{p-1})$ is finitely generated and there is a number d such that $\mathcal{H}_{p+q}(X_p, X_{p-1})$ vanishes for $|p+q| \geq d$, then we can conclude that $\mathcal{H}_n(X)$ is finitely generated for every n and vanishes for $|n| \geq d$, and get a computation of the Euler characteristic with respect to $\mathcal{H}_*$ of X by

$$\begin{aligned} \chi_{\mathcal{H}_*}(X) &:= \sum_n (-1)^n \cdot \text{rk}_R(\mathcal{H}_n(X)) \\ &= \sum_n (-1)^n \cdot (\text{rk}_R(E_{1,n-1}^\infty) + \text{rk}_R(E_{0,n}^\infty)) \\ &= \sum_n (-1)^n \cdot (\text{rk}_R(E_{0,n}^\infty) - \text{rk}_R(E_{1,n}^\infty)) \\ &= \sum_n (-1)^n \cdot \chi(C[n]_*) \\ &= \sum_n (-1)^n \cdot (\text{rk}_R(C[n]_0) - \text{rk}_R(C[n]_1)) \\ &= \sum_n (-1)^n \cdot \text{rk}_R(\mathcal{H}_n(X_0, X_{-1})) - \text{rk}_R(\mathcal{H}_{n+1}(X, X_0)) \\ &= \sum_n (-1)^n \cdot \text{rk}_R(\mathcal{H}_n(X_0, X_{-1})) + \sum_n (-1)^n \cdot \text{rk}_R(\mathcal{H}_n(X, X_0)) \\ &= \sum_{p,q} (-1)^{p+q} \cdot \text{rk}_R(\mathcal{H}_{p+q}(X_p, X_{p-1})). \end{aligned}$$

So we can compute the invariant $\chi_{\mathcal{H}_*}(X)$ directly from the numbers $\text{rk}_R(\mathcal{H}_{p+q}(X_p, X_{p-1}))$, and we do not have to solve the extension problem (18.4) and to determine the differentials and the homology of the chain complex $C[q]_*$.

Next we consider the general case. The extension of Example 18.3 to the general case is essentially a problem of a good book keeping.

We define for $p \in \mathbb{Z}^{\geq 0}$, $q \in \mathbb{Z}$, and $r \in \mathbb{Z}^{\geq 1}$, where we put $X_m = \emptyset$ for $m \leq -1$,

$$(18.7) \quad Z_{p,q}^r = \text{im}(\mathcal{H}_{p+q}(X_p, X_{p-r}) \rightarrow \mathcal{H}_{p+q}(X_p, X_{p-1}));$$

$$(18.8) \quad B_{p,q}^r = \text{im}(\Delta_{p+q+1}: \mathcal{H}_{p+q+1}(X_{p+r-1}, X_p) \rightarrow \mathcal{H}_{p+q}(X_p, X_{p-1}));$$

$$(18.9) \quad Z_{p,q}^\infty = \text{im}(\mathcal{H}_{p+q}(X_p) \rightarrow \mathcal{H}_{p+q}(X_p, X_{p-1}));$$

$$(18.10) \quad B_{p,q}^\infty = \text{im}(\Delta_{p+q+1}: \mathcal{H}_{p+q+1}(X, X_p) \rightarrow \mathcal{H}_{p+q}(X_p, X_{p-1}));$$

$$(18.11) \quad F_{p,q} = \text{im}(\mathcal{H}_{p+q}(X_p) \rightarrow \mathcal{H}_{p+q}(X)).$$

Here Δ_{p+q+1} is the boundary operator of the corresponding triple. Recall that for a triple (Z_2, Z_1, Z_0) given by $Z_0 \subseteq Z_1 \subseteq Z_2$ its n -th boundary operator is defined to be the composite

$$\Delta_n = \Delta[Z_2, Z_1, Z_0]_n: \mathcal{H}_n(Z_2, Z_1) \xrightarrow{\partial[Z_2, Z_1]_n} \mathcal{H}_{n-1}(Z_1) \rightarrow \mathcal{H}_{n-1}(Z_1, Z_0),$$

where $\partial[Z_2, Z_1]_n$ is the n -th boundary operator of the pair (Z_2, Z_1) , and that we have the long exact triple sequence

$$(18.12) \quad \dots \xrightarrow{\Delta[Z_2, Z_1, Z_0]_{n+1}} \mathcal{H}_n(Z_1, Z_0) \rightarrow \mathcal{H}_n(Z_2, Z_0) \rightarrow \mathcal{H}_n(Z_2, Z_1) \\ \xrightarrow{\Delta[Z_2, Z_1, Z_0]_n} \mathcal{H}_{n-1}(Z_1, Z_0) \rightarrow \mathcal{H}_{n-1}(Z_2, Z_0) \rightarrow \mathcal{H}_{n-1}(Z_2, Z_1) \xrightarrow{\Delta[Z_2, Z_1, Z_0]_{n-1}} \dots$$

Obviously we have the inclusions

$$\{0\} = B_{p,q}^1 \subseteq B_{p,q}^2 \subseteq \dots \subseteq B_{p,q}^r \subseteq B_{p,q}^{r+1} \subseteq \dots \subseteq B_{p,q}^\infty \\ \subseteq Z_{p,q}^\infty \subseteq \dots \subseteq Z_{p,q}^{r+1} \subseteq Z_{p,q}^r \subseteq \dots \subseteq Z_{p,q}^1.$$

So we can define

$$(18.13) \quad E_{p,q}^r = Z_{p,q}^r / B_{p,q}^r;$$

$$(18.14) \quad E_{p,q}^\infty = Z_{p,q}^\infty / B_{p,q}^\infty.$$

Lemma 18.15. *There are canonical isomorphisms*

$$Z_{p,q}^r / Z_{p,q}^{r+1} \xrightarrow{\cong} B_{p-r,q+r-1}^{r+1} / B_{p-r,q+r-1}^r.$$

Proof. This follows from the following commutative diagram

$$\begin{array}{ccccc} & & \mathcal{H}_{p+q}(X_p, X_{p-r-1}) & & \\ & & \downarrow & \searrow & \\ \mathcal{H}_{p+q}(X_{p-1}, X_{p-r}) & \longrightarrow & \mathcal{H}_{p+q}(X_p, X_{p-r}) & \longrightarrow & \mathcal{H}_{p+q}(X_p, X_{p-1}) \\ & \searrow \Delta[X_p, X_{p-1}, X_{p-r}]_{p+q} & \downarrow \Delta[X_p, X_{p-r}, X_{p-r-1}]_{p+q} & & \\ & & \mathcal{H}_{p+q-1}(X_{p-r}, X_{p-r-1}) & & \end{array}$$

and the long exact sequence (18.12) of the triple (X_p, X_{p-1}, X_{p-r}) and the triple $(X_p, X_{p-r}, X_{p-r-1})$. $\square$

Because of Lemma 18.15 we can define a map

$$(18.16) \quad d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$$

by the composite

$$E_{p,q}^r := Z_{p,q}^r / B_{p,q}^r \rightarrow Z_{p,q}^r / Z_{p,q}^{r+1} \xrightarrow{\cong} B_{p-r,q+r-1}^{r+1} / B_{p-r,q+r-1}^r \\ \rightarrow Z_{p-r,q+r-1}^r / B_{p-r,q+r-1}^r =: E_{p-r,q+r-1}^r.$$

Note that the first map appearing in the composite above is the canonical projection and hence surjective and the third map is the canonical inclusion and hence injective. So we get

$$\ker(d_{p,q}^r) = Z_{p,q}^{r+1} / B_{p,q}^r; \\ \operatorname{im}(d_{p,q}^r) = B_{p-r,q+r-1}^{r+1} / B_{p-r,q+r-1}^r.$$

Hence we have $\operatorname{im}(d_{p+r,q-r+1}^r) \subseteq \ker(d_{p,q}^r)$ and we get canonical R -isomorphisms

$$\ker(d_{p,q}^r) / \operatorname{im}(d_{p+r,q-r+1}^r) \cong (Z_{p,q}^{r+1} / B_{p,q}^r) / (B_{p-r,q+r-1}^{r+1} / B_{p-r,q+r-1}^r) \cong Z_{p,q}^{r+1} / B_{p,q}^{r+1} =: E_{p,q}^{r+1}.$$

Hence we have shown the following lemma.

Lemma 18.17. *We have $d_{p,q}^r \circ d_{p+r,q-r+1}^r = 0$ and therefore obtain an R -chain complex $C^r[q]_*$ if we define the l th chain module by $E_{p-(l-p)r, (l-p)(1-r)+q}^r$ and the l th-differential by $d_{p,q}^r$. Moreover, there are canonical R -isomorphisms*

$$H_p(C^r[q]_*) \xrightarrow{\cong} E_{p,q}^{r+1}.$$

Lemma 18.18. *We have the obvious inclusions*

$$\{0\} = F_{-1,p+q+1} \subseteq F_{0,p+q} \subseteq \cdots \subseteq F_{p,q} \subseteq \cdots \subseteq \mathcal{H}_{p+q}(X)$$

satisfying

$$\mathcal{H}_{p+q}(X) = \bigcup_{r=-1}^{\infty} F_{r,p+q-r}.$$

Moreover, there are canonical isomorphisms

$$F_{p,q}/F_{p-1,q+1} \xrightarrow{\cong} E_{p,q}^{\infty}.$$

Proof. Since $X = \text{colim}_{r \rightarrow \infty} X_r$, the canonical map

$$(18.19) \quad \text{colim}_{r \rightarrow \infty} \mathcal{H}_{p+q}(X_r) \xrightarrow{\cong} \mathcal{H}_{p+q}(X)$$

is bijective by Proposition 16.11 (i). This implies $\mathcal{H}_{p+q}(X) = \bigcup_{r=-1}^{\infty} F_{r,p+q-r}$, since $F_{r,p+q-r}$ is by definition the image of $\mathcal{H}_{p+q}(X_r) \rightarrow \mathcal{H}_{p+q}(X)$.

This desired isomorphisms $F_{p,q}/F_{p-1,q+1} \xrightarrow{\cong} E_{p,q}^{\infty}$ come from the following commutative diagram

$$\begin{array}{ccccc} & & \mathcal{H}_{p+q+1}(X, X_p) & & \\ & & \downarrow & \searrow \Delta[X, X_p, X_{p-1}]_{p+q} & \\ \mathcal{H}_{p+q}(X_{p-1}) & \longrightarrow & \mathcal{H}_{p+q}(X_p) & \longrightarrow & \mathcal{H}_{p+q}(X_p, X_{p-1}) \\ & \searrow & \downarrow & & \\ & & \mathcal{H}_{p+q}(X) & & \end{array}$$

and the long exact sequences of the triple (X, X_p, X_{p-1}) . □

Finally we relate the $E_{p,q}^{\infty}$ to the modules $E_{p,q}^r$.

Lemma 18.20. *We have*

$$\begin{aligned} Z_{p,q}^r &= Z_{p,q}^{\infty} \quad \text{if } r > p; \\ B_{p,q}^{\infty} &= \bigcup_{r \geq 1} B_{p,q}^r. \end{aligned}$$

Therefore we have for $r > p$ epimorphism $E_{p,q}^r \rightarrow E_{p,q}^{\infty}$ and we get

$$E_{p,q}^{\infty} = \text{colim}_{r \rightarrow \infty} E_{p,q}^r$$

Proof. Since $X_{p-r} = \emptyset$ holds for $p > r$, we get $Z_{p,q}^r = Z_{p,q}^{\infty}$ if $r > p$.

Since $X = \text{colim}_{r \rightarrow \infty} X_r$, the canonical map

$$(18.21) \quad \text{colim}_{r \rightarrow \infty} \mathcal{H}_{p+q+1}(X_{p+r-1}, X_p) \xrightarrow{\cong} \mathcal{H}_{p+q}(X, X_p)$$

is bijective by Proposition 16.11 (i). We have the obvious map

$$\begin{aligned} \text{colim}_{r \rightarrow \infty} \Delta[X_{p+r-1}, X_p, X_{p-1}]_{p+q+1} : \text{colim}_{r \rightarrow \infty} \mathcal{H}_{p+q+1}(X_{p+r-1}, X_p) \\ \rightarrow \mathcal{H}_{p+q}(X_p, X_{p-1}). \end{aligned}$$

It factorizes as the composite

$$\text{colim}_{r \rightarrow \infty} \mathcal{H}_{p+q+1}(X_{p+r-1}, X_p) \xrightarrow{\cong} \mathcal{H}_{p+q}(X, X_p) \rightarrow \mathcal{H}_{p+q}(X_p, X_{p-1}).$$

where the first map is the isomorphism (18.21). This implies $B_{p,q}^{\infty} = \bigcup_{r \geq 1} B_{p,q}^r$. Now the other claims are obviously true. □

Remark 18.22 (General strategy). Let X be a space coming with a filtration by subspaces

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X$$

such that $X = \operatorname{colim}_{p \rightarrow \infty} X_p$. Let $\mathcal{H}_*$ be a homology theory $\mathcal{H}_*$ with values in R -modules satisfying the disjoint union axiom and the WHE-axiom. Let us summarize how we can use the construction above to compute $\mathcal{H}_n(X)$ for an element $n \in \mathbb{Z}$ by trying to carry out the following program, where $p \in \mathbb{Z}^{\geq 0}$ and $q \in \mathbb{Z}$.

- (i) Identify $E_{p,q}^1 = \mathcal{H}_{p+q}(X_p, X_{p-1})$;
- (ii) Identify the differentials $d_{p,q}^1$;
- (iii) Compute the homology $E_{p,q}^2 = H_p(C^1[q]_*)$ of the R -chain complexes $C^1[q]_*$;
- (iv) Identify the differentials $d_{p,q}^2$;
- (v) Compute the homology $E_{p,q}^3 = H_p(C^2[q]_*)$ of the R -chain complexes $C^2[q]_*$;
- (vi) Repeat this process thus identifying $E_{p,q}^r$ for all $r \geq 1$;
- (vii) Compute $E_{p,q}^\infty = \operatorname{colim}_{r \rightarrow \infty} E_{p,q}^r$;
- (viii) Put $F_{0,n} = E_{0,n}^\infty$. Solve the extension problems $0 \rightarrow F_{p-1,n-p+1} \rightarrow F_{p,n-p} \rightarrow E_{p,n-p}^\infty \rightarrow 0$, thus determining $F_{p,n-p}$ for all $p \in \mathbb{Z}^{\geq 0}$;
- (ix) Then $\mathcal{H}_n(X) = \bigcup_{p \geq 0} F_{p,n-p}$ for the filtration $F_{0,n} \subseteq F_{1,n-1} \subseteq F_{2,n-2} \subseteq F_{3,n-3} \subseteq \cdots$.

Of course in this generality one cannot carry out this program completely, but we will see that in many interesting cases one gets very useful information about $\mathcal{H}_n(X)$.

19. BASIC NOTIONS AND FACTS ABOUT HOMOLOGICAL SPECTRAL SEQUENCES

Next we describe the abstract setting underlying the homological Atiyah-Hirzebruch spectral sequence.

Definition 19.1 (Homological spectral sequence with values in R -modules). A *homological spectral sequence with values in R -modules starting at d* for $d \in \mathbb{Z}^{\geq 1}$ $E_{*,*}^*$ consists of the following data:

- A family of R -modules $\{E_{p,q}^r\}$ for $r \in \mathbb{Z}^{\geq d}$, $p, q \in \mathbb{Z}$ such that $E_{p,q}^r = \{0\}$ for $p \leq -1$;
- Maps $d_{p,q}^r: E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$ called differentials for $r \in \mathbb{Z}^{\geq d}$, $p, q \in \mathbb{Z}$ such that the composite $d_{p,q}^r \circ d_{p-r,q-r+1}^r$ vanishes for $r \in \mathbb{Z}^{\geq d}$, $p, q \in \mathbb{Z}$.
In particular we get R -chain complexes $C^r[q]_*$ for $r \in \mathbb{Z}^{\geq d}$ and $p \in \mathbb{Z}$ whose q -th differential is $d_{p,q}^r$, in other words it is given by the lines of slope $-\frac{r-1}{r}$;
- R -isomorphisms

$$\alpha_{p,q}^r: H_p(C^r[q]_*) \xrightarrow{\cong} E_{p,q}^{r+1}$$

for $r \in \mathbb{Z}^{\geq d}$, $p, q \in \mathbb{Z}$.

We call the data given by $\{E_{p,q}^r, d_{p,q}^r \mid p, q \in \mathbb{Z}\}$ for $r \in \mathbb{Z}^{\geq s}$ the r -th page of the spectral sequence. Note that the r -th page determines the underlying modules of the $(r+1)$ -th page but not the differentials appearing on the $(r+1)$ th page.

Here is a picture of the first page:

(19.2)

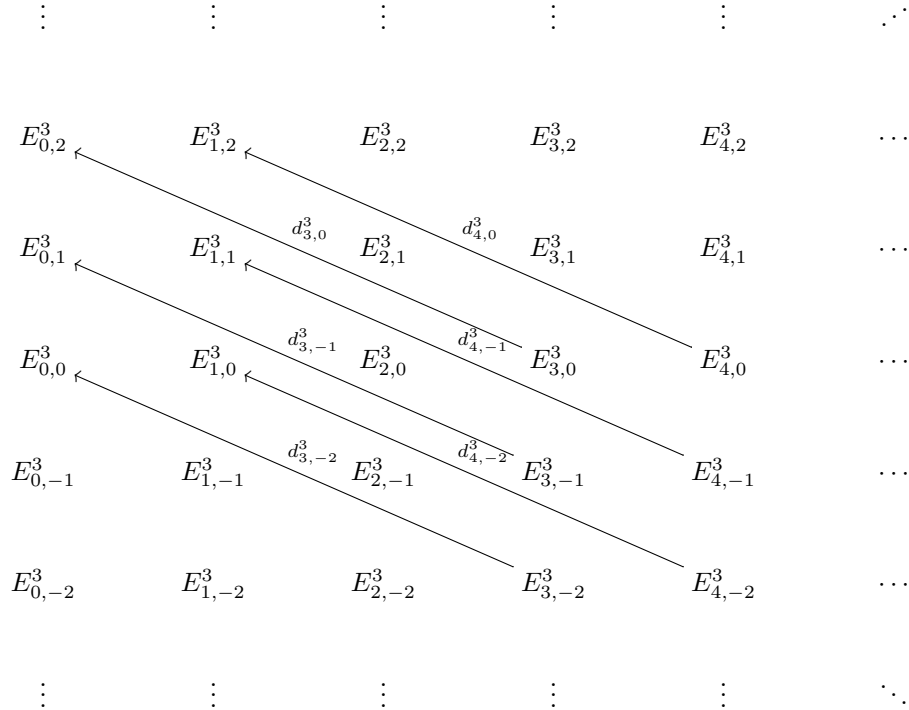
$$\begin{array}{ccccccccc}
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\
 E_{0,2}^1 \xleftarrow{d_{1,2}^1} E_{1,2}^1 \xleftarrow{d_{2,2}^1} E_{2,2}^1 \xleftarrow{d_{3,2}^1} E_{3,2}^1 \xleftarrow{d_{4,2}^1} E_{4,2}^1 & & \cdots \\
 E_{0,1}^1 \xleftarrow{d_{1,1}^1} E_{1,1}^1 \xleftarrow{d_{2,1}^1} E_{2,1}^1 \xleftarrow{d_{3,1}^1} E_{3,1}^1 \xleftarrow{d_{4,1}^1} E_{4,1}^1 & & \cdots \\
 E_{0,0}^1 \xleftarrow{d_{1,0}^1} E_{1,0}^1 \xleftarrow{d_{2,0}^1} E_{2,0}^1 \xleftarrow{d_{3,0}^1} E_{3,0}^1 \xleftarrow{d_{4,0}^1} E_{4,0}^1 & & \cdots \\
 E_{0,-1}^1 \xleftarrow{d_{1,-1}^1} E_{1,-1}^1 \xleftarrow{d_{2,-1}^1} E_{2,-1}^1 \xleftarrow{d_{3,-1}^1} E_{3,-1}^1 \xleftarrow{d_{4,-1}^1} E_{4,-1}^1 & & \cdots \\
 E_{0,-2}^1 \xleftarrow{d_{1,-2}^1} E_{1,-2}^1 \xleftarrow{d_{2,-2}^1} E_{2,-2}^1 \xleftarrow{d_{3,-2}^1} E_{3,-2}^1 \xleftarrow{d_{4,-2}^1} E_{4,-2}^1 & & \cdots \\
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots
 \end{array}$$

Here is a picture of the second page:

(19.3)

$$\begin{array}{ccccccccc}
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\
 E_{0,2}^2 \xleftarrow{d_{2,1}^2} E_{1,2}^2 \xleftarrow{d_{3,1}^2} E_{2,2}^2 \xleftarrow{d_{4,1}^2} E_{3,2}^2 \xleftarrow{d_{4,2}^2} E_{4,2}^2 & & \cdots \\
 E_{0,1}^2 \xleftarrow{d_{2,0}^2} E_{1,1}^2 \xleftarrow{d_{3,0}^2} E_{2,1}^2 \xleftarrow{d_{4,0}^2} E_{3,1}^2 \xleftarrow{d_{4,1}^2} E_{4,1}^2 & & \cdots \\
 E_{0,0}^2 \xleftarrow{d_{2,-1}^2} E_{1,0}^2 \xleftarrow{d_{3,-1}^2} E_{2,0}^2 \xleftarrow{d_{4,-1}^2} E_{3,0}^2 \xleftarrow{d_{4,0}^2} E_{4,0}^2 & & \cdots \\
 E_{0,-1}^2 \xleftarrow{d_{2,-2}^2} E_{1,-1}^2 \xleftarrow{d_{3,-2}^2} E_{2,-1}^2 \xleftarrow{d_{4,-2}^2} E_{3,-1}^2 \xleftarrow{d_{4,-1}^2} E_{4,-1}^2 & & \cdots \\
 E_{0,-2}^2 \xleftarrow{d_{2,-3}^2} E_{1,-2}^2 \xleftarrow{d_{3,-3}^2} E_{2,-2}^2 \xleftarrow{d_{4,-3}^2} E_{3,-2}^2 \xleftarrow{d_{4,-2}^2} E_{4,-2}^2 & & \cdots \\
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots
 \end{array}$$

(19.4)



Note that for given (p, q) we have

$$E_{p,q}^r = \{0\} \implies E_{p,q}^{r+1} = 0 \implies E_{p,q}^\infty = \{0\}.$$

We call the spectral sequence a *first quadrant spectral sequence* if $E_{p,q}^d = 0$ holds

A morphism $f_{*,*}^*: E_{*,*}^* \rightarrow E_{*,*}'$ of homological spectra sequences is a family of homomorphism $\{f_{r,p,q}^*: E_{p,q}^r \rightarrow E_{p,q}'^r\}$ for $r \in \mathbb{Z}^{\geq d}$, $p, q \in \mathbb{Z}$ which is compatible with the differentials $d_{p,q}^r: E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$ and $d_{p,q}'^r: E_{p,q}'^r \rightarrow E_{p-r,q+r-1}'^r$ and with the isomorphisms $\alpha_{p,q}^r: H_p(C^r[q]_*) \xrightarrow{\cong} E_{p,q}^{r+1}$ and $\alpha_{p,q}'^r: H_p(C'^r[q]_*) \xrightarrow{\cong} E_{p,q}'^{r+1}$.

Lemma 19.5. *Let $f_{*,*}^*: E_{*,*}^* \rightarrow E_{*,*}'^*$ be a morphism of homological spectral sequences starting at $d \in \mathbb{Z}^{\geq 1}$. Suppose that $f_{p,q}^d: E_{p,q}^d \rightarrow E_{p,q}'^d$ is bijective for all $p, q \in \mathbb{Z}$. Then $f_{p,q}^r: E_{p,q}^r \rightarrow E_{p,q}'^r$ is bijective for all $r \in \mathbb{Z}^{\geq d}$, $p, q \in \mathbb{Z}$.*

So in the favourite case one hopes for instance that many of the entries $E_{p,q}^r$ are zero so that all differentials starting or ending at such a place must be trivial. Sometimes one knows that a differential $d_{p,q}^r$ is trivial, since its source is a finite abelian group and its target is a free abelian group. Computing the differentials in general is difficult. Often one needs to know some clever tricks or some previous knowledge about the groups $E_{p,q}^\infty$.

Note that we obtain for every $p, q \in \mathbb{Z}$ a sequence of epimorphisms of R -modules

$$E_{p,q}^r \rightarrow E_{p,q}^{r+1} \rightarrow E_{p,q}^{r+2} \rightarrow$$

for $r \in \mathbb{Z}^{\geq p+1}$, since the differential $d_{p,q}^r: E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$ has $\{0\}$ as target and hence vanishes for $r \geq p+1$. Therefore we can define

$$E_{p,q}^\infty = \operatorname{colim}_{r \rightarrow \infty} E_{p,q}^r.$$

Consider a $\mathbb{Z}$ -graded R -module $\mathcal{H}_* = \{\mathcal{H}_n \mid n \in \mathbb{Z}\}$. It is called *filtered* if for every $n \in \mathbb{Z}$ there is a preferred ascending filtration

$$\{0\} = F_{-1,n+1} \subseteq F_{0,n} \subseteq F_{1,n-1} \subseteq F_{2,n-2} \subseteq \cdots \subseteq \mathcal{H}_n$$

with $\mathcal{H}_n = \bigcup_{p \geq 0} F_{n-p,p}$. We say that a homological spectral sequence converges to the filtered $\mathbb{Z}$ -graded R -module $\{\mathcal{H}_n \mid n \in \mathbb{Z}\}$ if there is for every $p, q \in \mathbb{Z}$ a preferred R -isomorphism

$$\beta_{p,q}: F_{p,q}/F_{p-1,q+1} \xrightarrow{\cong} E_{p,q}^\infty.$$

We say that *there are no extension problems* for the homological spectral sequence $E_{*,*}^*$ converging to the filtered $\mathbb{Z}$ -graded R -module $\mathcal{H}_*$ if we have

$$\mathcal{H}_n \cong \bigoplus_{p \geq 0} F_{n-p,p}/F_{n-p-1,p+1}.$$

We say that the homological spectral sequence $E_{*,*}^*$ converging to the filtered $\mathbb{Z}$ -graded R -module $\mathcal{H}_*$ *strongly collapses* if it collapses and there are no extension problems. In this rare and favourite cases we get

$$\mathcal{H}_n = \bigoplus_{p \geq 0} E_{n-p,p}^d.$$

20. THE ATIYAH-HIRZEBRUCH SPECTRAL SEQUENCE FOR HOMOLOGY

There is the following favourite case, namely, where X is a CW -complex and $\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X$ is the skeletal filtration. Then we get an identification

$$(20.1) \quad E_{p,q}^2 = H_p(X; \mathcal{H}_q(\{\bullet\})),$$

where $\mathcal{H}_*$ is a homology theory with values in R -module satisfying the disjoint union axiom. Next we sketch the proof of (20.1). Choose for every p a pushout

$$\begin{array}{ccc} \coprod_{i_p \in I_p} S^{p-1} & \xrightarrow{\coprod_{i_p \in I_p} c_i^p} & X_{p-1} \\ \downarrow & & \downarrow p \\ \coprod_{i \in I_p} D^p & \xrightarrow{\coprod_{i_p \in I_p} C_i^p} & X_p. \end{array}$$

Then we get isomorphisms

$$\varphi_p: \bigoplus_{i_p \in I_p} \mathcal{H}_q(\{\bullet\}) \xrightarrow{\cong} \mathcal{H}_{p+q}(X_p, X_{p-1}).$$

Given two indices $i_p \in I_p$ and $i_{p-1} \in I_{p-1}$, we define the following endomorphism of S^{p-1}

$$S^{p-1} \xrightarrow{c_{i_p}^p} X_p \xrightarrow{\text{pr}} X_p/(X_p)_{i_{p-1}} \xrightarrow{(\overline{C}_{i_p, i_{p-1}}^{p-1})^{-1}} D^{p-1}/S^{p-2} \xrightarrow{\mu_{p-1}} S^{p-1},$$

where $(X_p)_{i_{p-1}}$ is the union of X_{p-1} and all the closed cells associated to $i \in I_p$ with $i \neq i_p$, the homeomorphism $\overline{C}_{i_p, i_{p-1}}^{p-1}: D^{p-1}/S^{p-2} \xrightarrow{\cong} X_p/(X_p)_{i_{p-1}}$ is induced by the characteristic map $C_{i_p}^{p-1}$, and $\mu_{p-1}: D^{p-1}/S^{p-2} \rightarrow S^{p-1}$ is the standard

homeomorphism. Let $d_{I_p, i_{p-1}} \in \mathbb{Z}$ be the degree of this map. Then the following diagram commutes

$$\begin{array}{ccc} \bigoplus_{i_p \in I_p} \mathcal{H}_q(\{\bullet\}) & \xrightarrow{D_p} & \bigoplus_{i_{p-1} \in I_{p-1}} \mathcal{H}_q(\{\bullet\}) \\ \varphi_p \downarrow \cong & & \varphi_{p-1} \downarrow \cong \\ \mathcal{H}_{p+q}(X_p, X_{p-1}) & \xrightarrow{d_{p,q}^1} & \mathcal{H}_{p+q-1}(X_{p-1}, X_{p-2}) \end{array}$$

where D_p is given by the collection of integers $d_{i_p, i_{p-1}}$. The latter definition makes sense, since for fixed i_p there are only finitely many $i_{p-1} \in \mathbb{Z}$ with $d_{i_p, i_{p-1}} \neq 0$. Hence the chain complex $C^2[1]_*$ can be identified with $C_*(X) \otimes_{\mathbb{Z}} \mathcal{H}_q(\{\bullet\})$ for the cellular chain complex $C_*(X)$. This finishes the proof of (20.1).

Notation 20.2. Let R be a commutative ring and $\mathcal{H}_*$ be a homology theory taking values in R -modules satisfying the disjoint union axiom. If we use the skeletal filtration, we call the spectral sequence constructed in Section 19 the *Atiyah-Hirzebruch spectral sequence for homology converging to $\mathcal{H}_*$* .

Note that the Atiyah-Hirzebruch spectral sequence converges to the filtered $\mathbb{Z}$ -graded R -module $\mathcal{H}_*(X)$, where the filtration is described in (18.11), namely $F_{p,q}$ is the image of $\mathcal{H}_{p+q}(X_p) \rightarrow \mathcal{H}_{p+q}(X)$. It starts at 2. If we abbreviate $\mathcal{H}_q = \mathcal{H}_q(\{\bullet\})$, its second page looks like

$$\begin{array}{ccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\ H_0(X; \mathcal{H}_2) & \xleftarrow{d_{2,1}^2} & H_1(X; \mathcal{H}_2) & \xleftarrow{d_{3,1}^2} & H_2(X; \mathcal{H}_2) & \xleftarrow{d_{4,1}^2} & H_3(X; \mathcal{H}_2) & \xleftarrow{\quad} & H_4(X; \mathcal{H}_2) & \xleftarrow{\quad} & \cdots \\ H_0(X; \mathcal{H}_1) & \xleftarrow{d_{2,0}^2} & H_1(X; \mathcal{H}_1) & \xleftarrow{d_{3,0}^2} & H_2(X; \mathcal{H}_1) & \xleftarrow{d_{4,0}^2} & H_3(X; \mathcal{H}_1) & \xleftarrow{\quad} & H_4(X; \mathcal{H}_1) & \xleftarrow{\quad} & \cdots \\ H_0(X; \mathcal{H}_0) & \xleftarrow{d_{2,-1}^2} & H_1(X; \mathcal{H}_0) & \xleftarrow{d_{3,-1}^2} & H_2(X; \mathcal{H}_0) & \xleftarrow{d_{4,-1}^2} & H_3(X; \mathcal{H}_0) & \xleftarrow{\quad} & H_4(X; \mathcal{H}_0) & \xleftarrow{\quad} & \cdots \\ H_0(X; \mathcal{H}_{-1}) & \xleftarrow{d_{2,-2}^2} & H_1(X; \mathcal{H}_{-1}) & \xleftarrow{d_{3,-2}^2} & H_2(X; \mathcal{H}_{-1}) & \xleftarrow{d_{4,-2}^2} & H_3(X; \mathcal{H}_{-1}) & \xleftarrow{\quad} & H_4(X; \mathcal{H}_{-1}) & \xleftarrow{\quad} & \cdots \\ H_0(X; \mathcal{H}_{-2}) & & H_1(X; \mathcal{H}_{-2}) & & H_2(X; \mathcal{H}_{-2}) & & H_3(X; \mathcal{H}_{-2}) & & H_4(X; \mathcal{H}_{-2}) & & \cdots \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \end{array}$$

Example 20.3 (Homology satisfying the dimension axiom.). Let $\mathcal{H}_*$ be a homology theory taking values in R -modules satisfying the disjoint union axiom, the WHE-axiom, and the dimension axiom. Then the E^2 -terms of the Atiyah-Hirzebruch spectral sequence satisfies

$$E_{p,q}^2 \cong \begin{cases} H_p(X; \mathcal{H}_0(\{\bullet\})) & p \in \mathbb{Z}^{\geq 0}, q = 0; \\ \{0\} & \text{otherwise.} \end{cases}$$

So the second page looks like

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\
 0 & & 0 & & 0 & & 0 & & \dots \\
 0 & & 0 & & 0 & & 0 & & \dots \\
 0 & & 0 & & 0 & & 0 & & \dots
 \end{array}$$

$$H_0(X; \mathcal{H}_0(\{\bullet\})) \quad H_1(X; \mathcal{H}_0(\{\bullet\})) \quad H_2(X; \mathcal{H}_0(\{\bullet\})) \quad H_3(X; \mathcal{H}_0(\{\bullet\})) \quad \dots$$

Hence it strongly collapses and yields for $n \in \mathbb{Z}$ the well-know isomorphism

$$H_n(X; \mathcal{H}_0(\{\bullet\})) \cong \mathcal{H}_n(X).$$

Example 20.4. Let p be an odd prime. Consider a homology theory $\mathcal{H}_*$ with values in $\mathbb{F}_p$ -modules satisfying the disjoint union axiom. Let M be any $\mathbb{F}_p$ -module. Then an easy calculation using the cellular $\mathbb{Z}$ -chain complex of $\mathbb{RP}^\infty$ and the fact that $\bar{2} \in \mathbb{F}_p$ is a unit shows that $H_p(\mathbb{RP}^\infty; M)$ vanishes for $p \in \mathbb{Z}^{\geq 1}$ and is M if $p = 0$. Hence the second page of the Atiyah-Hirzebruch spectral sequence looks like

$$E_{p,q}^2 = H_p(\mathbb{RP}^\infty; \mathcal{H}_q(\{\bullet\})) = \begin{cases} \mathcal{H}_q(\{\bullet\}) & \text{if } p = 0; \\ \{0\} & \text{otherwise.} \end{cases}$$

So the second page is concentrated in the 0th column and looks like

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\
 \mathcal{H}_3(\{\bullet\}) & & 0 & & 0 & & 0 & & \dots \\
 \mathcal{H}_2(\{\bullet\}) & & 0 & & 0 & & 0 & & \dots \\
 \mathcal{H}_1(\{\bullet\}) & & 0 & & 0 & & 0 & & \dots \\
 \mathcal{H}_0(\{\bullet\}) & & 0 & & 0 & & 0 & & \dots
 \end{array}$$

Hence it collapses strongly and we get for $n \in \mathbb{Z}^{\geq 0}$

$$\mathcal{H}_n(\mathbb{RP}^\infty) \cong \mathcal{H}_n(\{\bullet\}).$$

Example 20.5 (Topological K -homology of complex projective spaces). Let K_* be the homology theory satisfying the disjoint union axiom given by complex K -homology, see Subsection 16.7. For the purpose here we only need to know that $K_n(\{\bullet\})$ is $\mathbb{Z}$ for n even and $\{0\}$ for n odd. Let $\mathbb{CP}^d$ be the complex projective space of dimension d for $d \in \mathbb{Z}^{\geq 0}$ II ∞ . Recall that it carries the structure of a

CW -complex which has precisely one cell of dimension $2n$ for $n \in \mathbb{Z}$ with $n \leq d$ and no cells in any other dimension. Hence we get for the E^2 -page

$$E_{p,q}^2 = H_p(\mathbb{CP}^d; K_q(\{\bullet\})) \cong \begin{cases} \mathbb{Z} & \text{if } p \text{ and } q \text{ are even and } p \leq 2d; \\ \{0\} & \text{otherwise.} \end{cases}$$

So the second page looks like:

$$\begin{array}{cccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\ & \swarrow & & \swarrow & & \swarrow & & \swarrow & & \swarrow & \\ \mathbb{Z} & & 0 & & \mathbb{Z} & & 0 & & \mathbb{Z} & & \dots \\ & \swarrow & & \swarrow & & \swarrow & & \swarrow & & \swarrow & \\ 0 & & 0 & & 0 & & 0 & & 0 & & \dots \\ & \swarrow & & \swarrow & & \swarrow & & \swarrow & & \swarrow & \\ \mathbb{Z} & & 0 & & \mathbb{Z} & & 0 & & \mathbb{Z} & & \dots \\ & \swarrow & & \swarrow & & \swarrow & & \swarrow & & \swarrow & \\ 0 & & 0 & & 0 & & 0 & & 0 & & \dots \\ & \swarrow & & \swarrow & & \swarrow & & \swarrow & & \swarrow & \\ \mathbb{Z} & & 0 & & \mathbb{Z} & & 0 & & \mathbb{Z} & & \dots \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \end{array}$$

This implies that for every $r \geq 2$ we have $E_{p,q}^r = 0$ if p or q is odd. Since the differential $d_{p,q}^r$ has as source $E_{p,q}^r$ and as target $E_{p-r,q+r-1}^r$, either its source or its target is trivial and hence the differential itself is trivial. This implies for all $r \geq 2$

$$E_{p,q}^2 = E_{p,q}^r = E_{p,q}^\infty \cong \begin{cases} \mathbb{Z} & \text{if } p \text{ and } q \text{ are even and } p \leq 2d; \\ \{0\} & \text{otherwise.} \end{cases}$$

Fix $n \in \mathbb{Z}$. Then we have the filtration

$$\{0\} = F_{-1,n+1} \subseteq F_{0,n} \subseteq F_{1,n-1} \subseteq F_{2,n-2} \subseteq F_{3,n-3} \subseteq \dots \subseteq K_n(\mathbb{CP}^d)$$

satisfying

$$\begin{aligned} K_n(\mathbb{CP}^d) &= \bigcup_{p \geq 0} F_{p,n-p} \\ F_{2d,0} &= K_n(\mathbb{CP}^d) \quad \text{if } d < \infty; \\ F_{p,n-p}/F_{p-1,n-p+1} &\cong \begin{cases} \mathbb{Z} & \text{if } p \text{ and } n-p \text{ are even and } p \leq 2d; \\ \{0\} & \text{otherwise.} \end{cases} \end{aligned}$$

Hence we get

$$K_n(\mathbb{CP}^d) \cong \begin{cases} \mathbb{Z}^{d+1} & \text{if } n \text{ is even and } d < \infty; \\ \bigoplus_{i=0}^\infty \mathbb{Z} & \text{if } n \text{ is even and } d = \infty; \\ \{0\} & \text{if } n \text{ is odd.} \end{cases}$$

So in this case the Atiyah-Hirzebruch spectral sequence strongly collapses.

Example 20.6 (Topological K -homology of $\mathbb{RP}^4$). We get for the E^2 -page

$$E_{p,q}^2 = H_p(\mathbb{RP}^4; K_q(\{\bullet\})) \cong \begin{cases} \mathbb{Z} & \text{if } p = 0 \text{ and } q \text{ is even;} \\ \mathbb{Z}/2 & \text{if } p = 1, 3 \text{ and } q \text{ is even;} \\ \{0\} & \text{otherwise.} \end{cases}$$

So the second page looks like:

$$\begin{array}{cccccc}
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\
 & & & & & & & & & & \\
 \mathbb{Z} & \leftarrow & \mathbb{Z}/2 & \leftarrow & 0 & \leftarrow & \mathbb{Z}/2 & \leftarrow & 0 & \leftarrow & \dots \\
 & & & & & & & & & & \\
 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \dots \\
 & & & & & & & & & & \\
 \mathbb{Z} & \leftarrow & \mathbb{Z}/2 & \leftarrow & 0 & \leftarrow & \mathbb{Z}/2 & \leftarrow & 0 & \leftarrow & \dots \\
 & & & & & & & & & & \\
 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \dots \\
 & & & & & & & & & & \\
 \mathbb{Z} & \leftarrow & \mathbb{Z}/2 & \leftarrow & 0 & \leftarrow & \mathbb{Z}/2 & \leftarrow & 0 & \leftarrow & \dots \\
 & & & & & & & & & & \\
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots
 \end{array}$$

Hence all the second differentials are trivial. So the third page looks like:

$$\begin{array}{cccccc}
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\
 & & & & & & & & & & \\
 \mathbb{Z} & \leftarrow & \mathbb{Z}/2 & \leftarrow & 0 & \leftarrow & \mathbb{Z}/2 & \leftarrow & 0 & \leftarrow & \dots \\
 & & & & & & & & & & \\
 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \dots \\
 & & & & & & & & & & \\
 \mathbb{Z} & \leftarrow & \mathbb{Z}/2 & \leftarrow & 0 & \leftarrow & \mathbb{Z}/2 & \leftarrow & 0 & \leftarrow & \dots \\
 & & & & & & & & & & \\
 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \dots \\
 & & & & & & & & & & \\
 \mathbb{Z} & \leftarrow & \mathbb{Z}/2 & \leftarrow & 0 & \leftarrow & \mathbb{Z}/2 & \leftarrow & 0 & \leftarrow & \dots \\
 & & & & & & & & & & \\
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots
 \end{array}$$

Hence all the third differentials are trivial. So the fourth page looks like:

$$\begin{array}{cccccc}
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\
 & & & & & & & & & & \\
 \mathbb{Z} & \leftarrow & \mathbb{Z}/2 & & 0 & & \mathbb{Z}/2 & & 0 & & \dots \\
 & & & & & & & & & & \\
 0 & \leftarrow & 0 & & 0 & & 0 & & 0 & & \dots \\
 & & & & & & & & & & \\
 \mathbb{Z} & & \mathbb{Z}/2 & & 0 & & \mathbb{Z}/2 & & 0 & & \dots \\
 & & & & & & & & & & \\
 0 & & 0 & & 0 & & 0 & & 0 & & \dots \\
 & & & & & & & & & & \\
 \mathbb{Z} & & \mathbb{Z}/2 & & 0 & & \mathbb{Z}/2 & & 0 & & \dots \\
 & & & & & & & & & & \\
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots
 \end{array}$$

Hence we get inductively over r

$$E_{p,q}^\infty = \begin{cases} \mathbb{Z} & \text{if } p = 0 \text{ and } q \text{ is even;} \\ \mathbb{Z}/2 & \text{if } p = 1, 3 \text{ and } q \text{ is even;} \\ \{0\} & \text{otherwise.} \end{cases}$$

This implies

$$K_0(\mathbb{RP}^4) \cong \mathbb{Z}$$

and that there is a short exact sequence

$$0 \rightarrow \mathbb{Z}/2 \rightarrow K_1(\mathbb{RP}^4) \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

Hence we know that $K_1(\mathbb{RP}^4)$ is isomorphic to either $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ or $\mathbb{Z}/4$ but we cannot decide whether it is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ or $\mathbb{Z}/4$ from the spectral sequence alone. Actually, it is $\mathbb{Z}/4$. So in this case the Atiyah-Hirzebruch spectral sequence collapses but does not collapse strongly.

20.1. Chern characters. Fix a commutative ring R with $\mathbb{Q} \subseteq R$.

For every pointed pair (Z, B, z) we have the Hurewicz homomorphism

$$\text{hur}_n(Z, B, z): \pi_n(Z, B, z) \rightarrow H_n(Z; B),$$

see (12.9). It induces for any pair (Z, B) a homomorphism of abelian groups

$$\text{hur}_n^s(Z, B): \pi_n^s(Z, B) \rightarrow H_n(Z, B).$$

It is not hard to check that this defines a transformation of homology theories with values in $\mathbb{Z}$ -modules. Note that $\text{hur}_n^s(\{\bullet\}): \pi_n^s(\{\bullet\}) \rightarrow H_n(\{\bullet\})$ is bijective for $p = 0$ and obviously surjective for all $p \geq 1$ but is not injective in general for $p \geq 1$, since $H_n(\{\bullet\})$ vanishes for $p \geq 1$. We will later show that $\pi_n^s(\{\bullet\})$ is a finite group for all $p \geq 1$, see Theorem 32.4, and we will use this fact for the remainder of this subsection. Hence the R -homomorphism $\text{hur}_n^s(\{\bullet\}) \otimes_{\mathbb{Z}} \text{id}_R: \pi_n^s(\{\bullet\}) \otimes_{\mathbb{Z}} R \rightarrow H_n(\{\bullet\}) \otimes_{\mathbb{Z}} R$ is bijective for $n \in \mathbb{Z}^{\geq 0}$. Since R is flat over $\mathbb{Z}$, we get by $\pi_n^s(Z, B) \otimes_{\mathbb{Z}} R$ a homology theory with values in R -modules satisfying the disjoint union axiom

and the WHE-axiom. This is also true for $H_n(Z, B) \otimes_{\mathbb{Z}} R \cong H_n(X, B; R)$. We conclude from Theorem 16.12 ((i))

Lemma 20.7. *For every pair (Z, B) the map*

$$\text{hur}_n^s(Z, B) \otimes_{\mathbb{Z}} \text{id}_R: \pi_n^s(Z, B) \otimes_{\mathbb{Z}} R \rightarrow H_n(Z, B) \otimes_{\mathbb{Z}} R$$

is bijective.

Next we describe a construction due to Dold [8]. Consider a homology theory $\mathcal{H}_*$ with values in R -modules. Then the Chern character for a CW -complex X is given by the following composite

$$\begin{aligned} \text{ch}_n(X): \bigoplus_{p+q=n} H_p(X; \mathcal{H}_q(\{\bullet\})) &\xrightarrow{\alpha_n} \bigoplus_{p+q=n} H_p(X; R) \otimes_R \mathcal{H}_q(\{\bullet\}) \\ &\xrightarrow{\bigoplus_{p+q=n} \text{hur}_p^s(X)^{-1} \otimes \text{id}} \bigoplus_{p+q=n} \pi_p^s(X) \otimes_{\mathbb{Z}} R \otimes_R \mathcal{H}_q(\{\bullet\}) \\ &\xrightarrow{\bigoplus_{p+q=n} D_{p,q}} \mathcal{H}_n(X). \end{aligned}$$

Here the canonical map α_n is bijective by the Universal Coefficient Theorem, since any R -module is flat over $\mathbb{Z}$ because of the assumption $\mathbb{Q} \subset R$. The second bijective map comes from the Hurewicz isomorphisms of Lemma 20.7. The map $D_{p,q}$ is defined as follows. For an element $a \otimes b \in \pi_p^s(X) \otimes_{\mathbb{Z}} \mathcal{H}_q(\{\bullet\})$ choose a representative $f: S^{p+k} \rightarrow S^k \wedge X_+$ of a . Define $D_{p,q}(a \otimes b)$ to be the image of b under the composite

$$\begin{aligned} \mathcal{H}_q(\{\bullet\}) &\xrightarrow{\cong} \mathcal{H}_q(S^0, *) \xrightarrow{\sigma'} \mathcal{H}_{p+q+k}(S^{p+k}, *) \\ &\xrightarrow{\mathcal{H}_{p+q+k}(f)} \mathcal{H}_{p+q+k}(S^k \wedge X_+, *) \xrightarrow{\sigma^{-1}} \mathcal{H}_{p+q}(X) \end{aligned}$$

where σ' and σ denote iterated suspension isomorphism. We leave it to the reader to check that this homomorphism $D_{p,q}$ is well-defined. It is not hard to show that the definition of the map $\text{ch}_n(X)$ extends to pairs (X, A) , and we thus get a transformation χ_* of homology theories with values in R -modules from $\bigoplus_{p+q=*} H_p(-; \mathcal{H}_q(\{\bullet\}))$ to $\mathcal{H}_*$, which induces an isomorphism for $X = \{\bullet\}$. Hence the following theorem follows from Theorem 16.12 (i).

Theorem 20.8. *Let R be a commutative ring with $\mathbb{Q} \subseteq R$ and $\mathcal{H}_*$ be a homology theory with values in R -modules satisfying the disjoint union axiom and the WHE-axiom.*

We get by ch_* an equivalence of homology theories with values in R -modules. In particular we get for every pair (X, A) and $n \in \mathbb{Z}$ an isomorphism, natural in (X, A)

$$\text{ch}_n(X, A): \bigoplus_{p+q=n} H_p(X, A; \mathcal{H}_q(\{\bullet\})) \xrightarrow{\cong} \mathcal{H}_n(X, A).$$

Corollary 20.9.

- (i) *Let R be a commutative ring with $\mathbb{Q} \subseteq R$ and $\mathcal{H}_*$ be a homology theory with values in R -modules satisfying the disjoint union axiom. Then the Atiyah-Hirzebruch spectral sequence converging to $\mathcal{H}_*$ collapses strongly;*
- (ii) *Let $\mathcal{H}_*$ be a homology theory with values in $\mathbb{Z}$ -modules satisfying the disjoint union axiom. Then every differential $d_{p,q}^r$ vanishes rationally, i.e., $\text{id}_{\mathbb{Q}} \otimes_{\mathbb{Z}} d_{p,q}^r$ vanishes, or, equivalently, its image consists only of torsion elements.*

Proof. (i) Since the Chern character is a transformation of homology theories, it induced an isomorphism of the Atiyah-Hirzebruch spectral sequence converging to homology theory $\bigoplus_{p+q=*} H_p(-; \mathcal{H}_q(\{\bullet\}))$ to $\mathcal{H}_*$ and the one converging to $\mathcal{H}_*$. The

one for $\bigoplus_{p+q=*} H_p(-; \mathcal{H}_q(\{\bullet\}))$ strongly collapses, since the Atiyah-Hirzebruch spectral sequence is compatible with direct sums of homology theories and obvious strongly collapses for singular homology, see Example 20.3

(ii) If $\mathcal{H}_*$ is a homology theory with values in $\mathbb{Z}$ -modules satisfying the disjoint union axiom, then $\mathcal{H}_* \otimes_{\mathbb{Z}} \mathbb{Q}$ is a homology theory with values in $\mathbb{Q}$ -modules satisfying the disjoint union axiom. Since $\mathbb{Q}$ is flat over $\mathbb{Z}$, the Atiyah-Hirzebruch spectral sequence converging to $\mathcal{H}_* \otimes_{\mathbb{Z}} \mathbb{Q}$ is obtained from the Atiyah-Hirzebruch spectral sequence converging to $\mathcal{H}_*$ by applying the functor $- \otimes_{\mathbb{Z}} \mathbb{Q}$. Now apply assertion (i). $\square$

21. (CO)HOMOLOGY WITH LOCAL COEFFICIENTS

21.1. Modules over a category. Let $\mathcal{C}$ be a small category. Our main example for $\mathcal{C}$ will be the fundamental groupoid $\Pi(X)$ of a space X . Let R be a commutative ring. A *covariant or contravariant RC-module* is a covariant or contravariant functor M from $\mathcal{C}$ to the category $R\text{-Mod}$ of R -modules. A morphism between a covariant or contravariant RC -module is a natural transformation of such functors. Let $RC\text{-Mod}$ and $\text{Mod-}RC$ respectively be the category of covariant and contravariant respectively RC -modules. One easily checks that $RC\text{-Mod}$ and $\text{Mod-}RC$ inherits from $R\text{-Mod}$ the structure of an abelian category. For instance a sequence of RC -modules $L \xrightarrow{i} M \xrightarrow{p} N$ is exact at M if for any object $x \in \mathcal{C}$ the sequence of R -modules $L(x) \xrightarrow{i(x)} M(x) \xrightarrow{p(x)} N(x)$ is exact at $M(x)$. The kernel of a morphism $f: M \rightarrow N$ of RC -modules is defined to be the RC -module whose value at an object $x \in \mathcal{C}$ is the kernel of the R -homomorphism $f(x): M(x) \rightarrow N(x)$.

Given a contravariant RC -module M and a covariant RC -module N , their *tensor product* $M \otimes_{RC} N$ is defined to be the following R -module. Consider the R -module $\bigoplus_{x \in \text{ob}(\mathcal{C})} M(x) \otimes N(x)$. Let T be the R -submodule of M generated by the subset $\{mf \otimes n - m \otimes fn \mid x, y \in \text{ob}(\mathcal{C}), f \in \text{mor}_{\mathcal{C}}(x, y), m \in M(y), n \in N(x)\}$, where mf stands for $M(f)(m)$ and fn for $N(f)(n)$. Define

$$(21.1) \quad M \otimes_{RC} N := \left(\bigoplus_{x \in \text{ob}(\mathcal{C})} M(x) \otimes N(x) \right) / T.$$

Given two contravariant RC -modules M and N , define the R -module $\text{hom}_{RC}(M, N)$ to be the R -module whose underlying set is $\text{mor}_{RC}(M, N)$, i.e., the set of transformations from $M \rightarrow N$. The R -module structure comes from

$$(r_1 \cdot f_1 + r_2 \cdot f_2)(x) = r_1 \cdot f_1(x) + r_2 \cdot f_2(x)$$

for $r_1, r_2 \in R, f_1, f_2 \in \text{mor}_{RC}(M, N)$, and $x \in \text{ob}(\mathcal{C})$.

Let $\mathcal{C}$ and $\mathcal{D}$ be two small categories. An RC - RD -bimodule B is a covariant $R(\mathcal{C} \times \mathcal{D}^{\text{op}})$ -module. Let M be a contravariant RC -module. Then for any object $d \in \mathcal{D}$ we obtain a covariant RC -module $B(?, d)$ by freezing the variable in $\mathcal{D}$ and hence an R -module $M \otimes_{RC} B(?, d)$. This becomes in the obvious way a contravariant RD -module by assigning to a morphism $u: d \rightarrow d'$ the R -homomorphism $M \otimes_{RC} B(?, u): M \otimes_{RC} B(?, d') \rightarrow M \otimes_{RC} B(?, d)$. We denote this contravariant RD -module by $M \otimes_{RC} B$. Let N be a contravariant RD -module. For any object $c \in \mathcal{C}$ we obtain a contravariant RD -module $B(c, ??)$ and can consider the R -module $\text{hom}_{RD}(B(c, ??), N)$. Using functoriality in c we obtain a contravariant RC -module which we will denote by $\text{hom}_{RD}(B, N)$. Define a R -homomorphism

$$(21.2) \quad \text{ad}: \text{hom}_{RD}(M \otimes_{RC} B, N) \rightarrow \text{hom}_{RC}(M, \text{hom}_{RD}(B, N))$$

by sending the homomorphism of contravariant RD -modules $\phi: M \otimes_{RC} B \rightarrow N$ to the homomorphism of contravariant RC -modules $\text{ad}(\phi): M \rightarrow \text{hom}_{RD}(B, N)$ which assigns to $m \in M(x)$ for $x \in \text{ob}(\mathcal{C})$ the RD -homomorphism $B(x, ??) \rightarrow N(??)$

sending $b \in B(x, ??)$ to $\phi(??)(m \otimes b)$, where $m \otimes b$ is the element in $(M \otimes_{RC} B)(??)$ represented by $m \otimes b \in M(x) \otimes_R B(x, ??)$. The elementary proof of the next lemma is left to the reader.

Lemma 21.3. *The R -homomorphism ad of (21.2) is bijective and natural in M , N , and B .*

Let $C_* = (C_*, c_*)$ be a contravariant RC -chain complex. If M is a covariant RC -module, we have the R -chain complex $C_* \otimes_{RC} M$. If N is a contravariant RC -module, we have the R -cochain complex $\text{hom}_{RC}(C_*, N)$.

Given an object c in $\mathcal{C}$, we define a covariant or contravariant RC -module respectively by $R\text{mor}_{\mathcal{C}}(c, ?)$ and $R\text{mor}_{\mathcal{C}}(?, c)$, where here and in the sequel we denote for a set S by RS the free R -module with R -basis, i.e., the R -module of all maps $u: S \rightarrow Z$ for which $\{s \in S \mid u(s) \neq 0\}$ is finite.

Lemma 21.4 (Yoneda Lemma). *Let c be an object, M be a contravariant, and N be covariant RC -module. Then the R -homomorphisms*

$$\text{hom}_{RC}(R\text{mor}_{\mathcal{C}}(?, c), M) \rightarrow M(c), \quad \varphi \mapsto \varphi(c)(\text{id}_c)$$

and

$$R\text{mor}_{\mathcal{C}}(?, c) \otimes_{RC} N \rightarrow N(c), \quad u \otimes n \mapsto \varphi(u)(n)$$

are bijective. The analogous statement holds for $R\text{mor}_{\mathcal{C}}(c, ?)$.

A systematic study of RC -modules can be found in [16, Section 9 and 10]. We will only be interested in the special and easy case where $\mathcal{C}$ is a groupoid $\mathcal{G}$, i.e., a small category in which every morphism is an isomorphism.

Example 21.5. Let H and G be groups and R a commutative ring. Denote by $\mathcal{I}(G)$ the groupoid having precisely one object whose automorphism group is G . Then the category of covariant $R\mathcal{I}(G)$ -modules is the category of left RG -modules, whereas the category of contravariant $R\mathcal{I}(G)$ -modules is the category of right RG -modules. Given a contravariant $R\mathcal{I}(G)$ -module M and a covariant $R\mathcal{I}(G)$ -module N , the R -module $M \otimes_{R\mathcal{I}(G)} N$ and $M \otimes_{RG} N$ agree. Given contravariant $R\mathcal{I}(G)$ -modules M and N , the R -modules $\text{hom}_{R\mathcal{I}(G)}(M, N)$ and $\text{hom}_{RG}(M, N)$ agree. Given a contravariant $R\mathcal{I}(G)$ -module M , a contravariant $R\mathcal{I}(H)$ -module N , and a $R\mathcal{I}(G)$ - $R\mathcal{I}(H)$ -bimodule B , then the adjunction isomorphism ad of (21.2) reduces to the well-known adjunction isomorphism

$$\text{hom}_{RH}(M \otimes_{RG} B, N) \xrightarrow{\cong} \text{hom}_{RG}(M, \text{hom}_{RH}(B, N)).$$

Remark 21.6. Let $\mathcal{G}$ be a connected groupoid, where connected means that between any two objects there exists a morphism. The fundamental groupoid $\Pi(X)$ is connected if and only if X is path connected. Consider an object x in $\mathcal{G}$. Let $\text{aut}_{\mathcal{G}}(x)$ be the group of automorphisms of x in $\mathcal{G}$. We denote the group ring $R[\text{aut}_{\mathcal{G}}(x)]$ by $R[x]$. Let C_* be a contravariant $R\mathcal{G}$ -chain complex. Let M be a covariant $R\mathcal{G}$ -module and let N be a contravariant $R\mathcal{G}$ -module. Then $C_*(x)$ is a chain complex of right $R[x]$ -modules, $M(x)$ is a left $R[x]$ -module, and $N(x)$ is a right $R[x]$ -module and we obtain obvious isomorphisms of R -chain complexes

$$\begin{aligned} C_*(x) \otimes_{R[x]} M(x) &\xrightarrow{\cong} C_* \otimes_{RG} M; \\ \text{hom}_{R[x]}(C_*(x), N(x)) &\xrightarrow{\cong} \text{hom}_{RG}(C_*, N). \end{aligned}$$

21.2. (Co)homology for local coefficient systems coming from a fibration.

Let X be a connected CW -complex. Recall that its fundamental groupoid $\Pi(X)$ has as objects elements in X and a morphism from x to y is a homotopy class relative endpoints $[w]$ of a path $w: [0, 1] \rightarrow X$ with $w(0) = x$ and $w(1) = y$. Composition

comes from concatenation of paths. Next we define for a connected CW -complex X a contravariant functor

$$(21.7) \quad \tilde{X}: \Pi(X) \rightarrow CW\text{-compl.}$$

to the category of CW -complexes. Define $\tilde{X}(x)$ to be the set of morphisms in $\Pi(X)$ with x as source and arbitrary target. The constant path c_x at x defines a preferred base point $\tilde{x} \in \tilde{X}(x)$. Let $\text{ev}_x: (\tilde{X}(x), \tilde{x}) \rightarrow (X, x)$ be the pointed map sending a morphism $[w]: x \rightarrow y$ to y . Now there is precisely one topology on $\tilde{X}(x)$ for which $\text{ev}_x: \tilde{X} \rightarrow X$ is a model for the universal covering of X . A CW -structure on $\tilde{X}$ is given by defining the n -th skeleton to be the preimage of the n -skeleton of X under ev_x . Given a morphism $[v]: x \rightarrow y$ in $\Pi(X)$, we obtain a cellular map $\tilde{X}(y) \rightarrow \tilde{X}(x)$ by precomposition in $\Pi(X)$, i.e., by sending $[w]: y \rightarrow z$ in $\tilde{X}(y)$ to $[w] \circ [v]: x \rightarrow z$.

Now we can compose this functor with the functor from $CW\text{-compl.}$ to the category of R -chain complexes given by taking the cellular chain complex with R -coefficients and obtain a contravariant $R\Pi(X)$ -chain complex called *cellular $R\Pi(X)$ -chain complex*

$$(21.8) \quad C_*^\Pi(X; R): \Pi(X) \rightarrow R\text{-Ch. Compl.}, \quad x \mapsto C_*(\tilde{X}(x)).$$

For $n \in \mathbb{Z}^{\geq 0}$ consider a pushout

$$(21.9) \quad \begin{array}{ccc} \coprod_{i \in I_n} S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & X_n. \end{array}$$

Recall that the existence of such a pushout is required for a CW -structure on X but a choice of it is not part of the CW -structure on X . Note that I_n can be identified with the set of open n -cells of X . Put $x_i^n = Q_i^n(0)$ for the origin $0 \in D^n$. There is precisely one map $\tilde{q}_i^n: D^n \rightarrow \tilde{X}(x_i^n)$ such that $\tilde{q}_i^n$ sends the origin $0 \in D^n$ to $\tilde{x}_i^n$ and satisfies $\text{ev}_{x_i} \circ \tilde{q}_i^n = q_i^n$. Let $\sigma_n: H_0(\{\bullet\}; R) \xrightarrow{\cong} H_n(D^n, S^{n-1}; R)$ be the suspension isomorphism. Then the image of the element in $H_0(\{\bullet\}; R)$ represented by the singular 0-simplex given by the homeomorphism $\Delta_0 \rightarrow \{\bullet\}$ under σ_n is denoted by $[D^n, S^{n-1}] \in H_n(D^n, S^{n-1}; R)$ and is called the standard generator. The map $\tilde{Q}_i^n$ yields maps of pairs $\tilde{Q}_i^n: (D^n, S^{n-1}) \rightarrow (\tilde{X}(x_i^n)_n, \tilde{X}(x_i^n)_{n-1})$. Define the element $a_i^n \in C_n^\Pi(X; R)(x_i^n)$ to be the image of $[D^n, S^{n-1}]$ under the induced homomorphisms $H_n(\tilde{Q}_i^n; R): H_n(D^n, S^{n-1}; R) \rightarrow H_n(\tilde{X}(x_i^n)_n, \tilde{X}(x_i^n)_{n-1}; R) = C_n^\Pi(X; R)(x_i^n)$. Because of Lemma 21.4 there is precisely one $R\Pi(X)$ -homomorphism $\overline{a}_i^n: R\text{mor}_{\Pi(X)}(?, x_i^n) \rightarrow C_n^\Pi(X; R)$ sending $\text{id}_{x_i^n} \in R\text{mor}_{\Pi(X)}(?, x_i^n)$ to a_i^n . We leave the elementary proof of the next lemma to the reader.

Lemma 21.10. *The $R\Pi(X)$ -homomorphism*

$$\bigoplus_{i \in I_n} R\text{mor}_{\Pi(X)}(?, x_i^n) \rightarrow C_n^\Pi(X; R)$$

is bijective for all $n \in \mathbb{Z}^{\geq 0}$.

Definition 21.11. Let X be a CW -complex and M be a covariant $R\Pi(X)$ -module and N be a contravariant $R\Pi(X)$ -module. Then we define the (co)homology of X with coefficients in M or N to be

$$\begin{aligned} H_n^\Pi(X; M) &= H_n(C_*^\Pi(X; R) \otimes_{R\Pi(X)} M); \\ H_n^\Pi(X; N) &= H^n(\text{hom}_{R\Pi(X)}(C_*^\Pi(X; R), N)). \end{aligned}$$

Lemma 21.12. *Let M be an R -module. Let $C_M: \Pi(X) \rightarrow R\text{-Mod}$ be the constant functor with value M . We can view C_M as a covariant and as a contravariant $R\Pi(X)$ -module. Then we get isomorphism of R -(co)chain complexes*

$$\begin{aligned} C_*^\Pi(X; R) \otimes_{R\Pi(X)} C_M &\xrightarrow{\cong} C_*(X; R) \otimes_R M; \\ \text{hom}_{R\Pi(X)}(C_*^\Pi(X; R), C_M) &\xrightarrow{\cong} \text{hom}_R(C_*(X; R), M), \end{aligned}$$

where $C_*(X; R)$ is the cellular R -chain complex of the CW-complex X .

Proof. The desired isomorphisms are induced by the cellular maps $\text{ev}_x: \tilde{X}(x) \rightarrow X$. $\square$

Example 21.13 (Coefficient system of a fibration). Let $p: E \rightarrow B$ be a fibration over a connected CW-complex B . We have assigned to it a covariant functor called fiber transport

$$\tau: \Pi(B) \rightarrow \mathbf{h}\text{-Top}, \quad b \rightarrow F_b = p^{-1}(b)$$

in (9.14). Let $\mathcal{H}_*$ be a homology theory with values in R -modules. Then we get for every $q \in \mathbb{Z}$ a covariant $R\Pi(B)$ -module

$$(21.14) \quad \mathcal{H}_q^{\pi(X)}(F): \Pi(B) \rightarrow R\text{-Mod}, \quad b \rightarrow \mathcal{H}_q(\tau(b)).$$

Hence we can consider the R -chain complex $C_*^\Pi(B) \otimes_{R\Pi(B)} \mathcal{H}_q^{\pi(X)}(F)$. We define

$$(21.15) \quad H_p^\Pi(B; \mathcal{H}_q(F)) := H_p(C_*^\Pi(B) \otimes_{R\Pi(B)} \mathcal{H}_q^{\pi(X)}(F)).$$

This notion will be relevant for us as it appears as the E^2 -term in the Leray-Serre spectral sequence.

We conclude directly from Remark 21.6.

Remark 21.16. Consider the situation of Example 21.13. Choose a base point $b \in B$. Let $\pi = \pi_1(B, b)$ and consider any model for the universal covering $\tilde{B} \rightarrow B$ of B . Recall that the cellular chain complex $C_*^c(\tilde{B})$ is a free $R\pi$ -chain complex. We get a $R\pi$ -module $\mathcal{H}_q(F_b)$ from the R -module $\mathcal{H}_q(F_b)$ and the π -action coming from the fiber transport. We conclude directly from Remark 21.6 that we obtain an isomorphism of R -modules

$$H_p^\Pi(B; \mathcal{H}_q(F)) \cong H_p(C_*^c(\tilde{B}) \otimes_{R\pi} \mathcal{H}_q(F_b))$$

So we can compute $H_p^\Pi(B; \mathcal{H}_q(F))$ in terms of the group ring $R\pi$.

The situation simplifies considerably in the following favourite cases, where we just have to look at classical singular homology with coefficients in an R -module and do not have to pass to group rings and the universal covering.

Lemma 21.17. *Let $p: E \rightarrow B$ be a fibration over a connected CW-complex B . Let $\mathcal{H}_*$ be a homology theory with values in R -modules. Suppose that one of the following conditions is satisfied:*

- (i) *For one (and hence all) $b \in B$ the following is true: For every loop w in B at b the map $\mathcal{H}_q(\tau([w])): \mathcal{H}_q(F_b) \rightarrow \mathcal{H}_q(F_b)$ is the identity;*
- (ii) *The fibration is orientable, i.e., for one (and hence all) $b \in B$ the following is true: The fiber transport satisfies $\tau([w]) = [\text{id}_{F_b}]$ for every loop w in B at b ;*
- (iii) *$p: E \rightarrow B$ is a principal G -bundle for a path connected topological group G .*
- (iv) *The space B is simply connected.*

Then there is an isomorphism

$$H_p^\Pi(X; \mathcal{H}_q(F)) \xrightarrow{\cong} H_p(X; \mathcal{H}_q(F))$$

where $H_p(X; \mathcal{H}_q(F))$ is the singular homology of X with coefficients in the R -module $\mathcal{H}_q(F_b)$ for some $b \in B$.

Proof. If condition (i) holds, this follows from Lemma 21.12. One easily checks the implications (iii) $\implies$ (ii) $\implies$ (i), and (iv) $\implies$ (ii). $\square$

21.3. Poincaré duality for non-orientable closed manifolds. Before we turn to the Leray-Serre spectral sequence, we revisit Poincaré duality. Let M be a connected closed manifold of dimension d . Let $TM \rightarrow M$ be its tangent bundle. Then we get a 1-dimensional real vector bundle $\Lambda^d TM \rightarrow M$. It defines $\mathbb{Z}/2$ -principal bundle $o_M: \overline{M} \rightarrow M$ called *orientation covering*, if we define $\overline{M}$ to be the quotient of $\Lambda^d TM \setminus \{0\}$ under the equivalence relation $v \sim w$ generated by identifying $v, w \in T_x M$ for $x \in M$ if there exists $r \in \mathbb{R}^{>0}$ with $v = r \cdot w$. If we choose a Riemannian metric on TM , we can consider the associated sphere bundle $STM \rightarrow M$. The choice of the Riemannian metric will not matter in the sequel, since the isomorphism type of the locally trivial bundle $p_{STM}: STM \rightarrow M$ with typical fiber S^{d-1} is independent of it. Since $p_{STM}: STM \rightarrow M$ is in particular a fibration, we have the fiber transport

$$\tau: \Pi(M) \rightarrow \mathbf{h}\text{-Top}, \quad b \rightarrow p_{STM}^{-1}(b)$$

In particular we get a $\mathbb{Z}\Pi(M)$ -module $O_M: \Pi(M) \rightarrow \mathbb{Z}\text{-Mod}$ by sending $x \in M$ to $H_{d-1}(p^{-1}(b))$.

Definition 21.18 (First Stiefel-Whitney class). A covariant or contravariant $\mathbb{Z}\Pi(M)$ -module O is called *infinite cyclic* if $O(x)$ is an infinite cyclic group for all $x \in M$.

Define the *Stiefel-Whitney class of an infinite cyclic $\mathbb{Z}\Pi(M)$ -module O* to be the element

$$w_1(O) \in H^1(M; \mathbb{Z}/2)$$

coming after a choice of an element $x \in X$ from the group homomorphism $\pi_1(X, x) \rightarrow \mathbb{Z}/2$ which sends a loop to zero, if $O(w): O(x) \xrightarrow{\cong} O(x)$ is the identity, and to the non-trivial element otherwise, the isomorphism $H^1(M; \mathbb{Z}/2) \xrightarrow{\cong} \text{hom}_{\mathbb{Z}}(H_1(M; \mathbb{Z}), \mathbb{Z}/2)$, and the Hurewicz homomorphism $\pi_1(X, x) \rightarrow H_1(M; \mathbb{Z})$.

Define the first *Stiefel-Whitney class of M* to be

$$w_1(M) = w_1(O_M) \in H^1(M; \mathbb{Z}/2).$$

We leave it to the reader figure out the elementary proofs of the following three lemmas.

Lemma 21.19. *Two infinite cyclic (covariant or contravariant) $\mathbb{Z}\Pi(X)$ -modules O and O' are isomorphic as $\mathbb{Z}\Pi(X)$ -modules if and only if $w_1(O) = w_1(O')$. Given an infinite cyclic (covariant or contravariant) $\mathbb{Z}\Pi(X)$ -module O , there are precisely two automorphisms of it, namely $\pm \text{id}$.*

Consider the orientation covering $o_M: \overline{M} \rightarrow M$. Then there is up to homotopy precisely one map $c_{o_M}: M \rightarrow \mathbb{RP}^\infty$ such that o_M is isomorphic as principal $\mathbb{Z}/2$ -bundle to the pullback with c_M with the universal covering $S^\infty \rightarrow \mathbb{RP}^\infty$. Recall that $H^1(\mathbb{RP}^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

Lemma 21.20. *Then $w_1(M)$ is the image of the generator of $H^1(\mathbb{RP}^\infty; \mathbb{Z}/2)$ under the homomorphism $H^1(c_{o_M}; \mathbb{Z}/2): H^1(\mathbb{RP}^\infty; \mathbb{Z}/2) \rightarrow H^1(M; \mathbb{Z}/2)$.*

Lemma 21.21. *The following assertions are equivalent:*

- (i) M is $\mathbb{Z}$ -orientable in the sense of [17, Definition 8.2 on page 129];

- (ii) $H_d(M; \mathbb{Z})$ is infinite cyclic;
- (iii) $H_d(M; \mathbb{Z})$ is non-trivial;
- (iv) The $\mathbb{Z}\pi(X)$ -module O_M is isomorphic to the constant $\mathbb{Z}\Pi(M)$ -module $C_{\mathbb{Z}}$ with value $\mathbb{Z}$;
- (v) The first Stiefel-Whitney class $w_1(M) \in H^1(M; \mathbb{F}_2)$ vanishes;
- (vi) The tangent bundle TM is orientable as a real vector bundle;
- (vii) The 1-dimensional real vector bundle $\Lambda^d TM \rightarrow M$ is orientable;
- (viii) The 1-dimensional real vector bundle $\Lambda^d TM \rightarrow M$ has a nowhere vanishing section;
- (ix) The 1-dimensional real vector bundle $\Lambda^d TM \rightarrow M$ is trivial;
- (x) The orientation covering o_M is trivial as principal $\mathbb{Z}/2$ -bundle;
- (xi) $\overline{M}$ is not connected.

Next we explain that there is a notion of Poincaré duality for non-orientable manifolds. Let M be a connected closed manifold of dimension d . We have the R -chain complex $C_*^\Pi(M) \otimes_{\mathbb{Z}\Pi(M)} O_M$. Define the R -chain complex $\text{hom}_{\mathbb{Z}\Pi(M)}(C_{d-*}^\Pi(\widetilde{M}), O_M)$ by defining its n -th differential to be

$$(-1)^{d-n+1} \cdot \text{hom}_{\mathbb{Z}\Pi(M)}(c_{d-(n-1)}^\Pi, O_M): \text{hom}_{\mathbb{Z}\Pi(M)}(C_{d-n}^\Pi(\widetilde{M}), O_M) \rightarrow \text{hom}_{\mathbb{Z}\Pi(M)}(C_{d-(n-1)}^\Pi(\widetilde{M}), O_M)$$

for $c_{d-(n-1)}^\Pi: C_{d-(n-1)}^\Pi(\widetilde{M}) \rightarrow C_{d-n}^\Pi(\widetilde{M})$ the $d - (n - 1)$ -differential of the $R\Pi(M)$ -chain complex $C_*^\Pi(\widetilde{M})$. Define

$$H_n^\Pi(M; O_M) := H_n(C_*^\Pi(\widetilde{M}) \otimes_{\mathbb{Z}\Pi(M)} O_M);$$

$$H_{\Pi}^n(M; O_M) := H_{d-n}(\text{hom}_{\mathbb{Z}\Pi(M)}(C_{d-*}^\Pi(\widetilde{M}), O_M)).$$

An element $u \in H_d^\Pi(M; O_M)$ defines R -chain maps unique up to R -chain homotopy

$$\begin{aligned} \text{hom}_{\mathbb{Z}\Pi(M)}(C_{d-*}^\Pi(\widetilde{M}), O_M) &\rightarrow C_*^c(M) \\ \text{hom}_{\mathbb{Z}}(C_{d-*}^c(M), \mathbb{Z}) &\rightarrow C_*^\Pi(M) \otimes_{\mathbb{Z}\Pi(M)} O_M; \end{aligned}$$

where $C_*^c(M)$ is the cellular $\mathbb{Z}$ -chain complex of M and hence for every $n \in \mathbb{Z}$ R -homomorphisms

$$(21.22) \quad - \cap u: H_{\Pi}^{d-n}(M; O_M) \rightarrow H_n(M; \mathbb{Z});$$

$$(21.23) \quad - \cap u: H^{d-n}(M; \mathbb{Z}) \rightarrow H_n^\Pi(M; O_M).$$

Theorem 21.24 (Poincaré duality). *Let M be a connected closed manifold of dimension d . Then $H_d^\Pi(M; O_M)$ is infinite cyclic. Let $[M] \in H_d^\Pi(M; O_M)$ be a generator. Then the $\mathbb{Z}$ -homomorphisms (21.22) and (21.23) for $u = [M]$*

$$\begin{aligned} - \cap u: H_{\Pi}^{d-n}(M; O_M) &\rightarrow H_n(M; \mathbb{Z}); \\ - \cap u: H^{d-n}(M; \mathbb{Z}) &\rightarrow H_n^\Pi(M; O_M), \end{aligned}$$

are bijective for $n \in \mathbb{Z}$.

Remark 21.25. If M is orientable, then $H_d^\Pi(M; O_M)$ reduces to $H_d(M, \mathbb{Z})$ and $[M] \in H_d(M; \mathbb{Z})$ is the classical fundamental class. Moreover the $\mathbb{Z}$ -homomorphisms (21.22) and (21.23) reduce by Lemma 21.12 to the classical Poincaré duality isomorphisms

$$- \cap [M]: H^{d-n}(M; \mathbb{Z}) \xrightarrow{\cong} H_n(M; \mathbb{Z}).$$

The proof of Theorem 21.24 is a variation of the proof in the oriented case.

So Theorem 21.24 essentially says that Poincaré duality holds for (not necessarily $\mathbb{Z}$ -oriented) connected closed manifolds, one has just to replace $[M] \in H_d(M; \mathbb{Z})$ by

$[M] \in H_d^\Pi(M; O_M)$ and the cohomology or the homology by the versions twisted by the infinite cyclic $\mathbb{Z}\Pi(M)$ -module O_M .

Example 21.26 (Real projective spaces). Consider the d -dimensional real projective space $\mathbb{RP}^d$ for $d \in \mathbb{Z}^{\geq 1}$. Recall that its fundamental group is $\mathbb{Z}/2$. Its universal covering is the principal $\mathbb{Z}/2$ -bundle $p: S^n \rightarrow \mathbb{RP}^n$, where $\mathbb{Z}/2$ acts on S^n by the antipodal map $a_n: S^n \rightarrow S^n$ coming from $-\text{id}_{\mathbb{R}^{n+1}}: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$. The cellular $\mathbb{Z}[\mathbb{Z}/2]$ -chain complex $C_*(\mathbb{RP}^d) = C_*(S^d)$ is d -dimensional. Its n -th chain module is $\mathbb{Z}[\mathbb{Z}/2]$ for $0 \leq n \leq d$ and trivial otherwise. Let $t \in \mathbb{Z}/2$ be the generator. The n -th differential is $t - 1: \mathbb{Z}[\mathbb{Z}/2] \rightarrow \mathbb{Z}[\mathbb{Z}/2]$ for $1 \leq n \leq d$ and odd n and is $t + 1: \mathbb{Z}[\mathbb{Z}/2] \rightarrow \mathbb{Z}[\mathbb{Z}/2]$ for $1 \leq n \leq d$ and even n . Hence $C_*(S^d)$ looks like

$$\begin{aligned} \cdots \rightarrow \{0\} \rightarrow \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{t+(-1)^d} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{t-(-1)^d} \cdots \rightarrow \cdots \\ \xrightarrow{t+1} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/2] \rightarrow \{0\} \rightarrow \cdots \end{aligned}$$

Denote by $\mathbb{Z}$ and $\mathbb{Z}^-$ respectively the $\mathbb{Z}[\mathbb{Z}/2]$ -module whose underlying abelian group is $\mathbb{Z}$ and on which the generator of $\mathbb{Z}/2$ acts by $\text{id}_{\mathbb{Z}}$ and $-\text{id}_{\mathbb{Z}}$ respectively. Then $C_*(S^d) \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \mathbb{Z}$ looks like

$$\cdots \rightarrow \{0\} \rightarrow \mathbb{Z} \xrightarrow{1+(-1)^d} \mathbb{Z} \xrightarrow{1-(-1)^d} \cdots \rightarrow \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow \{0\} \rightarrow \cdots,$$

$C_*(S^d) \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \mathbb{Z}^-$ looks like

$$\cdots \rightarrow \{0\} \rightarrow \mathbb{Z} \xrightarrow{-1+(-1)^d} \mathbb{Z} \xrightarrow{-1-(-1)^d} \cdots \rightarrow \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{-2} \mathbb{Z} \rightarrow \{0\} \rightarrow \cdots,$$

$\text{hom}_{\mathbb{Z}[\mathbb{Z}/2]}(C_{d-*}(S^d), \mathbb{Z})$ looks like

$$\cdots \rightarrow \{0\} \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{-2} \cdots \rightarrow \cdots \xrightarrow{1-(-1)^d} \mathbb{Z} \xrightarrow{1+(-1)^d} \mathbb{Z} \rightarrow \{0\} \rightarrow \cdots,$$

and $\text{hom}_{\mathbb{Z}[\mathbb{Z}/2]}(C_{d-*}(S^d), \mathbb{Z}^-)$ looks like

$$\cdots \rightarrow \{0\} \rightarrow \mathbb{Z} \xrightarrow{-2} \mathbb{Z} \xrightarrow{0} \cdots \rightarrow \cdots \xrightarrow{-1-(-1)^d} \mathbb{Z} \xrightarrow{-1+(-1)^d} \mathbb{Z} \rightarrow \{0\} \rightarrow \cdots.$$

Let T be the constant infinite cyclic $\mathbb{Z}\Pi(\mathbb{RP}^n)$ -module with value $\mathbb{Z}$ and let T^- be the infinite cyclic $\mathbb{Z}\Pi(\mathbb{RP}^n)$ -module which is not isomorphic to T . We get from Lemma 21.6 and Lemma 21.12 isomorphism of $\mathbb{Z}$ -chain complexes

$$\begin{aligned} C_*^\Pi(\widetilde{\mathbb{RP}^d}) \otimes_{\mathbb{Z}\Pi(\mathbb{RP}^d)} T &\cong C_*(\mathbb{RP}^d); \\ C_*^\Pi(\widetilde{\mathbb{RP}^d}) \otimes_{\mathbb{Z}\Pi(\mathbb{RP}^d)} T^- &\cong C_*(S^d) \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \mathbb{Z}^- \\ \text{hom}_{\mathbb{Z}\Pi(\mathbb{RP}^d)}(C_{d-*}(\widetilde{\mathbb{RP}^d}), T) &\cong \text{hom}_{\mathbb{Z}[\mathbb{Z}/2]}(C_{d-*}(S^d), \mathbb{Z}); \\ \text{hom}_{\mathbb{Z}\Pi(\mathbb{RP}^d)}(C_{d-*}(\widetilde{\mathbb{RP}^d}), T^-) &\cong \text{hom}_{\mathbb{Z}}(C_{d-*}(\mathbb{RP}^d), \mathbb{Z}^-). \end{aligned}$$

Hence we get

$$\begin{aligned}
H_n(C_*^\Pi(\widetilde{\mathbb{RP}^d}) \otimes_{\mathbb{Z}\Pi(\mathbb{RP}^d)} \otimes T) &\cong \begin{cases} \mathbb{Z} & \text{if } n = 0; \\ \mathbb{Z} & \text{if } n = d \text{ and } d \text{ is odd;} \\ \mathbb{Z}/2 & \text{if } 1 \leq n \leq d-1 \text{ and } n \text{ is odd;} \\ \{0\} & \text{otherwise.} \end{cases} \\
H_n(C_*^\Pi(\widetilde{\mathbb{RP}^d}) \otimes_{\mathbb{Z}\Pi(\mathbb{RP}^d)} \otimes T^-) &\cong \begin{cases} \mathbb{Z}/2 & \text{if } 0 \leq n \leq d \text{ and } n \text{ is even;} \\ \mathbb{Z} & \text{if } n = d \text{ and } d \text{ is even;} \\ \{0\} & \text{otherwise.} \end{cases} \\
H^n(\text{hom}_{\mathbb{Z}\Pi(\mathbb{RP}^d)}(C_{d-*}(\widetilde{\mathbb{RP}^d}), T)) &\cong \begin{cases} \mathbb{Z} & \text{if } n = d; \\ \mathbb{Z} & \text{if } n = 0 \text{ and } d \text{ is odd;} \\ \mathbb{Z}/2 & \text{if } 1 \leq n \leq d-1 \text{ and } d-n \text{ is odd;} \\ \{0\} & \text{otherwise.} \end{cases} \\
H^n(\text{hom}_{\mathbb{Z}\Pi(\mathbb{RP}^d)}(C_{d-*}(\widetilde{\mathbb{RP}^d}), T^-)) &\cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \text{ and } d \text{ is even;} \\ \mathbb{Z}/2 & \text{if } 0 \leq n \leq d \text{ and } d-n \text{ is odd;} \\ \{0\} & \text{otherwise.} \end{cases}
\end{aligned}$$

The following facts are essentially consequences of Lemma 21.21. Since $H_d(\mathbb{RP}^d; \mathbb{Z})$ is $\mathbb{Z}$ if d is even, and is $\{0\}$ if d is odd, $\mathbb{RP}^d$ is $\mathbb{Z}$ -orientable if and only if d is even. This is consistent with the fact that the antipodal map $a: S^d \rightarrow S^d$ has degree 1 if d is odd and degree -1 if d is even. The orientation covering of $\mathbb{RP}^d$ is the universal covering $S^d \rightarrow \mathbb{RP}^d$ if d is even, and the trivial covering $\mathbb{Z}/2 \times \mathbb{RP}^d \rightarrow \mathbb{RP}^d$ if d is odd. The first Stiefel-Whitney class $w_1(\mathbb{RP}^d) \in H^d(\mathbb{RP}^d; \mathbb{Z}/2) \cong \mathbb{Z}/2$ is trivial if and only if d is odd. The infinite cyclic $\mathbb{Z}\pi(\mathbb{RP}^n)$ -module $O_{\mathbb{RP}^d}$ is isomorphic to T , if d is odd, and to T^- if d is even.

One easily checks that the computations above are compatible with Theorem 21.24 which predicts for $n \in \mathbb{Z}$

$$\begin{aligned}
H_\Pi^{d-n}(\mathbb{RP}^d; O_{\mathbb{RP}^d}) &\cong H_n(\mathbb{RP}^d; \mathbb{Z}) && \text{if } d \text{ is even;} \\
H^{d-n}(\mathbb{RP}^d; \mathbb{Z}) &\cong H_n^\Pi(\mathbb{RP}^d; O_{\mathbb{RP}^d}) && \text{if } d \text{ is even;} \\
H^{d-n}(\mathbb{RP}^d; \mathbb{Z}) &\cong H_n(\mathbb{RP}^d; \mathbb{Z}) && \text{if } d \text{ is odd.}
\end{aligned}$$

22. (CO)HOMOLOGY THEORIES OVER A SPACE

22.1. Homology theories over a space. Next we describe the category $\text{Top} \downarrow B$ of spaces over B . An object is a pair (X, u) consisting of a space X and a map $u: X \rightarrow B$. A *morphism or map of spaces over B* $f: (X, u) \rightarrow (Y, v)$ is a map $f: X \rightarrow Y$ satisfying $v \circ f = u$, i.e., the following diagram commutes

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
& \searrow u & \swarrow v \\
& B &
\end{array}$$

Given two maps $f_0, f_1: (X, u) \rightarrow (Y, v)$ of spaces over B , we call them *homotopic* if there is a map $h: X \times I \rightarrow Y$ such that we have $h_k = f_k$ for $k = 0, 1$ and $v \circ h_t = u$ for $t \in I$. A map $i: (X, u) \rightarrow (Y, v)$ of spaces over B is called a *cofibration* if the underlying map $i: X \rightarrow Y$ is a cofibration. A *CW-complex over B* is a space (X, u) over B such that X is a CW-complex. One easily checks that a diagram of spaces

over B

$$\begin{array}{ccc} (X_0, u_0) & \xrightarrow{i_1} & (X_1, u_1) \\ i_2 \downarrow & & \downarrow j_1 \\ (X_2, u_2) & \xrightarrow{j_2} & (X, u) \end{array}$$

is a pushout in $\mathbf{Top} \downarrow B$ if and only if the underlying diagram of spaces

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ i_2 \downarrow & & \downarrow j_1 \\ X_2 & \xrightarrow{j_2} & X \end{array}$$

is a pushout in $\mathbf{Top}$. A pair of spaces (X, A, u) over B is a pair (X, A) together with a reference map $u: X \rightarrow B$. Note that we can assign to it the spaces over B given by (X, u) and $(A, u|_A)$. Thus we get the category $\mathbf{Top}^2 \downarrow B$.

Let R be a commutative ring. The definition of a *homology theory* $\mathcal{H}_*$ for pairs over B with values in R -modules is the following variation of the notion of a homology for pairs with values in R -modules. It is a covariant functor from $\mathbf{Top}^2 \downarrow B$ to the category of $\mathbb{Z}$ -graded R -modules together with a natural transformation $\partial_*(X, A, u): \mathcal{H}_*(X, A, u) \rightarrow \mathcal{H}_{*-1}(A, u|_A)$ such that the obvious versions of the long exact sequence of a pair and of excision hold and we have homotopy invariance in the following sense: Given a map $f: (X, u) \rightarrow (Y, v)$ of spaces over B such that the underlying map of spaces $f: X \rightarrow Y$ is a homotopy equivalence, then $\mathcal{H}_n(f): \mathcal{H}_n(X, u) \rightarrow \mathcal{H}_n(Y, v)$ is bijective for all $n \in \mathbb{Z}$. Note that by the long exact sequence of a pair and the Five Lemma this implies that for a map $f: (X, A, u) \rightarrow (Y, B, v)$ of pairs over B such that the underlying map $f: (X, A) \rightarrow (Y, B)$ is a homotopy equivalence of maps of pairs, $\mathcal{H}_n(f): \mathcal{H}_n(X, A, u) \rightarrow \mathcal{H}_n(Y, B, v)$ is bijective for all $n \in \mathbb{Z}$.

Note that this is a stronger condition than the following version which the reader may have expected, namely, that for two homotopic maps $f_0, f_1: (X, u) \rightarrow (Y, v)$ of pairs over B we have $\mathcal{H}_n(f_0) = \mathcal{H}_n(f_1)$ for $n \in \mathbb{Z}$. This latter version follows from the homotopy invariance defined above as follows. Choose a homotopy $h: f \simeq g$ of maps of pairs over B . Recall that h is given by a map $h: X \times I \rightarrow Y$ of spaces such that $h_t(A) \subseteq B$ and $v \circ h_t = u$ holds for $t \in [0, 1]$ and we have $h_k = f_k$ for $k = 0, 1$. Now we have the inclusion $j_k: (X, A, u) \rightarrow (X \times I, A \times I, u \circ \text{pr}_X)$ for $k = 0, 1$, where $\text{pr}_X: X \times I \rightarrow X$ is the projection and the map $j_k: X \rightarrow X \times [0, 1]$ sends x to (x, k) . We also get a map of pairs over B by $\text{pr}_X: (X \times I, A \times I, u \circ \text{pr}_X) \rightarrow (X, A, u)$. As we have $\text{pr}_X \circ j_k = \text{id}_{(X, A, u)}$ we get $\mathcal{H}_n(\text{pr}_X) \circ \mathcal{H}_n(j_k) = \text{id}_{\mathcal{H}_n(X, A, u)}$ for $k = 0, 1$. Since $\text{pr}_X: (X \times I, A \times I) \rightarrow (X, A)$ is a homotopy equivalence of pairs, the map $\mathcal{H}_n(\text{pr}_X)$ is by assumption bijective for $n \in \mathbb{Z}$. This implies $\mathcal{H}_n(j_0) = \mathcal{H}_n(j_1)$. We have the map $h: (X \times I, A \times I, u \circ \text{pr}_X) \rightarrow (Y, v)$ of pairs over B . Since $h \circ j_k = f_k$ holds for $k = 0, 1$, we get

$$\mathcal{H}_n(f_0) = \mathcal{H}_n(h \circ j_0) = \mathcal{H}_n(h) \circ \mathcal{H}_n(j_0) = \mathcal{H}_n(h) \circ \mathcal{H}_n(j_1) = \mathcal{H}_n(h \circ j_1) = \mathcal{H}_n(f_1).$$

The notion of the disjoint union axiom and the WHE-axiom for homology theories for pairs over B with values in R -modules is obvious.

It is now interesting to figure out what the coefficients of a homology theory for pairs over B with values in R -modules are, since instead of the one point space $\{\bullet\}$ we have to consider all maps $u: \{\bullet\} \rightarrow B$. The answer is given by the construction of the following covariant functor

$$(22.1) \quad \mathcal{H}_q(\{\bullet\}; ?): \Pi(B) \rightarrow R\text{-Mod}$$

called the q -th coefficient system of $\mathcal{H}_*$ for $q \in \mathbb{Z}$

It sends an element $b \in B$ to $\mathcal{H}_q(\{\bullet\}, c_b)$ for the map $c_b: \{\bullet\} \rightarrow B$ given by b . Now consider a morphism $[w]: b_0 \rightarrow b_1$ in $\Pi(X)$. It is represented by path $w: I \rightarrow B$ from $b_0 = w(0)$ to $b_1 = w(1)$. It defines an object (I, w) in $\text{Top} \downarrow B$. Let $j_k: (\{\bullet\}, c_{b_k}) \rightarrow (I, w)$ be the morphism in $\text{Top} \downarrow B$ given by the map $j_k: \{\bullet\} \rightarrow I$ with image $\{k\}$ for $k = 0, 1$. Since j_k is a homotopy equivalence, we get isomorphisms $\mathcal{H}_n(j_k): \mathcal{H}_q(\{\bullet\}, c_{b_k}) \xrightarrow{\cong} \mathcal{H}_q(I, w)$. We want to define

$$\mathcal{H}_q(\{\bullet\}; ?)([w]): \mathcal{H}_q(\{\bullet\}, c_{b_0}) \xrightarrow{\mathcal{H}_n(j_0)} \mathcal{H}_q(I, w) \xrightarrow{\mathcal{H}_n(j_1)^{-1}} \mathcal{H}_q(\{\bullet\}, c_{b_1}).$$

We have to show that $\mathcal{H}_q(\{\bullet\}, c_{b_0}) \xrightarrow{\mathcal{H}_n(j_0)} \mathcal{H}_q(I, w) \xrightarrow{\mathcal{H}_n(j_1)^{-1}} \mathcal{H}_q(\{\bullet\}, c_{b_1})$ depends only on the homotopy class of w relative endpoints. Consider two paths $w_k: I \rightarrow B$ from b_0 to b_1 for $k = 0, 1$ and a homotopy $H: I \times I \rightarrow B$ of such paths relative endpoints from w_0 to w_1 . Let $l_k: I \rightarrow I \times I$ be the map sending t to (t, k) . Then $H \circ l_0 \circ j_k = H \circ l_1 \circ j_k = b_k$ holds for $k = 0, 1$. Hence the following diagram commutes for $k = 0, 1$

$$\begin{array}{ccc} \mathcal{H}_q(X, c_{b_k}) & \xrightarrow[\cong]{\mathcal{H}_q(j_k)} & \mathcal{H}_q(I, w_0) \\ \mathcal{H}_q(j_k) \downarrow \cong & & \cong \downarrow \mathcal{H}_q(l_0) \\ \mathcal{H}_q(I, w_1) & \xrightarrow[\mathcal{H}_q(l_1)]{\cong} & \mathcal{H}_q(I \times I, H). \end{array}$$

We conclude that the following diagram commutes

$$\begin{array}{ccccc} & & \mathcal{H}_q(I, w_0) & & \\ & \nearrow \mathcal{H}_q(j_0) & \downarrow \mathcal{H}_q(l_0) \cong & \nwarrow \mathcal{H}_q(j_1) & \\ \mathcal{H}_q(\{\bullet\}, c_{b_0}) & & \mathcal{H}_q(I \times I, H) & & \mathcal{H}_q(\{\bullet\}, c_{b_1}) \\ & \nwarrow \mathcal{H}_q(j_0) & \uparrow \mathcal{H}_q(l_1) \cong & \nearrow \mathcal{H}_q(j_1) & \\ & & \mathcal{H}_q(I, w_1) & & \end{array}$$

This implies that $\mathcal{H}_q(\{\bullet\}, c_{b_0}) \xrightarrow{\mathcal{H}_n(j_0)} \mathcal{H}_q(I, w_k) \xrightarrow{\mathcal{H}_n(j_1)^{-1}} \mathcal{H}_q(\{\bullet\}, c_{b_1})$ is independent of k . Hence $\mathcal{H}_q(\{\bullet\}; ?)([w])$ is well-defined. We leave it to the reader to check that we indeed get a covariant functor $\mathcal{H}_q(\{\bullet\}; ?)$ as announced in (22.1).

22.2. Cohomology theories over a space. We leave it to the reader to figure out the obvious notion of a cohomology theory with values in R -modules over a space B and the associated contravariant coefficient system

$$(22.2) \quad \mathcal{H}^q(\{\bullet\}; ?): \Pi(B) \rightarrow R\text{-Mod}.$$

Let $\mathcal{H}^*$ be a cohomology theory with values in R -modules. Recall that *multiplicative structure* assigns to a CW -complex X with CW -subcomplexes $A, B \subseteq X$ natural R -homomorphisms

$$(22.3) \quad \cup_{n, n'}: \mathcal{H}^n(X, A) \otimes_R \mathcal{H}^{n'}(X, B) \rightarrow \mathcal{H}^{n+n'}(X, A \cup B).$$

This product is required to be compatible with the boundary homomorphism of the long exact sequence of a pair, to be graded commutative, to be associative, and to have a unit $1 \in \mathcal{H}^0(\{\bullet\})$. The cup product on singular cohomology $H^*(-; R)$ is an example.

The definition of a multiplicative structure has an obvious analog for cohomology theory with values in R -modules over a space B .

22.3. The Leray-Hirsch Theorem.

Theorem 22.4 (Leray-Hirsch Theorem). *Let $\mathcal{H}^*$ be a cohomology theory with values in R -modules satisfying the disjoint union axiom which comes with a multiplicative structure. Let B be a connected CW-complex with a base point $b \in B$. Let $(p, p_0): (E, E_0) \rightarrow B$ be a pair of fibrations over B with $(F, F_0) = (p^{-1}(b), p_0^{-1}(b_0))$ as fiber over b . Suppose that $\{e_1, e_2, \dots, e_r\}$ is a subset of $H^*(E, E_0)$ such that $\mathcal{H}^*(F, F_0)$ is a free graded $\mathcal{H}^*(\{\bullet\})$ -module with basis $\{i^*e_1, i^*e_2, \dots, i^*e_r\}$ for the inclusion $i: (F, F_0) \rightarrow (E, E_0)$. Consider $\mathcal{H}^*(E, E_0)$ as a graded $\mathcal{H}^*(B)$ -module with respect to the action given by $b \cdot e := p^*(b) \cup e$ for $b \in \mathcal{H}^*(B)$ and $e \in \mathcal{H}^*(E, E_0)$.*

Then $\mathcal{H}^*(E, E_0)$ is a free graded $\mathcal{H}^*(B)$ -module with basis $\{e_1, e_2, \dots, e_r\}$.

Proof. Consider $w \in \pi = \pi_1(B, b)$ and let $f: (F, F_0) \rightarrow (F, F_0)$ be a representative of the fiber transport $\tau(w)$. Then the following diagram commutes up to homotopy

$$\begin{array}{ccc} (F, F_0) & & (E, E_0) \\ \downarrow f & \searrow i & \uparrow i \\ (F, F_0) & & (E, E_0) \end{array}$$

Hence $f^* \circ i^*(e_k) = i^*(e_k)$ for $k = 1, 2, \dots, r$. Since $\{i^*e_1, i^*e_2, \dots, i^*e_r\}$ is a basis for the graded $\mathcal{H}^*(\{\bullet\})$ -module $\mathcal{H}^*(F, F_0)$, the map $f^*: \mathcal{H}^*(F, F_0) \rightarrow \mathcal{H}^*(F, F_0)$ is the identity. Hence we conclude for two paths $w_0, w_1: [0, 1] \rightarrow B$ from b_1 to b_0 that the maps $\tau([w_0])^* = \tau([w_1])^*: H^*(p^{-1}(b_0), p_0^{-1}(b_0)) \rightarrow H^*(p^{-1}(b_1), p_0^{-1}(b_1))$ agree for the element $\tau([w]) \in [(p^{-1}(b_1), p_0^{-1}(b_1)), (p^{-1}(b_0), p_0^{-1}(b_0))]$ given by the fiber transport. Hence we can define for two points b_0, b_1 a homomorphism of graded $\mathcal{H}^*(\{\bullet\})$ -modules

$$\eta(b_0, b_1) \in H^*(p^{-1}(b_0), p_0^{-1}(b_0)) \rightarrow H^*(p^{-1}(b_1), p_0^{-1}(b_1))$$

by $\eta(b_0, b_1) = \tau([w])^*$ for any path $w: [0, 1] \rightarrow B$ from b_1 to b_0 . We get $\eta(b_1, b_2) \circ \eta(b_0, b_1) = \eta(b_0, b_2)$ and $\eta(b_0, b_0) = \text{id}_{H^*(p^{-1}(b_0), p_0^{-1}(b_0))}$ for $b_0, b_1, b_2 \in B$.

Define a cohomology theory $\mathcal{K}^*$ with values in R -modules satisfying the disjoint union axiom

$$\mathcal{K}^*(X, A) = \mathcal{H}^*(X, A) \otimes_{\mathcal{H}^*(\{\bullet\})} \mathcal{H}^*(F, F_0).$$

Since $\mathcal{H}^*(F, F_0)$ is a free graded $\mathcal{H}^*(\{\bullet\})$ -module with finite basis, the functor $-\otimes_{\mathcal{H}^*(\{\bullet\})} \mathcal{H}^*(F, F_0)$ is exact and compatible with direct sums over arbitrary index sets. Hence the axioms of a cohomology theory with values in R -modules satisfying the disjoint union axiom are satisfied. We can consider it as a cohomology theory $\mathcal{K}^*$ with values in R -modules over B satisfying the disjoint union axiom by ignoring the reference maps u to B .

We have the pullbacks

$$\begin{array}{ccc} u^*E & \xrightarrow{\bar{u}} & E \\ \bar{p}_u \downarrow & & \downarrow p \\ X & \xrightarrow{u} & B \end{array}$$

and

$$\begin{array}{ccc} u^*E_0 & \xrightarrow{\bar{u}_0} & E_0 \\ \bar{p}_{0,u} \downarrow & & \downarrow p_0 \\ X & \xrightarrow{u} & B. \end{array}$$

We get another cohomology theory $\mathcal{L}^*$ with values in R -modules over B satisfying the disjoint union axiom for pairs by defining

$$\mathcal{L}^*(X, A, u) = \mathcal{H}^*(\bar{p}_u^{-1}(X), \bar{p}_u^{-1}(A) \cup \bar{p}_{0u}^{-1}(X)).$$

We define a transformation of cohomology theories with values in R -modules over B satisfying the disjoint union axiom

$$T^*: \mathcal{K}^* \rightarrow \mathcal{L}^*$$

by assigning to a pair (X, A, u) over B the homomorphism

$$\begin{aligned} T^*(X, A, u): \mathcal{H}^*(X, A) \otimes_{\mathcal{H}^*(\{\bullet\})} \mathcal{H}^*(F, F_0) &\rightarrow \mathcal{H}^*(\bar{p}_u^{-1}(X), \bar{p}_u^{-1}(A) \cup \bar{p}_{0u}^{-1}(X)), \\ (y \otimes z) &\mapsto \bar{p}_u^*(y) \cup j_{u(x)}^* \circ \eta(b_0, u(x))^*(z) \end{aligned}$$

for any $x \in X$ and $j_{u(x)}: (p^{-1}(u(x)), p_0^{-1}(u(x))) \rightarrow (\bar{p}_u^{-1}(X), \bar{p}_u^{-1}(A) \cup \bar{p}_{0u}^{-1}(X))$ the inclusion. It is not hard to check that this is independent of the choice of $x \in X$ using the assumption that the space B is path connected. The homomorphism $T^*(\{\bullet\}, u): \mathcal{H}^*(\{\bullet\}) \otimes_{\mathcal{H}^*(\{\bullet\})} \mathcal{H}^*(F, F_0) \rightarrow \mathcal{H}^*(\bar{p}_u^{-1}(\{\bullet\}))$ can be identified with $\eta(b, u(\{\bullet\})): \mathcal{H}^*(F, F_0) \rightarrow \mathcal{H}^*(p^{-1}(u(\{\bullet\})), p_0^{-1}(u(\{\bullet\})))$ and hence is bijective for any space over B of the shape $(\{\bullet\}, u)$. There is an obvious version of Proposition 16.12 (ii) for cohomology theories with values in R -modules satisfying the disjoint union axiom over B , whose proof is analogous to the one of Proposition 16.12 (ii). Hence $T^*: \mathcal{L}^* \rightarrow \mathcal{K}^*$ is an equivalence of cohomology theories with values in R -modules satisfying the disjoint union axiom over B . If we apply this to the space (B, id_B) over B , we obtain an isomorphism of $\mathcal{H}^*(B)$ -modules

$$\mathcal{H}^*(B) \otimes_{\mathcal{H}^*(\{\bullet\})} \mathcal{H}^*(F, F_0) \rightarrow \mathcal{H}^*(E, E_0), \quad (y \otimes z) \mapsto p^*(y) \cup i^*(z).$$

Hence $\mathcal{H}^*(E, E_0)$ is a free graded $\mathcal{H}^*(B)$ -module with basis $\{e_1, e_2, \dots, e_r\}$ $\square$

22.4. The Thom Isomorphism. Let $p: E \rightarrow B$ be a $(k-1)$ -spherical fibration, i.e., a fibration with fiber S^{k-1} . The associated disk fibration is defined by $Dp: DE := \text{cyl}(p) \rightarrow B$, where $\text{cyl}(p)$ is the mapping cylinder of p and Dp is the obvious map. A simple application of [6, Proposition 1.3] shows that Dp is a fibration.

Define the *Thom space* $\text{Th}(p)$ of p to be the pointed space $\text{cone}(p)$, where $\text{cone}(p)$ is the mapping cone of p with its canonical base point, or, equivalently, put $\text{Th}(p) = DE/E$. If $k = 0$, then $DE = B$ and $\text{Th}(p) = B_+$. If $p_\xi: E \rightarrow B$ is the projection of a k -dimensional vector bundle ξ over B with sphere bundle $p_{SE}: SE \rightarrow B$, then we can identify the disk bundle of ξ with the mapping cylinder of p_{SE} , so that $DE = \text{cyl}(p_{SE})$. Note that the canonical inclusion of B in $\text{cyl}(p_{SE})$ is a homotopy equivalence, which is analogous to the fact that the inclusion defined by the zero-section of ξ is a homotopy equivalence. The canonical inclusion of E into $\text{cyl}(p)$ corresponds to the inclusion of $SE \subset DE$. Hence the definition $\text{Th}(p) = DE/E = \text{cone}(p)$ for a $(k-1)$ -spherical fibration p corresponds to $\text{Th}(\xi) = DE/SE$ for the k -dimensional vector bundle ξ .

Put $F_x := p^{-1}(x)$. It comes with a preferred infinite cyclic local coefficient system

$$(22.5) \quad \mathcal{O}_p: \Pi(B) \rightarrow \mathbb{Z}\text{-MOD.}, \quad b \mapsto H_k(\text{cone}(F_b), F_b; \mathbb{Z}).$$

It sends a morphism $[w]: b_0 \rightarrow b_1$ to the isomorphism of infinite cyclic groups $H_k(\text{cone}(t_{[w]}), t_{[w]}; \mathbb{Z}): H_k(\text{cone}(F_{b_0}), F_{b_0}; \mathbb{Z}) \rightarrow H_k(\text{cone}(F_{b_1}), F_{b_1}; \mathbb{Z})$ induced by the homotopy class $\tau([w])$ of maps $F_{b_0} \rightarrow F_{b_1}$ coming from the fibre transport along w , see (9.14).

For $b \in B$ let $i(b): \{b\} \rightarrow B$ be the inclusion. Recall that $p_{DE}: DE \rightarrow X$ is the canonical projection. Hence we get an infinite local coefficient system $p_{DE}^* \mathcal{O}_p$. We get from $i(b)$ an isomorphism

$$H^k(DE, E; p_{DE}^* \mathcal{O}_p) \xrightarrow{\cong} H^k(\text{cone}(F_b), F_b; i(b)^* p_{DE}^* \mathcal{O}_p).$$

The elementary proof that it is bijective can be found in [19, Lemma 4.2 (iii) on page 156]. Since $i(b)^* p_{DE}^* \mathcal{O}_p$ is the constant infinite cyclic local coefficient system on $\text{cone}(E_x)$ with value $\mathcal{O}_p(b)$, we get a canonical isomorphism

$$H^k(\text{cone}(F_b), F_b; i(b)^* p_{DE}^* \mathcal{O}_p) \xrightarrow{\cong} H^k(\text{cone}(F_b), F_b; \mathcal{O}_p(b)).$$

The universal coefficient theorem yields a natural isomorphism

$$\begin{aligned} H^k(\text{cone}(F_b), F_b; \mathcal{O}_p(b)) &\xrightarrow{\cong} \text{hom}_{\mathbb{Z}}(H_k(\text{cone}(F_b), F_b), \mathcal{O}_p(b)) \\ &\xrightarrow{\cong} \text{hom}_{\mathbb{Z}}(\mathcal{O}_p(b), \mathcal{O}_p(b)). \end{aligned}$$

Putting these isomorphisms together yields a isomorphism

$$(22.6) \quad \alpha_p(b): H^k(DE, E; p_{DE}^* \mathcal{O}_p) \xrightarrow{\cong} \text{hom}_{\mathbb{Z}}(\mathcal{O}_p(b), \mathcal{O}_p(b)).$$

Suppose from now on that B is a connected CW -complex. Then the map α_p of (22.6) turns out to be an isomorphism and for path $w: I \rightarrow B$ with $w(0) = b_0$ and $w(1) = b_1$ the following diagram commutes

$$\begin{array}{ccc} & & \text{hom}_{\mathbb{Z}}(\mathcal{O}_p(b_0), \mathcal{O}_p(b_0)) \\ & \nearrow \alpha_p(b_0) & \downarrow \cong \\ H^k(DE, E; p_{DE}^* \mathcal{O}_p) & & \text{hom}_{\mathbb{Z}}(\mathcal{O}_p([w])^{-1}, \mathcal{O}_p([w])) \\ & \searrow \alpha_p(b_1) & \downarrow \\ & & \text{hom}_{\mathbb{Z}}(\mathcal{O}_p(b_1), \mathcal{O}_p(b_1)). \end{array}$$

This is proved for instance in [19, Lemma 6.42 (iii) on page 156]. If we compose $\alpha_p(b)$ with the inverse of the isomorphism $\mathbb{Z} \xrightarrow{\cong} \text{hom}_{\mathbb{Z}}(\mathcal{O}_p(b), \mathcal{O}_p(b))$ sending n to $n \cdot \text{id}_{\mathcal{O}_p(b)}$, we get an isomorphism

$$(22.7) \quad \beta_p: H^k(DE, E; p_{DE}^* \mathcal{O}_p) \xrightarrow{\cong} \mathbb{Z},$$

which is, by the observations above, indeed independent of the choice of $b \in B$.

Definition 22.8. Let $p: E \rightarrow X$ be a $(k-1)$ -spherical fibration. Its *Thom class*

$$U_p \in H^k(DE, E; p_{DE}^* \mathcal{O}_p)$$

is defined to be the preimage of $1 \in \mathbb{Z}$ under the isomorphism β_p of (22.7).

Theorem 22.9 (Thom Isomorphism Theorem). *Let $p: E \rightarrow B$ be a $(k-1)$ -spherical fibration of connected finite CW -complexes with first Stiefel-Whitney class $w: \pi = \pi_1(B) \rightarrow \{\pm 1\}$ for $k \in \mathbb{Z}^{\geq 1}$.*

Then the composites

$$\begin{aligned} H_n(DE, E; \mathbb{Z}) &\xrightarrow{U_p \cap -} H_{n-k}^{\pi}(DE; \mathbb{Z}^w) \xrightarrow{H_{n-k}^{\pi}(p_{DE})} H_{n-k}^{\pi}(B; \mathbb{Z}^w); \\ H_n^{\pi}(DE, E; \mathbb{Z}^w) &\xrightarrow{U_p \cap -} H_{n-k}(DE; \mathbb{Z}) \xrightarrow{H_{n-k}(p_{DE})} H_{n-k}(B; \mathbb{Z}); \\ H^n(B; \mathbb{Z}) &\xrightarrow{H^n(p_{DE})} H^n(DE; \mathbb{Z}) \xrightarrow{- \cup U_p} H_{\pi}^{n+k}(DE, E; \mathbb{Z}^w); \\ H_{\pi}^n(B; \mathbb{Z}^w) &\xrightarrow{H_{\pi}^n(p_{DE})} H_{\pi}^n(DE; \mathbb{Z}^w) \xrightarrow{- \cup U_p} H^{n+k}(DE, E; \mathbb{Z}), \end{aligned}$$

are bijective for $n \in \mathbb{Z}^{\geq 0}$. (These maps are called Thom isomorphisms.)

Proof. The proof can be found in [19, Theorem 6.45 on page 160]. We indicate its proof for cohomology in the special case where w is trivial. Choose $b_0 \in B$ and fix a generator $[F_b]$ of the infinite cyclic group $H_k(\text{cone}(F_{b_0}), F_{b_0}; \mathbb{Z})$. This yields an isomorphism of infinite local coefficient systems between $p_{DE}^* \mathcal{O}_p$ and the constant system with values $\mathbb{Z}$ which in turn induces an identification $H^k(DE, E; p_{DE}^* \mathcal{O}_p) \cong H^k(DE, E; \mathbb{Z})$. Hence the Thom class U_p of Definition 22.8 is an element in $H^k(DE, E; \mathbb{Z})$ such that for any element $b \in B$ the homomorphism $i(b)^*: H^k(DE, E) \rightarrow H^k(\text{cone}(F_b), F_b; \mathbb{Z})$ induced by the inclusion $i(b): F_b \rightarrow W$ sends U_p to a generator of the infinite cyclic group $H^k(\text{cone}(F_b), F_b; \mathbb{Z})$. Now we conclude from the Leray-Hirsch Theorem 22.4 applied to p and to $\mathcal{H}^* = H^*(-; \mathbb{Z})$ that the composite

$$H^n(B; \mathbb{Z}) \xrightarrow{H^n(p_{DE})} H^n_p(DE; \mathbb{Z}) \xrightarrow{-\cup U_p} H^{n+k}(DE, E; \mathbb{Z})$$

is bijective. $\square$

Example 22.10 (Singular Cohomology ring of $\mathbb{RP}^\infty$). Consider the $(1-1)$ -spherical fibration $S^0 \rightarrow S^\infty \xrightarrow{p} \mathbb{RP}^\infty$. Let $U \in H^1(DS^\infty, S^\infty; p_{DS^\infty}^* \mathcal{O}_p)$ be its Thom class. We obtain from Theorem 22.9 isomorphisms

$$H^n(\mathbb{RP}^\infty; \mathbb{Z}) \xrightarrow{H^n(p_{DE})} H^n(DS^\infty; \mathbb{Z}) \xrightarrow{-\cup U_p} H^{n+1}_\pi(DS^\infty, S^\infty; \mathbb{Z}^w).$$

As S^∞ is contractible, the map $H^{n+1}_\pi(DS^\infty, S^\infty; \mathbb{Z}^w) \rightarrow H^{n+1}(DS^\infty, \mathbb{Z}^w)$ is bijective for $n \geq 1$. Since p_{DE} is a homotopy equivalence, it induces an isomorphism $H^{n+1}_\pi(\mathbb{RP}^\infty, \mathbb{Z}^w) \xrightarrow{\cong} H^{n+1}_\pi(DS^\infty, \mathbb{Z}^w)$. Therefore we obtain an isomorphism for $n \geq 1$

$$H^n(\mathbb{RP}^\infty; \mathbb{Z}) \xrightarrow{\cong} H^{n+1}(\mathbb{RP}^\infty; \mathbb{Z}^w).$$

Analogously one obtains an isomorphism

$$H^{n+1}(\mathbb{RP}^\infty; \mathbb{Z}^w) \xrightarrow{\cong} H^{n+2}(\mathbb{RP}^\infty; \mathbb{Z}).$$

So we get for $n \geq 1$ an isomorphism

$$H^n(\mathbb{RP}^\infty; \mathbb{Z}) \xrightarrow{\cong} H^{n+2}(\mathbb{RP}^\infty; \mathbb{Z}),$$

which turns out to be the cup product with an element $u \in H^2(\mathbb{RP}^\infty; \mathbb{Z})$. Since $H^2(\mathbb{RP}^\infty; \mathbb{Z}) \cong \mathbb{Z}/2$, $H^0(\mathbb{RP}^\infty) \cong \mathbb{Z}$, and $H^1(\mathbb{RP}^\infty) \cong \{0\}$ hold, we conclude that for $i \geq 1$ we have $H^{2i}(\mathbb{RP}^\infty; \mathbb{Z}) \cong \mathbb{Z}/2$ with u^i as generator and $H^{2i-1}(\mathbb{RP}^\infty; \mathbb{Z}) \cong \{0\}$.

We leave it to the reader to figure out using the Bockstein sequence associated to $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2$ that $H^*(\mathbb{RP}^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x]$ for $|x| = 1$ holds.

23. THE CONSTRUCTION OF THE LERAY-SERRE SPECTRAL SEQUENCE

Theorem 23.1. *Let $p: E \rightarrow B$ be a fibration over a CW-complex B . Consider a commutative ring R . Let $\mathcal{H}_*$ be a homology theory with values in R -modules satisfying the disjoint union axiom and the WHE-axiom. Then there exists the Leray-Serre spectral sequence converging to $\mathcal{H}_n(E)$ whose E^2 -page satisfies*

$$E^2_{p,q} \cong H_p^\Pi(B; \mathcal{H}_q(F))$$

where $H_p^\Pi(B; \mathcal{H}_q(F))$ has been defined in (21.15).

We at least sketch the construction of this spectral sequence. This needs some preparations.

Now the construction of the Atiyah-Hirzebruch spectral sequence of Sections 18 and 20 carries over to the homology theory $\mathcal{H}_*$ for pairs over B with values in

R -modules. So we get for any CW -complex (X, u) over B a spectral sequence converging to $\mathcal{H}_n(X, u)$ whose E^1 -term is given by

$$E_{p,q}^1 = \mathcal{H}_p(X_q, X_{q-1}, u|_{X_q})$$

whose differential $d_{p,q}^1$ are given by the composite

$$d_{p,q}^1: \mathcal{H}_p(X_q, X_{q-1}, u|_{X_q}) \xrightarrow{\partial_q} \mathcal{H}_p(X_{q-1}, u|_{X_{q-1}}) \rightarrow \mathcal{H}_p(X_{q-1}, X_{q-2}, u|_{X_{q-1}}),$$

and whose filtration of $\mathcal{H}_{p+q}(X, u)$ is given by

$$F_{p,q} = \text{im}(\mathcal{H}_q(X_p, u|_{X_p}) \rightarrow \mathcal{H}_{p+q}(X, u)).$$

Lemma 23.2. *There is a canonical R -isomorphism*

$$E_{p,q}^2 \cong H_p^\Pi(X; \mathcal{H}_q^u(\{\bullet\}; ?))$$

where the covariant $R\Pi(X)$ -module $\mathcal{H}_q^u(\{\bullet\}; ?)$ is obtained from the covariant $R\Pi(B)$ -module $\mathcal{H}_q(\{\bullet\}; ?)$ of (22.1) by precomposition with $\Pi(u): \Pi(X) \rightarrow \Pi(B)$ and the R -module $H_p^\Pi(X; \mathcal{H}_q^u(\{\bullet\}; ?))$ has been introduced in Definition 21.11.

Proof. Consider $n \geq \mathbb{Z}^{\geq 0}$.

Consider a cellular pushout as described in (21.9), where we here replace $n \in \mathbb{Z}^{\geq 0}$ by $p \in \mathbb{Z}^{\geq 0}$ in (21.9) and consider any $q \in \mathbb{Z}$.

Then we obtain by excision and the disjoint union axiom for every $q \in \mathbb{Z}$ an isomorphism

$$\bigoplus_{i \in I_p} \mathcal{H}_{p+q}(D^p, S^{p-1}, u \circ Q_i^n) \xrightarrow{\cong} \mathcal{H}_{p+q}(X_p, X_{p-1}, u|_{X_p}).$$

Put $x_i^n = Q_i^n(0)$ and $y_i^n = u \circ Q_i^n(0)$ for the origin $0 \in D^p$. We denote by $c_{x_i^n}$ and $c_{u(x_i^n)}$ the constant maps $\{\bullet\} \rightarrow X$ and $\{\bullet\} \rightarrow B$ with image $\{x_i^n\}$ and $\{u(x_i^n)\}$. Then there is an suspension isomorphism

$$\sigma_i: \mathcal{H}_q^u(\{\bullet\}, c_{x_i^n}) = \mathcal{H}_q(\{\bullet\}, c_{u(x_i^n)}) \xrightarrow{\cong} \mathcal{H}_{p+q}(D^p, S^{p-1}, u \circ Q_i^n).$$

Thus we obtain an isomorphism

$$U_{p,q}: \bigoplus_{i \in I_p} \mathcal{H}_q^u(\{\bullet\}, c_{x_i^n}) \xrightarrow{\cong} \mathcal{H}_{p+q}(X_p, X_{p-1}, u|_{X_p}).$$

We obtain from Lemma 21.4 and Lemma 21.10 isomorphisms

$$V_{p,q}: \bigoplus_{i \in I_p} \mathcal{H}_q^u(\{\bullet\}, c_{x_i^n}) \xrightarrow{\cong} C_p^{\Pi(X)}(X; R) \otimes_{R\Pi(X)} \mathcal{H}_q^u(\{\bullet\}; ?).$$

Thus we obtain an isomorphism of R -modules

$$W_{p,q} = V_{p,q} \circ U_{p,q}^{-1}: \mathcal{H}_{p+q}(X_p, X_{p-1}, u|_{X_p}) \xrightarrow{\cong} C_p^{\Pi(X)}(X; R) \otimes_{R\Pi(X)} \mathcal{H}_q^u(\{\bullet\}; ?).$$

Note that both $U_{p,q}$ and $V_{p,q}$ depend on the choice of the cellular pushout (21.9). Recall that such a choice is not part of the structure of a CW -complex on X . However, one can show that $W_{p,q}$ does not depend on the choice of the cellular pushout (21.9) and hence depends only on the CW -structure on X as follows.

Suppose we have made another choice of a cellular pushout

$$(23.3) \quad \begin{array}{ccc} \coprod_{i \in I_n} S^{n-1} & \xrightarrow{\coprod_{i \in I_n} \bar{q}_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} D^n & \xrightarrow{\coprod_{i \in I_n} \bar{Q}_i^n} & X_n. \end{array}$$

resulting in isomorphism $\overline{U}_{p,q}$ and $\overline{V}_{p,q}$. Choose for every $i \in I_n$ a path $w_i: I \rightarrow X$ from $x_i^n = Q_i^n(0)$ to $\overline{x}_i^n = \overline{Q}_i^n(0)$. Then there is for every $i \in I_n$ an element $\epsilon_i \in \{\pm 1\}$ such that for the isomorphism

$$T_{p,q} = \bigoplus_{i \in I_n} \epsilon_i \cdot \mathcal{H}_q^u(\{\bullet\}; [w_i]): \bigoplus_{i \in I_n} \mathcal{H}_q^u(\{\bullet\}, c_{x_i^n}) \xrightarrow{\cong} \mathcal{H}_q^u(\{\bullet\}, c_{\overline{x}_i^n})$$

we have

$$\begin{aligned} \overline{U}_{p,q} \circ T_{p,q} &= U_{p,q}; \\ \overline{V}_{p,q} \circ T_{p,q} &= V_{p,q}. \end{aligned}$$

This implies

$$V_{p,q} \circ U_{p,q}^{-1} = \overline{V}_{p,q} \circ \overline{U}_{p,q}^{-1}.$$

Moreover, one can show that the collection of the maps $W_{p,q}$ is compatible with the differentials and hence we obtain for every $q \in \mathbb{Z}$ an isomorphisms of R -chain complexes

$$T_{*,q}: \mathcal{H}_{*+q}(X_*, X_{*-1}, u|_{X_*}) \xrightarrow{\cong} C_*^\Pi(X) \otimes_{R\Pi(X)} \mathcal{H}_q^u(\{\bullet\}; ?).$$

Now the desired isomorphism

$$E_{p,q}^2 \xrightarrow{\cong} H_p^\Pi(X; \mathcal{H}_q^u(\{\bullet\}; ?))$$

is given by applying $H_p(-)$ to $T_{*,q}$. $\square$

Obviously a homology theory for pairs over B with values in R -modules is the same as a homology theory of pairs with values in R -modules if $B = \{\bullet\}$. Moreover, given a homology theory of pairs with values in R -modules, we obtain a homology theory of pairs over B with values in R -modules by forgetting the reference maps to B , i.e., by assigning to (X, A, u) the $\mathbb{Z}$ -graded R -module $\mathcal{H}_*(X, A)$. Here is our main example of a homology theory for pairs over B with values in R -modules.

Example 23.4. Let R be a commutative ring and $\mathcal{H}_*$ be a homology theory for pairs over B with values in R -modules satisfying the WHE-axiom. Let $p: E \rightarrow B$ be a fibration. Then we obtain a homology theory $\mathcal{H}_*^p$ for pairs over B with values in R -modules by defining for a pair (X, A, u) over B

$$\mathcal{H}_n^p(X, A, u) = \mathcal{H}_n(u^*E, (u|_A)^*E)$$

where u^*E is defined by the pullback

$$\begin{array}{ccc} u^*E & \xrightarrow{\overline{u}} & E \\ \overline{p} \downarrow & & \downarrow p \\ X & \xrightarrow{u} & B \end{array}$$

and analogously for $(u|_A)^*E$. We omit the proof that $\mathcal{H}_*^p$ is a homology theory for pairs over B with values in R -modules which is essentially a direct consequence of Proposition 9.8 (iv) and Theorem 9.18.

Obviously $\mathcal{H}_*^p$ satisfies the disjoint union axiom or the WHE-axiom, if $\mathcal{H}_*$ does.

One easily checks that the covariant $\mathbb{Z}\Pi(X)$ -module $\mathcal{H}_q^p(\{\bullet\}; ?)$ of (22.1) is isomorphic to the covariant $\mathbb{Z}\Pi(X)$ -module $\mathcal{H}_q(F)$ of (21.14) and hence we get an identification of $H_p^\Pi(B; \mathcal{H}_q(\{\bullet\}; ?))$ and $H_p^\Pi(B; \mathcal{H}_q(F))$ introduced in (21.15).

Now Theorem 23.1 follows from the Atiyah-Hirzebruch spectral sequence for spaces over B applied to $X = B$ and the situation of Example 23.4.

24. SOME APPLICATIONS OF THE LERAY-SERRE SPECTRAL SEQUENCE

As preliminary we mention the following result due to Maschke and Wedderburn.

Theorem 24.1. *Let F be a field and G be a finite group such that $|G|$ is invertible in F . Then:*

- (i) *The group ring FG is semisimple, i.e., every FG -module is projective;*
- (ii) *Every irreducible FG -module is isomorphic to an ideal in FG and every FG -module is a direct sum of simple modules, where simple means that the only submodules are $\{0\}$ and the module itself;*
- (iii) *There are only finitely many pairwise non-isomorphic simple ideals $I_1, I_2, \dots, I_n$ in FG ;*
- (iv) *Let D_j be $\text{End}_{FG}(I_j)$ for $j = 1, 2, \dots, n$. Then each D_i is a skewfield and there are elements $d_i \in \mathbb{Z}^{\geq 1}$ such that we get an identification of rings*

$$FG \cong \prod_{j=1}^n M_{d_i, d_i}(D_i).$$

Example 24.2. Let p be an odd prime. Consider a homology theory $\mathcal{H}_*$ with values in $\mathbb{F}_p$ -modules satisfying the disjoint union axiom. Let $F \xrightarrow{i} E \xrightarrow{p} \mathbb{RP}^\infty$ be a fibration.

Let M be any $\mathbb{F}_p[\mathbb{Z}/2]$ -module. Let $\mathbb{F}_p$ or $\mathbb{F}_p^-$ respectively be the $\mathbb{F}_p[\mathbb{Z}/2]$ -module whose underlying $\mathbb{F}_p$ -module is $\mathbb{F}_p$ and on which the generator of $\mathbb{Z}/2$ acts trivially or by $-\text{id}_{\mathbb{F}_p}$ respectively. We conclude from Theorem 24.1 that any FG -module is a direct sum of copies of $\mathbb{F}_p$ and $\mathbb{F}_p^-$. An easy calculation using the cellular $\mathbb{Z}[\mathbb{Z}/2]$ -chain complex of $\mathbb{RP}^\infty$ and the fact that $\bar{2} \in \mathbb{F}_p$ is a unit shows that $H_k^\pi(\mathbb{RP}^\infty; M)$ vanishes for $k \in \mathbb{Z}^{\geq 1}$ and is $\mathbb{F}_p \otimes_{\mathbb{F}_p[\mathbb{Z}/2]} M$ if $k = 0$. So the second page is concentrated in the 0th column and looks like

$$\begin{array}{cccc} \vdots & \vdots & \vdots & \ddots \\ \mathbb{F}_p \otimes_{\mathbb{F}_p[\mathbb{Z}/2]} \mathcal{H}_3(F) & 0 & 0 & \dots \\ \mathbb{F}_p \otimes_{\mathbb{F}_p[\mathbb{Z}/2]} \mathcal{H}_2(F) & 0 & 0 & \dots \\ \mathbb{F}_p \otimes_{\mathbb{F}_p[\mathbb{Z}/2]} \mathcal{H}_1(F) & 0 & 0 & \dots \\ \mathbb{F}_p \otimes_{\mathbb{F}_p[\mathbb{Z}/2]} \mathcal{H}_0(F) & 0 & 0 & \dots \end{array}$$

where the $\pi_1(\mathbb{RP}^\infty) = \mathbb{Z}/2$ -action on $\mathcal{H}_q(F)$ comes from the fiber transport. Hence it strongly collapses and yields for $n \in \mathbb{Z}$ the isomorphism

$$\mathbb{F}_p \otimes_{\mathbb{F}_p[\mathbb{Z}/2]} \mathcal{H}_n(F) \cong \mathcal{H}_n(E).$$

It comes from the map $\mathcal{H}_n(i): \mathcal{H}_n(F) \xrightarrow{\cong} \mathcal{H}_n(E)$ induced by the inclusion $i: F \rightarrow E$ which factorizes through the projection $\mathcal{H}_n(F) \rightarrow \mathbb{F}_p \otimes_{\mathbb{F}_p[\mathbb{Z}/2]} \mathcal{H}_n(F)$, since for the automorphism induced by the fiber transport $\tau: F \rightarrow F$ we have $i \circ \tau \simeq i$. This claim follows from the naturality of the Leray-Serre spectral sequence by inspecting

the map of fibrations

$$\begin{array}{ccc}
 F & \xrightarrow{\text{id}_F} & F \\
 \downarrow \text{id}_F & & \downarrow i \\
 F & \xrightarrow{i} & E \\
 \downarrow & & \downarrow \\
 \{\bullet\} & \longrightarrow & B
 \end{array}$$

Example 24.3 (Fibrations over S^l for $l \geq 2$). Let $F \rightarrow E \xrightarrow{f} S^l$ be a fibration over the sphere S^l for $l \in \mathbb{Z}^{\geq 2}$. Then S^l is simply connected. We conclude

$$H_p(S^l; \mathcal{H}_q(\{\bullet\})) = \begin{cases} \mathcal{H}_q(\{\bullet\}) & p = 0, l; \\ \{0\} & \text{otherwise,} \end{cases}$$

from Lemma 21.17. So the E^2 -term and hence each E^r -term and the E^∞ -term of the Leray Serre spectral sequence associated to f has non-trivial entries only in the columns for $q = 0$ and $q = l$. So we get $E_{p,q}^2 = E_{p,q}^r$ for $r \leq l$ and $E_{p,q}^r = E_{p,q}^\infty$ for $r \geq l + 1$, and the only non-trivial differentials occur on the l th page, where they look like $d_{p,q}^l: E_{p,q}^l \rightarrow E_{p-l,q+l-1}^l$. So for $l = 3$ the l -th page looks like

$$\begin{array}{cccccc}
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\
 \mathcal{H}_2(F) & & 0 & & 0 & & \mathcal{H}_2(F) & & 0 & & \cdots \\
 \mathcal{H}_1(F) & & 0 & & 0 & & \mathcal{H}_1(F) & & 0 & & \cdots \\
 \mathcal{H}_0(F) & & 0 & & 0 & & \mathcal{H}_0(F) & & 0 & & \cdots \\
 \mathcal{H}_{-1}(F) & & 0 & & 0 & & \mathcal{H}_{-1}(F) & & 0 & & \cdots \\
 \mathcal{H}_{-2}(F) & & 0 & & 0 & & \mathcal{H}_{-2}(F) & & \cdots & & \\
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots
 \end{array}$$

So we get exact sequences

$$0 \rightarrow E_{l,q}^\infty \rightarrow \mathcal{H}_q(F) \xrightarrow{d_{l,q}^l} \mathcal{H}_{q+l-1}(F) \rightarrow E_{0,q+l-1}^\infty \rightarrow 0$$

and filtrations

$$\{0\} \subseteq F_{0,n} \subseteq F_{l,n-l} = \mathcal{H}_n(E)$$

satisfying $F_{0,n} = E_{0,n}^\infty$ and $\mathcal{H}_n(E)/F_{0,n} = E_{l,n-l}^\infty$. These data can be spliced together to a long exact sequence

$$\begin{aligned} \cdots \rightarrow \mathcal{H}_{n-l+2}(F) &\xrightarrow{d_{l,n-l+2}^l} \mathcal{H}_{n+1}(F) \xrightarrow{\mathcal{H}_{n+1}(i)} \mathcal{H}_{n+1}(E) \xrightarrow{\partial_{n+1}} \\ &\mathcal{H}_{n+1-l}(F) \xrightarrow{d_{l,n-l+1}^l} \mathcal{H}_n(F) \xrightarrow{\mathcal{H}_n(i)} \mathcal{H}_n(E) \xrightarrow{\partial_n} \mathcal{H}_{n-l}(F) \xrightarrow{\mathcal{H}_{n-l}(i)} \cdots \end{aligned}$$

where $i: F \rightarrow E$ is the inclusion.

Example 24.4 (Fibrations over S^1). Since $\pi_1(S^1)$ infinite cyclic, the fiber transport is given by a selfhomotopy equivalence $\tau: F \rightarrow F$. So $H_p^\Pi(S^1; \mathcal{H}_q(F))$ fits into the exact sequence

$$0 \rightarrow H_1^\Pi(S^1; \mathcal{H}_q(F)) \rightarrow \mathcal{H}_q(F) \xrightarrow{\text{id}_{\mathcal{H}_q(F)} - \mathcal{H}_q(\tau)} \mathcal{H}_q(F) \rightarrow H_0^\Pi(S^1; \mathcal{H}_q(F)) \rightarrow 0.$$

Each E^r -term and the E^∞ -term of the Leray Serre spectral sequence associated to p has non-trivial entries only in the columns for $q = 0$ and $q = 1$ and hence

$$E_{p,q}^\infty = E_{p,q}^2 = \begin{cases} H_p^\Pi(S^1; \mathcal{H}_q(F)) & p = 0, 1; \\ \{0\}. & \text{otherwise.} \end{cases}$$

Hence we obtain a long exact sequence

$$(24.5) \quad \begin{aligned} \cdots \xrightarrow{\partial_{n+1}} \mathcal{H}_n(F) &\xrightarrow{\text{id}_{\mathcal{H}_n(F)} - \mathcal{H}_n(\tau)} \mathcal{H}_n(F) \xrightarrow{\mathcal{H}_n(i)} \mathcal{H}_n(E) \\ &\xrightarrow{\partial_n} \mathcal{H}_{n-1}(F) \xrightarrow{\text{id}_{\mathcal{H}_{n-1}(F)} - \mathcal{H}_{n-1}(f)} \mathcal{H}_{n-1}(F) \xrightarrow{\mathcal{H}_{n-1}(i)} \cdots \end{aligned}$$

Consider the pushout

$$\begin{array}{ccc} \{0, 1\} & \longrightarrow & \{\bullet\} \\ \downarrow & & \downarrow \\ [0, 1] & \longrightarrow & S^1 \end{array}$$

We conclude from Proposition 9.8 (iv) and Theorem 9.11 that the pullback construction yields a pushout

$$\begin{array}{ccc} F \amalg F & \xrightarrow{\text{id} \amalg \tau} & F \\ \downarrow & & \downarrow \\ F \times [0, 1] & \longrightarrow & E \end{array}$$

Hence E is homotopy equivalent to the mapping torus and the sequence 24.5 is the so called Wang sequence which can be obtained from the pushout above by the Mayer-Vietoris sequence.

Example 24.6 (Fibrations with S^l as fiber). Consider a fibration $S^l \rightarrow E \xrightarrow{f} B$ over a connected CW-complex B with fiber S^l for $l \geq 1$. Let $H_*(-; G)$ be singular homology with coefficients in the abelian group G . Then $H_q(S^l; G)$ is G for $q = 0, l$ and trivial otherwise. The G -action of $\pi = \pi_1(B)$ on $H_0(S^l; G)$ is trivial and hence $H_p^\Pi(B; H_0(S^l; G)) = H_p(B, G)$. We get $H_p^\Pi(B; H_l(S^l; G)) = H_p^\pi(B; G_p)$ for the π -action on G given by $w \cdot g = \rho(w) \cdot g$ for $w \in \pi$ and $g \in G$ for the homomorphism $\rho: \pi \rightarrow \{\pm 1\}$ sending $w \in \pi$ to the degree of the map $S^l \rightarrow S^l$ given by the fiber transport applied to w . Then the E^2 -term and hence the E^r -term for $r \in \mathbb{Z}^{\geq 2} \amalg \{\infty\}$ of Leray-Serre spectral sequence for f and $H_*(-; G)$ has non-trivial entries only in

the 0-th row and in the l th row. The only non-trivial differentials appear on the $(l+1)$ -th page and are given

$$d_{p,0}^{l+1}: E_{p,0}^{l+1} = H_p^\pi(B; G) \rightarrow E_{p-l-1,l}^{l+1} = H_{p-l-1}(B; G_\rho).$$

So for $l = 2$ the $(l+1)$ -th page looks like

$$\begin{array}{ccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\ 0 & & 0 & & 0 & & 0 & & 0 & & \dots \\ & & & & & & & & & & \\ H_0^\pi(B; G_\rho) & H_1^\pi(B; G_\rho) & H_2^\pi(B; G_\rho) & H_3^\pi(B; G_\rho) & H_4^\pi(B; G_\rho) & & \dots & & & & \\ & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & \dots & & & \\ & & d_{3,0}^3 & & d_{4,0}^3 & & & & & & \\ H_0(B; G) & H_1(B; G) & H_2(B; G) & H_3(B; G) & H_4(B; G) & & \dots & & & & \end{array}$$

We obtain for every $n \in \mathbb{Z}^{\geq 0}$ a filtration

$$\{0\} \subseteq F_{n-l,l} \subseteq F_{n,0} = H_n(E; G)$$

and an exact sequence

$$0 \rightarrow E_{n-l,l}^\infty \rightarrow H_n(E; G) \rightarrow E_{n,0}^\infty \rightarrow 0.$$

These data can be spliced together to a long exact sequence

$$\begin{aligned} \dots \xrightarrow{\partial_{n-l+1}} H_{n+1}(E; G) \xrightarrow{H_{n+1}(f; G)} H_{n+1}(B; G) \xrightarrow{d_{n+1,0}^{l+1}} H_{n-l}^\pi(B; G_\rho) \xrightarrow{\partial_{n-l}} \\ H_n(E; G) \xrightarrow{H_n(f; G)} H_n(B; G) \xrightarrow{d_{n,0}^{l+1}} H_{n-l-1}^\pi(B; G_\rho) \xrightarrow{\partial_{n-l-1}} \dots \end{aligned}$$

Let R be a principal ideal domain. We call a space X *R -homological finite* if $H_m(X; R)$ is a finitely generated R -module for every $m \in \mathbb{Z}^{\geq 0}$ and is non-trivial for only finitely many elements $m \in \mathbb{Z}^{\geq 0}$. In this case we define its *R -homological Euler characteristic* by

$$(24.7) \quad \chi(X; R) := \sum_{m \geq 0} (-1)^m \cdot \text{rk}_R(H_m(X; R)) \in \mathbb{Z}$$

If X is a finite *CW-complex* then it is *R -homological finite* and $\chi_R(X; R)$ agrees with $\text{ch}(M) := \sum_{m \geq 0} (-1)^m \cdot |I_m|$ for I_m the set of m -cells of X .

Theorem 24.8 (Multiplicativity of the Euler characteristic). *Let R be a principal ideal domain. Let $F \rightarrow E \xrightarrow{f} B$ be a fibration such that F and B are R -homological finite and the action of $\pi_1(B)$ on $H_m(F; R)$ induced by the fiber transport is trivial.*

Then E is R -homological finite and we get

$$\chi(E; R) = \chi(F; R) \cdot \chi(B; R).$$

Proof. Consider the Leray-Serre spectral sequence for the fibration f and singular homology $H_*(-; R)$ with R -coefficients. Then there exists a number d such that $E_{p,q}^2 = H_p(B; H_q(F; R)) \neq 0 \implies p, q \leq d$. The Universal Coefficient Theorem yields an exact sequence of R -modules $0 \rightarrow H_p(B; R) \otimes_R H_q(F; R) \rightarrow H_p(B; H_q(F; R)) \rightarrow \text{Tor}_1^R(H_{p-1}(B; R), H_q(F; R)) \rightarrow 0$. By assumption $H_p(B; R)$

and $H_q(F; R)$ are finitely generated for $p, q \in \mathbb{Z}^{\geq 0}$. This implies that the R -module $\text{Tor}_1^R(H_{p-1}(B; R), H_q(F; R))$ is a finitely generated R -module of rank zero and $H_p(B; R) \otimes_R H_q(F; R)$ is a finitely generated R -module satisfying

$$\text{rk}_R(H_p(B; R) \otimes_R H_q(F; R)) = \text{rk}_R(H_p(B; R)) \cdot \text{rk}_R(H_q(F; R)).$$

We conclude that $H_p(B; H_q(F; R))$ is a finitely generated R -module satisfying

$$(24.9) \quad \text{rk}_R(H_p(B; H_q(F; R))) = \text{rk}_R(H_p(B; R)) \cdot \text{rk}_R(H_q(F; R)).$$

Hence we conclude for every $r \in \mathbb{Z}^{\geq 1} \amalg \{\infty\}$ that $E_{p,q}^r$ is a finitely generated R -module for $p, q \in \mathbb{Z}^{\geq 0}$ and that $E_{p,q}^2 = H_p(B, H_q(F; R)) \neq 0 \implies p, q \leq d$ holds. Hence we can define for $r \in \mathbb{Z}^{\geq 2} \amalg \{\infty\}$

$$\chi^r = \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \cdot \text{rk}_R(E_{p,q}^r) = \sum_{\substack{p,q \in \mathbb{Z} \\ 0 \leq p,q \leq d}} (-1)^{p+q} \cdot \text{rk}_R(E_{p,q}^r) \in \mathbb{Z}.$$

We compute

$$\begin{aligned} \chi(B; R) \cdot \chi(F; R) &= \left(\sum_{p \in \mathbb{Z}} (-1)^p \cdot \text{rk}_R(H_p(F; R)) \right) \cdot \left(\sum_{q \in \mathbb{Z}} (-1)^q \cdot \text{rk}_R(H_q(B; R)) \right) \\ &= \sum_{p,q \in \mathbb{Z}} (-1)^p \cdot \text{rk}_R(H_p(F; R)) \cdot (-1)^q \cdot \text{rk}_R(H_q(B; R)) \\ &= \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \cdot \text{rk}_R(H_p(F; R)) \cdot \text{rk}_R(H_q(B; R)) \\ &\stackrel{(24.9)}{=} \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \cdot \text{rk}_R(H_p(B; H_q(F; R))) \\ &= \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \cdot \text{rk}_R(E_{p,q}^2) \\ &= \chi^2. \end{aligned}$$

If C_* is a chain complex of finitely generated R -module which is bounded, i.e., $C_m \neq \{0\}$ only for finitely many $m \in \mathbb{Z}$. Then $H_m(C_*)$ is finitely generated for $m \in \mathbb{Z}$ and non-trivial only for finitely many $m \in \mathbb{Z}$ and we get

$$(24.10) \quad \sum_{m \in \mathbb{Z}} (-1)^m \cdot \text{rk}_R(C_m) = \sum_{m \in \mathbb{Z}} (-1)^m \cdot \text{rk}_R(H_m(C_*)).$$

We compute for $r \in \mathbb{Z}^{\geq 2}$.

$$\begin{aligned}
\chi^{r+1} &= \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \cdot \text{rk}_R(E_{p,q}^{r+1}) \\
&= \sum_{p \in \mathbb{Z}} \sum_{i=0}^{r-2} \sum_{s \in \mathbb{Z}} (-1)^{p-l \cdot s + i + (l-1) \cdot s} \cdot \text{rk}_R(E_{p-l \cdot s, i + (l-1) \cdot s}^{r+1}) \\
&= \sum_{p \in \mathbb{Z}} \sum_{i=0}^{r-2} \sum_{s \in \mathbb{Z}} (-1)^{p+i+s} \cdot \text{rk}_R(H_s(E_{p-l \cdot s, i + (l-1) \cdot s}^r, d_{p-l \cdot s, i + (l-1) \cdot s}^r)) \\
&= \sum_{p \in \mathbb{Z}} \sum_{i=0}^{r-2} (-1)^{p+i} \cdot \left(\sum_{s \in \mathbb{Z}} (-1)^s \cdot \text{rk}_R((H_s(E_{p-l \cdot s, i + (l-1) \cdot s}^r, d_{p-l \cdot s, i + (l-1) \cdot s}^r)) \right) \\
&\stackrel{(24.10)}{=} \sum_{p \in \mathbb{Z}} \sum_{i=0}^{r-2} (-1)^{p+i} \cdot \left(\sum_{s \in \mathbb{Z}} (-1)^s \cdot \text{rk}_R(E_{p-l \cdot s, i + (l-1) \cdot s}^r) \right) \\
&= \sum_{p \in \mathbb{Z}} \sum_{i=0}^{r-2} \sum_{s \in \mathbb{Z}} (-1)^{p+i+s} \cdot \text{rk}_R(E_{p-l \cdot s, i + (l-1) \cdot s}^r) \\
&= \sum_{p \in \mathbb{Z}} \sum_{i=0}^{r-2} \sum_{s \in \mathbb{Z}} (-1)^{p-l \cdot s + i + (l-1) \cdot s} \cdot \text{rk}_R(E_{p-l \cdot s, i + (l-1) \cdot s}^r) \\
&= \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \cdot \text{rk}_R(E_{p,q}^r) \\
&= \chi^r.
\end{aligned}$$

Since $E_{p,q}^\infty = E_{p,q}^{d+1}$ holds, we get

$$\chi(B; R) \cdot \chi(F; R) = \chi^2 = \chi^\infty.$$

From the filtration

$$\{0\} \subseteq F_{0,n} \subseteq F_{1,n-1} \subseteq F_{2,n-2} \subseteq \dots \subseteq F_{0,n} = H_n(E; R)$$

with filtration quotients $F_{p,q}/F_{p-1,q+1} = E_{p,q}^\infty$ we conclude

$$\begin{aligned}
\chi(E; R) &= \sum_{n \in \mathbb{Z}} (-1)^n \cdot \text{rk}_R(H_n(E; R)) \\
&= \sum_{n \in \mathbb{Z}} (-1)^n \cdot \left(\sum_{t \in \mathbb{Z}} \text{rk}_R(E_{n-t,t}^\infty) \right) \\
&= \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \cdot \text{rk}_R(E_{p,q}^\infty) \\
&= \chi^\infty \\
&= \chi(B; R) \cdot \chi(F; R).
\end{aligned}$$

□

25. NATURALITY PROPERTIES OF SPECTRAL SEQUENCES

Let $f: X \rightarrow Y$ be a cellular map of CW -complexes. Let $\mathcal{H}_*$ be a homology theory with values in R -modules satisfying the disjoint union axiom. Then we get a morphism from the Atiyah-Hirzebruch spectral sequences for X to the one for Y . Hence Lemma 19.5 implies the next result.

Theorem 25.1. *Suppose that $H_n(f; R): H_n(X; R) \rightarrow H_n(Y; R)$ is bijective for all $n \in \mathbb{Z}^{\geq 0}$.*

Then the map $\mathcal{H}_n(f): \mathcal{H}_n(X) \rightarrow \mathcal{H}_n(Y)$ is bijective for all $n \in \mathbb{Z}$.

Consider a pullback of fibrations with CW -complexes as base space

$$\begin{array}{ccc} E_0 & \xrightarrow{\bar{f}} & E_1 \\ p_0 \downarrow & & \downarrow p_1 \\ B_0 & \xrightarrow{f} & B_1. \end{array}$$

Theorem 25.2. *Suppose that for any $n \in \mathbb{Z}^{\geq 0}$, any $b_1 \in B_1$, and any loop w in B_1 at b_1 the automorphism $\mathcal{H}_n(p_1^{-1}(b_k)) \rightarrow \mathcal{H}_n(p_1^{-1}(b_k))$ induced by the element $\tau_1(w_1) \in [p_1^{-1}(b), p_1^{-1}(b)]$ given by the fiber transport is the identity. Assume that $H_n(f; R): H_n(B_0; R) \rightarrow H_n(B_1; R)$ is bijective for all $n \in \mathbb{Z}^{\geq 0}$.*

Then the map $\mathcal{H}_n(\bar{f}): \mathcal{H}_n(E_0) \rightarrow \mathcal{H}_n(E_1)$ is bijective for every $n \in \mathbb{Z}$.

Proof. This follows from the Universal Coefficient Theorem and Lemma 19.5 using the Leray-Serre spectral sequence. $\square$

We also mention that one can feed in pairs into the Atiyah-Hirzebruch spectral sequence and pairs of fibrations over the same base space into the Leray-Serre spectral sequence.

26. BASIC NOTIONS AND FACTS ABOUT COHOMOLOGICAL SPECTRAL SEQUENCES

Next we discuss what can be said if one considers cohomology. The basic setup concerning pages and differentials yielding finally the terms $E_{\infty}^{p,q}$ is essentially the same, if one takes into account that the differentials now have the opposite slope and the E^r -term is given by cohomology of the r -terms and the computations of the E^1 - or E^2 -terms will now be in cohomological terms. However, the convergence issue is much more complicated. Fortunately, there are some favourite situations, where the convergence is as good as in the homological case, but now with respect to descending filtrations. Cohomology theories often come with a valuable multiplicative structure and the cohomological spectral sequences do respect these.

Next we describe the abstract setting of a cohomological spectral sequence.

Definition 26.1 (Cohomological spectral sequence with values in R -modules). A cohomological spectral sequence with values in R -modules starting at d for $d \in \mathbb{Z}^{\geq 1}$ $E_r^{*,*}$ consists of the following data:

- A family of R -modules $\{E_r^{p,q}\}$ for $r \in \mathbb{Z}^{\geq d}$, $p, q \in \mathbb{Z}$ such that $E_r^{p,q} = \{0\}$ for $p \leq -1$;
- Maps $d_r^{p,q}: E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ called differentials for $r \in \mathbb{Z}^{\geq d}$, $p, q \in \mathbb{Z}$ such that the composite $d_{p,q}^r \circ d_{p-r, q+r-1}^r$ vanishes for $r \in \mathbb{Z}^{\geq d}$, $p, q \in \mathbb{Z}$.
In particular we get R -chain complexes $C_r[q]^*$ for $r \in \mathbb{Z}^{\geq d}$ and $p \in \mathbb{Z}$ whose p -th differential is $d_r^{p,q}$, in other words it is given by the lines of slope $-\frac{r-1}{r}$;
- R -isomorphism

$$\alpha_r^{p,q}: H^p(C_r[q]^*) \xrightarrow{\cong} E_{r+1}^{p,q}$$

for $r \in \mathbb{Z}^{\geq d}$, $p, q \in \mathbb{Z}$.

We call the data given by $\{E_r^{p,q}, d_r^{p,q} \mid p, q \in \mathbb{Z}\}$ for $r \in \mathbb{Z}^{\geq s}$ the r -th page of the spectral sequence. Note that the r -th pages determines the underlying modules of the $(r+1)$ th page but not the differentials appearing on the $(r+1)$ th page.

Here is a picture of the first page:

(26.2)

$$\begin{array}{ccccccccc}
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\
 E_1^{0,2} & \xrightarrow{d_1^{0,2}} & E_1^{1,2} & \xrightarrow{d_1^{1,2}} & E_1^{2,2} & \xrightarrow{d_1^{2,2}} & E_1^{3,2} & \xrightarrow{d_1^{3,2}} & E_1^{4,2} & & \dots \\
 E_1^{0,1} & \xrightarrow{d_1^{0,1}} & E_1^{1,1} & \xrightarrow{d_1^{1,1}} & E_1^{2,1} & \xrightarrow{d_1^{2,1}} & E_1^{3,1} & \xrightarrow{d_1^{3,1}} & E_1^{4,1} & & \dots \\
 E_1^{0,0} & \xrightarrow{d_1^{0,0}} & E_1^{1,0} & \xrightarrow{d_1^{1,0}} & E_1^{2,0} & \xrightarrow{d_1^{2,0}} & E_1^{3,0} & \xrightarrow{d_1^{3,0}} & E_1^{4,0} & & \dots \\
 E_1^{0,-1} & \xrightarrow{d_1^{0,-1}} & E_1^{1,-1} & \xrightarrow{d_1^{1,-1}} & E_1^{2,-1} & \xrightarrow{d_1^{2,-1}} & E_1^{3,-1} & \xrightarrow{d_1^{3,-1}} & E_1^{4,-1} & & \dots \\
 E_1^{0,-2} & \xrightarrow{d_1^{0,-2}} & E_1^{1,-2} & \xrightarrow{d_1^{1,-2}} & E_1^{2,-2} & \xrightarrow{d_1^{1,-2}} & E_1^{3,-2} & \xrightarrow{d_1^{1,-2}} & E_1^{4,-2} & & \dots \\
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots
 \end{array}$$

Here is a picture of the second page:

(26.3)

$$\begin{array}{ccccccccc}
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\
 E_2^{0,2} & & E_2^{1,2} & & E_2^{2,2} & & E_2^{3,2} & & E_2^{4,2} & & \dots \\
 & \searrow d_2^{0,2} & & \searrow d_2^{1,2} & & \searrow d_2^{2,2} & & \searrow d_2^{3,2} & & \searrow d_2^{4,2} & \\
 E_2^{0,1} & & E_2^{1,1} & & E_2^{2,1} & & E_2^{3,1} & & E_2^{4,1} & & \dots \\
 & \searrow d_2^{0,1} & & \searrow d_2^{1,1} & & \searrow d_2^{2,1} & & \searrow d_2^{3,1} & & \searrow d_2^{4,1} & \\
 E_2^{0,0} & & E_2^{1,0} & & E_2^{2,0} & & E_2^{3,0} & & E_2^{4,0} & & \dots \\
 & \searrow d_2^{0,0} & & \searrow d_2^{1,0} & & \searrow d_2^{2,0} & & \searrow d_2^{3,0} & & \searrow d_2^{4,0} & \\
 E_2^{0,-1} & & E_2^{1,-1} & & E_2^{2,-1} & & E_2^{3,-1} & & E_2^{4,-1} & & \dots \\
 & \searrow d_2^{0,-1} & & \searrow d_2^{1,-1} & & \searrow d_2^{2,-1} & & \searrow d_2^{3,-1} & & \searrow d_2^{4,-1} & \\
 E_2^{0,-2} & & E_2^{1,-2} & & E_2^{2,-2} & & E_2^{3,-2} & & E_2^{4,-2} & & \dots \\
 & \searrow d_2^{0,-2} & & \searrow d_2^{1,-2} & & \searrow d_2^{2,-2} & & \searrow d_2^{3,-2} & & \searrow d_2^{4,-2} & \\
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots
 \end{array}$$

Here is a picture of the third page:

(26.4)

$$\begin{array}{cccccc}
 \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
 E_3^{0,2} & E_3^{1,2} & E_3^{2,2} & E_3^{3,2} & E_3^{4,2} & \cdots \\
 E_3^{0,1} & E_3^{1,1} & \xrightarrow{d_3^{0,2}} E_3^{2,1} & \xrightarrow{d_3^{1,2}} E_3^{3,1} & E_3^{4,1} & \cdots \\
 E_3^{0,0} & E_3^{1,0} & \xrightarrow{d_3^{0,1}} E_3^{2,0} & \xrightarrow{d_3^{1,1}} E_3^{3,0} & E_3^{4,0} & \cdots \\
 E_3^{0,-1} & E_3^{1,-1} & \xrightarrow{d_3^{0,0}} E_3^{2,-1} & E_3^{3,-1} & E_3^{4,-1} & \cdots \\
 E_3^{0,-2} & E_3^{1,-2} & E_3^{2,-2} & E_3^{3,-2} & E_3^{4,-2} & \cdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
 \end{array}$$

So the differential $d_r^{p,q}$ starts at (p, q) and points to the place which is r steps to the right and $(r-1)$ steps downwards. The relevant cochain complexes $C_r[p]^*$ appear in the picture in the obvious way by the corresponding lines of slope $-\frac{r-1}{r}$.

Note that all entries at (p, q) for $p < 0$ are $\{0\}$. So differentials beginning in the region (p, q) with $p < 0$ are automatically trivial. This implies that all differentials of the r -th page ending at (p, q) are trivial for $p < r$. Hence we get for $r > p$ an inclusion $E_{r+1}^{p,q} \rightarrow E_r^{p,q}$. Hence we can define

$$E_\infty^{p,q} = \operatorname{invlim}_{r \rightarrow \infty} E_r^{p,q} = \bigcap_{r \geq p} E_r^{p,q}.$$

Note that for given (p, q) we have

$$E_r^{p,q} = \{0\} \implies E_{r+1}^{p,q} = 0 \implies E_\infty^{p,q} = \{0\}.$$

We say that the spectral sequence *collapses* if all differentials are trivial. Note that this implies $E_d^{p,q} = E_r^{p,q} = E_\infty^{p,q}$ for all $r \in \mathbb{Z}^{\geq d}$, $p, q \in \mathbb{Z}$.

We call the spectral sequence a *first quadrant spectral sequence* if $E_r^{p,q} = 0$ holds for $q \leq -1$ holds for $r \geq 1$.

A *morphism* $f_*^{*,*}: E_*^{*,*} \rightarrow E'^{*,*}$ of cohomological spectral sequences is a family of R -homomorphism $\{f_r^{p,q}: E_r^{p,q} \rightarrow E'^{p,q}\}$ for $r \in \mathbb{Z}^{\geq d}$, $p, q \in \mathbb{Z}$ which is compatible with the differentials $d_r^{p,q}: E_r^{p,q} \rightarrow E_r^{p+1,q-r+1}$ and $(d')_r^{p,q}: E'^{p,q}_r \rightarrow E'^{p+1,q-r+1}_r$ and with the isomorphisms $\alpha_r^{p,q}: H_q(C_r[p]^*) \xrightarrow{\cong} E_{r+1}^{p,q}$ and $(\alpha')_r^{p,q}: H_q((C')_r[p]^*) \xrightarrow{\cong} E'^{p,q}_{r+1}$.

The next lemma is a direct consequence of the Five-Lemma.

Lemma 26.5. *Let $f_*^{*,*}: E_*^{*,*} \rightarrow E'^{*,*}$ be a morphism of cohomological spectral sequences starting at $d \in \mathbb{Z}^{\geq 1}$. Suppose that $f_d^{p,q}: E_d^{p,q} \rightarrow E'^{p,q}_d$ is bijective for all $p, q \in \mathbb{Z}$. Then $f_r^{p,q}: E_r^{p,q} \rightarrow E'^{p,q}_r$ is bijective for all $r \in \mathbb{Z}^{\geq d} \amalg \{\infty\}$, $p, q \in \mathbb{Z}$.*

Consider a $\mathbb{Z}$ -graded R -module $\mathcal{H}^* = \{\mathcal{H}^n \mid n \in \mathbb{Z}\}$. It is called *descendingly filtered* if for every $n \in \mathbb{Z}$ there is a preferred descending filtration

$$\mathcal{H}^n = F^{-1,n+1} \supseteq F^{0,n} \supseteq F^{1,n-1} \supseteq F^{2,n-2} \supseteq \dots$$

with $\{0\} = \bigcap_{p \geq 0} F^{n-p,p}$. We say that a cohomological spectral sequence converges to the descendingly filtered $\mathbb{Z}$ -graded R -module $\{\mathcal{H}^n \mid n \in \mathbb{Z}\}$ if there are for every $p, q \in \mathbb{Z}$ preferred R -isomorphisms

$$\beta^{p,q}: F^{p,q}/F^{p-1,q+1} \xrightarrow{\cong} E_{\infty}^{p,q}.$$

We say that *there are no extension problems* for the homological spectral sequence $E_{*,*}^*$ converging to the descendingly filtered $\mathbb{Z}$ -graded R -module $\mathcal{H}_*$ if we have

$$\mathcal{H}^n \cong \prod_{p \geq 0} F^{n-p,p}/F^{n-p+1,q-1}.$$

We say that the cohomological spectral sequence $E_{*,*}^*$ converging to the filtered $\mathbb{Z}$ -graded R -module $\mathcal{H}_*$ *strongly collapses* if it collapses and there are no extension problems. In this rare and favourite cases we get

$$\mathcal{H}^n = \prod_{p \geq 0} E_{n-p,p}^d.$$

26.1. The convergence problem cohomology. Recall that the homological versions of the Atiyah-Hirzebruch spectral sequence converges to $\mathcal{H}_{p+q}(X)$ and the homological versions of the Leray-Serre spectral sequence converges to $\mathcal{H}_{p+q}(E)$. The main technical reason for this is Proposition 16.11 (i) which essentially says that homology commutes with colimits indexed by $\mathbb{N}$. Recall that this is not true for cohomology, a correction term given by $\lim^1$ -terms comes in, see Proposition 16.11 (ii). Therefore one needs extra assumptions to guarantee convergence. The easiest way to circumvent this problem is to arrange that the descending filtration reaches already after finitely many steps $\{0\}$, or, equivalently, for every $n \in \mathbb{Z}$ there exists a number $d(n)$ such that $F_{p,n-p} = \{0\}$ holds for $p \geq d(n)$. This leads to the following result.

Theorem 26.6. *Let $\mathcal{H}^*$ be a cohomology theory with values in R -modules satisfying the disjoint union axiom. Let X be a CW-complex and let $p: E \rightarrow B$ be a fibration over a CW-complex B .*

Then cohomological Atiyah-Hirzebruch spectral sequence converges to $\mathcal{H}^{p+q}(X)$ and the cohomological Leray-Serre spectral sequence converges to $\mathcal{H}^{p+q}(E)$ if one of the following conditions is satisfied:

- (i) *The CW-complex X respectively B is finite-dimensional;*
- (ii) *The cohomology theory $\mathcal{H}^*$ is bounded from below, i.e., there is an integer d such that $\mathcal{H}^q(\{\bullet\})$ vanishes for $q \leq d$;*
- (iii) *The spectral sequence is bounded in the sense that for any $n \in \mathbb{Z}$ the set $\{p \in \mathbb{Z}^{\geq 0} \mid E_{p,n-p}^2 \neq 0\}$ is finite.*

Remark 26.7. Recall from Proposition 16.11 (ii) that there is an exact sequence

$$(26.8) \quad 0 \rightarrow \operatorname{invlim}_{k \rightarrow \infty}^1 \mathcal{H}^{n-1}(X_k) \rightarrow \mathcal{H}^n(X) \rightarrow \operatorname{invlim}_{k \rightarrow \infty} \mathcal{H}^n(X_k) \rightarrow 0.$$

In the general case one can only expect that the Atiyah-Hirzebruch spectral sequence converges in some sense to $\operatorname{invlim}_{p \rightarrow \infty} \mathcal{H}^n(X_p)$ and that it gives no information about $\operatorname{invlim}_{p \rightarrow \infty}^1 \mathcal{H}^n(X_p)$. Actually, there exists a descending filtration $F^{p,n-p} \operatorname{invlim}_{n \rightarrow \infty} H_{\mathcal{C}}^n(X_n; \mathbf{E})$ for $p = 0, 1, 2, \dots$ of $\operatorname{invlim}_{n \rightarrow \infty} H_{\mathcal{C}}^n(X_n; \mathbf{E})$ and

the following exact sequence, see [33, Theorem XIII.3.4 on page 615 and Theorem XIII.3.5 and Theorem XIII.3.6 on page 616],

$$\begin{aligned} 0 \rightarrow F^{p,q} \operatorname{invlim}_{n \rightarrow \infty} H_{\mathcal{C}}^{p+q}(X_n) / F^{p+1,q-1} \operatorname{invlim}_{n \rightarrow \infty} H_{\mathcal{C}}^{p+q}(X_n) &\rightarrow E_{\infty}^{p,q} \\ &\rightarrow \operatorname{invlim}_{m \rightarrow \infty}^1 H_{\mathcal{C}}^{p+q}(X_{p+m}, X_p) \rightarrow \operatorname{invlim}_{m \rightarrow \infty}^1 H_{\mathcal{C}}^{p+q}(X_{p+m}, X_{p-1}). \end{aligned}$$

Recall that the Mittag-Leffler condition is a good criterion to guarantee that invlim^1 -terms are trivial.

If none of the conditions appearing in Lemma 26.6 are satisfied the general strategy is to compute by the Atiyah-Hirzebruch spectral sequence $\mathcal{H}^n(X_p)$ for each $p \in \mathbb{Z}^{\geq 0}$ including the obvious map $\mathcal{H}^n(X_p) \rightarrow \mathcal{H}^n(X_{p-1})$ and then to apply the exact sequence (26.8). In good situations each map $\mathcal{H}^n(X_p) \rightarrow \mathcal{H}^n(X_{p-1})$ is surjective, or, more generally, the Mittag-Leffler condition is satisfied, which implies that the canonical map $\mathcal{H}^n(X) \xrightarrow{\cong} \operatorname{invlim}_{p \rightarrow \infty} \mathcal{H}^n(X_p)$ for $n \in \mathbb{Z}$ is an isomorphism.

Example 26.9 (Topological K -theory of $\mathbb{RP}^{\infty}$). Let us describe what happens for the topological K -theory of $\mathbb{RP}^{\infty}$. So first one tries to compute $K^*(\mathbb{RP}^d)$ for $d \in \mathbb{Z}^{\geq 1}$. The E^2 -term of the Atiyah-Hirzebruch spectral sequence looks like

$$E_2^{p,q} = H^p(\mathbb{RP}^d; K^q(\{\bullet\})) \begin{cases} \mathbb{Z} & \text{if } p = 0 \text{ and } q \text{ even;} \\ \mathbb{Z} & \text{if } p = d \text{ and } d \text{ is odd and } q \text{ even;} \\ \mathbb{Z}/2 & \text{if } 2 \leq p \leq d \text{ and } p \text{ is even and } q \text{ even;} \\ \{0\} & \text{otherwise.} \end{cases}$$

Hence all the second differentials are obviously trivial except the differentials starting at the place $(0, p)$ for p even. These are also trivial by the following argument. Consider the map $\operatorname{pr}: \mathbb{RP}^d \rightarrow \{\bullet\}$. It induces maps of cohomological spectral sequences from the Atiyah-Hirzebruch spectral sequence of $\{\bullet\}$ to the one of $\mathbb{RP}^d$. Hence the following diagram commutes

$$\begin{array}{ccc} H^0(\{\bullet\}, K^q(\{\bullet\})) & \xrightarrow{d_2^{0,q}} & H^2(\{\bullet\}; K^{q-1}(\{\bullet\})) = \{0\} \\ \downarrow H^0(\operatorname{pr}; K^q(\{\bullet\})) \cong & & \downarrow H^2(\operatorname{pr}; K^{q-1}(\{\bullet\})) \\ H^0(\mathbb{RP}^d, K^q(\{\bullet\})) & \xrightarrow{d_2^{0,q}} & H^2(\mathbb{RP}; K^{q-1}(\{\bullet\})). \end{array}$$

This implies that the map $d_2^{0,q}: H^0(\mathbb{RP}^d; K^q(\{\bullet\})) \rightarrow H^2(\{\bullet\}; K^{q-1}(\{\bullet\}))$ is trivial. Using induction over r one can show by a similar argument that all differentials $d_{p,q}^r$ vanish. In other words the cohomological Atiyah-Hirzebruch spectral collapses. This implies

$$E_{\infty}^{p,q} = H^p(\mathbb{RP}^d; K^q(\{\bullet\})) \begin{cases} \mathbb{Z} & \text{if } p = 0 \text{ and } q \text{ even;} \\ \mathbb{Z} & \text{if } p = d \text{ and } d \text{ is odd and } q \text{ even;} \\ \mathbb{Z}/2 & \text{if } 2 \leq p \leq d \text{ and } p \text{ is even and } q \text{ even;} \\ \{0\} & \text{otherwise.} \end{cases}$$

This implies

$$K^1(\mathbb{RP}^d) \cong \begin{cases} \mathbb{Z} & \text{if } d \text{ is odd;} \\ \{0\} & \text{if } d \text{ is even;} \end{cases}$$

and there is descending filtration

$$K^0(\mathbb{RP}^d) = F^{-1,d+1} \supseteq F^{0,d} \supseteq F^{1,d-1} \supseteq F^{2,d-2} \supseteq \dots \supseteq F^{0,d} \supseteq F^{-1,d+1} = \{0\}$$

such that

$$F^{p,d-p}/F^{p+1,d-p-1} \cong \begin{cases} \mathbb{Z}/2 & \text{if } p \text{ is even and satisfies } 2 \leq p \leq d; \\ \mathbb{Z} & \text{if } p = 0; \\ \{0\} & \text{otherwise.} \end{cases}$$

Now one has to solve the extension problems and that requires additional input going beyond the spectral sequence which we will not explain here. At least we state what the result is. Namely, one gets

$$K^0(\mathbb{RP}^d) \cong \mathbb{Z} \oplus \mathbb{Z}/2^k \quad \text{for } d = 2k \text{ or } d = 2k + 1$$

and that the maps $K^0(\mathbb{RP}^{d+1}) \rightarrow K^0(\mathbb{RP}^d)$ are given under these identification by the identity if $d = 2k$ is even and by $\text{id}_{\mathbb{Z}} \oplus \text{pr}$ for the projection $\mathbb{Z}/2^{k+1} \rightarrow \mathbb{Z}/2^k$ if $d = 2k + 1$. Hence these maps are always surjective and the Mittag-Leffler condition is satisfied. This implies $\text{invlim}_{k \rightarrow \infty}^1 \mathcal{H}^{n-1}(\mathbb{RP}^k) = \{0\}$. We get from the exact sequence (26.8) an isomorphism

$$K^0(\mathbb{RP}^\infty) \cong \text{invlim}_{k \rightarrow \infty} (\mathbb{Z} \oplus \mathbb{Z}/2^k) = \mathbb{Z} \oplus \mathbb{Z}_2^\wedge,$$

where $\mathbb{Z}_2^\wedge = \text{invlim}_{k \rightarrow \infty} \mathbb{Z}/2^k$ is the ring of the 2-adic integers. Moreover, we get

$$K^1(\mathbb{RP}^\infty) = \{0\}.$$

26.2. Outlook: The Completion Theorem of Atiyah and Segal. Let G be a finite group. We denote by $R_{\mathbb{C}}(G)$ the *complex representation ring* of G . The underlying abelian group is the Grothendieck construction applied to the abelian semigroup of isomorphism classes of finite dimensional complex G -representations under direct sum. The ring structure comes from the tensor product of complex vector spaces and the diagonal G -action. The unit element is given by $\mathbb{C}$ with the trivial G -action. Consider the ring homomorphism $\dim_{\mathbb{C}}: R(G) \rightarrow \mathbb{Z}$ sending a finite-dimensional complex G -representation to the dimension of the underlying complex vector space. The augmentation ideal $\mathbb{I} = \mathbb{I}(G)$ of $R(G)$ is its kernel. Taking the k -th fold product $\mathbb{I}^k$ of $\mathbb{I}$ yields a descending filtration

$$R(G) \supseteq \mathbb{I} \supseteq \mathbb{I}^2 \supseteq \mathbb{I}^3 \supseteq \dots$$

The $\mathbb{I}$ -adic completion of $R_{\mathbb{C}}(G)$ is defined to be the ring

$$R_{\mathbb{C}}(G)_{\mathbb{I}}^\wedge = \text{invlim}_{k \rightarrow \infty} R_{\mathbb{C}}(G)/\mathbb{I}^k.$$

The Atiyah-Segal Completion Theorem, see [2], says:

$$K^n(BG) \cong \begin{cases} R_{\mathbb{C}}(G)_{\mathbb{I}}^\wedge & n \text{ even;} \\ \{0\} & n \text{ odd.} \end{cases}$$

This yields to the following explicit calculation of $K^0(BG)$, see [18, Theorem 0.3]. If $G_p \subseteq G$ is a p -Sylow subgroup, restriction defines a map $\mathbb{I}(G) \rightarrow \mathbb{I}(G_p)$. Let $\mathbb{I}_p(G)$ be the quotient of $\mathbb{I}(G)$ by the kernel of this map. This is independent of the choice of the p -Sylow subgroup since two p -Sylow subgroups of G are conjugate. There is an obvious isomorphism $\mathbb{I}_p(G) \xrightarrow{\cong} \text{im}(\mathbb{I}(G) \rightarrow \mathbb{I}(G_p))$. For a prime p denote by $r(p) = |\text{con}_p(G)|$ the number of conjugacy classes (g) of elements $g \in G$ whose order $|g|$ is p^d for some integer $d \geq 1$. Then there are isomorphisms of abelian groups

$$\begin{aligned} K^0(BG) &\cong \mathbb{Z} \times \prod_{p \text{ prime}} \mathbb{I}_p(G) \otimes_{\mathbb{Z}} \mathbb{Z}_p^\wedge \cong \mathbb{Z} \times \prod_{p \text{ prime}} (\mathbb{Z}_p^\wedge)^{r(p)}; \\ K^1(BG) &\cong 0. \end{aligned}$$

The isomorphism $K^0(BG) \xrightarrow{\cong} \mathbb{Z} \times \prod_{p \text{ prime}} \mathbb{I}_p(G) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_p$ is compatible with the standard ring structure on the source and the ring structure on the target given by

$$(m, u_p \otimes a_p) \cdot (n, v_p \otimes b_p) = (mn, m \cdot v_p \otimes b_p + n \cdot u_p \otimes a_p + (u_p v_p) \otimes (a_p b_p))$$

for $m, n \in \mathbb{Z}$, $u_p, v_p \in \mathbb{I}_p(G)$ and $a_p, b_p \in \widehat{\mathbb{Z}}_p$ and the obvious multiplication in $\mathbb{Z}$, $\mathbb{I}_p(G)$, and $\widehat{\mathbb{Z}}_p$.

This computation is remarkable since one does not have such a formula for the singular cohomology $H^*(BG; \mathbb{Z})$. If $G = \mathbb{Z}/2$, then $B\mathbb{Z}/2$ is $\mathbb{P}\mathbb{P}^\infty$ and we rediscover the computation above based on the Atiyah-Hirzebruch spectral sequence.

Example 26.10 ($G = \mathbb{Z}/2$). Note that $R_{\mathbb{Z}}(\mathbb{Z}/2)$ is as abelian group isomorphic to $\mathbb{Z}^2$ with the class $[\mathbb{C}]$ of the trivial $\mathbb{Z}/2$ -representation $\mathbb{C}$ and the class $[\mathbb{C}^-]$ of the $\mathbb{Z}/2$ -representation $\mathbb{C}^-$ given by $-\text{id}_{\mathbb{C}}: \mathbb{C} \rightarrow \mathbb{C}$ as basis. The augmentation ideal is $\mathbb{I} = \{n \cdot ([\mathbb{C}^-] - [\mathbb{C}]) \mid n \in \mathbb{Z}\}$. Since we have $([\mathbb{C}^-] - [\mathbb{C}])^2 = 2 \cdot ([\mathbb{C}^-] - [\mathbb{C}])$, we see that $\mathbb{I}^k = 2^k \cdot \mathbb{I}$. We get an isomorphism of abelian group $\mathbb{Z} \times \mathbb{I} \xrightarrow{\cong} R_{\mathbb{C}}(\mathbb{Z}/2)$ sending $(n, x) \rightarrow n \cdot [\mathbb{C}] + x$. Using this isomorphism the quotient $R_{\mathbb{C}}(\mathbb{Z}/2)/\mathbb{I}^{k+1}$ can be identified with $\mathbb{Z} \times \mathbb{Z}/2^k$ and the projection $R_{\mathbb{C}}(\mathbb{Z}/2)/\mathbb{I}^{k+1} \rightarrow R_{\mathbb{C}}(\mathbb{Z}/2)/\mathbb{I}^k$ becomes $\text{id}_{\mathbb{Z}} \times \text{pr}_k$ for the projection $\text{pr}_k: \mathbb{Z}/2^k \rightarrow \mathbb{Z}/2^{k-1}$. Hence we get

$$R_{\mathbb{C}}(\mathbb{Z}/2)_{\mathbb{I}} \cong \mathbb{Z} \times \widehat{\mathbb{Z}}_2.$$

This confirms all the computations and statements above in the special case $G = \mathbb{Z}/2$.

26.3. Multiplicative structures. The Atiyah-Hirzebruch spectral sequence inherits the following multiplicative structure if we apply it to a cohomology theory with values in R -modules coming with a multiplicative structure. For every $r \in \mathbb{Z}^{\geq 2}$ the page $E_r^{*,*}$ becomes a bi-graded ring, i.e., for $p_0, p_1, \in \mathbb{Z}^{\geq 0}$ and $q_0, q_1 \in \mathbb{Z}$ we have a R -bilinear maps

$$E_r^{p_0, q_0} \times E_r^{p_1, q_1} \rightarrow E_r^{p_0+p_1, q_0+q_1}, \quad x, y \mapsto x \cdot y$$

and an element $1 \in E_r^{0,0}$ such that the following holds for $x_i \in E_r^{p_i, q_i}$ for $i = 0, 1, 2$:

$$\begin{aligned} (x_0 \cdot x_1) \cdot x_2 &= x_0 \cdot (x_1 \cdot x_2); \\ x_0 \cdot x_1 &= (-1)^{(p_0+q_0) \cdot (p_1+q_1)} x_1 \cdot x_0; \\ 1 \cdot x_0 &= x_0. \end{aligned}$$

The differentials satisfy

$$d_r^{p_0+p_1, q_0+q_1}(x_0 \cdot x_1) = d_r^{p_0, q_0}(x_0) \cdot x_1 + (-1)^{p_0+q_0} \cdot x_0 \cdot d_r^{p_1, q_1}(x_1).$$

The identification of the E^2 -term

$$E_2^{p,q} \xrightarrow{\cong} H^p(X; \mathcal{H}_q)$$

is compatible the products up to a sign $(-1)^{(p_0+q_0)p_1}$. The descending filtration $F^{p,q}$ of $\mathcal{H}^{p+q}$ is compatible with the multiplicative structure of $\mathcal{H}^*$ in the sense that for $x_0 \in F^{p_0, q_0}$ and $x_1 \in F^{p_1, q_1}$ we have $x_0 \cup x_1 \in F^{p_0+p_1, q_0+q_1}$, and the identification

$$F^{p,q}/F^{p-1, q+1} \cong E_{\infty}^{p,q}$$

is compatible with the induced products.

All the analogous statements hold for the Leray-Serre spectral sequence, see [33, Theorem XIII.8.10 on page 668].

27. EDGE HOMOMORPHISMS

Consider a homological spectral sequence $E_{*,*}^*$ starting at $d \in \mathbb{Z}^{\geq 2}$ and converging to the ascendingly filtered $\mathbb{Z}$ -graded module $\mathcal{H}^*$. Then each differential $d_{0,q}^r$ starting at $(0, q)$ has zero as target and hence is trivial. Therefore $E_{0,q}^{r+1}$ is a quotient of $E_{0,q}^r$. Hence we get a surjective homomorphism $E_{0,q}^d \rightarrow E_{0,q}^\infty$. Recall that we have an identification $E_\infty^{0,q} = F^{0,q}$, since $F^{-1,q+1} = \{0\}$. As $F^{0,q} \subseteq \mathcal{H}_q$, we obtain a canonical homomorphism, called *qth edge homomorphism at $(0, q)$* ,

$$\text{edge}_{0,q}: E_{0,q}^d \rightarrow \mathcal{H}_q$$

which is the composite $E_{0,q}^d \rightarrow E_{0,q}^\infty = F^{0,q} \subseteq \mathcal{H}_q$.

Consider a cohomological spectral sequence $E_{*,*}^*$ starting at $d \in \mathbb{Z}^{\geq 2}$ converging to the descendingly filtered $\mathbb{Z}$ -graded module $\mathcal{H}^*$. Then each differential $d_{r,q+r-1}^r$ ending at $(0, q)$ has zero as source and hence is trivial. Hence $E_{r+1}^{0,q}$ is a submodule of $E_r^{0,q}$. Therefore we get an injective homomorphism $E_\infty^{0,q} \rightarrow E_d^{0,q}$. Recall that we have an identification $E_\infty^{0,q} = \mathcal{H}^q / F^{0,q}$. Hence we obtain a canonical homomorphism, called *edge homomorphism at $(0, q)$* ,

$$\text{edge}^{0,q}: \mathcal{H}^q \rightarrow E_d^{0,q}$$

which is the composite $\mathcal{H}^q \rightarrow \mathcal{H}^q / F^{0,q} = E_\infty^{0,q} \subseteq E_d^{0,q}$.

Consider a homological spectral sequence $E_{*,*}^*$ starting at $d \in \mathbb{Z}^{\geq 2}$ converging to the ascendingly filtered $\mathbb{Z}$ -graded module $\mathcal{H}_n$ which is a first quadrant spectral sequence. Then each differential $d_{p+r,-r+1}^r$ ending at $E_{p,0}^r$ is trivial. Hence $E_{p,0}^{r+1}$ is a submodule of $E_{p,0}^r$ for $r \geq d$. This implies that $E_{p,0}^\infty$ is a submodule of $E_{p,0}^d$. Recall that we have an identification $\mathcal{H}_p / F_{p,0} = E_{p,0}^\infty$. Hence we get a homomorphism

$$\text{edge}_{p,0}: \mathcal{H}_p \rightarrow E_{p,0}^d$$

called *edge homomorphism at $(p, 0)$* .

Consider a cohomological spectral sequence $E_{*,*}^*$ starting at $d \in \mathbb{Z}^{\geq 2}$ converging to the descendingly filtered $\mathbb{Z}$ -graded module $\mathcal{H}^n$ which is a first quadrant spectral sequence. Then each differential $d_{p,0}^r$ starting at $(p, 0)$ has zero target and hence is trivial. Hence $E_{p+1}^{p,0}$ is a quotient module of $E_p^{p,0}$. Therefore we get a surjective homomorphism $E_d^{p,0} \rightarrow E_\infty^{p,0}$. Recall that we have an identification $E_\infty^{p,0} = F^{p,0} \subseteq \mathcal{H}^p$. Hence we obtain a canonical homomorphism

$$\text{edge}^{p,0}: E_d^{p,0} \rightarrow \mathcal{H}^p$$

called *edge homomorphism at $(p, 0)$* .

For the homological Atiyah-Hirzebruch spectral sequence $\text{edge}_{0,q}$ can be identified with the canonical map

$$\bigoplus_{C \in \pi_0(X)} \mathcal{H}_q(i_C): H_0(X; \mathcal{H}_q(\{\bullet\})) = \bigoplus_{C \in \pi_0(X)} \mathcal{H}_q(\{\bullet\}) \rightarrow \mathcal{H}_q(X).$$

where $i_C: \{\bullet\} \rightarrow X$ is any map with image in C .

For the cohomological Atiyah-Hirzebruch spectral sequence $\text{edge}^{0,q}$ can be identified with the canonical map

$$\prod_{C \in \pi_0(X)} \mathcal{H}^q(i_C): \mathcal{H}^q(X) \rightarrow H^0(X; \mathcal{H}^q(\{\bullet\})) = \prod_{C \in \pi_0(X)} \mathcal{H}^q(\{\bullet\}).$$

The edge homomorphism $\text{edge}_{p,0}: \mathcal{H}_p(X) \rightarrow H_p(X; \mathcal{H}_0(\{\bullet\}))$ does not have in general an explicit description but can be very interesting in special cases, see Example 27.2. The same comment applies to the cohomological $\text{edge}^{p,0}$.

For the homological Leray-Serre spectral sequence for the fibration $F \rightarrow E \xrightarrow{p} B$ with a connected CW -complex B as base space $\text{edge}_{0,q}$ can be identified with the canonical map

$$H_0(B; \mathcal{H}_q(F)) = \mathbb{Z} \otimes_{\mathbb{Z}\pi} \mathcal{H}_q(F_b) \rightarrow \mathcal{H}_q(E)$$

where b is any point in B , $F_b = p^{-1}(b)$ is the fiber over b , the $\pi = \pi_1(B, b)$ action on $\mathcal{H}_q(F_b)$ comes from the fiber transport, and the map is induced by $\mathcal{H}_q(j_b): \mathcal{H}_q(F_b) \rightarrow \mathcal{H}_q(E)$ for the inclusion $j_b: F_b \rightarrow E$.

For the cohomological version $\text{edge}^{0,q}$ can be identified with the map

$$\mathcal{H}^q(E) \rightarrow H^0(B; \mathcal{H}^q(F)) = \mathcal{H}^q(F_b)^\pi$$

induced by $\mathcal{H}^q(j_b): \mathcal{H}^q(E) \rightarrow \mathcal{H}^q(F_b)$.

For the homological Leray-Serre spectral sequence for the fibration $F \rightarrow E \xrightarrow{p} B$ with a connected CW -complex B and a path connected F and singular homology $H_*(-; R)$, the edge homomorphism $\text{edge}_{p,0}$ can be identified with the canonical map

$$H_p(p; R): H_p(E; R) \rightarrow H_p^\Pi(B, H_0(F, R)) = H_p(B; R)$$

For the cohomological Leray-Serre sequence for the fibration $F \rightarrow E \xrightarrow{p} B$ with a connected CW -complex B and a path connected F and singular cohomology $H^*(-; R)$, the edge homomorphism $\text{edge}^{p,0}$ can be identified with the canonical map

$$H^p(p; R): H_\Pi^p(B, H^0(F, R)) = H^p(B; R) \rightarrow H^p(E; R)$$

These identifications of the edge homomorphisms at $(0, q)$ follow for the Atiyah-Hirzebruch spectral sequence by naturality applied to the inclusions $\{\bullet\} \rightarrow X$. For the Leray Serre spectral sequence one uses naturality applied to the map of fibrations from $p: E \rightarrow B$ to $\text{id}_B: B \rightarrow B$ induced by p and id_B and from $\text{pr}: F \rightarrow \{\bullet\}$ to $p: E \rightarrow B$ induced by i and the inclusion $\{\bullet\} \rightarrow F$.

Example 27.1. Consider the homological Atiyah-Serre spectral sequence and assume (for simplicity) that the CW -complex X is connected. Let $i: \{\bullet\} \rightarrow X$ be the inclusion and $\text{pr}: X \rightarrow \{\bullet\}$ be the projection. Then the following diagram commutes

$$\begin{array}{ccc} H_0(\{\bullet\}; \mathcal{H}_q(\{\bullet\})) & \xrightarrow[\cong]{\text{edge}(\{\bullet\})_{0,q}} & \mathcal{H}_q(\{\bullet\}) \\ \downarrow \cong & & \downarrow \mathcal{H}_q(i) \\ H_0(X; \mathcal{H}_q(\{\bullet\})) & \xrightarrow{\text{edge}(X)_{0,q}} & \mathcal{H}_q(X) \end{array}$$

The right vertical arrow is injective as it has the retraction $\mathcal{H}_q(\text{pr})$. The left vertical and the upper horizontal arrows are obviously bijective, Hence the lower horizontal arrow is injective. This implies for the homological Atiyah-Serre spectral sequence for X that $E_{0,q}^2 = E_{0,q}^\infty$ holds and all differentials $d_{r,q-r+1}^r$ ending at $(0, q)$ are trivial.

Analogously one can show for the cohomological Atiyah-Hirzebruch spectral sequence that $E_2^{0,q} = E_\infty^{0,q}$ holds and all differentials starting at $(0, q)$ are trivial.

Now suppose that the CW -complex X is d -dimensional. Then the same considerations yield for the Atiyah-Hirzebruch spectral sequence edge homomorphism

$$\begin{aligned} \text{edge}_{d,q}: \mathcal{H}_{q+d}(X) &\rightarrow H_d(X; \mathcal{H}_q(\{\bullet\})); \\ \text{edge}^{d,q}: H^d(X; \mathcal{H}^q) &\rightarrow \mathcal{H}^{q+d}(X). \end{aligned}$$

Now suppose that the CW -complex B is d -dimensional. Then the same considerations yield for the Leray-Serre spectral sequence edge homomorphism

$$\begin{aligned} \text{edge}_{q,d}: \mathcal{H}_{q+d}(E) &\rightarrow H_{q+d}^\Pi(B; \mathcal{H}_q(F)); \\ \text{edge}^{q,d}: H_\Pi^d(B; \mathcal{H}^q(F)) &\rightarrow \mathcal{H}^{q+d}(E). \end{aligned}$$

Next we suppose that $\mathcal{H}_q(\{\bullet\}) = \{0\}$ for $q < 0$. Then we obtain for the Atiyah-Hirzebruch spectral sequence edge homomorphisms

$$\begin{aligned} \text{edge}_{p,0}: \mathcal{H}_p(X) &\rightarrow H_p(X; \mathcal{H}_0(\{\bullet\})); \\ \text{edge}^{p,0}: H^p(X; \mathcal{H}^0(\{\bullet\})) &\rightarrow \mathcal{H}^p(X). \end{aligned}$$

and for the Leray-Serre spectral sequences edge homomorphisms

$$\begin{aligned} \text{edge}_{p,0}: \mathcal{H}_p(E) &\rightarrow H_p^\Pi(B; \mathcal{H}_0(F)); \\ \text{edge}^{p,0}: H_p^\Pi(B; \mathcal{H}^0(F)) &\rightarrow \mathcal{H}^p(E). \end{aligned}$$

Example 27.2 (Edge homomorphisms for bordism theory). Consider the homology theory oriented bordism Ω_* . It satisfies $\Omega_q(\{\bullet\}) = \{0\}$ for $q \leq -1$. We obtain an isomorphism $\Omega_0(\{\bullet\}) \xrightarrow{\cong} \mathbb{Z}$ by sending a point $\{\bullet\}$ with the orientation $\pm$ to ± 1 . Hence we obtain an edge homomorphism

$$\text{edge}_{p,0}: \Omega_p(X) \rightarrow H_p(X; \Omega_0(\{\bullet\})) \cong H_p(X; \mathbb{Z}).$$

It sends the class of a closed oriented manifold M with reference map $f: M \rightarrow X$ to the image of its fundamental class $[M]$ under the induced homomorphism $H_p(f, \mathbb{Z}): H_p(M; \mathbb{Z}) \rightarrow H_p(X; \mathbb{Z})$.

Consider the homology theory unoriented bordism $\mathcal{N}_*$. It satisfies $\mathcal{N}_q(\{\bullet\}) = \{0\}$ for $q \leq -1$. We obtain an isomorphism $\mathcal{N}_0(\{\bullet\}) \xrightarrow{\cong} \mathbb{Z}/2$ by sending a closed manifold M to the image of its cardinality under the projection $\mathbb{Z} \rightarrow \mathbb{Z}/2$. Hence we obtain an edge homomorphism

$$\text{edge}_{p,0}: \mathcal{N}_p(X) \rightarrow H_p(X; \mathcal{N}_0(\{\bullet\})) \cong H_p(X; \mathbb{Z}/2).$$

It sends the class of a closed manifold M of dimension p with reference map $f: M \rightarrow X$ to the image of the element in $H_p(M; \mathbb{Z}/2) \cong H^0(M; \mathbb{Z}/2) = \text{map}(\pi_0(M), \mathbb{Z}/2)$ given by the constant function with value 1 under the induced homomorphism $H_p(f, \mathbb{Z}/2): H_p(M; \mathbb{Z}/2) \rightarrow H_p(X; \mathbb{Z}/2)$.

Example 27.3 ($\Omega_n(X)$ for $n \leq 4$). We have already explained that $\Omega_0(\{\bullet\}) \cong \mathbb{Z}$ in Example 27.2. We have $\Omega_q(\{\bullet\}) = \{0\}$ for $q = 1, 2, 3$. The signature defines an isomorphism $\text{sign}: \Omega_4(\{\bullet\}) \xrightarrow{\cong} \mathbb{Z}$.

Consider a connected CW-complex. We conclude from the Atiyah-Hirzebruch spectral sequence and Example 27.2 that we obtain an isomorphism

$$\text{sign} \times \text{edge}_{4,0}: \Omega_4(X) \xrightarrow{\cong} \mathbb{Z} \times H_4(X; \mathbb{Z})$$

sending the class of a closed oriented manifold M with reference map $f: M \rightarrow X$ to $(\text{sign}(M), H_4(f; \mathbb{Z})([M]))$, that we obtain for $n = 1, 2, 3$ isomorphisms

$$\text{edge}_{n,0}: \Omega_n(X) \xrightarrow{\cong} H_n(X; \mathbb{Z}), \quad (M, f) \rightarrow H_n(f; \mathbb{Z})([M])$$

and that the projection $\text{pr}: X \rightarrow \{\bullet\}$ induces an isomorphism

$$\Omega_0(\text{pr}): \Omega_0(X) \rightarrow \Omega_0(\{\bullet\}) \cong \mathbb{Z}.$$

Proposition 27.4 (Serre sequence). *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration over a connected CW-complex B with path connected fiber F . Suppose the following conditions for $k, l \in \mathbb{Z}^{\geq 1}$:*

- (i) *The action of $\pi_1(B)$ on $H_q(F)$ induced by the fiber transport is trivial for $q \in \mathbb{Z}^{\geq 0}$;*
- (ii) *$H_i(B)$ vanishes for $0 < i < k$;*
- (iii) *$H_i(F)$ vanishes for $0 < i < l$.*

Then there exists an exact sequence

$$\begin{aligned} H_{k+l-1}(F) &\xrightarrow{H_{k+l-1}(i)} H_{k+l-1}(E) \xrightarrow{H_{k+l-1}(p)} H_{k+l-1}(B) \xrightarrow{\partial_{k+l-1}} H_{k+l-2}(F) \\ &\xrightarrow{H_{k+l-2}(i)} \dots \xrightarrow{\partial_2} H_1(F) \xrightarrow{H_1(i)} H_1(E) \xrightarrow{H_1(p)} H_1(B) \rightarrow 0. \end{aligned}$$

Proof. We apply the Leray-Serre spectral sequence. We have the exact sequence

$$0 \rightarrow H_p(B) \otimes_{\mathbb{Z}} H_q(F) \rightarrow E_{p,q}^2 = H_p(B; H_q(F)) \rightarrow \text{Tor}_1(H_{p-1}(B), H_q(F)) \rightarrow 0.$$

Hence $E_{p,q}^2$ vanishes if one of the following conditions are satisfied

- $0 < p < k$ and $q \geq 0$;
- $p \geq 0$ and $0 < q < l$.

So for $k = l = 3$ the second page looks in the range $0 \leq p \leq 5$ and $0 \leq q \leq 5$ like

$H_5(F)$	0	0	$E_{3,5}^n$	$E_{4,5}^n$	$E_{5,5}^n$
$H_4(F)$	0	0	$E_{3,4}^n$	$E_{4,4}^n$	$E_{5,4}^n$
$H_3(F)$	0	0	$E_{3,3}^n$	$E_{4,3}^n$	$E_{5,3}^n$
0	0	0	0	0	0
0	0	0	0	0	0
$\mathbb{Z}$	0	0	$H_3(B)$	$H_4(B)$	$H_5(B)$

Hence the differential $d_{p,0}^r: E_{p,0}^r \rightarrow E_{p-r,r-1}^r$ vanishes for $2 \leq r < p \leq k+l$ and the differential $d_{r,q+1-r}^r: E_{r,q+1-r}^r \rightarrow E_{0,q}^r$ is trivial for $2 \leq r < q+1 \leq k+l-1$. This implies $E_{n,0}^n = E_{n,0}^2 = H_n(B)$ and $E_{0,n-1}^n = E_{0,n-1}^2 = H_{n-1}(F)$ for $n \leq k+l-1$. We conclude $E_{n,0}^\infty = \ker(d_{n,0}^n)$ and $E_{0,n-1}^\infty = \text{cok}(d_{n,0}^n)$ for the differential $d_{n,0}^n: E_{n,0}^n = H_n(B) \rightarrow E_{0,n-1}^n = H_{n-1}(F)$ if $n \leq k+l-1$ holds. Now we get from the construction and the identification of the edge homomorphism an exact sequence

$$\begin{aligned} H_{k+l-1}(F) &\xrightarrow{H_{k+l-1}(i)} H_{k+l-1}(E) \xrightarrow{H_{k+l-1}(p)} H_{k+l-1}(B) \xrightarrow{\partial_{k+l-1}} H_{k+l-2}(F) \\ &\xrightarrow{H_{k+l-2}(i)} \dots \xrightarrow{\partial_2} H_1(F) \xrightarrow{H_1(i)} H_1(E) \xrightarrow{H_1(p)} H_1(B) \rightarrow 0. \end{aligned}$$

if we put $\partial_n = d_{n,0}^n$.

□

28. APPLICATIONS OF THE COHOMOLOGICAL LERAY-SERRE SPECTRAL SEQUENCE

28.1. The Gysin sequence. Consider a fibration $S^l \rightarrow E \xrightarrow{f} B$ over a connected CW-complex B with fiber S^l for $l \geq 1$. Let $H^*(-; G)$ be singular cohomology with coefficients in the abelian group G . Then $H^q(S^l; G)$ is G for $q = 0, l$ and trivial otherwise. The G -action of $\pi = \pi_1(B)$ on $H^0(S^l; G)$ is trivial and hence $H_{\Pi}^p(B; H^0(S^l; G)) = H^p(B, G)$. We get $H_{\Pi}^p(B; H^0(S^l; G)) = H_{\pi}^p(B; G_{\rho})$ for the

π -action on G given by $w \cdot g = \rho(w) \cdot g$ for $w \in \pi$ and $g \in G$ for the homomorphism $\rho: \pi \rightarrow \{\pm 1\}$ sending $w \in \pi$ to the degree of the map $S^l \rightarrow S^l$ given by the fiber transport applied to w . Then the E_2 -term and hence the E_r -term for $r \in \mathbb{Z}^{\geq 2} \amalg \{\infty\}$ of Leray-Serre spectral sequence for f and $H^*(-; G)$ has non-trivial entries only in the 0-th row and in the l -th row. The only non-trivial differentials appear on the $(l+1)$ -th page and are given by

$$d_{l+1}^{p,l}: E_{l+1}^{p,l} = H_{\pi}^p(B; G_{\rho}) \rightarrow E_{p-l+1,0}^{l+1} = H_{p+l+1}(B; G).$$

So for $l = 2$ the $(l+1)$ -th page looks like

$$\begin{array}{ccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\ 0 & & 0 & & 0 & & 0 & & 0 & & \dots \\ & & & & & & & & & & \\ H_{\pi}^0(B; G_{\rho}) & H_{\pi}^1(B; G_{\rho}) & H_{\pi}^2(B; G_{\rho}) & H_{\pi}^3(B; G_{\rho}) & H_{\pi}^4(B; G_{\rho}) & \dots \\ & \searrow & & \searrow & & \\ 0 & 0 & d_3^{0,3} & 0 & d_3^{1,3} & 0 & 0 & \dots \\ & & & & & & & \\ H^0(B; G) & H^1(B; G) & H^2(B; G) & H^3(B; G) & H^4(B; G) & \dots \end{array}$$

So we get exact sequences

$$0 \rightarrow E_{\infty}^{p,l} \rightarrow H_{\pi}^p(B; G_{\rho}) \xrightarrow{d_{l+1}^{p,l}} H^{p+l+1}(B; G) \rightarrow E_{\infty}^{p+l+1,0} \rightarrow \{0\}.$$

We have the ascending filtration

$$H^n(E, G) = F^{n-l,l} \supseteq F^{n,0} \supseteq \{0\}$$

with filtration quotients $H^n(E, G)/F^{n,0} = E_{\infty}^{n-l,l}$ and $F^{n,0} = E_{\infty}^{n,0}$. We can splice these data together to a long exact sequence

$$(28.1) \quad \dots \xrightarrow{\delta^{n-1}} H^{n-1-l}(B; G) \xrightarrow{d_{l+1}^{n-1-l,l}} H_{\pi}^n(B; G_{\rho}) \xrightarrow{H^n(f; G)} H^n(E; G) \\ \xrightarrow{\delta^n} H^{n-l}(B; G) \xrightarrow{d_{l+1}^{n-l,l}} H_{\pi}^{n+1}(B; G_{\rho}) \xrightarrow{H^{n+1}(f; G)} H^{n+1}(E; G) \xrightarrow{d_{l+1}^{n+1-l+1,l}} \dots$$

So far this is just dual as the in the homological case. But now one can use the multiplicative structure on singular cohomology and the induced multiplicative structure on the Leray-Serre spectral sequence to get a better understanding of the differentials $d_{l+1}^{n-1-l,l}$ in the case, where ρ is trivial and G is actually a commutative ring R . Let $1 \in H^0(B; R) \cong R$ be the generator given by the constant function from the set of singular zero-simplices in B to R with value $1 \in R$. Let $e_f \in H^{l+1}(B; R)$ be the image of 1 under the differential $d_{l+1}^{0,l}: H_{\pi}^0(B; R) \rightarrow H^{l+1}(B; R)$, where we consider 1 an element in $E_{l+1}^{0,l}$. Then we compute for $x \in H^{n-1-l}(B; G)$ which we can think of as an element in both $E_{l+1}^{n-1-l,l}$ and in $E_{l+1}^{n-1-l,0}$.

$$\begin{aligned} d_{l+1}^{n-1-l,l}(x) &= d_{l+1}^{n-1-l,l}(1 \cup x) = d_{l+1}^{0,l}(1) \cup x + (-1)^{n-1} \cdot 1 \cup d_{l+1}^{n-1-l,0}(x) \\ &= e_f \cup x + (-1)^{n-1} \cdot 1 \cup 0 = e_f \cup x. \end{aligned}$$

Hence the long exact sequence (28.1) becomes

$$(28.2) \quad \begin{aligned} \dots &\xrightarrow{\delta^{n-1}} H^{n-1-l}(B; R) \xrightarrow{e_f \cup -} H^n_\pi(B; R) \xrightarrow{H^n(f; R)} H^n(E; R) \\ &\xrightarrow{\delta^n} H^{n-l}(B; R) \xrightarrow{e_f \cup -} H^{n+1}_\pi(B; R) \xrightarrow{H^{n+1}(f; R)} H^{n+1}(E; R) \xrightarrow{d^{n+1-l+1, l}} \dots \end{aligned}$$

Consider the universal principal S^1 -bundle $f: ES^1 \xrightarrow{f} BS^1$. An explicit model is given by $S^1 \rightarrow S^\infty \rightarrow \mathbb{CP}^\infty$. Recall that $\mathbb{CP}^\infty$ is a model for $K(\mathbb{Z}, 2)$ and we get for any CW -complex B an isomorphism

$$[B, \mathbb{CP}^\infty] \xrightarrow{\cong} H^2(B; \mathbb{Z}), \quad [g] \mapsto H^2(g; \mathbb{Z})(c_f).$$

Note that f is orientable, i.e., has trivial fiber transport. Since ES^1 is contractible, we get from the Gysin sequence (28.1) isomorphisms for $n \in \mathbb{Z}^{\geq 0}$.

$$e_f \cup -: H^n(\mathbb{CP}^\infty; \mathbb{Z}) \xrightarrow{\cong} H^{n+2}(\mathbb{CP}^\infty; \mathbb{Z}).$$

Since $\mathbb{CP}^\infty$ is simply connected $H^1(\mathbb{CP}^\infty; \mathbb{Z})$ vanishes. Hence we obtain the following theorem.

Theorem 28.3. *The cohomology ring $H^*(\mathbb{CP}^\infty; \mathbb{Z})$ is isomorphic to the free polynomial $\mathbb{Z}[x]$ for a generator x in degree 2 which is given by $e_f \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$.*

28.2. The first Chern class of a principal S^1 -bundle. Let $p: E \rightarrow B$ be a principal S^1 -bundle. Recall that up to homotopy there is a map $c_p: B \rightarrow \mathbb{CP}^\infty$ which is up to homotopy uniquely determined by the property that the pullback of the universal principal S^1 -bundle $f: ES^1 \rightarrow BS^1$ with c_p is isomorphic as a principal S^1 -bundle is isomorphic to p . Denote by

$$(28.4) \quad c_1(p) \in H^2(B; \mathbb{Z})$$

the preimage of $e_f \in H^2(\mathbb{CP}^\infty; \mathbb{Z}/2)$. This element is called the *first Chern class* of p . It has the following properties.

Theorem 28.5. (i) *Consider a map $f: A \rightarrow B$ between CW -complexes. Let $p: E \rightarrow B$ be a principal S^1 -bundle over a CW -complex B . Let the principal S^1 -bundle f^*p over A be the pullback of p with f . Then we get*

$$c_1(f^*p) = f^*(c_1(p));$$

(ii) *Let $p_1: E_1 \rightarrow B$ and $p_2: E_2 \rightarrow B$ be principal S^1 -bundles over the CW -complex B . Then they are isomorphic as principal S^1 -bundles if and only if $c_1(p_1) = c_2(p_2)$ holds;*

(iii) *Consider a CW -complex B and an element $y \in H^2(B; \mathbb{Z})$. Then there exists a principal S^1 -bundle $p: E \rightarrow B$ with*

$$c_1(p) = y.$$

28.3. A variation of the Leray-Hirsch Theorem. We mention the following variation of the Leray-Hirsch Theorem.

Theorem 28.6 (Leray-Hirsch Theorem for singular cohomology with field coefficients). *Let k be a field and $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration over a connected CW -complex B . Suppose that the following conditions are satisfied:*

- (i) *The $\pi_1(B)$ -action on the singular cohomology $H^*(F; k)$ with coefficients in k coming from the fiber transport is trivial;*
- (ii) *The k -module $H^p(B; k)$ is finitely generated for $p \in \mathbb{Z}^{\geq 0}$;*
- (iii) *The map $i^*: H^*(E; k) \rightarrow H^*(F; k)$ is surjective.*

Then we obtain an isomorphism of graded $H^*(B; k)$ -modules

$$H^*(B; k) \otimes_k H^*(F; k) \xrightarrow{\cong} H^*(E; k)$$

where we equip $H^*(E; k)$ with the $H^*(B; k)$ -module structure coming from the cup product on $H^*(E; k)$ and the homomorphism of graded k -algebras $p^*: H^*(B; k) \rightarrow H^*(E; k)$ and we equip $H^*(B; k) \otimes_k H^*(F; k)$ with the obvious $H^*(B; k)$ -module structure given by $u \otimes (v \otimes w) = uv \otimes w$ for $u, v \in H^*(B; k)$ and $w \in H^*(F; k)$.

If we furthermore assume that $H^*(F; k)$ is a free graded commutative k -algebra, then we obtain an isomorphism of graded commutative k -algebras

$$H^*(B; k) \otimes_k H^*(F; k) \xrightarrow{\cong} H^*(E; k)$$

Proof. Since $H^p(B; k)$ is finitely generated as k -module and k is a field, we can identify the second page of the Leray-Serre spectral sequence using the Universal Coefficient Theorem to be

$$\begin{aligned} E_2^{p,q} &= H^p(B; H^q(F; k)) \cong \text{hom}_k(H_p(B; k), H^q(F; k)) \\ &\cong \text{hom}_k(H^p(B; k), k) \otimes_k H^q(F; k) \cong H^p(B; k) \otimes_k H^q(F; k). \end{aligned}$$

This identification is compatible with the multiplicative structure coming from the cup product on singular cohomology and is natural in B and F .

Consider $p, q \in \mathbb{Z}^{\geq 0}$. We conclude that the multiplication map

$$E_2^{p,0} \otimes_k E_2^{0,q} \rightarrow E_2^{p,q}$$

is bijective for $p, q \in \mathbb{Z}^{\geq 0}$. Since $i^*: H^q(F; k) \rightarrow H^q(E; k)$ is by assumption surjective, the differential $d_2^{0,q}$ is trivial. Since the differential $d_2^{p,q}$ is a derivation, we get for $a \in E_2^{p,0}$ and $b \in E_2^{0,q}$

$$d_2^{p,q}(a \cdot b) = d_2^{p,0}(a) \cdot b + (-1)^p \cdot a \cdot d_2^{0,q}(b) = 0 \cdot b + (-1)^p \cdot a \cdot 0 = 0.$$

Hence all second differentials are trivial and we have $E_2^{p,q} = E_3^{p,q}$. This argument can be repeated and yields the equality $E_2^{p,q} = E_r^{p,q}$ for all $p, q \in \mathbb{Z}^{\geq 0}$ and $r \in \mathbb{Z}^{\geq 3}$. Hence we get an identification

$$E_\infty^{p,q} = H^p(B; k) \otimes_k H^q(F; k).$$

Fix a section $s^*: H^*(F; k) \rightarrow H^*(E; k)$ of the surjective map of graded k -modules $i^*: H^*(E; k) \rightarrow H^*(F; k)$. By inspecting the filtrations of $H^*(E; k)$ appearing in the Leray-Serre spectral sequence and the Five Lemma, one easily checks the map

$$\Phi^*: H^*(B; k) \otimes_k H^*(F; k) \rightarrow H^*(E; k), \quad a \otimes b \mapsto p^*(a) \cup s^*(b)$$

is bijective. Obviously it is a map of graded $H^*(B; k)$ -modules.

Now suppose additionally that $H^*(F; k)$ is a free graded k -algebra. Then we can choose s^* such that it becomes an map of k -algebras and hence Φ^* is a map of graded k -algebras. $\square$

If we replace in Theorem 28.6 the condition that the k -module $H^p(B; k)$ is finitely generated for $p \in \mathbb{Z}^{\geq 0}$ by the condition that there exists an element $d \in \mathbb{Z}^{\geq 1}$ such that the k -module $H^q(F; k)$ is finitely generated for $0 \leq p \leq d$ and vanishes for $p \geq (d + 1)$, then Theorem 28.6 follows from Theorem 22.4. The proof of Theorem 28.6 in general is an obvious variation of the one for Theorem 22.4.

28.4. An example concerning the Leray-Hirsch Theorem 28.6. One may ask whether one can replace in Theorem 28.6 the condition that $H^*(i; k)$ is surjective by the condition that $H^*(p; k): H^*(B; k) \rightarrow H^*(E; k)$ is injective. This is not the case, as we will show by constructing a fibration $S^3 \times S^2 \rightarrow E \xrightarrow{p} S^2$ for which $H^*(p): H^*(B) \rightarrow H^*(E)$ is injective but the Leray-Serre spectral sequence has a non trivial differential and hence does not strongly collapses as predicted by Theorem 28.6. Figuring out this example is a good exercise for working with spectral sequence and fibrations.

Let $S^3 \rightarrow S^7 \xrightarrow{q} S^4$ be the quaternionic Hopf-fibration for $\mathbb{H}^2$ using the identification $S^4 \cong \mathbb{H}\mathbb{P}^1$. Note that S^3 inherits from its embedding into the skewfield of quaternions $\mathbb{H}$ the structure of a topological group and the quaternionic Hopf-fibration is actually a principal S^3 -bundle. Let $c: S^2 \times S^2 \rightarrow S^4$ be the map obtained by collapsing the 2-skeleton of $S^2 \times S^2$ to a point. This is a map of closed oriented smooth 4-manifolds of degree 1. Consider the pull back

$$\begin{array}{ccc} E & \xrightarrow{\bar{c}} & S^7 \\ \bar{q} \downarrow & & \downarrow q \\ S^2 \times S^2 & \xrightarrow{c} & S^4 \end{array}$$

Then $\bar{q}: E \rightarrow S^2 \times S^2$ is a fibration. Define

$$p: E \xrightarrow{\bar{q}} S^2 \times S^2 \xrightarrow{\text{pr}_1} S^2$$

where pr_1 is the projection onto the first coordinate. This is a fibration as $\bar{q}$ and pr_1 are fibrations. Let $s \in S^2$ be the standard base point. Put $F = p^{-1}(s) = \bar{q}^{-1}(\{s\} \times S^2)$. We have the pullback

$$\begin{array}{ccc} F & \longrightarrow & S^7 \\ \downarrow & & \downarrow q \\ S^2 & \xrightarrow{c \circ i_2} & S^4 \end{array}$$

for the inclusion $i_2: S^2 = \{s\} \times S^2 \rightarrow S^2 \times S^2$. Since $c \circ i_2$ is the constant map and q is a fibration with fiber S^3 , we get a homotopy equivalence $F \simeq S^2 \times S^3$. This finishes the construction of the fibration $S^3 \times S^2 \rightarrow E \xrightarrow{p} S^2$. Since S^2 is simply connected, its fiber transport is trivial. The Leary-Serre spectral sequence of it has the following second page and there is at most one differential, namely $d_{3,0}^2$, which

may be nontrivial:

$$\begin{array}{ccc}
H^0(S^2; H^5(S^3 \times S^2)) \cong \mathbb{Z} & 0 & H^2(S^2; H^5(S^3 \times S^2)) \cong \mathbb{Z} \\
H^0(S^2; H^4(S^3 \times S^2)) \cong 0 & 0 & H^2(S^2; H^4(S^3 \times S^2)) \cong 0 \\
H^0(S^2; H^3(S^3 \times S^2)) \cong \mathbb{Z} & 0 & H^2(S^2; H^3(S^3 \times S^2)) \cong \mathbb{Z} \\
H^0(S^2; H^2(S^3 \times S^2)) \cong \mathbb{Z} & 0 & H^2(S^2; H^2(S^3 \times S^2)) \cong \mathbb{Z} \\
H^0(S^2; H^1(S^3 \times S^2)) \cong 0 & 0 & H^2(S^2; H^1(S^3 \times S^2)) \cong 0 \\
H^0(S^2; H^0(S^3 \times S^2)) \cong \mathbb{Z} & 0 & H^2(S^2; H^0(S^3 \times S^2)) \cong \mathbb{Z}
\end{array}$$

$\xrightarrow{d_2^{3,0}}$

We obtain from the Leray-Serre spectral sequence that $H^*(p): H^*(S^2) \rightarrow H^*(E)$ is injective and we have the exact sequence

$$\{0\} \rightarrow H^3(E) \rightarrow H^0(S^2; H^3(S^3 \times S^2)) \cong \mathbb{Z} \xrightarrow{d_2^{3,0}} H^2(S^2; H^2(S^3 \times S^2)) \cong \mathbb{Z}.$$

Hence $d_2^{3,0}$ is non-trivial if and only if $H^3(E)$ vanishes. Now the Gysin sequences of $S^3 \rightarrow E \xrightarrow{\bar{q}} S^2 \times S^2$ and $S^3 \rightarrow S^7 \xrightarrow{q} S^4$ fit together to a commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^3(S^2 \times S^2) = \{0\} & \longrightarrow & H^3(E) & \longrightarrow & H^0(S^2 \times S^2) & \longrightarrow & H^4(S^4) \\
\downarrow H^3(c) & & \downarrow H^3(\bar{c}) & & \cong \downarrow H^0(c) & & \cong \downarrow H^4(c) \\
H^3(S^4) = \{0\} & \longrightarrow & H^4(S^7) = \{0\} & \longrightarrow & H^0(S^4) & \longrightarrow & H^4(S^4)
\end{array}$$

The Five Lemma implies that $H^3(E)$ vanishes and hence $d_2^{3,0}$ is non-trivial. We get

$$H^n(E) \cong \begin{cases} \mathbb{Z} & n = 0, 7; \\ \mathbb{Z}^2 & n = 2, 5; \\ \{0\} & n = 1, 3, 6, ; \text{ or } n \geq 8; \\ \text{cok}(d_2^{3,0}) & n = 4. \end{cases}$$

for a finite cyclic group $\text{cok}(d_2^{3,0})$. So we get by the Universal Coefficient Theorem

$$H^n(E; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & n = 0, 7; \\ \mathbb{Q}^2 & n = 2, 5; \\ \{0\} & n = 1, 3, 4, 6, ; \text{ or } n \geq 8, \end{cases}$$

whereas

$$(H^*(S^2; \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(S^3 \times S^2; \mathbb{Q}))^n \cong \begin{cases} \mathbb{Q} & n = 0, 3, 4, 7; \\ \mathbb{Q}^2 & n = 2, 5; \\ \{0\} & n = 1, 6, \text{ or } n \geq 8. \end{cases}$$

28.5. Computation of the singular cohomology ring with rational coefficients of $U(n)$.

Theorem 28.7. *The singular cohomology ring $H^*(U(n); \mathbb{Q})$ with rational coefficients of the Lie group $U(n)$ is the exterior algebra $\Lambda_{\mathbb{Q}}(x_1, x_2, \dots, x_n)$, where the degree $|x_i|$ of i is $2i - 1$.*

Proof. We use induction over n .

In the induction beginning we have $U(1) = S^1$ and we know already the equality $H^*(S^{2n-1}; \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(x_n)$ for $|x_n| = 2n - 1$.

In the induction step from $(n-1)$ to n for $n \geq 2$ we use the fibration $U(n-1) \rightarrow U(n) \rightarrow S^{2n-1}$ coming from the obvious $U(n)$ -action on S^{2n-1} . Since S^{2n-1} is $(2n-2)$ -connected, the map $U(n-1) \rightarrow U(n)$ is $(2n-2)$ -connected. The relative Hurewicz Theorem 12.11 implies that $H_k(U(n-1); \mathbb{Z}) \rightarrow H_k(U(n); \mathbb{Z})$ is bijective for $k \leq 2n-3$ and surjective for $k = 2n-2$.

We conclude from Universal Coefficient Theorem that the map $i^*: H^*(U(n); \mathbb{Q}) \rightarrow H^*(U(n-1); \mathbb{Q})$ is surjective, since $H^k(U(n-1); \mathbb{Q})$ vanishes for $k \geq (2n-1)$. Now Theorem 28.6 yields an isomorphism of graded k -algebras

$$\begin{aligned} H^*(U(n); \mathbb{Q}) &\cong H^*(U(n-1); \mathbb{Q}) \otimes_k H^*(S^{2n-1}; \mathbb{Q}) \\ &\cong \Lambda_{\mathbb{Q}}(x_1, x_2, \dots, x_{n-1}) \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}(x_n) \cong \Lambda_{\mathbb{Q}}(x_1, x_2, \dots, x_n). \end{aligned}$$

□

28.6. Singular Cohomology of ΩS^3 . Recall that we have $H^*(S^3) \cong \Lambda_{\mathbb{Z}}(x) = \mathbb{Z}[x]/(x^2)$ for $|x| = 3$. We apply the Leray-Serre spectral sequence to the fibration $\Omega S^3 \rightarrow P(S^3) \rightarrow S^3$. Then the third page agrees with the second page and looks like

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\ \\ H^6(\Omega S^3) & 0 & & 0 & & H^6(\Omega S^3) & 0 & & \cdots \\ & \searrow & & & \searrow & & & & \\ H^5(\Omega S^3) & 0 & d_3^{0,6} & 0 & & H^5(\Omega S^3) & 0 & & \cdots \\ & \searrow & & & \searrow & & & & \\ H^4(\Omega S^3) & 0 & d_3^{0,5} & 0 & & H^4(\Omega S^3) & 0 & & \cdots \\ & \searrow & & & \searrow & & & & \\ H^3(\Omega S^3) & 0 & d_3^{0,4} & 0 & & H^3(\Omega S^3) & 0 & & \cdots \\ & \searrow & & & \searrow & & & & \\ H^2(\Omega S^3) & 0 & d_3^{0,3} & 0 & & H^2(\Omega S^3) & 0 & & \cdots \\ & \searrow & & & \searrow & & & & \\ H^1(\Omega S^3) & 0 & d_3^{0,2} & 0 & & H^1(\Omega S^3) & 0 & & \cdots \\ & \searrow & & & \searrow & & & & \\ H^0(\Omega S^3) & 0 & & 0 & & H^0(\Omega S^3) & 0 & & \cdots \end{array}$$

The fourth page agrees with the r th page for $r \in \mathbb{Z}^{\geq 4}$. Since the Leray-Serre spectral sequence converges to $P(S^3)$ which is contractible and hence satisfies $H^n(P(S^3)) = \{0\}$ for $n \geq 1$, all the differentials $d_3^{0,q}$ for $q \geq 3$ are isomorphisms and we have $H^1(\Omega S^3) = \{0\}$. Hence $H^n(\Omega S^3) = \{0\}$ for odd n and $H^n(\Omega S^3) = \mathbb{Z}$ for even n . Therefore the E^3 -page reduces to

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \vdots \\
 & & & & & & \ddots \\
 H^6(\Omega S^3) \cong \mathbb{Z} & 0 & & 0 & H^6(\Omega S^3) \cong \mathbb{Z} & 0 & \dots \\
 & \searrow & & & & & \\
 0 & 0 & d_3^{0,6} & 0 & 0 & 0 & \dots \\
 & & \cong & & & & \\
 H^4(\Omega S^3) \cong \mathbb{Z} & 0 & & 0 & H^4(\Omega S^3) \cong \mathbb{Z} & 0 & \dots \\
 & \searrow & & & & & \\
 0 & 0 & d_3^{0,4} & 0 & 0 & 0 & \dots \\
 & & \cong & & & & \\
 H^2(\Omega S^3) \cong \mathbb{Z} & 0 & & 0 & H^2(\Omega S^3) \cong \mathbb{Z} & 0 & \dots \\
 & \searrow & & & & & \\
 0 & 0 & d_3^{0,2} & 0 & 0 & 0 & \dots \\
 & & \cong & & & & \\
 H^0(\Omega S^3) \cong \mathbb{Z} & 0 & & 0 & H^0(\Omega S^3) \cong \mathbb{Z} & 0 & \dots
 \end{array}$$

The obvious identification

$$E_3^{p,q} = E_2^{p,q} = H^p(S^3; H^q(\Omega^3 S^3)) = H^p(S^3) \otimes H^q(\Omega S^3)$$

is compatible with the obvious product structure on $H^p(S^3) \otimes H^q(\Omega S^3)$. Hence the pairing

$$E_3^{p,0} \otimes E_3^{0,q} \rightarrow E_3^{p,q}$$

is an isomorphism and we can choose generators $z_q \in E_3^{0,2q}$ for $q \in \mathbb{Z}^{\geq 0}$ such that $z_q \cdot d_3^{0,2}(z_1) = d_3^{0,2(q+1)}(z_{q+1})$ holds for $q \in \mathbb{Z}^{\geq 0}$ and $z_0 \cdot u = u$ holds for all $u \in E_3^{p,q}$. Then we denote the E^3 -page as follows indicating generators

$$\begin{array}{cccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\
\mathbb{Z}\langle z_3 \rangle & 0 & & 0 & & \mathbb{Z} & 0 & & \dots & & \\
& \searrow & & \searrow & & \searrow & & \searrow & & \searrow & \\
0 & 0 & \xrightarrow[\cong]{d_3^{0,6}} & 0 & 0 & 0 & 0 & & \dots & & \\
\mathbb{Z}\langle z_2 \rangle & 0 & & 0 & & \mathbb{Z} & 0 & & \dots & & \\
& \searrow & & \searrow & & \searrow & & \searrow & & \searrow & \\
0 & 0 & \xrightarrow[\cong]{d_3^{0,4}} & 0 & 0 & 0 & 0 & & \dots & & \\
\mathbb{Z}\langle z_1 \rangle & 0 & & 0 & & \mathbb{Z} & 0 & & \dots & & \\
& \searrow & & \searrow & & \searrow & & \searrow & & \searrow & \\
0 & 0 & \xrightarrow[\cong]{d_3^{0,2}} & 0 & 0 & 0 & 0 & & \dots & & \\
\mathbb{Z}\langle z_0 \rangle & 0 & & 0 & & \mathbb{Z} & 0 & & \dots & &
\end{array}$$

Next we show by induction over $n \in \mathbb{Z}^{\geq 1}$ that $z_1^n = n! \cdot z_n$ holds. The induction begin $n = 1$ is trivial. The induction step from n to $n + 1$ for $n \in \mathbb{Z}^{\geq 1}$ follows from the following calculation, since $d_3^{0,2(n+1)}$ is bijective.

$$\begin{aligned}
d_3^{0,2(n+1)}(z_1^{n+1}) &= d_3^{0,2(n+1)}(z_1 \cdot z_1^n) \\
&= d_3^{0,2}(z_1) \cdot z_1^n + z_1 \cdot d_3^{0,2n}(z_1^n) \\
&= d_3^{0,2}(z_1) \cdot n! \cdot z_n + z_1 \cdot d_3^{0,2n}(n! \cdot z_n) \\
&= n! \cdot z_n \cdot d_3^{0,2}(z_1) + n! \cdot z_1 \cdot d_3^{0,2n}(z_n) \\
&= n! \cdot d_3^{0,2(n+1)}(z_{n+1}) + n! \cdot z_1 \cdot d_3^{0,2}(z_1) \cdot z_{n-1} \\
&= n! \cdot d_3^{0,2(n+1)}(z_{n+1}) + n \cdot z_1 \cdot (n-1)! \cdot z_{n-1} \cdot d_3^{0,2}(z_1) \\
&= n! \cdot d_3^{0,2(n+1)}(z_{n+1}) + n \cdot z_1 \cdot z_1^{n-1} \cdot d_3^{0,2}(z_1) \\
&= n! \cdot d_3^{0,2(n+1)}(z_{n+1}) + n \cdot z_1^n \cdot d_3^{0,2}(z_1) \\
&= n! \cdot d_3^{0,2(n+1)}(z_{n+1}) + n \cdot n! \cdot z_n \cdot d_3^{0,2}(z_1) \\
&= n! \cdot d_3^{0,2(n+1)}(z_{n+1}) + n \cdot n! \cdot d_3^{0,2(n+1)}(z_{n+1}) \\
&= (n+1)! \cdot d_3^{0,2(n+1)}(z_{n+1}) \\
&= d_3^{0,2(n+1)}((n+1)! \cdot z_{n+1}).
\end{aligned}$$

Hence the graded ring $H^*(\Omega^3 S^3)$ agrees with the so called *divided power algebra* which is denoted by

$$\mathbb{Z} \left[y, \frac{y^2}{2!}, \frac{y^3}{3!}, \frac{y^4}{4!} \dots \right].$$

where y has degree two and corresponds to z_1 in the notation above. Explicitly the underlying graded $\mathbb{Z}$ -module has $\mathbb{Z}$ as entry in degree n if $n \in \mathbb{Z}^{\geq 0}$ and n is even,

and $\{0\}$ otherwise. If z_n is the standard generator of the $2n$ th module, then the multiplication is given by the formula

$$z_m \cdot z_n = \frac{(m+n)!}{m! \cdot n!} \cdot z_{m+n}.$$

Note that the divided power algebra is isomorphic to $\mathbb{Z}[y]$ for $|y| = 2$ as graded abelian group, but not as graded commutative ring.

Note that in this argument we have computed the cohomology of the fiber of the fibration $\Omega S^3 \rightarrow P(S^3) \rightarrow S^3$ from the knowledge of the cohomology of the total space and the base space, whereas before we have always computed the cohomology of the total space from the one of the fiber and the base space.

28.7. Computation of π_1^s . We give another proof that $\pi_1^s \cong \mathbb{Z}/2$ holds. Because of the Freudenthal Suspension Theorem 11.16 it suffices to prove $\pi_4(S^3) \cong \mathbb{Z}/2$.

Choose a map $f: S^3 \rightarrow K(\mathbb{Z}, 3)$ which induces an isomorphism $\pi_3(f)$. We can turn it into fibration $\bar{f}: E \rightarrow K(\mathbb{Z}, 3)$. Let F be its fiber. We conclude from the long homotopy sequence of $\bar{f}$ that F is 3-connected and $\pi_4(F) \cong \pi_4(S^3)$. By the Hurwicz Theorem 12.5 we get $H_4(F) \cong \pi_4(S^3)$. So it remains to prove $H_4(F) \cong \mathbb{Z}/2$.

We have the fibration $\Omega K(\mathbb{Z}, 3) \rightarrow F \rightarrow E$. Since E is homotopy equivalent to S^3 and $\mathbb{CP}^\infty$ is homotopy equivalent to $\Omega K(\mathbb{Z}, 3)$, there exists a fibration $\mathbb{CP}^\infty \rightarrow F \xrightarrow{p} S^3$. Recall that the cohomology ring $H^*(\mathbb{CP}^\infty)$ is $\mathbb{Z}[c]$ for $|c| = 2$ and the cohomology ring of $H^*(S^3)$ is $\mathbb{Z}[x]/(x^2)$ for $|x| = 3$.

Next we apply the Leray-Serre spectral sequence to the fibration p and singular cohomology with $\mathbb{Z}$ -coefficients. The second page is concentrated in the two columns over 0 and 3. Hence it agrees with the the third page:

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\
 \mathbb{Z}\langle c^3 \rangle & & 0 & & 0 & & \mathbb{Z}\langle xc^3 \rangle & & 0 & & \dots \\
 & \searrow & & & & & & & & & \\
 0 & & 0 & & d_3^{0,6} & & 0 & & 0 & & \dots \\
 & \searrow & & & & & & & & & \\
 \mathbb{Z}\langle c^2 \rangle & & 0 & & 0 & & \mathbb{Z}\langle xc^2 \rangle & & 0 & & \dots \\
 & \searrow & & & & & & & & & \\
 0 & & 0 & & d_3^{0,4} & & 0 & & 0 & & \dots \\
 & \searrow & & & & & & & & & \\
 \mathbb{Z}\langle c \rangle & & 0 & & 0 & & \mathbb{Z}\langle xc \rangle & & 0 & & \dots \\
 & \searrow & & & & & & & & & \\
 0 & & 0 & & d_3^{0,2} & & 0 & & 0 & & \dots \\
 & \searrow & & & \cong & & & & & & \\
 \mathbb{Z}\langle c^0 \rangle & & 0 & & 0 & & \mathbb{Z}\langle x \rangle & & 0 & & \dots
 \end{array}$$

Here we also indicate the generators of the infinite cyclic groups $\mathbb{Z}$. The differential $d_3^{0,2}$ is an isomorphism as $H^2(F)$ vanishes. Without loss of generality we can assume

$d_3^{0,2}(c) = x$. We claim

$$d_3^{0,2k}(c^k) = k \cdot c^{k-1} \cdot x.$$

The induction begin $k = 1$ has already been taken care of. The induction step from $k \geq 1$ to $(k + 1)$ follows from the following computation using the fact that differentials are derivations.

$$\begin{aligned} d_3^{0,2(k+1)}(c^{k+1}) &= d_3^{0,2k}(c \cdot c^k) = d_3^{0,2}(c) \cdot c^k + c \cdot d_3^{0,2k}(c^k) \\ &= x \cdot c^k + c \cdot k \cdot c^{k-1} \cdot x = (k + 1) \cdot c^k \cdot x. \end{aligned}$$

This implies

$$H^i(F) \cong \begin{cases} \mathbb{Z}/k & \text{if } i = 2k + 1 \geq 5; \\ \mathbb{Z} & i = 0; \\ \{0\} & \text{otherwise.} \end{cases}$$

An easy application of the homological Leray-Serre spectral sequence to the fibration $\mathbb{CP}^\infty \rightarrow F \rightarrow S^3$ shows that the singular homology $H_i(F)$ of F is finitely generated for all $i \in \mathbb{Z}^{\geq 0}$. We obtain from the Universal Coefficient Theorem the exact sequence

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_{i-1}(F), \mathbb{Z}) \rightarrow H^i(F) \rightarrow \text{hom}_{\mathbb{Z}}(H_i(F); \mathbb{Z}) \rightarrow 0.$$

Since $H_i(F)$ is finitely generated, $\text{hom}_{\mathbb{Z}}(H_i(F); \mathbb{Z})$ is a finitely generated free abelian group. Since $H^i(F)$ is a finite abelian group for $i \geq 1$, we conclude for $i \geq 1$ that $\text{tors}(H_{i-1}(F)) \cong H_{i-1}(F)$ holds and we get

$$H^i(F) \cong \text{Ext}_{\mathbb{Z}}^1(H_{i-1}(F), \mathbb{Z}) \cong \text{Ext}_{\mathbb{Z}}^1(\text{tors}(H_{i-1}(F)), \mathbb{Z}) \cong \text{tors}(H_{i-1}(F)) \cong H_{i-1}(F).$$

This implies

$$H_i(F) \cong \begin{cases} \mathbb{Z}/k & \text{if } i = 2k \geq 4; \\ \mathbb{Z} & i = 0; \\ \{0\} & \text{otherwise.} \end{cases}$$

This finishes the proof that $\pi_1^s \cong \mathbb{Z}/2$ holds.

28.8. The rational singular cohomology of $K(\mathbb{Z}, n)$. Next want to show the following result.

Theorem 28.8. *We get isomorphisms of graded $\mathbb{Q}$ -algebras*

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q}[x_n] & \text{if } n \text{ is even;} \\ \Lambda_{\mathbb{Q}}(x_n) & \text{if } n \text{ is odd,} \end{cases}$$

where $|x_n| = n$.

Proof. We have already taken care of the cases $n = 1, 2$. Next we handle the case $n = 3$. We want to apply the Leray-Serre spectral sequence to the path fibration $K(\mathbb{Z}, 2) \rightarrow P(K(\mathbb{Z}, 3)) \rightarrow K(\mathbb{Z}, 3)$. Since $K(\mathbb{Z}, 3)$ is 2 connected, the first and second column of the second page are trivial. Since $H^3(K(\mathbb{Z}, 3); \mathbb{Q}) \cong \mathbb{Q}$ the third

page looks in the region $0 \leq p \leq 3$ and $0 \leq q \leq 6$ as follows

$$\begin{array}{ccccccc}
 \mathbb{Q}\langle x_2^3 \rangle & & 0 & & 0 & & \mathbb{Q}\langle x_2^3 x_3 \rangle \\
 & \searrow & & & & & \\
 0 & & 0 & & 0 & & 0 \\
 & & & & d_3^{0,6} & & \\
 \mathbb{Q}\langle x_2^2 \rangle & & 0 & & 0 & & \mathbb{Q}\langle x_2^2 x_3 \rangle \\
 & \searrow & & & & & \\
 0 & & 0 & & 0 & & 0 \\
 & & & & d_3^{0,4} & & \\
 \mathbb{Q}\langle x_2 \rangle & & 0 & & 0 & & \mathbb{Q}\langle x_2 x_3 \rangle \\
 & \searrow & & & & & \\
 0 & & 0 & & 0 & & 0 \\
 & & & & d_3^{0,2} & & \\
 \mathbb{Q}\langle x_2^0 \rangle & & 0 & & 0 & & \mathbb{Q}\langle x_3 \rangle
 \end{array}$$

As $H^2(P(K(\mathbb{Z}, 3)); \mathbb{Q})$ vanishes, the differential $d_3^{0,2}$ is bijective. Since the differentials $d_3^{0,2n}$ are derivations and the pairings $E_r^{0,2q} \otimes E_r^{3,0} \rightarrow E_r^{3,q}$ are isomorphism for $q \in \mathbb{Z}^{\geq 1}$ and $r = 2, 3$, we compute for $a \in E_r^{0,2q}$ and $m \in \mathbb{Z}$ and $q \in \mathbb{Z}^{\geq 1}$

$$d_3^{0,2(q+1)}(a \cdot (m \cdot x_3)) = d_3^{0,2q}(a) \cdot (m \cdot x_3) + a \cdot d_3^{3,0}(m \cdot x_3) = m \cdot d_3^{0,2q}(a) \cdot x_3$$

Since the map $E_3^{2q,0} \xrightarrow{\cong} E_3^{2(q+1),0}$ sending b to $b \cdot x_3$ is bijective for $q \in \mathbb{Z}^{\geq 1}$, $d_3^{0,2(q+1)}$ is bijective if and only if $d_3^{0,2q}$ is bijective. This implies that the differentials $d_3^{0,2q}$ are bijective for $q \in \mathbb{Z}^{\geq 0}$. Hence the fourth page looks in the region $0 \leq p \leq 4$ and

$0 \leq q \leq 6$ like

0	0	0	0	$E_4^{4,6}$
0	0	0	0	0
0	0	0	0	$E_4^{4,4}$
0	0	0	0	0
0	0	0	0	$E_4^{4,2}$
0	0	0	0	0
$\mathbb{Q}$	0	0	0	$E_4^{4,0}$

Note that $E_4^{4,0} = E_2^{4,0} = H^4(K(\mathbb{Z}, 3); H^0(\mathbb{CP}^\infty)) \cong H^4(K(\mathbb{Z}, 3))$ holds, since no differentials of the previous pages can hit the entry at $(4, 0)$. Obviously we have $E_4^{4,0} = E_\infty^{4,0}$. Since $H^4(P(K(\mathbb{Z}, 3))) = \{0\}$, we must have $\{0\} = E_\infty^{4,0} \cong H^4(K(\mathbb{Z}, 3))$. We conclude

$$E_2^{4,q} = H^4(K(\mathbb{Z}, 3); H^q(\mathbb{CP}^\infty)) \cong H^4(K(\mathbb{Z}, 3); \mathbb{Q}) = \{0\}.$$

This implies $E_4^{4,q} = \{0\}$ for $q \in \mathbb{Z}^{\geq 0}$. Hence we have shown that $H^i(K(\mathbb{Z}, 3); \mathbb{Q})$ vanishes for $i = 1, 2, 4$ and that the fifth page looks in the range $0 \leq p \leq 5$ and

$0 \leq q \leq 6$ like

0	0	0	0	0	$E_5^{4,6}$
0	0	0	0	0	0
0	0	0	0	0	$E_5^{5,4}$
0	0	0	0	0	0
0	0	0	0	0	$E_5^{5,2}$
0	0	0	0	0	0
$\mathbb{Q}$	0	0	0	0	$E_5^{5,0}$

Now one repeats the argument over and over again and concludes that $H^i(K(\mathbb{Z}, 3); \mathbb{Q})$ vanishes for $i \neq 0, 3$ and is $\mathbb{Q}$ for $i = 0, 3$. This implies $H^*(K(\mathbb{Z}, 3); \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(x_3)$ for $|x_3| = 3$.

Next we explain why $H^*(K(\mathbb{Z}, 4); \mathbb{Q}) \cong \mathbb{Q}[x_4]$ for $|x_4| = 4$ holds. Since the argument is similar to the one, where we computed $H^*(K(\mathbb{Z}, 2); \mathbb{Q})$, we will give only a sketch. The fourth page of the Leray Serre spectral sequence for the path fibration $K(\mathbb{Z}, 3) \rightarrow P(K(\mathbb{Z}, 4)) \rightarrow K(\mathbb{Z}, 4)$ has only non-trivial entries in the 0th and 3rd row and looks in the range $0 \leq p \leq 8$ and $0 \leq q \leq 3$ like

$\mathbb{Q}\langle x_3 \rangle$	0	0	0	$\mathbb{Q}\langle x_3 x_4 \rangle$	0	0	0	$\mathbb{Q}\langle x_3 x_4^2 \rangle$
0	0	$d_4^{0,3}$	0	0	0	$d_4^{4,3}$	0	0
0	0	0	0	0	0	0	0	0
$\mathbb{Q}\langle x_4^0 \rangle$	0	0	0	$\mathbb{Q}\langle x_4 \rangle$	0	0	0	$\mathbb{Q}\langle x_4^2 \rangle$

As the Leray-Serre spectral sequence converges to $H^*(P(K(\mathbb{Z}, 4)); \mathbb{Q}) \cong H^*(\{\bullet\}; \mathbb{Q})$, the differential $d_4^{4p,3}$ is an isomorphism for $p \in \mathbb{Z}^{\geq 0}$ and we can define $x_4 \in E_4^{4,0} = H^4(K(\mathbb{Z}, 4); \mathbb{Q})$ by $x_4 = d_4^{0,3}(x_3)$. Since the differentials are derivations and each pairing $E_4^{0,3} \otimes_{\mathbb{Q}} E_4^{4p,0} \xrightarrow{\cong} E_4^{4p,3}$ is an isomorphism, we get $d_4^{4p,3}(x_3 x_4^p) = x_4^{p+1}$ for $p \in \mathbb{Z}^{\geq 0}$ and that $x_3 x_4^p$ generates $E_4^{4p,3}$ and x_4^p generates $E_4^{4p,0}$ for $p \in \mathbb{Z}^{\geq 0}$. This implies $H^*(K(\mathbb{Z}, 4); \mathbb{Q}) \cong \mathbb{Q}[x_4]$ for $|x_4| = 4$.

Now one repeats the argument again to conclude $H^*(K(\mathbb{Z}/5); \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(x_5)$ for $|x_5| = 5$, $H^*(K(\mathbb{Z}, 6); \mathbb{Q}) \cong \mathbb{Q}[x_6]$ for $|x_6| = 6$, and so on. $\square$

29. SERRE CLASSES

29.1. Basics about Serre classes. Let R be a principal ideal domain, e.g., $\mathbb{Z}$ or a field.

Definition 29.1 (Serre class). A non-empty full subcategory $\mathcal{C}$ of the category of R -modules $R\text{-Mod}$ is called a *Serre class* if for any exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ we have

$$A, C \in \mathcal{C} \iff B \in \mathcal{C}.$$

Remark 29.2. A non-empty full subcategory $\mathcal{C}$ of the category of R -modules $R\text{-Mod}$ is a Serre class, if it is closed under extensions, taking quotient modules, and taking submodules. A Serre class $\mathcal{C}$ contains $\{0\}$ and satisfies $M, N \in \mathcal{C} \implies M \oplus N \in \mathcal{C}$.

Example 29.3 (Examples of Serre classes). Let R be a principal ideal domain. Then the following full subcategories of $R\text{-Mod}$ are Serre classes:

- (i) $\mathcal{C} = R\text{-Mod}$;
- (ii) The full subcategory $R\text{-Triv}$ consisting of the trivial R -module $\{0\}$;
- (iii) The full subcategory $R\text{-Mod}_{\text{fg}}$ of finitely generated R -modules;
- (iv) The full subcategory $R\text{-Tors}$ of torsion R -modules, i.e., R -modules M such that for every $m \in M$ there exists $r \in R$ with $r \neq 0$ and $rm = 0$;
- (v) Let $R = \mathbb{Z}$ and $\mathcal{P}$ be a collection of prime numbers. Then

$$R\text{-Tors}_{\mathcal{P}} = \{M \in \mathbb{Z}\text{-Mod} \mid M \text{ is a } \mathcal{P}\text{-primary torsion module}\}$$

is a Serre class in $\mathbb{Z}\text{-Mod}$, where an R -module M is called $\mathcal{P}$ -primary if for every $m \in M$ there exists $r \in \mathbb{Z}^{\geq 1}$, a finite subset $\{p_1, p_2, \dots, p_r\} \subseteq \mathcal{P}$ and elements $n_1, n_2, \dots, n_r \in \mathbb{Z}^{\geq 1}$ such that $(p_1^{n_1} \cdot p_2^{n_2} \cdot \dots \cdot p_r^{n_r}) \cdot m = 0$ holds.

If $\mathcal{P}$ consists of one element p only, i.e., $\mathcal{P} = \{p\}$, then we write $R\text{-Tors}_p$ instead of $R\text{-Tors}_{\mathcal{P}}$ and $\mathcal{P}$ -primary means that for every $m \in M$ there exists $n \in \mathbb{Z}^{\geq 1}$ satisfying $p^n \cdot m = 0$;

If $\mathcal{P}$ consists of all primes different from a given prime p , then we write $R\text{-Tors}_{(p)}$ instead of $R\text{-Tors}_{\mathcal{P}}$. A module in $R\text{-Tors}_{(p)}$ is sometimes called *p-coprimary*.

- (vi) The full subcategory $R\text{-Mod}_{\text{fin}}$ of $R\text{-Mod}$ of R -modules whose underlying set is finite is a Serre class;
- (vii) The intersection of two Serre classes of $R\text{-Mod}$ is again a Serre class.

Example 29.4 (Non-examples of Serre classes). Let R be a principal ideal domain. Then the following full subcategories of $R\text{-Mod}$ given by projective R -modules, finitely generated projective R -module, free R -modules, and finitely generated free R -modules are in general not Serre classes in $R\text{-Mod}$. They are Serre classes if R is a field.

Lemma 29.5. Let R be a principal ideal domain and $\mathcal{C} \subseteq R\text{-Mod}$ a Serre class. Then:

- (i) If $L \xrightarrow{f} M \xrightarrow{g} N$ is an exact sequence of R -modules and L and N belong to $\mathcal{C}$, then M belongs to $\mathcal{C}$;
- (ii) Let C_* be an R -chain complex. Consider $n \in \mathbb{Z}$ with the property C_n belongs to $\mathcal{C}$. Then $H_n(C_*)$ belongs to $\mathcal{C}$;
- (iii) Consider a finite filtration of the R -module M

$$\{0\} = F_0M \subseteq F_1M \subseteq F_2M \subseteq \dots \subseteq F_kM = M$$

for $k \in \mathbb{Z}^{\geq 1}$. Then the quotient R -module $F_iM/F_{i-1}M$ lies in $\mathcal{C}$ for $i = 1, 2, \dots, k$ if and only if M lies in $\mathcal{C}$.

Proof. (i) This follows from Remark 29.2, the factorization of $f = i \circ \bar{f}$ for the inclusion $i: \text{im}(f) \rightarrow M$ and the epimorphism $\bar{f}: L \rightarrow \text{im}(f)$ induced by f , the factorization of g as $j \circ \bar{g}$ for the inclusion $j: \text{im}(g) \rightarrow N$ and the epimorphism $\bar{g}: M \rightarrow \text{im}(g)$ induced by g , and the fact that we obtain an exact sequence $0 \rightarrow \text{im}(f) \xrightarrow{i} M \xrightarrow{\bar{g}} \text{im}(g) \rightarrow 0$.

(ii) This follows from Remark 29.2 since $H_n(C_*)$ is a subquotient of C_n .

(iii) Suppose that the quotient R -module $F_i M / F_{i-1} M$ lies in $\mathcal{C}$ for $i = 1, 2, \dots, k$. We want to prove that M lies in $\mathcal{C}$. This is done by induction over $k = 1, 2, \dots$. The induction beginning $k = 1$ is obvious. The induction step from $(k-1)$ to $k \geq 2$ follows from the observation that by the induction hypothesis $F_{k-1} M$ belongs to $\mathcal{C}$, by assumption $M / F_{k-1} M$ belongs to $\mathcal{C}$, and we have the exact sequence $0 \rightarrow F_{k-1} M \rightarrow M \rightarrow M / F_{k-1} M \rightarrow 0$.

Suppose that M lies in $\mathcal{C}$. Consider $i \in \{1, 2, \dots, k\}$. Since $F_i M / F_{i-1} M$ is a subquotient of M , it belongs to $\mathcal{C}$ by Remark 29.2. $\square$

29.2. Some applications using spectral sequences.

Lemma 29.6.

- (i) Let $(E_*^{*,*}, d_*^{*,*})$ be a homological first quadrant spectral sequence starting at d for $d \in \mathbb{Z}^{\geq 1}$ which converges to the graded R -module $\mathcal{H}_*$. Consider any $r \in \mathbb{Z}^{\geq d}$. Then:
 - (a) If $E_{p,q}^r$ belongs to $\mathcal{C}$ for every $p, q \in \mathbb{Z}^{\geq 0}$, then $E_{p,q}^{r+1}$ belongs to $\mathcal{C}$ for every $p, q \in \mathbb{Z}^{\geq 0}$;
 - (b) The R -module $E_{p,q}^\infty$ belongs to $\mathcal{C}$ for every $p, q \in \mathbb{Z}^{\geq 0}$ if $E_{p,q}^r$ belongs to $\mathcal{C}$ for every $p, q \in \mathbb{Z}^{\geq 0}$.
 - (c) The R -module $\mathcal{H}_n$ belongs to $\mathcal{C}$ for every $n \in \mathbb{Z}$ if and only if $E_{p,q}^\infty$ belongs to $\mathcal{C}$ for every $p, q \in \mathbb{Z}^{\geq 0}$.
- (ii) The analogous statements hold for cohomological first quadrant spectral sequence.

Proof. We give only the proof of assertion (i), the one for assertion (ii) is completely analogous.

(i)a Since $E_{p,q}^{r+1}$ is computed as the homology modules of an appropriate R -chain complex whose chain modules are given by the modules $E_{s,t}^r$ and hence belong to $\mathcal{C}$, each $E_{p,q}^{r+1}$ belongs to $\mathcal{C}$ by Lemma 29.5 (ii).

(i)b Consider $p, q \in \mathbb{Z}^{\geq 0}$. Since the sequence is assumed to be first quadrant spectral sequence, there exists $s \in \mathbb{Z}^{\geq r}$ such that $E_{p,q}^s = E_{p,q}^\infty$ holds. By assertion (i)a we get $E_{p,q}^s \in \mathcal{C}$ and hence $E_{p,q}^\infty \in \mathcal{C}$.

Since the spectral sequence is a first quadrant spectral sequence, the ascending filtration

$$\{0\} = F_{-1,n+1} \subseteq F_{0,n} \subseteq F_{1,n-1} \subseteq F_{2,n-2} \subseteq \dots \subseteq \mathcal{H}_n$$

is finite, since $F_{n,0} = \mathcal{H}_n$ holds. Recall that $F_{p,q} / F_{p-1,q+1} \cong E_{p,q}^\infty$ holds for $p, q \in \mathbb{Z}^{\geq 0}$. Now the claim follows from Lemma 29.5 (iii). $\square$

The next corollary is a direct consequence of the Leray-Serre spectral sequence and Lemma 29.6.

Corollary 29.7. Let $F \rightarrow EE \rightarrow B$ be a fibration over a path connected CW-complex. Let $\mathcal{C}$ be a Serre class in $R\text{-Mod}$. Suppose that $H_p^\Pi(B; H_q(F; R))$ belongs to $\mathcal{C}$ for $p, q \in \mathbb{Z}^{\geq 0}$.

Then $H_n(E; R)$ belongs to $\mathcal{C}$ for every $n \in \mathbb{Z}^{\geq 0}$.

Example 29.8. Let $F \rightarrow E \rightarrow E \rightarrow B$ be a fibration over a path connected CW -complex. Suppose that R is a field and that $\mathcal{C}$ is the Serre class of finitely generated R -modules. Assume that the $\pi_1(B)$ -action on $H_q(F; R)$ is trivial and R is a field. Then $H_p^\Pi(B; H_q(F; R))$ is R -isomorphic to $H_p(B; R) \otimes_R H_q(F; R)$. Hence Corollary 29.7 implies that $H_n(E; R)$ is finitely generated for every $n \in \mathbb{Z}$, provided that $H_p(B; R)$ for every $p \in \mathbb{Z}^{\geq 0}$ and $H_q(F; R)$ for every $q \in \mathbb{Z}^{\geq 0}$ are finitely generated.

Next we will prove a stronger version of Corollary 29.7. For this purpose we need the following additional assumption.

Definition 29.9 (Property (TT)). We say that a Serre class $\mathcal{C}$ in $R\text{-Mod}$ has property (TT) if for two objects A and B of $\mathcal{C}$ the objects $A \otimes_R B$ and $\text{Tor}_1^R(A, B)$ lie in $\mathcal{C}$.

Proposition 29.10. Let $F \rightarrow E \rightarrow E \rightarrow B$ be a fibration over a path connected CW -complex such that F has only finitely many path components. Assume that the $\pi_1(B)$ -action on $H_q(F; R)$ is trivial. Let $\mathcal{C}$ be a Serre class in $R\text{-Mod}$ satisfying property (TT). For $n \in \mathbb{Z}^{\geq 1}$ consider the following assertions:

- (B_n) $H_k(B; R) \in \mathcal{C}$ for $1 \leq k \leq n$;
- (F_n) $H_k(F; R) \in \mathcal{C}$ for $1 \leq k \leq n$;
- (E_n) $H_k(E; R) \in \mathcal{C}$ for $1 \leq k \leq n$.

Then we have the following implications:

- (i) If (F_n) and (B_n) hold, then so does (E_n);
- (ii) If (F_{n-1}) and (E_n) hold, then so does (B_n);
- (iii) If (E_n) and (B_{n+1}) hold, then so does (F_n).

Proof. (i) Since the $\pi_1(B)$ -action on $H_q(F; R)$ is trivial, the R -module $E_{p,q}^2 = H_p^\Pi(B; H_q(F; R))$ is isomorphic to $H_p(B; H_q(F; R))$. The Universal Coefficient Theorem yields an exact sequence

$$\begin{aligned} 0 \rightarrow H_p(B; R) \otimes_R H_q(F; R) \rightarrow E_{p,q}^2 &= H_p(B; H_q(F; R)) \\ &\rightarrow \text{Tor}_1^R(H_{p-1}(B; R), H_q(F; R)) \rightarrow 0. \end{aligned}$$

The term $H_p(B; R) \otimes_R H_q(F; R)$ belongs to $\mathcal{C}$ for $p = 1$ and $1 \leq q \leq n$ because of property (TT). Moreover, the term $\text{Tor}_1^R(H_{p-1}(B; R), H_q(F; R))$ vanishes for $p = 1$ and $1 \leq q \leq n$, as $H_0(B; R) \cong R$. Hence $E_{p,q}^2$ belongs to $\mathcal{C}$ for $p = 1$ and $1 \leq q \leq n$. Analogously one can show that $E_{p,q}^2$ belongs to $\mathcal{C}$ if $1 \leq p \leq n$ and $q = 0$ holds, since $H_0(F; R)$ is a finitely generated free R -module. For $2 \leq p \leq n$ and $1 \leq q \leq n$ the first and the third term belong to $\mathcal{C}$ because of (TT) and hence $E_{p,q}^2$ is in $\mathcal{C}$. Thus we have shown that $E_{p,q}^2$ belongs to $\mathcal{C}$ for $(p, q) \in \mathbb{Z}^2$ if $(p, q) \neq 0$ and $0 \leq p, q \leq n$ hold. Given $p, q \in \mathbb{Z}^{\geq 0}$, we have $E_{p,q}^r = E_{p,q}^\infty$ for large enough r and $E_{p,q}^{r+1}$ is a subquotient of $E_{p,q}^r$ for $r \in \mathbb{Z}^{\geq 2}$. Hence Remark 29.2 implies that $E_{p,q}^\infty$ belongs to $\mathcal{C}$ for $p, q \in \mathbb{Z}^{\geq 0}$ with $1 \leq p + q \leq n$. For $1 \leq k \leq n$ we have the finite filtration

$$0 \subseteq F_{0,k} \subseteq F_{1,k-1} \subseteq \cdots \subseteq F_{k,0} = H_k(E; R)$$

with $E_{p,q}/F_{p-1,q+1} \cong E_{p,q}^\infty$. Lemma 29.5 (iii) implies $H_k(E; R) \in \mathcal{C}$ for $1 \leq k \leq n$.

(ii) Since $H_k(E; R)$ lies in $\mathcal{C}$ for $1 \leq k \leq n$, we conclude from Lemma 29.5 (iii) that $E_{p,q}^\infty$ belongs to $\mathcal{C}$ for $p, q \in \mathbb{Z}^{\geq 0}$ with $1 \leq p + q \leq n$.

Next we show by induction for $k = 1, 2, \dots, n$ that $H_l(B; R)$ belongs to $\mathcal{C}$ for $1 \leq l \leq k$. The induction beginning $k = 1$ follows from $H_1(B; R) \cong E_{1,0}^2 = E_{1,0}^\infty$. The induction step from $(k-1)$ to k for $2 \leq k \leq n$ is done as follows.

By the induction hypothesis we know that $H_l(B; R)$ belongs to $\mathcal{C}$ for $1 \leq l \leq k-1$. Next we show by induction that $E_{k,0}^r$ lies in $\mathcal{C}$ for all $r \in \mathbb{Z}^{\geq 2}$. We have $E_{k,0}^r = E_{k,0}^\infty$

and hence $E_{k,0}^r \in \mathcal{C}$ for $r \geq k+1$. Hence it remains to show by downward induction that $E_{k,0}^r \in \mathcal{C}$ holds for $r = (k+1), k, (k-1), \dots, 2$. The induction step from $(r+1)$ to $2 \leq r \leq k$ is done as follows. By the induction hypothesis $E_{k,0}^{r+1}$ belongs to $\mathcal{C}$. There is a short exact sequence

$$0 \rightarrow E_{k,0}^{r+1} \rightarrow E_{k,0}^r \rightarrow \text{im}(d_{k,0}^r: E_{k,0}^r \rightarrow E_{k-r,r-1}^r) \rightarrow 0.$$

Obviously $\text{im}(d_{k,0}^r)$ is a submodule of $E_{k-r,r-1}^r$ which is a subquotient of $E_{k-r,r-1}^2 \cong H_{k-r}(B; H_{r-1}(F; R))$. We have $0 \leq k-r < k \leq n$ and $1 \leq r-1 < k \leq n$ and $H_0(B; H_{r-1}(F; R)) \cong H_{r-1}(F; R)$. As we assume conditions (F_{n-1}) , (E_n) , and (TT) and (B_{n-1}) holds by the induction hypothesis, the Universal Coefficient implies that $H_{k-r}(B; H_{r-1}(F; R))$ and hence $\text{im}(d_{k,0}^r)$ lie in $\mathcal{C}$. Therefore $E_{k,0}^r$ lies in $\mathcal{C}$. This finishes the proof that $E_{k,0}^2$ lies in $\mathcal{C}$. Since $E_{k,0}^2 \cong H_k(B, H_0(F; R))$ holds and $H_0(F; R)$ is a finitely generated free R -module, $H_k(B; R)$ is a direct summand in $E_{k,0}^2$ and hence belongs to $\mathcal{C}$. This finishes the proof of assertion (ii).

(iii) This proof is omitted, since it is completely analogous to the one of assertion(ii). $\square$

Remark 29.11. Obviously Proposition 29.10 holds also for $n = \infty$ if we interpret $\infty \pm 1 = \infty$.

Example 29.12. Let $\mathcal{C}$ be a Serre class in $R\text{-Mod}$ satisfying property (TT) . Let X be a simply connected CW -complex. Then we can apply Proposition 29.10 to the path fibration $\Omega X \rightarrow P(X) \rightarrow X$. Since $P(X)$ is contractible, $H_n(P(X); R)$ lies in $\mathcal{C}$ for $n \in \mathbb{Z}^{\geq 1}$.

Hence $H_n(\Omega X; R)$ lies in $\mathcal{C}$ for every $n \in \mathbb{Z}^{\geq 1}$ if and only if $H_n(X; R)$ lies in $\mathcal{C}$ for every $n \in \mathbb{Z}^{\geq 1}$.

Definition 29.13 (Properties (K) and (TTK)). A Serre class $\mathcal{C}$ in $R\text{-Mod}$ has property (K) if for every $A \in \mathcal{C}$ and $n \in \mathbb{Z}^{\geq 1}$ the R -module $H_n(K(A, 1); R)$ lies in $\mathcal{C}$.

A Serre class in $R\text{-Mod}$ has property (TTK) if it has property both (TT) and property (K).

The next corollary follows directly from Example 29.12 and the path fibration $K(A, n-1) \rightarrow P(K(A, n)) \rightarrow K(A, n)$ for $n \in \mathbb{Z}^{\geq 2}$.

Corollary 29.14. Suppose that $\mathcal{C}$ is a Serre class in $R\text{-Mod}$ having property (TTK) . Consider $A \in \mathcal{C}$.

Then for $n, k \in \mathbb{Z}^{\geq 1}$ we have $H_k(K(A, n); R) \in \mathcal{C}$.

Lemma 29.15. The Serre classes in $\mathbb{Z}\text{-Mod}$ given by $\mathbb{Z}\text{-Mod}$, $\mathbb{Z}\text{-Triv}$, $\mathbb{Z}\text{-Mod}_{\text{fg}}$, $\mathbb{Z}\text{-Mod}_{\text{fin}}$, $\mathbb{Z}\text{-Tors}$, and $\mathbb{Z}\text{-Tors}_{\mathcal{P}}$ for any set of primes $\mathcal{P}$ satisfy the condition (TTK) .

Proof. Obviously $\mathbb{Z}\text{-Mod}$ and $\mathbb{Z}\text{-Triv}$ satisfy (TTK) , since $K(\{0\}, 1)$ is contractible.

The classes $\mathbb{Z}\text{-Tors}$ and $\mathbb{Z}\text{-Tors}_{\mathcal{P}}$ are closed under tensor products since for two $\mathbb{Z}$ -modules M and N , $x \in M$, $y \in N$ and $n \in \mathbb{Z}$ we have $n \cdot (x \otimes y) = (nx) \otimes y = x \otimes (ny)$. The category $\mathbb{Z}\text{-Mod}_{\text{fg}}$ is closed under tensor products since $\mathbb{Z}^m \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Z}^{m \cdot n}$ holds and for two epimorphisms of $\mathbb{Z}$ -modules $f: \mathbb{Z}^m \rightarrow M$ and $g: N \rightarrow \mathbb{Z}^n$ the $\mathbb{Z}$ -homomorphism $f \otimes g: \mathbb{Z}^m \otimes_{\mathbb{Z}} \mathbb{Z}^n \rightarrow M \otimes_{\mathbb{Z}} N$ is surjective. Since $\mathbb{Z}\text{-Mod}_{\text{fin}} = \mathbb{Z}\text{-Mod}_{\text{fg}} \cap \mathbb{Z}\text{-Tors}$, also $\mathbb{Z}\text{-Mod}_{\text{fin}}$ is closed under tensor products.

If M belongs to $\mathbb{Z}\text{-Mod}_{\text{fg}}$, we can find a 1-dimensional resolution by finitely generated free $\mathbb{Z}$ -modules $0 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$. Hence we get for $M \in \mathbb{Z}\text{-Mod}_{\text{fg}}$ and any $N \in \mathbb{Z}\text{-Mod}$ an exact sequence

$$0 \rightarrow \text{Tors}_{\mathbb{Z}}^1(M, N) \rightarrow F_0 \otimes_{\mathbb{Z}} N \rightarrow F_1 \otimes_{\mathbb{Z}} N.$$

If N belongs to the Serre class $\mathcal{C}$ of $\mathbb{Z}\text{-Mod}$, then we conclude $\text{Tors}_1^{\mathbb{Z}}(M, N) \in \mathcal{C}$ for every $M \in \mathbb{Z}\text{-Mod}_{\text{fg}}$ by Remark 29.2. Hence the classes $\mathbb{Z}\text{-Mod}_{\text{fg}}$ and $\mathbb{Z}\text{-Mod}_{\text{fin}}$ are closed under the passage to $\text{Tor}_1^{\mathbb{Z}}(-, -)$. Since $\text{Tors}_1^{\mathbb{Z}}(-, N)$ commutes with directed colimits, every $\mathbb{Z}$ -module is the directed union of its finitely generated $\mathbb{Z}$ -submodules and $\mathbb{Z}\text{-Tors}$ and $\mathbb{Z}\text{-Tors}_{\mathcal{P}}$ are closed under directed colimits, also $\mathbb{Z}\text{-Tors}$ and $\mathbb{Z}\text{-Tors}_{\mathcal{P}}$ are closed under the passage to $\text{Tor}_1^{\mathbb{Z}}(-, -)$. Hence the Serre classes $\mathbb{Z}\text{-Mod}_{\text{fg}}$, $\mathbb{Z}\text{-Mod}_{\text{fin}}$, $\mathbb{Z}\text{-Tors}$, and $\mathbb{Z}\text{-Tors}_{\mathcal{P}}$ for any set of primes $\mathcal{P}$ satisfy condition (TT).

A model for $K(\mathbb{Z}, 1)$ is S^1 . A model for $K(\mathbb{Z}/m, 1)$ is given by the quotient $S^\infty/\mathbb{Z}/m$ where the free $\mathbb{Z}/m$ -action on S^∞ is given by restricting the canonical S^1 -action to the subgroup $\mathbb{Z}/m \subseteq \mathbb{Z}$. Let $t \in \mathbb{Z}/m$ be the generator and put $N = \sum_{i=0}^{m-1} t^i$. Denote by $\epsilon: \mathbb{Z}[\mathbb{Z}/m] \rightarrow \mathbb{Z}$ the augmentation homomorphism sending $\sum_{i=0}^{m-1} n_i \cdot t_i$ to $\sum_{i=0}^{m-1} n_i$. One easily checks that

$$\cdots \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/m] \xrightarrow{N} \mathbb{Z}[\mathbb{Z}/m] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/m] \xrightarrow{N} \mathbb{Z}[\mathbb{Z}/m] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/m] \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0$$

is a free $\mathbb{Z}[\mathbb{Z}/m]$ -resolution F_* of the trivial $\mathbb{Z}[\mathbb{Z}/m]$ -module $\mathbb{Z}$. (Actually this is the $\mathbb{Z}$ -chain complex of S^∞ for an appropriate CW -structure on S^∞ coming from one on $S^\infty/\mathbb{Z}/m$.) By the Fundamental Lemma of Homological algebra we get for any CW -structure on $S^\infty/\mathbb{Z}/m$ that $H_n(K(\mathbb{Z}/m, 1); \mathbb{Z}) \cong H_n(\mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} F_*)$ holds for $n \in \mathbb{Z}^{\geq 0}$. Since $\mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} F_*$ looks like

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z},$$

we get

$$H_n(K(\mathbb{Z}/m, 1); \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 0; \\ \mathbb{Z}/m & n \geq 1 \text{ and } n \text{ is odd}; \\ \{0\} & \text{otherwise.} \end{cases}$$

If A is a finitely generated $\mathbb{Z}$ -module, we get an isomorphism

$$A \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^s \mathbb{Z}/n_i$$

for appropriate elements $r, s \in \mathbb{Z}^{\geq 0}$ and $n_i \in \mathbb{Z}^{\geq 2}$. Hence we obtain a model for $K(A, 1)$ by $\prod_{i=1}^r K(\mathbb{Z}, 1) \times \prod_{i=1}^s K(\mathbb{Z}/n_i, 1)$. Now the Künneth Theorem implies that $H_n(K(A, 1); \mathbb{Z})$ is finitely generated for $n \in \mathbb{Z}^{\geq 0}$. Hence the class $\mathbb{Z}\text{-Mod}_{\text{fg}}$ satisfies (TTK).

If A is finite, we get $r = 0$ in the decomposition above, and again by the Künneth Theorem we conclude that $H_n(K(A, 1); \mathbb{Z})$ is finite for $n \in \mathbb{Z}^{\geq 1}$. Hence the class $\mathbb{Z}\text{-Mod}_{\text{fin}}$ satisfies (TTK).

Let $\mathcal{P}$ be a set of primes and A be an element in $\mathbb{Z}\text{-Mod}_{\text{fg}} \cap \mathbb{Z}\text{-Tors}_{\mathcal{P}}$. Then in the decomposition above we get $r = 0$ and each number n_i can be written as $\prod_{k=1}^l p_k^{m_k}$ for some $l \in \mathbb{Z}^{\geq 1}$, $p_i \in \mathcal{P}$, and $n_i \in \mathbb{Z}^{\geq 1}$. Again by the Künneth Theorem we conclude that $H_n(K(A, 1); \mathbb{Z})$ belongs to $\mathbb{Z}\text{-Mod}_{\text{fg}} \cap \mathbb{Z}\text{-Tors}_{\mathcal{P}}$ for $n \in \mathbb{Z}^{\geq 1}$.

Next consider A in $\mathbb{Z}\text{-Tors}_{\mathcal{P}}$. Let I be the set of finitely generated $\mathbb{Z}$ -submodules of A , directed by inclusion. Then A is the directed union $\bigcup_{i \in I} A_i$. One can show for every $n \in \mathbb{Z}^{\geq 0}$

$$H_n(K(A, 1); \mathbb{Z}) = \text{colim}_{i \in I} H_n(K(A_i, 1); \mathbb{Z}).$$

Since $H_n(K(A_i, 1); \mathbb{Z})$ belongs to $\mathbb{Z}\text{-Tors}_{\mathcal{P}}$ for $i \in I$, the same is true for $H_n(K(A, 1); \mathbb{Z})$. Hence $\mathbb{Z}\text{-Tors}_{\mathcal{P}}$ satisfies the condition (TTK).

Since $\mathbb{Z}\text{-Tors}$ is $\mathbb{Z}\text{-Tors}_{\mathcal{P}}$ if $\mathcal{P}$ consist of all prime numbers, the proof of Lemma 29.15 is finished. $\square$

30. THE HUREWICZ THEOREM MODULO A SERRE CLASS

Definition 30.1. Let $\mathcal{C}$ be a Serre class in $R\text{-Mod}$. Consider a morphism of R -modules $f: M \rightarrow N$.

- (i) We call f a $\mathcal{C}$ -monomorphism if $\ker(f)$ belongs to $\mathcal{C}$;
- (ii) We call f a $\mathcal{C}$ -epimorphism if $\text{cok}(f)$ belongs to $\mathcal{C}$;
- (iii) We call f a $\mathcal{C}$ -isomorphism if $\ker(f)$ and $\text{cok}(f)$ belongs to $\mathcal{C}$.

Theorem 30.2 (Hurewicz Theorem modulo a Serre class). *Let X be a simply connected space. Consider $n \in \mathbb{Z}^{\geq 2}$ and a Serre class of $\mathbb{Z}\text{-Mod}$ satisfying condition (TTK).*

Then the following statements are equivalent:

- (i) $\pi_k(X)$ lies in $\mathcal{C}$ for $2 \leq k \leq (n-1)$;
- (ii) $H_k(X)$ lies in $\mathcal{C}$ for $2 \leq k \leq (n-1)$.

If one of the conditions above is satisfied, then the Hurewicz homomorphism

$$\text{hur}_n(X): \pi_n(X) \rightarrow H_n(X)$$

is a $\mathcal{C}$ -isomorphism.

Proof. We begin with the implication (i) $\implies$ (ii). We choose a Postnikov tower $\{\tau_{\leq k}X, p_k, \varphi_k\}$ for X in the sense of Section 15. Since X is simply connected, we have $\tau_{\leq 2}X \simeq K(\pi_2(X), 2)$. The fiber of $p_k: \tau_{\leq k}X \rightarrow \tau_{\leq k-1}X$ is $K(\pi_k(X), k)$ for $k \geq 3$. Corollary 29.14 implies that $H_i(K(\pi_k(X), k))$ belongs to $\mathcal{C}$ for $2 \leq k \leq (n-1)$ and $i \in \mathbb{Z}^{\geq 1}$. We conclude from Proposition 29.10 and Remark 29.11 that $H_i(\tau_{\leq n-1}X)$ belongs to $\mathcal{C}$ for $i \in \mathbb{Z}^{\geq 1}$.

Since $\varphi_{n-1}: X \rightarrow \tau_{\leq n-1}X$ is an n -connected map of simply connected spaces, the induced map $H_k(X) \rightarrow H_k(\tau_{\leq n-1}X)$ is bijective for $1 \leq k \leq (n-1)$ by the classical Hurewicz Theorem, see Proposition 12.12 (i). Hence $H_k(X)$ belong to $\mathcal{C}$ for $1 \leq k \leq (n-1)$.

Next we show that, if the condition (ii) is satisfied, the condition (i) holds and $\text{hur}_n(X): \pi_n(X) \rightarrow H_n(X)$ is a $\mathcal{C}$ -isomorphism. It suffices to show that $\text{hur}_k(X): \pi_k(X) \rightarrow H_k(X)$ is a $\mathcal{C}$ -isomorphism for $2 \leq k \leq n$. This will be done by induction over $k = 2, 3, \dots, n$. The induction beginning follows from the classical Hurewicz Theorem, see Proposition 12.12 (i). In the induction step from k to $k+1 \leq n$ we can assume by the induction hypothesis that $\text{hur}_i(X): \pi_i(X) \rightarrow H_i(X)$ a $\mathcal{C}$ -isomorphism for $2 \leq i \leq k \leq n-1$. From the arguments appearing in the proof of the the implication (i) $\implies$ (ii) we conclude that $H_i(\tau_{\leq k}X)$ and $H_i(\tau_{\leq k+1}X)$ belong to $\mathcal{C}$ for $i \in \mathbb{Z}^{\geq 1}$. We have $H_1(\tau_{\leq k}X) = H_1(\tau_{\leq k+1}X) = \{0\}$ since $\tau_{\leq k}X$ and $\tau_{\leq k+1}X$ are simply connected. From the Hurewicz Theorem 12.8 we conclude $H_i(K(\pi_{k+1}(X), k+1)) = \{0\}$ for $1 \leq i \leq k$. Now the Serre sequence, see Proposition 27.4, yields the exact sequence

$$\begin{aligned} H_{k+2}(\tau_{\leq k}X) &\rightarrow H_{k+1}(K(\pi_{k+1}(X), k+1)) \\ &\xrightarrow{j} H_{k+1}(\tau_{\leq k+1}X) \rightarrow H_{k+1}(\tau_{\leq k+1}X) \rightarrow \{0\}. \end{aligned}$$

Since $H_{k+2}(\tau_{\leq k}X)$ and $H_{k+1}(\tau_{\leq k+1}X)$ belong to $\mathcal{C}$, the R -homomorphism j is a $\mathcal{C}$ -isomorphism. The following diagram commutes and all maps marked with $\cong$ are $\mathcal{C}$ -isomorphisms

$$\begin{array}{ccccc} \pi_{k+1}(K(\pi_{k+1}(X), k+1)) & \xrightarrow{\cong} & \pi_{k+1}(\tau_{\leq k+1}X) & \xleftarrow[\cong]{\pi_{k+1}(\varphi_{k+1})} & \pi_{k+1}(X) \\ \cong \downarrow \text{hur}_{k+1}(K(\pi_{k+1}(X), k+1)) & & \downarrow \text{hur}_{k+1}(\tau_{\leq k+1}X) & & \downarrow \text{hur}_{k+1}(X) \\ H_{k+1}(K(\pi_{k+1}(X), k+1)) & \xrightarrow{j} & H_{k+1}(\tau_{\leq k+1}X) & \xleftarrow[\cong]{H_{k+1}(\varphi_{k+1})} & H_{k+1}(X). \end{array}$$

As j is a $\mathcal{C}$ -isomorphism, $\text{hur}_{k+1}(X)$ is a $\mathcal{C}$ -isomorphism. This finishes the proof of Theorem 30.2. $\square$

Corollary 30.3. *Let X be a simply connected CW-complex such that $H_n(X)$ is finitely generated for every $n \geq \mathbb{Z}$. Then $\pi_n(X)$ is finitely generated for every $n \geq \mathbb{Z}$.*

Proof. This follows from Theorem 30.2 applied to the Serre class $R\text{-Mod}_{\text{fg}}$. $\square$

31. THE WHITEHEAD THEOREM MODULO A SERRE IDEAL

Definition 31.1 (Serre ideal). A Serre class $\mathcal{C}$ in $R\text{-Mod}$ is called an *ideal* if for any object M in $\mathcal{C}$ and any R -module N both $M \otimes_R N$ and $\text{Tors}_1^R(M, N)$ lie in $\mathcal{C}$.

Obviously a Serre ideal has property (TT) and is closed under direct sums over arbitrary index sets.

Theorem 31.2 (Whitehead Theorem modulo a Serre ideal). *Let $f: X \rightarrow Y$ be a map of simply connected spaces such that $\pi_2(f): \pi_2(X) \rightarrow \pi_2(Y)$ is surjective. Consider $n \in \mathbb{Z}^{\geq 2}$ and a Serre ideal $\mathcal{C}$ in $\mathbb{Z}\text{-Mod}$ satisfying property (K). Then:*

- (i) *The following assertions are equivalent:*
 - (a) *The homomorphism $\pi_k(f): \pi_k(X) \rightarrow \pi_k(Y)$ is a $\mathcal{C}$ -isomorphism for $2 \leq k < n$ and a $\mathcal{C}$ -epimorphism for $k = n$;*
 - (b) *The homomorphism $H_k(f): H_k(X) \rightarrow \pi_k(Y)$ is a $\mathcal{C}$ -isomorphism for $2 \leq k < n$ and a $\mathcal{C}$ -epimorphism for $k = n$.*
- (ii) *The following assertions are equivalent:*
 - (a) *The homomorphism $\pi_k(f): \pi_k(X) \rightarrow \pi_k(Y)$ is a $\mathcal{C}$ -isomorphism for every $k \in \mathbb{Z}^{\geq 2}$;*
 - (b) *The homomorphism $H_k(f): H_k(X) \rightarrow \pi_k(Y)$ is a $\mathcal{C}$ -isomorphism for every $k \in \mathbb{Z}^{\geq 2}$.*

Proof. Obviously assertion (ii) follows from assertion (i) which is proved as follows.

Since we can turn f into a fibration, we can assume without loss of generality that f itself is a fibration. Let F be its fiber. The long exact homotopy sequence associated to f and the assumption that $\pi_2(f)$ is surjective imply that F is simply connected.

We start with the implication (i)a $\implies$ (i)b.

The long exact homotopy sequence of f and Lemma 29.5 (i) imply that $\pi_k(F)$ belongs to $\mathcal{C}$ for $k < n$. Hence $H_k(F)$ belongs to $\mathcal{C}$ for $2 \leq k < n$ by the Hurewicz Theorem modulo a Serre class, see Theorem 30.2.

Next we consider the Leray-Serre spectral sequence of the fibration $F \xrightarrow{i} X \xrightarrow{f} Y$ for singular homology. Note that this is a first quadrant spectral sequence.

We get from the Universal Coefficient Theorem a short exact sequence

$$\{0\} \rightarrow H_p(Y) \otimes_{\mathbb{Z}} H_q(F) \rightarrow H_p(Y; H_q(F)) \rightarrow \text{Tors}_1^{\mathbb{Z}}(H_{p-1}(Y), H_q(F)) \rightarrow \{0\}.$$

Since $\mathcal{C}$ is a Serre ideal, $E_{p,q}^2 = H_p(Y; H_q(F))$ belongs to $\mathcal{C}$ for $p, q \in \mathbb{Z}^{\geq 0}$ if $0 < q < n$ holds. This implies that $E_{p,q}^r = H_p(Y; H_q(F))$ belongs to $\mathcal{C}$ for $p, q \in \mathbb{Z}^{\geq 0}$ and $r \in \mathbb{Z}^{\geq 2} \amalg \{\infty\}$ if $0 < q < n$ holds.

Recall that we have the filtration

$$\{0\} = F_{-1,k+1} \subseteq F_{0,k} \subseteq F_{1,k-1} \subseteq \cdots \subseteq F_{k-1,1} \subseteq F_{k,0} = H_k(X)$$

such that $F_{i,k-i}/F_{i-1,k-i+1} \cong E_{i,k-i}^{\infty}$ holds for $i = 0, 1, 2, \dots, k$. Hence $F_{i,k-i}$ belongs to $\mathcal{C}$ for $i = 0, 1, 2, \dots, (k-1)$ by Lemma 29.5 (iii) if $k < n$ holds. In particular $F_{k-1,1}$ belongs to $\mathcal{C}$ if $k < n$ holds.

Moreover, we have the filtration

$$\{0\} \subseteq E_{k,0}^\infty \cong E_{k,0}^{k+1} \subseteq E_{k,0}^k \subseteq E_{k,0}^{k-1} \subseteq \cdots \subseteq E_{k,0}^2 \cong H_k(Y, H_0(F)) \cong H_k(Y).$$

The filtration quotients are given by $E_{k,0}^\infty, \text{im}(d_{k,0}^k), \text{im}(d_{k,0}^{k-1}), \dots, \text{im}(d_{k,0}^2)$. In particular we obtain a filtration of $H_k(Y)/E_{k,0}^\infty$ whose filtration quotients are $\text{im}(d_{k,0}^k), \text{im}(d_{k,0}^{k-1}), \dots, \text{im}(d_{k,0}^2)$. Since $\text{im}(d_{k,0}^r)$ is a submodule of $E_{k-r, r-1}^r$ which lies for $2 \leq r \leq k \leq n$ in $\mathcal{C}$, each of these filtration quotients lie in $\mathcal{C}$. Hence $H_k(Y)/E_{k,0}^\infty$ lies in $\mathcal{C}$ for $k \leq n$ by Lemma 29.5 (iii).

We get from the edge homomorphism $\text{edge}_{k,0}$ the exact sequence

$$\{0\} \rightarrow F_{k-1,1} \rightarrow H_k(X) \xrightarrow{H_k(f)} H_k(Y) \rightarrow H_k(Y)/E_{k,0}^\infty \rightarrow \{0\}$$

Since $F_{k-1,1}$ belongs to $\mathcal{C}$ if $k < n$ holds and $H_k(Y)/E_{k,0}^\infty$ belong to $\mathcal{C}$ for $k \leq n$, the homomorphism $H_k(X) \xrightarrow{H_k(f)} H_k(Y)$ is a $\mathcal{C}$ -isomorphism for $k < n$ and a $\mathcal{C}$ -epimorphism for $k = n$. Hence assertion (i)b holds.

Next we prove the implication (i)b $\implies$ (i)a. Thanks to the long homotopy sequence of f it suffices to show that $\pi_k(F)$ belongs to $\mathcal{C}$ for $k < n$. Assume that contrary. So we can choose $k \in \{2, 3, \dots, n-1\}$ such that $\pi_k(F)$ does not belong to $\mathcal{C}$ but $\pi_i(F)$ belongs to $\mathcal{C}$ for $i \in \{2, 3, \dots, k-1\}$. The Hurewicz Theorem modulo a Serre class, see Theorem 30.2 implies that $E_{0,k}^2 = H_0(Y; H_k(F)) \cong H_k(F)$ does not belong to $\mathcal{C}$.

The same argument as the first step shows that $E_{p,q}^r$ belong to $\mathcal{C}$ for $p, q \in \mathbb{Z}^{\geq 0}$ and $r \in \mathbb{Z}^{\geq 2}$ if $0 < q < r$ holds.

Next we show that $E_{0,k}^{k+1}$ does not belong to $\mathcal{C}$. The induction beginning $r = 2$ as already been taken care of. The induction step from r for $2 \leq r \leq k$ to $(r+1)$ is done as follows. We have the exact sequence

$$E_{r,k-r+1}^r \xrightarrow{d_{r,k-r+1}^r} E_{0,k}^r \rightarrow E_{0,k}^{r+1} \rightarrow \{0\}.$$

The middle term $E_{0,k}^r$ does not belong to $\mathcal{C}$ by the induction hypothesis. Since $2 \leq r \leq k$ implies $0 < k-r+1 < k$, we know already that $E_{r,k-r+1}^r$ belongs to $\mathcal{C}$. This implies that $E_{0,k}^{r+1}$ does not belong to $\mathcal{C}$. This finishes the proof that $E_{0,k}^{k+1}$ does not belong to $\mathcal{C}$.

In the next step we show $E_{0,k}^{k+2} = E_{0,k}^\infty$ does not belong to $\mathcal{C}$. Consider the exact sequence

$$E_{k+1,0}^{k+1} \xrightarrow{d_{k+1,0}^{k+1}} E_{0,k}^{k+1} \rightarrow E_{0,k}^{k+2} \rightarrow \{0\}.$$

The differential $d_{k+1,0}^{k+1}$ factorizes as the composite of a monomorphism and epimorphism by

$$E_{k+1,0}^{k+1} \rightarrow E_{0,k}^{k+1} / \ker(d_{k+1,0}^{k+1}) \rightarrow E_{0,k}^{k+2}.$$

Since $\ker(d_{k+1,0}^{k+1}) \cong E_{k+1,0}^{k+2} \cong E_{k+1,0}^\infty \cong \text{im}(H_{k+1}(f))$ holds, we get an injection

$$E_{0,k}^{k+1} / \ker(d_{k+1,0}^{k+1}) \rightarrow H_{k+1}(Y) / \text{im}(H_{k+1}(f)) \cong \text{cok}(H_{k+1}(f)).$$

Since $k < n$ and hence $H_{k+1}(f)$ is a $\mathcal{C}$ -epimorphism, $E_{0,k}^{k+1} / \ker(d_{k+1,0}^{k+1})$ belongs to $\mathcal{C}$. Since $E_{0,k}^{k+1}$ does not belong to $\mathcal{C}$, $E_{0,k}^{k+2} = E_{0,k}^\infty$ does not belong to $\mathcal{C}$.

On the other hand $E_{0,k}^\infty$ is contained in $F_{k-1,1} \subseteq H_k(X)$ and $F_{k-1,1}$ is the kernel of $H_k(f)$. As $k < n$ holds, $H_k(f)$ is a $\mathcal{C}$ -isomorphism by assumption. This implies that the kernel of $H_k(f)$ and hence also $E_{0,k}^\infty$ lies in $\mathcal{C}$, a contradiction. Hence we have shown that $\pi_k(F)$ belongs to $\mathcal{C}$ for $k \leq n$ which implies assertion (i)a. This finishes the proof of Theorem 31.2. $\square$

Remark 31.3. One can generalize Theorem 30.2 and Theorem 31.2 as follows. One can drop in Theorem 30.2 the assumption that X is simply connected by the assumption that X is a nilpotent space, i.e, the $\pi_1(X)$ -action on the higher homotopy groups is nilpotent. (This is a weaker condition than being a simple space). In Theorem 31.2 one can replace the assumptions that X and Y are simply connected and $\pi_2(f)$ is surjective by the assumption that X and Y are nilpotent spaces.

32. (STABLE) HOMOTOPY GROUPS OF SPHERES

Definition 32.1. Let $\mathcal{P}$ be a set of primes. Let $\mathbb{Z}[1/\mathcal{P}] \subseteq \mathbb{Q}$ be the subring containing $\mathbb{Z}$ and p^{-1} for every $p \in \mathcal{P}$.

If p is a prime and $\mathcal{P} = \{p\}$, then we write $\mathbb{Z}[1/p]$ instead of $\mathbb{Z}[1/\mathcal{P}]$. If p is a prime and $\mathcal{P}$ consists of all primes different from p , then we write $\mathbb{Z}_{(p)}$ instead of $\mathbb{Z}[1/\mathcal{P}]$.

If $\mathcal{P}$ is empty, then $\mathbb{Z}[1/\mathcal{P}]$ agrees with $\mathbb{Z}$. If $\mathcal{P}$ consists of all primes, then $\mathbb{Z}[1/\mathcal{P}]$ agrees with $\mathbb{Q}$.

The proof of the next result is left to the reader.

Lemma 32.2. Consider the functor $I: \mathbb{Z}\text{-Mod} \rightarrow \mathbb{Z}[1/\mathcal{P}]\text{-Mod}$ sending M to $M \otimes_{\mathbb{Z}} \mathbb{Z}[1/\mathcal{P}]$.

- (i) The functor I has a right adjoint given by restriction with the inclusion $\mathbb{Z} \rightarrow \mathbb{Z}[1/\mathcal{P}]$;
- (ii) The functor I commutes with colimits over directed systems of $\mathbb{Z}$ -modules;
- (iii) The functor I is exact;
- (iv) Let $f: M \rightarrow N$ be a $\mathbb{Z}$ -homomorphism of $\mathbb{Z}$ -modules. Then

$$\text{id}_{\mathbb{Z}[1/\mathcal{P}]} \otimes_{\mathbb{Z}} f: \mathbb{Z}[1/\mathcal{P}] \otimes_{\mathbb{Z}} M \rightarrow \mathbb{Z}[1/\mathcal{P}] \otimes_{\mathbb{Z}} N$$

is trivial if and only if f is a $\mathbb{Z}\text{-Tors}_{\mathcal{P}}$ -isomorphism.

32.1. Finite generation and rational computations.

Theorem 32.3. Consider $k, n \in \mathbb{Z}^{\geq 1}$. If n is odd and $k > n$, then $\pi_k(S^n)$ is finite.

Proof. For $n = 1$ we know already $\pi_k(S^1) = \{0\}$ for $k > n$. So we can assume without loss of generality that $n \geq 3$ holds. Choose a map $f: S^n \rightarrow K(\mathbb{Z}, n)$ inducing an isomorphism $\pi_n(f): \pi_n(S^n) \xrightarrow{\cong} \pi_n(K(\mathbb{Z}, n))$. We conclude from Theorem 28.8 that $H^k(f; \mathbb{Q}): H^k(S^n; \mathbb{Q}) \xrightarrow{\cong} H^k(K(\mathbb{Z}, n); \mathbb{Q})$ is bijective for all $k \in \mathbb{Z}^{\geq 0}$. The Universal Coefficient Theorem implies that $H_k(f; \mathbb{Q}): H_k(S^n; \mathbb{Q}) \xrightarrow{\cong} H_k(K(\mathbb{Z}, n); \mathbb{Q})$ is bijective for all $k \in \mathbb{Z}^{\geq 0}$. The Whitehead Theorem modulo a Serre ideal, see Theorem 31.2 (ii), and Lemma 32.2 imply that $\text{id}_{\mathbb{Q}} \otimes_{\mathbb{Z}} \pi_k(f): \mathbb{Q} \otimes_{\mathbb{Z}} \pi_k(S^n) \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \pi_k(K(\mathbb{Z}, n))$ is an isomorphism for $k \in \mathbb{Z}^{\geq 0}$. Therefore $\mathbb{Q} \otimes_{\mathbb{Z}} \pi_k(S^n)$ vanishes for $k > n$. Hence $\pi_k(S^n)$ belongs to the Serre class $\mathbb{Z}\text{-Tors}$. Since $H_k(S^n)$ is finitely generated for $k \in \mathbb{Z}^{\geq 0}$, we conclude from Corollary 30.3 that $\pi_k(S^n)$ is finitely generated for $k > n$. We conclude that $\pi_k(S^n)$ is finite for $k > n$. $\square$

Corollary 32.4. The abelian group $\pi_k^s(\{\bullet\})$ is finite for $k \geq 1$.

Proof. We conclude from the Freudenthal Suspension Theorem 11.16 that $\pi_k^s(\{\bullet\}) \cong \pi_{k+n}(S^n)$ holds for $n \geq k + 2$. Now apply Theorem 32.3. $\square$

Theorem 32.5. Consider $k, n \in \mathbb{Z}^{\geq 1}$ such that n is even. Then $\pi_k(S^n)$ is finite for $k \notin \{n, (2n - 1)\}$ and we have $\pi_n(S^n) \cong \mathbb{Z}$ and $\pi_{2n-1}(S^n) \cong \mathbb{Z} \oplus F$ for some finite abelian group F .

Proof. We get $H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}[x_n]$ with $|x_n| = n$ from Theorem 28.8. Choose a map $f: K(\mathbb{Z}, n) \rightarrow K(\mathbb{Q}, 2n)$ representing $x_n^2 \in H^n(K(\mathbb{Z}, n); \mathbb{Q})$ under the identification $[K(\mathbb{Z}, n), K(\mathbb{Q}, 2n)] = H^n(K(\mathbb{Z}, n); \mathbb{Q})$ appearing in Example 16.26. We can turn f into a fibration. Let F be its fiber. Then we obtain a fibration

$$K(\mathbb{Q}, 2n-1) = \Omega K(\mathbb{Q}, 2n) \rightarrow F \xrightarrow{p} K(\mathbb{Z}, n).$$

Next we compute $H^*(F; \mathbb{Q})$ using the Leray-Serre spectral sequence applied to the fibration above. We have shown $H^*(K(\mathbb{Z}, 2n-1); \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(y_{2n-1})$ with $|y_{2n-1}| = 2n-1$ in Theorem 28.8. Let $g: K(\mathbb{Z}, 2n-1) \rightarrow K(\mathbb{Q}, 2n-1)$ be the map representing the generator y_{2n-1} of $H^{2n-1}(K(\mathbb{Z}, 2n-1); \mathbb{Q})$. It induces an isomorphism $\text{id}_{\mathbb{Q}} \otimes_{\mathbb{Z}} \pi_k(g): \mathbb{Q} \otimes_{\mathbb{Z}} \pi_k(K(\mathbb{Z}, 2n-1)) \xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} \pi_k(K(\mathbb{Q}, 2n-1))$ for $k \in \mathbb{Z}^{\geq 1}$. The Whitehead Theorem modulo the Serre ideal $R\text{-Tors}$, see Theorem 31.2 together with Lemma 32.2 imply that $H_k(g; \mathbb{Q}): H_k(K(\mathbb{Z}, 2n-1); \mathbb{Q}) \rightarrow H_k(K(\mathbb{Q}, 2n-1); \mathbb{Q})$ is bijective for every $k \in \mathbb{Z}^{\geq 1}$. We conclude from the Universal Coefficient Theorem that we obtain for every $k \in \mathbb{Z}^{\geq 1}$ an isomorphism

$$H^k(g; \mathbb{Q}): H^k(K(\mathbb{Q}, 2n-1); \mathbb{Q}) \xrightarrow{\cong} H^k(K(\mathbb{Z}, 2n-1); \mathbb{Q}).$$

The $(2n)$ th page of the cohomological Leray-Serre spectral sequence for the fibration $p: F \rightarrow K(\mathbb{Z}, n)$ is the only page with possible non-trivial differentials. We have

$$\begin{aligned} E_{2n}^{p,q} &= E_2^{p,q} = H^p(K(\mathbb{Z}, n); H^q(K(\mathbb{Q}, 2n-1); \mathbb{Q})) \\ &= H^p(K(\mathbb{Z}, n); \mathbb{Q}) \otimes H^q(K(\mathbb{Q}, 2n-1); \mathbb{Q}) = \mathbb{Q}[x_n] \otimes \Lambda_{\mathbb{Q}}(y_{2n-1}). \end{aligned}$$

and the only possible non-trivial differentials are $d_{2n}^{pn, 2n-1}: E_{2n}^{pn, 2n-1} \rightarrow E_{2n}^{(p+2)n, 0}$ for $p \in \mathbb{Q}^{\geq 0}$.

For $n = 3$ it looks in the range $0 \leq p \leq 9$ and $0 \leq q \leq 5$ like

$\mathbb{Q}\langle y_5 \rangle$	0	0	$\mathbb{Q}\langle x_3 y_5 \rangle$	0	0	$\mathbb{Q}\langle x_3^2 y_5 \rangle$	0	0	$\mathbb{Q}\langle x_3^3 y_5 \rangle$
0	0	0	0	0	0	0	0	0	0
0	0	0	$d_3^{0,5}$	0	0	$d_3^{3,5}$	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
$\mathbb{Q}$	0	0	$\mathbb{Q}\langle x_3 \rangle$	0	0	$\mathbb{Q}\langle x_3^2 \rangle$	0	0	$\mathbb{Q}\langle x_3^3 \rangle$

The composite

$$H^{2n}K(\mathbb{Q}, 2n); \mathbb{Q} \xrightarrow{H^{2n}(f; \mathbb{Q})} H^{2n}(K(\mathbb{Z}, n); \mathbb{Q}) \xrightarrow{H^{2n}(p; \mathbb{Q})} H^{2n}(F; \mathbb{Q})$$

is trivial, since $f \circ p$ is null homotopic. The map $H^{2n}(f; \mathbb{Q})$ sends the canonical generator of $H^{2n}(K(\mathbb{Q}, 2n); \mathbb{Q}) \cong \mathbb{Q}$ to x_n^2 . Hence $H^n(p; \mathbb{Q}): H^n(K(\mathbb{Z}, n); \mathbb{Q}) \rightarrow H^n(F; \mathbb{Q})$ sends x_n^2 to zero and therefore is the trivial map. Recall from the discussions about the edge homomorphism $\text{edge}^{2n,0}$ and the fact that $E_{\infty}^{0, 2n-1} =$

$E_{2n+1}^{0,2n-1}$ holds that we obtain an exact sequence

$$E_{2n}^{0,2n-1} \xrightarrow{d_{2n}^{0,2n-1}} E_{2n}^{2n,0} \rightarrow E_{2n+1}^{2n,0} = E_{\infty}^{2n,0} \rightarrow \{0\}$$

and the equality

$$E_{\infty}^{2n,0} = \text{im}(H^{2n}(p; \mathbb{Q}): H^{2n}(K(\mathbb{Z}, n); \mathbb{Q}) \rightarrow H^{2n}(F; \mathbb{Q})).$$

Since $H^n(p; \mathbb{Q})$ is trivial, we conclude that the differential

$$d_{2n}^{0,2n-1}: E_{2n}^{0,2n-1} = \mathbb{Q}\langle y_{2n-1} \rangle \rightarrow E_{2n}^{2n,0} = \mathbb{Q}\langle x_n^2 \rangle$$

is surjective and hence bijective, as its source and its domain have the same dimension. Choose $r \in \mathbb{Q}$ with $r \neq 0$ and $d_{2n}^{0,2n-1}(y_{2n-1}) = r \cdot x_n^2$.

Next we show for $k \in 0, 1, 2, \dots$

$$d_{2n}^{kn,2n-1}(x_n^k y_{2n-1}) = (-1)^{kn} \cdot r \cdot x_n^{k+2}.$$

The induction beginning $k = 0$ is obvious. The induction step from k to $(k+1)$ follows from the computation

$$\begin{aligned} d_{2n}^{(k+1)n,2n-1}(x_n^{k+1} y_{2n-1}) &= d_{2n}^{(k+1)n,2n-1}(x_n(x_n^k y_{2n-1})) \\ &= d_{2n}^{n,0}(x_n)(x_n^k y_{2n-1}) + (-1)^n \cdot x_n \cdot d_{2n}^{kn,2n-1}(x_n^k y_{2n-1}) \\ &= 0 + (-1)^n x_n ((-1)^{kn} \cdot r \cdot x_n^{k+2}) = (-1)^{(k+1)n} \cdot r \cdot x_n^{(k+1)+2}. \end{aligned}$$

Hence for $k \in \mathbb{Z}^{\geq 0}$ the differential

$$d_{2n}^{kn,2n-1}: E_{2n}^{kn,2n-1} = \mathbb{Q}\langle x_n^k y_{2n-1} \rangle \rightarrow E_{2n}^{(k+2)n,0} = \mathbb{Q}\langle x_n^{k+2} \rangle$$

is bijective. Let z be the image of x_n under the map $H^n(p; \mathbb{Q}): H^n(K(\mathbb{Z}, n); \mathbb{Q}) \rightarrow H^n(F; \mathbb{Q})$. So we get

$$E_{\infty}^{p,q} = \begin{cases} \mathbb{Q} & p = q = 0; \\ \mathbb{Q}\langle x_n \rangle & p = n, q = 0; \\ \{0\} & \text{otherwise.} \end{cases}$$

This implies

$$H^*(F; \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}[z] \quad \text{for } |z| = n.$$

Let $s: S^n \rightarrow K(\mathbb{Z}, n)$ be a map representing a generator of $\pi_n(K(\mathbb{Z}, n)) \cong \mathbb{Z}$. Since its composite with the map $f: K(\mathbb{Z}, n) \rightarrow K(\mathbb{Q}, 2n)$ is nullhomotopic, we can find a map $t: S^n \rightarrow F$ whose composite with $p: F \rightarrow K(\mathbb{Z}, n)$ is homotopic to s . Then $H_k(t; \mathbb{Q}): H_k(S^n; \mathbb{Q}) \rightarrow H_k(F; \mathbb{Q})$ is bijective for all $k \in \mathbb{Z}^{\geq 0}$. The Whitehead Theorem modulo the Serre ideal $\mathbb{Z}\text{-Tors}$, see Theorem 31.2 together with Lemma 32.2 imply that $\text{id}_{\mathbb{Q}} \otimes_{\mathbb{Z}} \pi_k(t): \mathbb{Q} \otimes_{\mathbb{Z}} \pi_k(S^n) \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \pi_k(F)$ is bijective for all $k \in \mathbb{Z}^{\geq 0}$. The long exact homotopy sequence associated to the fibration $F \rightarrow K(\mathbb{Z}, n) \xrightarrow{f} K(\mathbb{Q}, 2n)$ implies

$$\pi_k(F) \cong \begin{cases} \mathbb{Z} & k = n; \\ \mathbb{Q} & k = 2n - 1; \\ \{0\} & \text{otherwise.} \end{cases}$$

Hence we get

$$\mathbb{Q} \otimes_{\mathbb{Z}} \pi_k(S^n) \cong \begin{cases} \mathbb{Q} & k \in \{n, 2n - 1\}; \\ \{0\} & \text{otherwise.} \end{cases}$$

We know already that $\pi_n(S^n) \cong \mathbb{Z}$ holds and that $\pi_k(S^n)$ is finitely generated for all $k \in \mathbb{Z}^{\geq 1}$, see Theorem 3.4 and Corollary 30.3. This finishes the proof of Theorem 32.5. $\square$

32.2. p -coprimary computations. Note that Corollary 30.3 implies that for fixed $n \in \mathbb{Z}^{\geq 1}$ we get

$$\pi_k(S^n) \cong \mathbb{Z}^r \oplus \bigoplus_{p \text{ prime}} \bigoplus_{i=1}^{s_p} \mathbb{Z}/p^{l_{i,p}}$$

for $s_p \in \mathbb{Z}^{\geq 0}$ and $l_{i,p} \in \mathbb{Z}^{\geq 1}$. We have figured out the value of r_i in Theorem 32.3 and Theorem 32.5, namely

$$r = \begin{cases} 1 & \text{if } k = n; \\ 1 & \text{if } k = 2n - 1 \text{ and } n \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

In general the values of s_p and $l_{i,p}$ are not known. In order to get some information, it is useful to attack this problem for each prime separately. The key idea is the formula

$$\mathbb{Z}_{(p)} \otimes \pi_k(S^n) \cong \mathbb{Z}_{(p)}^r \oplus \bigoplus_{i=1}^{s_p} \mathbb{Z}/p^{l_{i,p}}.$$

Proposition 32.6. *Let p be a prime. Then we get*

$$\pi_k(S^3)_{(p)} \cong \begin{cases} \mathbb{Z}_{(p)} & k = 3; \\ \{0\} & k < 2p, k \neq 3; \\ \mathbb{Z}/p & k = 2p. \end{cases}$$

Proof. Consider a map $s: S^3 \rightarrow K(\mathbb{Z}, 3)$ such that $\pi_3(s): \pi_3(S^3) \xrightarrow{\cong} \pi_3(K(\mathbb{Z}, 3))$ is bijective. We can turn it into a fibration. Let F be the fiber. The long exact homotopy sequence implies $\pi_k(F) \cong \pi_k(S^3)$ for $k \in \mathbb{Z}^{\geq 4}$. In Subsection 28.7 we have proved

$$H_k(F) \cong \begin{cases} \mathbb{Z}/l & \text{if } k = 2l \geq 4; \\ \mathbb{Z} & k = 0; \\ \{0\} & \text{otherwise.} \end{cases}$$

Hence $H_k(F)$ belong to the Serre class $\mathbb{Z}\text{-Tors}_{(p)}$ for $k < 2p$, or, equivalently $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} H_k(F)$ vanishes for $k < 2p$. The Hurewicz Theorem modulo $\mathbb{Z}\text{-Tors}_{(p)}$, see Theorem 30.2 implies that $\pi_k(F)$ belongs to $\mathbb{Z}\text{-Tors}_{(p)}$ for $k < 2p$ and that the Hurewicz homomorphism $\pi_{2p}(F) \rightarrow H_{2p}(F)$ is a $\mathbb{Z}\text{-Tors}_{(p)}$ -isomorphism. This finishes the proof of Proposition 32.6. $\square$

One interesting aspect of Proposition 32.6 is that $\pi_k(S^3)$ is non trivial for infinite many values of k . It also implies $\pi_4(S^3) \cong \mathbb{Z}/2$ and $\pi_1^s \cong \mathbb{Z}/2$.

Proposition 32.7. *Let l be a prime. Consider $n \in \mathbb{Z}^3$ which is odd. Then the double suspension homomorphism*

$$\sigma_k^2(S^n): \pi_k(S^n) \rightarrow \pi_{k+2}(S^{n+2})$$

is a $\mathbb{Z}\text{-Tors}_{(l)}$ -isomorphism for $k < l(n+1) - 3$ and a $\mathbb{Z}\text{-Tors}_{(l)}$ -epimorphism for $k = l(n+1) - 3$.

Proof. Let $\omega: S^n \rightarrow \Omega^2 S^{n+2}$ be the adjoint of $\text{id}_{S^{n+2}}: S^2 \wedge S^n = S^{n+2} \rightarrow S^{n+2}$. Since we have the commutative diagram

$$\begin{array}{ccc} \pi_k(S^n) & \xrightarrow{\pi_k(\omega)} & \pi_k(\Omega^2 S^{n+2}) \\ & \searrow \sigma_k^2(S^n) & \downarrow \cong \\ & & \pi_{k+2}(S^{n+1}) \end{array}$$

it suffices consider $\pi_k(\omega)$ instead of $\sigma_k^2(S^n)$. Because of the Whitehead Theorem modulo the Serre ideal $\mathbb{Z}\text{-Tors}_{(l)}$, see Theorem 31.2, it suffices to show that $H_k(\omega): H_k(S^n) \rightarrow H_k(\Omega^2 S^{n+2})$ is a $\mathbb{Z}\text{-Tors}_{(l)}$ -isomorphism for $k < l(n+1) - 3$ and a $\mathbb{Z}\text{-Tors}_{(l)}$ -epimorphism for $k = l(n+1) - 3$.

We have already proved that $\pi_k(\Omega^2 S^{n+2}) \cong \pi_k(S^k)$ is finitely generated for all $k \in \mathbb{Z}^{\geq 1}$, see Corollary 30.3. The Hurewicz Theorem modulo the Serre class $\mathbb{Z}\text{-Mod}_{\text{fg}}$, see Theorem 30.2, implies that $H_k(\Omega^2 S^{n+2})$ is finitely generated for $k \in \mathbb{Z}^{\geq 1}$. Hence the kernel and the cokernel of $H_k(\omega)$ are finitely generated abelian groups. Hence it suffices to show that $\mathbb{F}_l \otimes_{\mathbb{Z}} \ker(H_k(\omega))$ is trivial for $k < l(n+1) - 3$ and $\mathbb{F}_l \otimes_{\mathbb{Z}} \text{cok}(H_k(\omega))$ is trivial for $k \leq l(n+1) - 3$.

Since the homology of S^n is concentrated in dimension n and $H_n(\omega): H_n(S^n) \rightarrow H_n(\Omega^2 S^{n+2})$ is an isomorphism by the Hurewicz Theorem, see Proposition 12.12, we get $\ker(H_k(\omega)) = \{0\}$ for $k \in \mathbb{Z}^{\geq 1}$ and $\text{cok}(H_k(\omega)) = \{0\}$ for $k = n$. Hence it remains to show that $\mathbb{F}_l \otimes_{\mathbb{Z}} \text{cok}(H_k(\omega))$ is trivial for $k \leq l(n+1) - 3, k \neq n$.

Consider the commutative diagram

$$\begin{array}{ccc}
 \mathbb{F}_l \otimes_{\mathbb{Z}} H_k(S^n) & \xrightarrow{\cong} & H_k(S^n; \mathbb{F}_l) \\
 \text{id}_{\mathbb{F}_l} \otimes_{\mathbb{Z}} H_k(\omega) \downarrow & & \downarrow \text{id}_{H_k(\omega; \mathbb{F}_l)} \\
 \mathbb{F}_l \otimes_{\mathbb{Z}} H_k(\Omega^2 S^{n+2}) & \longrightarrow & H_k(\Omega^2 S^{n+2}; \mathbb{F}_l) \\
 \downarrow & & \downarrow \\
 \mathbb{F}_l \otimes_{\mathbb{Z}} \text{cok}(H_k(\omega)) & \longrightarrow & \text{cok}(H_k(\omega; \mathbb{F}_l)) \\
 \downarrow & & \downarrow \\
 \{0\} & & \{0\}
 \end{array}$$

whose rows are exact and whose top arrow and second top vertical arrow come from the Universal Coefficient Theorem. An easy diagram chase shows that it suffices to prove that $\text{cok}(H_k(\omega; \mathbb{F}_l))$ is trivial for $k \leq l(n+1) - 3, k \neq n$.

Next we compute $H^*(\Omega S^{n+2}; \mathbb{F}_l)$ in a range. We apply the Leray-Serre spectral sequence to the path fibration $\Omega S^{n+2} \rightarrow P(S^n) \rightarrow S^n$. The only nontrivial differentials occur on the $(n+2)$ th page. For $n = 3$ it looks in the range $0 \leq p \leq 5$ and

$0 \leq q \leq 8$ like

$$\begin{array}{cccccc}
 H^8(\Omega S^5; \mathbb{F}_l) & 0 & 0 & 0 & 0 & H^8(\Omega S^5; \mathbb{F}_l) \\
 & \searrow & & & & \\
 H^7(\Omega S^5; \mathbb{F}_l) & 0 & 0 & 0 & 0 & H^7(\Omega S^5; \mathbb{F}_l) \\
 & \searrow & & & & \\
 H^6(\Omega S^5; \mathbb{F}_l) & 0 & 0 & d_5^{0,8} & 0 & H^6(\Omega S^5; \mathbb{F}_l) \\
 & \searrow & & & & \\
 H^5(\Omega S^5; \mathbb{F}_l) & 0 & 0 & d_5^{0,7} & 0 & H^5(\Omega S^5; \mathbb{F}_l) \\
 & \searrow & & & & \\
 H^4(\Omega S^5; \mathbb{F}_l) & 0 & 0 & d_5^{0,6} & 0 & H^4(\Omega S^5; \mathbb{F}_l) \\
 & \searrow & & & & \\
 H^3(\Omega S^5; \mathbb{F}_l) & 0 & 0 & d_5^{0,5} & 0 & H^3(\Omega S^5; \mathbb{F}_l) \\
 & \searrow & & & & \\
 H^2(\Omega S^5; \mathbb{F}_l) & 0 & 0 & d_5^{0,4} & 0 & H^2(\Omega S^5; \mathbb{F}_l) \\
 & \searrow & & & & \\
 H^1(\Omega S^5; \mathbb{F}_l) & 0 & 0 & 0 & 0 & H^1(\Omega S^5; \mathbb{F}_l) \\
 & \searrow & & & & \\
 H^0(\Omega S^5; \mathbb{F}_l) & 0 & 0 & 0 & 0 & H^0(\Omega S^5; \mathbb{F}_l)
 \end{array}$$

Recall that $P(S^n)$ is contractible and hence $H^m(P(S^n); \mathbb{F}_l)$ vanishes for $m \geq 1$. Hence all the differentials $d_{n+2}^{p,0}$ must be isomorphisms. Pick a generator x of $E_n^{0,n+1} = H^{n+1}(\Omega S^n; \mathbb{F}_l) = \mathbb{F}_l$ and y of $E_n^{n+2,0} = H^0(\Omega S^{n+2}; \mathbb{F}_l) = \mathbb{F}_l$ such that $d_n^{n+2,0}$ maps y to x . Hence for $n = 3$ the n th page it looks in the range $0 \leq p \leq 5$

and $0 \leq q \leq 6$ like

$$\begin{array}{cccccc}
 \mathbb{F}_l & 0 & 0 & 0 & 0 & \mathbb{F}_l \\
 & \searrow & & & & \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 & & \searrow & & & \\
 0 & 0 & 0 & d_5^{0,8} & 0 & 0 \\
 & & & \searrow & & \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & \searrow & \\
 \mathbb{F}_l \langle y \rangle & 0 & 0 & 0 & 0 & \mathbb{F}_l \\
 & \searrow & & & & \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 & & \searrow & & & \\
 0 & 0 & 0 & d_5^{0,4} & 0 & 0 \\
 & & & \searrow & & \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & \searrow & \\
 \mathbb{F}_l & 0 & 0 & 0 & 0 & \mathbb{F}_l \langle x \rangle
 \end{array}$$

A computation similar to the one in Subsection 28.6 using the fact that the differentials are derivations shows that

$$(32.8) \quad d_{n+1}^{0, (n+1)^k}(y^k) = k! \cdot u_k \text{ holds for some generator } u_k \in \mathbb{F}_l$$

For $k < l$, the number $k!$ a unit in $\mathbb{F}_l$. Hence the map

$$\mathbb{F}_l[y] \rightarrow H^*(\Omega S^{n+2}; \mathbb{F}_l)$$

is an isomorphism in degrees $< l(n+1)$.

Next we consider the path fibration $\Omega^2 S^{n+2} \rightarrow P(\Omega S^{n+2}) \rightarrow \Omega S^{n+2}$ and we apply the Leray-Serre spectral sequence to it. We conclude from the Künneth Theorem

$$E_2^{p,q} = H^p(\Omega S^{n+2}; H^q(\Omega^2 S^{n+2}; \mathbb{F}_l)) = H^p(\Omega S^{n+2}; \mathbb{F}_l) \otimes_{\mathbb{F}_l} H^q(\Omega^2 S^{n+2}; \mathbb{F}_l).$$

Recall that $P(\Omega S^{n+2})$ is contractible and hence $H^m(P(\Omega S^{n+2}); \mathbb{F}_l) = \{0\}$ holds for $m \in \mathbb{Z}^{\geq 1}$. This implies that $E_\infty^{p,q}$ must be trivial if $(p, q) \neq (0, 0)$ holds. We have $E_2^{p,q} = H^p(\Omega S^{n+2}; \mathbb{F}_l) \otimes_{\mathbb{F}_l} H^q(\Omega^2 S^{n+2}; \mathbb{F}_l) = \{0\}$ Hence $E_r^{p,q}$ vanishes for every $r \in \mathbb{Z}^{\geq 2} \amalg \{\infty\}$, unless $q = m \cdot (n+1)$ holds for some $m \in \mathbb{Z}^{\geq}$.

Let us consider the $(n+1)$ th page which does agree with the second page. Every differential $d_r^{p,0}$ for $0 \leq p < n-1$ and $r \in \mathbb{Z}^{\geq 2} \amalg \{\infty\}$ has $\{0\}$ as target and hence is trivial. This implies that $E_2^{p,0} = H^p(\Omega S^{n+2}; \mathbb{F}_l)$ vanishes for $0 \leq p < n-1$. Hence $E_r^{p,q} = 0$ for every $r \in \mathbb{Z}^{\geq 2} \amalg \{\infty\}$, unless $q = m \cdot (n+1)$ holds for some $m \in \mathbb{Z}^{\geq}$ and $p \geq n$ holds. The differential

$$d_{n+1}^{0,n}: E_{n+1}^{0,n} = H^n(\Omega S^{n+2}; \mathbb{F}_l) \rightarrow E_{n+1}^{n+1,0} = H^{n+1}(\Omega S^{n+2}; \mathbb{F}_l)$$

must be an isomorphism, since $E_\infty^{0,n}$ vanishes and each differential $d_{n+1}^{0,n}$ for $r \geq n+2$ has trivial target and hence vanishes. This implies that $H^n(\Omega^2 S^{n+2}; \mathbb{F}_l) \cong \mathbb{F}_l$

holds and that we can choose a generator $z \in H^n(\Omega^2 S^{n+2}; \mathbb{F}_l)$ which is sent under $d_{n+1}^{0,n}$ to the generator $y \in H^{n+1}(\Omega S^{n+2}; \mathbb{F}_l)$. Recall the map $H^p(\Omega S^{n+2}; \mathbb{F}_l) \rightarrow H^{p+n+1}(\Omega S^{n+2}; \mathbb{F}_l)$ given by multiplication with y is bijective for $p < (l-1)(n+1)$ and the zero map for $p = (l-1)(n+1)$. Hence the map $E_2^{p,q} = E_{n+1}^{p,q} \rightarrow E_2^{p+n+1,q} = E_{n+1}^{p+n+1,q}$ is for all $q \in \mathbb{Z}^{\geq 0}$ an isomorphism for $p < (l-1)(n+1)$ and the zero map for $p = (l-1)(n+1)$.

So the $(n+1)$ th page looks schematically in the range $0 \leq p \leq l(n+1)$ and $0 \leq q \leq n$

$$\begin{array}{ccccccc}
 \mathbb{F}_l\langle z \rangle & \mathbb{F}_l\langle zy \rangle & \cdots & \mathbb{F}_l\langle zy^{l-3} \rangle & \mathbb{F}_l\langle zy^{l-2} \rangle & \mathbb{F}_l\langle zy^{l-1} \rangle & ? \\
 \searrow d_{n+1}^{n,0} & \searrow d_{n+1}^{n,n+1} & & \searrow d_{n+1}^{n,(l-3)(n+1)} & \searrow d_{n+1}^{n,(l-2)(n+1)} & \searrow d_{n+1}^{n,(l-1)(n+1)} & \\
 \mathbb{F}_l & \mathbb{F}_l\langle y \rangle & \mathbb{F}_l\langle y^2 \rangle & \cdots & \mathbb{F}_l\langle y^{l-2} \rangle & \mathbb{F}_l\langle y^{l-1} \rangle & 0 \rightarrow ?
 \end{array}$$

where we have listed the intersection of the 0-th and n -row and the columns over 0, $(n+1), \dots, p(n+2)$, all other entries are trivial. Since $d_{n+1}^{n+1,n}$ is an isomorphism and $d_{n+1}^{n+1,n} \circ d_{n+1}^{0,2n} = 0$, the differential $d_{n+1}^{0,2n}$ is trivial. Obviously the differentials $d_{n+1}^{0,p}$ vanish for $n < p < 2n$, since their targets are trivial. This implies that all differentials starting at $(p, 0)$ for $n < p < 2n$, are trivial. Hence $H^p(\Omega^2 S^{n+2}; \mathbb{F}_l)$ vanishes for $1 \leq p \leq 2n, p \neq n$. Note that this implies that $E_{n+1}^{p,q}$ for $q \leq 2n$ vanishes unless (p, q) is of the form $(0, m(n+1))$ or $(n, m(n+1))$ for some $m \in \mathbb{Z}^{\geq 0}$ and that $E_r^{p,q}$ for $q \leq 2n$ and $r \in \mathbb{Z}^{\geq (n+2)}$ vanishes unless $(p, q) = (0, 0)$. Now one easily shows inductively for $i = 2n, 2n+1, 2n+2, \dots, l(n+1)-3$

- All differentials starting at $(p, 0)$ for $n < p < i$ have $\{0\}$ as target and hence are trivial;
- $H^p(\Omega^2 S^{n+2}; \mathbb{F}_l)$ vanishes for $1 \leq p \leq i, p \neq n$;
- $E_{n+1}^{p,q}$ for $q \leq i$ vanishes unless (p, q) is of the form $(0, m(n+1))$ or $(n, m(n+1))$ for some $m \in \mathbb{Z}^{\geq 0}$;
- $E_r^{p,q}$ for $q \leq 2i$ and $r \in \mathbb{Z}^{\geq (n+2)}$ vanishes unless $(p, q) = (0, 0)$.

If we take $i = l(n+1) - 3$, we conclude that $H^k(\Omega^2 S^{n+1}; \mathbb{F}_l)$ is trivial for $k \leq l(n+1) - 3, k \neq n$. The Universal Coefficient Theorem implies that $H_k(\Omega S^2; \mathbb{F}_l)$ and hence $\text{cok}(H_k(\omega; \mathbb{F}_l))$ is trivial for $k \leq l(n+1) - 3, k \neq n$. This finishes the proof of Proposition 32.7.

As an addendum we mention that the fact that $d_{n+1}^{(l-1)(n+1),n}$ is trivial implies that the differential

$$d_{(l-1)(n+1)}^{0, l(n+1)-2}: E_{(l-1)(n+1)}^{0, l(n+1)-2} \rightarrow E_{(l-1)(n+1)}^{(l-1)(n+1), n} \cong \mathbb{F}_l$$

is surjective and hence $H_{l(n+1)-2}(\Omega^2 S^{n+2}; \mathbb{F}_l)$ is non-trivial. $\square$

Theorem 32.9. *Let p be a prime. We have*

$$\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \pi_n^s \cong \begin{cases} \{0\} & n < 2p-3; \\ \mathbb{Z}/p & n = 2p-3. \end{cases}$$

Proof. Consider the sequence of double suspension maps

$$\pi_{n+3}(S^3) \rightarrow \pi_{n+5}(S^5) \rightarrow \pi_{n+7}(S^7) \rightarrow \cdots$$

The first map $\pi_{n+3}(S^3) \rightarrow \pi_{n+5}(S^5)$ is a $\mathbb{Z}\text{-Tors}_{(p)}$ -isomorphism for $n+3 < 4p-3$, or, equivalently, for $n < 4p-6$. The other maps are $\mathbb{Z}\text{-Tors}_{(p)}$ -isomorphism in even larger ranges for n . These claims follow from Proposition 32.7. We conclude from Proposition 32.6 that $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \pi_{n+3}(S^3)$ vanishes for $3 < n+3 < 2p$, or, equivalently, for $0 < n < 2p-3$ and is $\mathbb{Z}/p$ for $n = 2p-3$. Since the double suspensions map above are $\mathbb{Z}\text{-Tors}_{(p)}$ -isomorphism for $n \leq 2p-3$, Theorem 32.9 follows. $\square$

An interesting aspect of Theorem 32.9 is that for any prime p there is an $n \in \mathbb{Z}^{\geq 1}$ such that π_n^s contains p -torsion. The reader should check that the first 19 explicit values of the stable stems from the table (11.28) are consistent with Theorem 32.9.

33. EXERCISES PART I

Exercise 1. Let M be a connected closed 3-manifold whose fundamental group G is perfect, i.e., G agrees with its commutator subgroup $[G, G]$, and non-trivial.

Prove or disprove that there is a map $f: M \rightarrow S^3$ which is not a homotopy equivalence and induces an isomorphism $H_n(f; A): H_n(M; A) \rightarrow H_n(S^3; A)$ for any abelian group A and any $n \geq 0$.

Exercise 2. Let $f: X \rightarrow Y$ be a homotopy equivalence. Show that for any $x \in X$ and $n \geq 1$ the induced map $\pi_n(f, x): \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$ is an isomorphism.

Exercise 3. Compute $\pi_n(T^k \times \mathbb{R}^l, x)$ for all $k, l, n \geq 1$, where T^k is the k -torus.

Exercise 4. Consider a path connected space X with base point $x \in X$ and $n \geq 1$.

- (i) Let $[S_n] \in H_n(S^n)$ be a generator. Show that we get a well-defined group homomorphism

$$\text{hur}_n: \pi_n(X, x) \rightarrow H_n(X)$$

by sending $[f]$ represented by the pointed map $f: (S^n, s) \rightarrow (X, x)$ to the image of the fundamental class $[S^n]$ under the map $H_n(f): H_n(S^n) \rightarrow H_n(X)$.

- (ii) Give for every $n \geq 2$ examples of closed connected orientable manifolds M, N of dimension n such that $\text{hur}_n: \pi_n(M, x) \rightarrow H_n(M)$ is surjective and $\text{hur}_n: \pi_n(N, x) \rightarrow H_n(N)$ is trivial.

Exercise 5. Let W be the *Warsaw circle*, i.e., the union of subsets of $\mathbb{R}^2$ given by the union of $\{(x, \sin(2\pi/x)) \mid x \in (0, 1]\}$, $\{(1, y) \mid y \in [-2, 0]\}$, $\{(x, -2) \mid x \in [0, 1]\}$ and $\{(0, y) \mid y \in [-2, 1]\}$.

Show that the projection $p: W \rightarrow \{\bullet\}$ is a weak homotopy equivalence but not a homotopy equivalence.

Exercise 6. Let $X \subseteq \mathbb{R}^{n+1}$ be the union $\bigcup_{k=0}^{\infty} Y_k$, where Y_k is the sphere around $(1/k, 0, \dots, 0)$ of radius $1/k$.

Prove or disprove that there is a surjective homomorphism $\pi_n(X, x) \rightarrow \prod_{i=0}^{\infty} \mathbb{Z}$ and hence $\pi_n(X, x)$ is uncountable for any base point $x \in X$.

Exercise 7. Compute the set of homotopy classes $[X, Y]$ of maps $X \rightarrow Y$ for the following cases:

- (i) $X = Y = S^n$ for $1 \leq n$;
- (ii) $X = S^m$ and $Y = S^n$ for $0 \leq m < n$;
- (iii) $X = S^n$ and $Y = T^n$ for $n \geq 2$;
- (iv) $X = \mathbb{CP}^n$ and $Y = S^{2n}$ for $n \geq 1$;
- (v) $X = \mathbb{CP}^n$ and $Y = S^1$ for $n \geq 1$.

Exercise 8. Let (X, A) be a topological pair such that A is $(n-1)$ -connected and X is n -connected for $n \in \{1, 2, \dots\} \amalg \{\infty\}$. Prove or disprove that (X, A) is n -connected.

Exercise 9. Prove or disprove for $n \geq 0$ that an n -connected CW -complex of dimension n is contractible.

Exercise 10. Let X be a CW -complex such that for every natural number m there is a natural number n with $m \leq n$ such that the inclusion $X_m \rightarrow X_n$ is nullhomotopic. Prove or disprove that X is contractible.

Exercise 11. Let G be a finitely generated abelian group G and $n \in \mathbb{Z}^{\geq 1}$. Construct a compact $(n+1)$ -dimensional path connected CW -complex X with $\pi_n(X) \cong G$.

Exercise 12. Let N be a path connected oriented closed smooth n -manifold which has the following property: For any path connected oriented closed smooth n -manifold M the degree defines a bijection $\deg: [M, N] \rightarrow \mathbb{Z}$.

Prove or disprove that N is oriented homotopy equivalent to S^n .

Exercise 13. Let X be a topological space which is dominated by a CW -complex Y , i.e., there exists a CW -complex Y with maps $i: X \rightarrow Y$ and $r: Y \rightarrow X$ with $r \circ i \simeq \text{id}_X$.

Prove or disprove that X has the homotopy type of a CW -complex.

Exercise 14. Prove or disprove that a compact metric space Y has a CW -approximation $f: X \rightarrow Y$ with compact X .

Exercise 15. Let X be a m -connected and Y be a n -connected CW -complex coming with base points. Prove or disprove that $X \wedge Y$ is $(m+n+1)$ -connected.

Exercise 16. Let X be the space obtained from $S^1 \subseteq \mathbb{R}^2$ by identifying the open subsets $\{(x, y) \in S^1 \mid y > 0\}$ and $\{(x, y) \in S^1 \mid y < 0\}$. Let $p: S^1 \rightarrow X$ be the projection. Then the set X has four points, namely, the images of $(0, 1)$, $(0, -1)$, $(1, 0)$, and $(0, -1)$ under p .

- (i) Describe the open subsets of X and show that X is not a Hausdorff space, is path connected, and is pre-compact, i.e., every open covering has a finite subcovering;
- (ii) Prove or disprove that X has a universal covering $p: \tilde{X} \rightarrow X$;
- (iii) Prove or disprove that $p: S^1 \rightarrow X$ is a CW -approximation;
- (iv) Prove or disprove that S^1 and X are homotopy equivalent.

Exercise 17. Show that any metric space and any locally compact Hausdorff space is compactly generated.

Exercise 18. Prove or disprove:

- (i) The composite of two cofibrations is again a cofibration;
- (ii) The product of two cofibrations is again a cofibration (i.e., if $f_1: A_1 \rightarrow X_1$ and $f_2: A_2 \rightarrow X_2$ are cofibrations, then $f_1 \times f_2: A_1 \times A_2 \rightarrow X_1 \times X_2$ is a cofibration);
- (iii) A cofibration with non-empty domain is surjective if and only if it is a homeomorphism.

Exercise 19. Let (X, A) be a NDR. Prove or disprove that the canonical projection $p: X \rightarrow X/A$ is a homotopy equivalence if A is contractible.

Exercise 20. Consider closed subspaces A and B of X . Suppose that the inclusions $A \rightarrow X$, $B \rightarrow X$, and $A \cap B \rightarrow X$ are cofibrations.

Prove or disprove that the inclusion $A \cup B \rightarrow X$ is a cofibration.

Exercise 21. Consider the pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow i & & \downarrow \tilde{i} \\ X & \xrightarrow{\tilde{f}} & Y. \end{array}$$

Suppose that i is the inclusion of a strong neighborhood deformation retraction (X, A) .

Prove or disprove that $\bar{i}$ is the inclusion of a strong neighborhood deformation retraction (Y, B) .

Exercise 22. Let $i: A \rightarrow X$ be a cofibration. Let $f: (X, A) \rightarrow (Y, B)$ be a map which is as a map of pairs homotopic to a map $g: (X, A) \rightarrow (Y, B)$ satisfying $g(X) \subseteq A$.

Prove or disprove that f is homotopic relative A to a map $g: (X, A) \rightarrow (Y, B)$ satisfying $g(X) \subseteq A$.

Exercise 23. Let X and Y be well-pointed spaces. Prove or disprove that their smash product $X \wedge Y$ is well-pointed.

Exercise 24. Consider the commutative diagram

$$\begin{array}{ccccccc} X_0 & \xrightarrow{i_0} & X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & \cdots \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \\ Y_0 & \xrightarrow{j_0} & Y_1 & \xrightarrow{j_1} & Y_2 & \xrightarrow{j_2} & \cdots \end{array}$$

Suppose that each horizontal arrow is a cofibration and each vertical arrow is a homotopy equivalence.

Prove or disprove that the induced map

$$\operatorname{colim}_{n \rightarrow \infty} f_n: \operatorname{colim}_{n \rightarrow \infty} X_n \rightarrow \operatorname{colim}_{n \rightarrow \infty} Y_n$$

is a homotopy equivalence.

Exercise 25. Consider the subspace $X = I \times \{0\} \cup \{0\} \times I$ of $\mathbb{R}^2$. Let $f: X \rightarrow I$ be the map sending (x, y) to x .

Prove or disprove that f is a fibration.

Exercise 26. Let F be a finite set equipped with the discrete topology. Put $X = \prod_{n \in \mathbb{Z}} F$ with respect to the classical product topology. Let X_d be the set X equipped with the discrete topology and let $p: X_d \rightarrow X$ be the map given by the identity. Prove:

- (i) X is a compact Hausdorff space;
- (ii) X is a compactly generated space;
- (iii) X is totally disconnected, i.e., each of its components contains only one point;
- (iv) Each path components of X contains only one point;
- (v) p is continuous and bijective;
- (vi) p is a not homeomorphism;
- (vii) p is a fibration.

Exercise 27. Prove or disprove:

- (i) The composite of two fibrations is again a fibration;
- (ii) The product of two fibrations is again a fibration;
- (iii) A fibration with non-empty domain and locally contractible codomain is injective if and only if it is a homeomorphism.

Exercise 28. Let $p: E \rightarrow S^1$ be a fibration. Let $F_s = p^{-1}(s)$ be the fiber of s . Let the homotopy equivalence $f: F_s \rightarrow F_s$ be a representative of the fiber transport associated to a generator of $\pi_1(S^1, s)$.

Prove that E is homotopy equivalent to the mapping torus T_f of f .

Exercise 29. Let G be a path connected topological group and $p: E \rightarrow B$ be a principal G -bundle. Prove or disprove that the fiber transport associated to p regarded as a fibration is trivial.

Exercise 30. Let $H: \mathbb{Z} \rightarrow \pi_3(S^2)$ be the isomorphism sending $1 \in \mathbb{Z}$ to the class $[p]$ of the Hopf fibration $p: S^3 \rightarrow S^2$. Let $f: S^3 \rightarrow S^3$ and $g: S^2 \rightarrow S^2$ be maps. Prove: $H(\deg(f)) = [p \circ f]$ and $H(\deg(g)^2) = [g \circ p]$.

Exercise 31. Decide for which $d \in \mathbb{Z}^{\geq 1}$ any principal G -bundle over any d -dimensional CW -complex is trivial, where G is $\mathbb{Z}$ with the discrete topology, S^1 , or S^3 with the multiplication coming from the embedding $S^3 \subseteq \mathbb{H}$ into the field of quaternions.

Exercise 32. Let $p: E \rightarrow B$ be a fibration over a path connected space B . Let $F = p^{-1}(b)$ for some $b \in B$. Recall that a space X is called aspherical if it is path connected and $\pi_n(X, x)$ vanishes for all base points $x \in X$ and $n \geq 2$. Prove or disprove:

- (i) If F and B are aspherical, then E is aspherical;
- (ii) If F and E are aspherical, then B is aspherical;
- (iii) If E and B are aspherical, then F is aspherical.

Exercise 33. Compute for $n \geq 2$ and $k \geq 1$:

- (i) $\pi_1(S^{n-1} \times \mathrm{SO}(n) \times \mathbb{R}\mathbb{P}^n \times \mathbb{C}\mathbb{P}^n)$;
- (ii) $\pi_k(T^n \times \mathbb{R}\mathbb{P}^\infty \times \mathbb{C}\mathbb{P}^\infty)$;
- (iii) $\pi_2(S^n \vee \mathbb{C}\mathbb{P}^n)$.

Exercise 34. Prove or disprove that the obvious map $\pi_3(D^2, S^1) \rightarrow \pi_3(D^2/S^1)$ is surjective.

Exercise 35. Consider $m, n \in \mathbb{Z}^{\geq -1}$. Let X and Y be spaces such that X is m connected and Y is n -connected, where (-1) -connected means that there is no condition. The join $X * Y$ of X and Y is defined by the pushout

$$\begin{array}{ccc} X \times Y & \longrightarrow & X \times \mathrm{cone}(Y) \\ \downarrow & & \downarrow \\ \mathrm{cone}(X) \times Y & \longrightarrow & X * Y. \end{array}$$

Prove that the join $X * Y$ is $(m + n + 2)$ -connected.

Exercise 36. Prove or disprove:

- (i) For every simply connected topological group G we have $\pi_1(\Omega BG) = \{1\}$;
- (ii) If G is a topological group, then $\pi_1(G)$ is abelian;
- (iii) If G is a compact connected Lie group and the universal principal G -bundle $p: EG \rightarrow BG$ has a section $s: BG \rightarrow EG$, then G is the trivial group.

Exercise 37. Let ξ be an n -dimensional vector bundle over the space B . For $l \in \mathbb{Z}^{\geq 0}$ an l -framing of ξ is a bundle isomorphism $(\mathrm{id}_B, \bar{u}): \mathbb{R}^{n+l} \xrightarrow{\cong} \xi \oplus \mathbb{R}^l$ over B . We call an l_0 -framing $(\mathrm{id}_B, \bar{u}_0): \mathbb{R}^{n+l_0} \xrightarrow{\cong} \xi \oplus \mathbb{R}^{l_0}$ and an l_1 -framing $(\mathrm{id}_B, \bar{u}_1): \mathbb{R}^{n+l_1} \xrightarrow{\cong} \xi \oplus \mathbb{R}^{l_1}$ equivalent if there exists $l \in \mathbb{Z}^{\geq 0}$ with $l \geq l_0, l_1$ such that for $i = 0, 1$ the two bundle isomorphisms over B

$$\mathbb{R}^{n+l} = \mathbb{R}^{n+l_i} \oplus \mathbb{R}^{l-l_i} \xrightarrow{(\mathrm{id}_B, \bar{u}_i) \oplus \mathrm{id}_{\mathbb{R}^{l-l_i}}} \xi \oplus \mathbb{R}^{l_i} \oplus \mathbb{R}^{l-l_i} = \xi \oplus \mathbb{R}^l$$

are homotopic through bundle isomorphisms over B . A stable framing on ξ is an equivalence class of l -framings.

- (i) Prove that the group $[B, \mathrm{SO}]$ acts transitively and freely on the set of stable framing of ξ if there exists a stable framing on ξ ;
- (ii) Show that the tangent bundle TS^2 has precisely one stable framing;
- (iii) Show that the tangent bundle TS^1 has precisely two stable framings;

- (iv) Construct explicit representatives for these stable framings on TS^2 and TS^1 .

Exercise 38. Prove that $\pi_0^s \cong \mathbb{Z}$ and that there is a surjection $\mathbb{Z} \rightarrow \pi_1^s$.

Exercise 39. Construct a natural isomorphism

$$\pi_n^s(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} H_n(X; \mathbb{Q})$$

for any space X using the fact that π_n^s is finite for every $n \in \mathbb{Z}^{\geq 0}$.

Exercise 40. Consider $n \geq 2$ and $X = S^1 \vee S^n$. Show that the $\mathbb{Z}[\pi_1(X)]$ -module $\pi_n(X)$ is free of rank 1.

Exercise 41. Decide which of the following spaces are Eilenberg-MacLane spaces of type (G, n) . If the answer is yes, specify the values for G and n :

- (i) S^d for $d \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$;
- (ii) $\mathbb{RP}^d$ for $d \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$;
- (iii) $\mathbb{CP}^d$ for $d \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$;
- (iv) $S^1 \vee S^1$;
- (v) T^d for $d \in \mathbb{Z}^{\geq 1}$.
- (vi) A simply connected 4-manifold.

Exercise 42. Let X be an Eilenberg-MacLane space of type (G, n) for $n \geq 2$. Prove or disprove that there is a CW -approximation $K(G, (n-1)) \rightarrow \Omega(X, x)$ for every $x \in X$.

Exercise 43. (i) Find simply connected pointed spaces X and Y such that the inclusion $X \vee Y \rightarrow X \times Y$ is not a weak homotopy equivalence;
(ii) Let $\mathbf{E}$ and $\mathbf{F}$ be spectra. Show that we get well-defined spectra $\mathbf{E} \vee \mathbf{F}$ and $\mathbf{E} \times \mathbf{F}$ satisfying $(\mathbf{E} \vee \mathbf{F})_n = E(n) \vee F(n)$ and $(\mathbf{E} \times \mathbf{F})_n = E(n) \times F(n)$ for $n \in \mathbb{Z}$, and that there is an obvious map of spectra $\mathbf{i}: \mathbf{E} \vee \mathbf{F} \rightarrow \mathbf{E} \times \mathbf{F}$.
Prove or disprove that $\mathbf{i}$ is a weak homotopy equivalence of spectra.

Exercise 44. Define the n th homology of a spectrum $\mathbf{E}$ for $n \in \mathbb{Z}$ by

$$H_n(\mathbf{E}) := \operatorname{colim}_{k \rightarrow \infty} H_{n+k}(E(k))$$

where the k -th structure map is the composite

$$\begin{aligned} H_{n+k}(E(k)) &\xrightarrow{\sigma_{n+k}(E(k))} H_{n+k+1}(S^1 \wedge E(k)) \\ &\xrightarrow{H_{n+k+1}(\text{flip})} H_{n+k+1}(E(k) \wedge S^1) \xrightarrow{H_{n+k+1}(\sigma(k))} H_{n+k+1}(E(k+1)). \end{aligned}$$

of the homological suspension isomorphism $\sigma_{n+k}(E(k))$, the map induced by the flip map flip and the homomorphism induced by the structure map $\sigma(k)$.

Decide whether for any abelian group G there is a spectrum $\mathbf{M}(G)$ such that $H_0(\mathbf{M}(G)) \cong G$ holds and $H_n(\mathbf{M}(G))$ vanishes for $n \neq 0$.

34. EXERCISES PART II

Exercise 1. Let M be a simply connected closed 4-manifold whose Euler characteristic $\chi(M)$ is 2. Prove or disprove that M is homotopy equivalent to S^4 .

Exercise 2. Let X and Y be CW -complexes and ξ and η vector bundles over X and Y . Prove that there are pointed homeomorphisms

$$\begin{aligned} \operatorname{Th}(\xi \times \eta) &\xrightarrow{\cong} \operatorname{Th}(\xi) \wedge \operatorname{Th}(\eta); \\ \operatorname{Th}(\xi \oplus \mathbb{R}^k) &\xrightarrow{\cong} S^k \wedge \operatorname{Th}(\xi). \end{aligned}$$

Exercise 3. Compute $\pi_0^s(X)$ for a connected CW -complex X .

Exercise 4. Prove:

- (i) There exists a self-homotopy equivalence $f: \mathbb{CP}^\infty \rightarrow \mathbb{CP}^\infty$ which is not homotopy to the identity.
- (ii) There exists a fibration $\mathbb{CP}^\infty \rightarrow E \rightarrow S^1$ such that E is homotopy equivalent to the mapping torus of f ;
- (iii) We have

$$\pi_n(T_f) = \begin{cases} \mathbb{Z} & \text{if } n = 1, 2; \\ \{0\} & \text{otherwise;} \end{cases}$$

- (iv) The mapping torus T_f is not homotopy equivalent to a product of Eilenberg-MacLane spaces.

Exercise 5. Show that we obtain a transformation of homology theories with values in $\mathbb{F}_2$ -modules $T_*: \mathcal{N}_* \rightarrow H_*(-; \mathbb{F}_2)$ by assigning to an element $[f: (M, \partial M) \rightarrow (X, A)] \in \mathcal{N}_n(X, A)$ the image of the fundamental class $[M, \partial M] \in H_n(M, \partial M; \mathbb{F}_2)$ under the homomorphism $H_n(M, \partial M; \mathbb{F}_2) \rightarrow H_n(X, A; \mathbb{F}_2)$ induced by f . Show that $T_n(X)$ is bijective for any 2-dimensional CW-complex X and any $n \in \{0, 1, 2\}$.

Exercise 6. Show that we obtain for $n \in \mathbb{Z}^{\geq 0}$ a surjective homomorphism $\mathcal{N}_{2n} \rightarrow \mathbb{Z}/2$ by sending an element $[M]$ to its Euler characteristic $\chi(M)$ modulo 2 and a surjective homomorphism $\Omega_{4n} \rightarrow \mathbb{Z}$ by sending an element $[M]$ to its signature $\text{sign}(M)$.

Exercise 7. Compute the topological K -theory $K^*(\mathbb{CP}^d)$ for $d \in \mathbb{Z}^{\geq 1}$.

Exercise 8. Let M be a connected closed $(n-1)$ -dimensional smooth submanifold of $\mathbb{R}^n$ for $n \in \mathbb{Z}^{\geq 1}$. Prove or disprove that its normal bundle $\nu(M \subseteq \mathbb{R}^n)$ is not trivial if and only if $H_n(M; \mathbb{Z})$ vanishes.

Exercise 9. Let (P) be a property of $\mathbb{Z}$ -modules. We say that a connected CW-complex X has property (P) if $H_n(X)$ has (P) for $n \in \mathbb{Z}^{\geq 1}$. We call property (P) compatible with products if for two connected finite CW-complexes X and Y all three spaces X , Y , and $X \times Y$ have property (P) if two of them have property (P).

Decide which of the following properties (P) is compatible with products:

- (i) The $\mathbb{Z}$ -module is trivial;
- (ii) The $\mathbb{Z}$ -module is finite;
- (iii) The $\mathbb{Z}$ -module is finitely generated;
- (iv) The $\mathbb{Z}$ -module is finitely generated free.

Exercise 10. Let M be a closed smooth manifold of dimension d . Let $\{U_1, U_2, \dots, U_n\}$ be a finite set of open subsets U_i of M such that every U_i is diffeomorphic to $\mathbb{R}^d$.

Construct an injective smooth map

$$f: M \rightarrow \mathbb{R}^{dn}$$

whose differential $T_x f$ is injective for every $x \in M$.

Exercise 11. Let μ be an n -dimensional system of vector bundles over the CW-complex X . Let μ' be the $n+1$ -dimensional system of vector bundles over X obtained from μ whose vector bundle in degree k is $\xi_k \oplus \underline{\mathbb{R}}$ if ξ_k is the vector bundle in degree k of μ , and whose structure maps are the obvious ones.

Give and prove a formula how to compute $\Omega_*(\mu')$ from $\Omega_*(\mu)$.

Exercise 12. Is there a fibration $F \rightarrow S^4 \rightarrow B$ for which F and B are closed connected orientable manifolds of dimension ≥ 1 ?

Exercise 13. Let X be a CW -complex which is of finite type, i.e., each i -skeleton is finite. Suppose that $\mathcal{H}_*$ is a homology theory with values in $\mathbb{Z}$ -modules which satisfies the disjoint union axiom and $\mathcal{H}_m(\{\bullet\}) = 0$ for $m < 0$. Suppose that $\mathcal{H}_m(\{\bullet\})$ is finitely generated for all $m \in \mathbb{Z}^{\geq 0}$.

Prove or disprove that $\mathcal{H}_n(X)$ is finitely generated for every $n \in \mathbb{Z}$ and vanishes for $n < 0$.

Exercise 14. Let $f: X \rightarrow Y$ be a weak homotopy equivalence of compactly generated Hausdorff spaces.

Prove or disprove that the induced map $\Omega_n(f): \Omega_n(X) \rightarrow \Omega_n(Y)$ is bijective for all $n \in \mathbb{Z}$.

Exercise 15. Prove or disprove:

- (i) Let $\mathcal{H}_*$ be homology theory with values in R -modules. Let $f: S^1 \rightarrow S^1$ be the map sending z to z^d for $d \in \mathbb{Z}$. Then the induced map $\mathcal{H}_n(f): \mathcal{H}_n(S^1, \{1\}) \rightarrow \mathcal{H}_n(S^1, \{1\})$ can be identified with the map $\mathcal{H}_{n-1}(\{\bullet\}) \rightarrow \mathcal{H}_{n-1}(\{\bullet\})$ given by multiplication with d ;
- (ii) We have:

$$\mathcal{N}_n(\mathbb{RP}^2) \cong_{\mathbb{F}_2} \mathcal{N}_n(\{\bullet\}) \oplus \mathcal{N}_{n-1}(\{\bullet\}) \oplus \mathcal{N}_{n-2}(\{\bullet\}).$$

Exercise 16. Let G be a compact Lie group. Show that its tangent bundle is trivial. Describe an (interesting) construction which assigns to G an element in the stable stem π_n^s for $n = \dim(G)$.

Exercise 17. Let $\mathcal{H}_*$ be a homology theory with values in R -modules satisfying the disjoint union axiom. Let X be a 2-dimensional CW -complex.

Prove or disprove that for every $n \in \mathbb{Z}$ there is a filtration

$$\{0\} \subseteq A \subseteq B \subseteq \mathcal{H}_n(X)$$

satisfying

$$\begin{aligned} A &\cong_R H_0(X; \mathcal{H}_{n-1}(\{\bullet\})); \\ B/A &\cong_R H_1(X; \mathcal{H}_{n-1}(\{\bullet\})); \\ \mathcal{H}_2(X)/B &\cong_R H_2(X; \mathcal{H}_{n-2}(\{\bullet\})). \end{aligned}$$

Exercise 18. Let X be a connected finite CW -complex with $\pi_1(X) \cong \mathbb{Z}/3$ whose universal covering is homeomorphic to S^3 .

- (i) Compute $H_n(X, \mathbb{Z})$ and $H_n(X; \mathbb{F}_2)$ for $n \in \mathbb{Z}^{\geq 0}$;
- (ii) Compute $K_n(X)$ for $n \in \mathbb{Z}$;
- (iii) Compute $\Omega_n(X)$ for $n \in \{0, 1, 2, 3, 4, 5\}$.

Exercise 19. Consider a homology theory $\mathcal{H}_*$ such that $\mathcal{H}_q(\{\bullet\})$ is finitely generated free for $q \in \mathbb{Z}$ and a finite CW -complex X with $H_p(X)$ finitely generated free for every $p \geq 0$

Prove or disprove that the Atiyah-Hirzebruch spectral sequence strongly collapses and yields isomorphisms

$$\mathcal{H}_n(X) \cong \bigoplus_{p+q=n} H_p(X) \otimes_{\mathbb{Z}} \mathcal{H}_q(\{\bullet\}).$$

Exercise 20. Let X be a CW -complex such that $H_n(X; \mathbb{F}_2)$ vanishes for $n \in \mathbb{Z}^{\geq 1}$. Prove or disprove that $\mathcal{N}_n(X)$ and $H_n(X; \mathbb{F}_2)$ are isomorphic $\mathbb{F}_2$ -modules.

Exercise 21. Let $\mathbb{Z} \rtimes \mathbb{Z}$ be the semidirect product with respect to the group homomorphism $\mathbb{Z} \rightarrow \text{aut}(\mathbb{Z})$ sending $m \in \mathbb{Z}$ to the automorphism $(-1)^m \cdot \text{id}_{\mathbb{Z}}$.

(Note that $\mathbb{Z} \rtimes \mathbb{Z}$ has the presentation $\langle t, s \mid sts^{-1} = t^{-1} \rangle$.) Let K be the Klein bottle K which is the quotient of $\mathbb{R}^2$ by the free $\mathbb{Z} \rtimes \mathbb{Z}$ -action for which t and s act by sending (r_1, r_2) to $(r_1 + 1)$ and $(-r_1, r_2 + 1)$ respectively.

- (i) Show that K is a closed 2-dimensional manifold;
- (ii) Compute $\pi_1(K)$, $H_n(K; \mathbb{Z})$, $H^n(K; \mathbb{Z})$, $H_n(K; \mathbb{F}_2)$, and $H^n(K; \mathbb{F}_2)$ for $n \geq 0$;
- (iii) Compute the first Stiefel Whitney class $w_1(M) \in H^1(K; \mathbb{F}_2)$;
- (iv) Decide whether K is orientable and determine its orientation covering.

Exercise 22. Prove Lemma 2.13 of the script saying that the adjunction homomorphism

$$\text{ad}: \text{hom}_{R\mathcal{D}}(M \otimes_{RC} B, N) \rightarrow \text{hom}_{RC}(M, \text{hom}_{R\mathcal{D}}(B, N))$$

is bijective and natural.

Exercise 23. Let X be a CW -complex such that $H_n(X; \mathbb{Z}) \cong_{\mathbb{Z}} H_n(\{\bullet\}; \mathbb{Z})$ holds for $n \in \mathbb{Z}^{\geq 0}$. Prove or disprove that for any homology theory $\mathcal{H}_*$ with values in R -modules satisfying the disjoint union axiom the R -modules $\mathcal{H}_n(X)$ and $\mathcal{H}_n(\{\bullet\})$ are isomorphic for $n \in \mathbb{Z}$.

Exercise 24. Consider $d \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$. Compute $\mathbb{Q} \otimes_{\mathbb{Z}} K_n(\mathbb{RP}^d)$ for $n \in \mathbb{Z}$ for the complex topological K -homology K_* .

Exercise 25. Let A be a $\mathbb{Z}$ -module. Let $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ be a nested sequence of $\mathbb{Z}$ -submodules of A such that $A = \bigcup_{i \in \mathbb{I}} A_i$ holds.

Prove or disprove that for every $n \in \mathbb{Z}^{\geq 0}$ we get

$$H_n(K(A, 1); \mathbb{Z}) = \text{colim}_{i \rightarrow \infty} H_n(K(A_i, 1); \mathbb{Z}).$$

Exercise 26. Let $f: X \rightarrow Y$ be a map of CW -complexes. Consider $d \in \mathbb{Z}^{\geq 1}$ and a homology theory $\mathcal{H}_*$ with values in R -modules satisfying the disjoint union axiom such that $\mathcal{H}_i(\{\bullet\}) = \{0\}$ holds for $i \leq -1$. Suppose that $H_i(f; \mathbb{Z}): H_i(X; \mathbb{Z}) \rightarrow H_i(Y; \mathbb{Z})$ is bijective for $i < d$ and surjective for $i = d$.

Prove or disprove that $\mathcal{H}_i(f; \mathbb{Z}): H_i(X; \mathbb{Z}) \rightarrow H_i(Y; \mathbb{Z})$ is bijective for $i < d$ and surjective for $i = d$.

Exercise 27. Let X be a finite CW -complex. Let $\mathcal{H}_*$ homology theory with values in $\mathbb{Q}$ -modules satisfying the disjoint union axiom such that $\mathcal{H}_i(\{\bullet\}) \neq \{0\}$ holds only for finitely many $i \in \mathbb{Z}$ and $\mathcal{H}_i(\{\bullet\})$ is finitely generated for every $i \in \mathbb{Z}$.

Prove or disprove that $\mathcal{H}_n(X)$ is finitely generated for all $i \in \mathbb{Z}$ and we get for the Euler characteristic

$$\chi(X) = \sum_{n \in \mathbb{Z}} (-1)^n \cdot \dim_{\mathbb{Q}}(\mathcal{H}_n(X)).$$

Exercise 28. Let $\mathcal{H}_*$ be any homology theory with values in $\mathbb{Z}$ -modules satisfying the disjoint union axiom such that $\mathcal{H}_i(\{\bullet\}) = \{0\}$ holds for $i \leq -1$. Let $f: X \rightarrow Y$ be a map of connected finite CW -complexes. Suppose that $\mathcal{H}_i(f): \mathcal{H}_i(X) \rightarrow \mathcal{H}_i(Y)$ is bijective for all $i \in \mathbb{Z}$.

Prove or disprove that $H_i(f; \mathbb{Z}): H_i(X; \mathbb{Z}) \rightarrow H_i(Y; \mathbb{Z})$ is bijective for all $i \in \mathbb{Z}^{\geq 0}$.

Exercise 29. Consider $k, l, n \in \mathbb{Z}^{\geq 0}$ for which there exists a fibration $S^k \rightarrow S^n \rightarrow S^l$. Prove or disprove that then $l = k + 1$ and $n = k + l$ holds.

Exercise 30. Consider fibration of closed connected smooth manifolds $F \rightarrow E \rightarrow B$. Prove or disprove:

- (i) We have $\dim(E) = \dim(F) + \dim(B)$;
- (ii) If E is orientable, then B and F are orientable;
- (iii) If B and F are orientable, then E is orientable;

- (iv) If E is the total space of principal S^1 -bundle $S^1 \rightarrow E \rightarrow X$, then $\chi(B)$ or $\chi(F)$ vanishes.

Exercise 31. Let $\mathcal{H}_*$ be a homology theory satisfying the disjoint union axiom. Consider a pullback of fibrations with CW -complexes as base space

$$\begin{array}{ccc} E_0 & \xrightarrow{\bar{f}} & E_1 \\ p_0 \downarrow & & \downarrow p_1 \\ B_0 & \xrightarrow{f} & B_1. \end{array}$$

Suppose that for any $n \in \mathbb{Z}^{\geq 0}$, any $b_1 \in B_1$, and any loop w in B_1 at b_1 the map $\mathcal{H}_n(p_1^{-1}(b_k)) \rightarrow \mathcal{H}_n(p_1^{-1}(b_k))$ induced by the element $\tau_1(w_1) \in [p_1^{-1}(b), p_1^{-1}(b)]$ given by the fiber transport is the identity and that $H_n(f): H_n(B_0) \rightarrow H_n(B_1)$ is bijective.

Prove or disprove that the map $\mathcal{H}_n(\bar{f}): \mathcal{H}_n(E_0) \rightarrow \mathcal{H}_n(E_1)$ is bijective for every $n \in \mathbb{Z}$.

Exercise 32. Let $F: E \rightarrow B$ be a fibration such that the fiber transport is trivial. Let R be a principal ideal domain. Suppose $H_i(F; R)$ and $H_i(B; R)$ are finitely generated for all $i \in \mathbb{Z}^{\geq}$ and non-trivial only for finitely many values of i .

- (i) Show that $H_i(E; R)$ is finitely generated for all $i \in \mathbb{Z}^{\geq}$ and non-trivial only for finitely many values of i ;
- (ii) Show for the Betti numbers, which are define by $b_i(X; R) := \text{rk}_R(H_i(X; R))$,

$$\sum_{i \leq 0} b_i(E; R) \leq \left(\sum_{j \geq 0} b_j(F; R) \right) \cdot \left(\sum_{k \geq 0} b_k(B; R) \right);$$

- (iii) Suppose that the inequality above is an equality and R is a field. Prove or disprove

$$H_n(E; R) = \bigoplus_{i \geq 0} H_i(F; R) \otimes_R H_{n-i}(B; R).$$

Exercise 33. Let F be a field and G be a finite group. Prove or disprove that the following assertions are equivalent:

- (i) FG is semisimple;
- (ii) The FG -module F whose underlying F -module is F and on which G acts trivial is projective;
- (iii) The order $|G|$ of G is invertible in F .

Exercise 34. Let $p: E \rightarrow B$ be a principal G -bundle for the discrete finite group G . Prove or disprove that $H^n(E; \mathbb{Q})^G$ is isomorphic to $H^n(B; \mathbb{Q})$ for $n \in \mathbb{Z}^{\geq 0}$.

Exercise 35. Let $F \rightarrow E \rightarrow B$ be a fibration where F and B are path connected closed nonorientable 2-manifolds. Suppose that the fiber transport is trivial. Prove or disprove:

$$H_3(E) \cong_{\mathbb{Z}} \text{Tors}_{\mathbb{Z}}^{\mathbb{Z}}(H_1(B), H_1(F)).$$

Exercise 36. Let X be a space. We call a class $a \in H_n(X, \mathbb{Z})$ *realisable* if there exists a map $f: M \rightarrow X$ with M a closed oriented n -manifold such that $a = f_*[M]$. Prove:

- (i) The homomorphisms

$$\Omega_n(X) \rightarrow H_n(X, \mathbb{Z}), \quad (f: M \rightarrow X) \mapsto f_*[M]$$

extend to a transformation of homology theories;

- (ii) Every class in degrees $n \leq 5$ is realisable;

- (iii) For every n , every class $a \in H_n(X)$ is rationally realisable, i.e., there exists $N > 0$ such that Na is realisable.

Exercise 37. Compute the first Chern class of the principal S^1 -bundle over S^2 given by the Hopf fibration.

Exercise 38. The infinite dihedral group D_∞ is defined by the presentation $\langle s, t \mid sts = t^{-1}, s^2 = 1 \rangle$.

- (i) Show that there is a fibration $S^1 \rightarrow BD_\infty \rightarrow \mathbb{RP}^\infty$;
- (ii) Compute $H_n(BD_\infty; \mathbb{Z})$ for $n \in \mathbb{Z}^{\geq 0}$.

Exercise 39. Let M be a closed connected 3-manifold whose fundamental group is perfect, i.e. $\pi = [\pi, \pi]$. Consider a prime p . Let $f: M \rightarrow S^3$ be a map of a degree which is prime to p . Consider a pullback

$$\begin{array}{ccc} \overline{E} & \xrightarrow{\bar{f}} & E \\ \downarrow & & \downarrow q \\ M & \xrightarrow{f} & S^3 \end{array}$$

where q is a fibration. Let $\mathcal{H}_*$ be a homology theory with values in $\mathbb{F}_p$ -modules satisfying the disjoint union axiom.

Prove or disprove that $\mathcal{H}_n(\bar{f}): \mathcal{H}_n(\overline{E}) \rightarrow \mathcal{H}_n(E)$ is an isomorphism for all $n \in \mathbb{Z}$.

Exercise 40. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration of path connected spaces with a CW-complex B as basis. Suppose that the action of $\pi_1(B)$ on $H_q(F)$ given by the fiber transport is trivial. Denote by $E_r^{p,q}$ be the Leray-Serre spectral sequence for the fibration p . Prove or disprove:

- (i) The map $H^n(i): H^n(E) \rightarrow H^n(F)$ factorizes as the composite $H^n(E) \xrightarrow{\alpha} E_\infty^{0,n} \xrightarrow{\beta} H^n(F)$ for an epimorphism α and a monomorphism β ;
- (ii) The map $H^n(p): H^n(B) \rightarrow H^n(E)$ factorizes as the composite $H^n(B) \xrightarrow{\alpha} E_\infty^{n,0} \xrightarrow{\beta} H^n(E)$ for an epimorphism α and a monomorphism β ;

Exercise 41. Let R be a torsionfree commutative ring. Prove or disprove that the divided power R -algebra $R\left[y, \frac{y^2}{2!}, \frac{y^3}{3!}, \frac{y^4}{4!}, \dots\right]$ and the R -algebra $R[x]$ for $|x|$ and $|y|$ even are isomorphic as graded R -algebras if and only if $|x| = |y|$ and $\mathbb{Q} \subseteq R$ hold.

Exercise 42. Let R be a commutative ring. Prove or disprove:

- (i) The full subcategory of $R\text{-Mod}$ given by finitely generated R -modules is a Serre class, if and only if R is Noetherian;
- (ii) The full subcategory of $R\text{-Mod}$ given by R -modules whose underlying set is finite is a Serre class;
- (iii) The full subcategory of $R\text{-Mod}$ given projective R -modules is a Serre class if and only if R is a semisimple, i.e., every R -module is projective.
- (iv) The full subcategory of $R\text{-Mod}$ given by free R -modules is a Serre class if and only if R is a field.

Exercise 43. (i) Prove that there is an isomorphism $H^*(BU(n), \mathbb{Z}) \simeq \mathbb{Z}[c_1, \dots, c_n]$ of graded rings with generators in degrees $|c_i| = 2i$. *Hint: You may use the fibration $S^{2n-1} \rightarrow BU(n-1) \rightarrow BU(n)$ without proof.*

- (ii) Consider a complex rank n vector bundle $\zeta: E \rightarrow B$ over a CW-complex B and denote its classifying map by $f: B \rightarrow BU(n)$. We can define its k th Chern class by $c_k(\zeta) = f^*c_k \in H^{2k}(B, \mathbb{Z})$ for $k \leq n$ and $c_k(\zeta) = 0$ for $k > n$. Prove the following:

- (a) $c_0(\zeta) = 1$.
- (b) $c_k(\zeta) = 0$ for $k \geq 1$ if ζ is trivial.
- (c) Compute $c_k(\gamma_n)$ where γ_n is the universal rank n bundle over $BU(n)$.
- (d) Show that this definition of $c_1(\zeta)$ agrees with the one from the lecture for $n = 1$.
- (e) Explain how $c_n(\zeta)$ can be identified with the Thom class of the associated sphere bundle $S(\zeta)$, sometimes also called its Euler class.

Exercise 44. (i) Show that the map

$$\text{edge}_{n,0}(S^n) \times \Omega_n(\text{pr}): \Omega_n(S^n) \rightarrow \Omega_n(\{\bullet\}) \times H_n(S^n; \mathbb{Z})$$

is bijective, where $\text{pr}: S^n \rightarrow \{\bullet\}$ is the projection;

- (ii) Show that $\text{edge}_{n,0}(X)$ sends the bordism class of $f: M \rightarrow X$ to the image of the fundamental class $[M]$ under the map $H_n(f; \mathbb{Z}): H_n(M; \mathbb{Z}) \rightarrow H_n(X; \mathbb{Z})$, provided this claim holds for $X = S^n$, without using exercise 36.

Exercise 45. Let X be a connected finite CW -complex with finite fundamental group. Prove or disprove that $\pi_n(X)$ is finitely generated for all $n \geq 1$.

Exercise 46. Let A be a finitely generated abelian group and $n \in \mathbb{Z}^{\geq 1}$. Prove or disprove:

- (i) Suppose that A has an element of infinite order. Then there exists $d \in \mathbb{Z}^{\geq 2}$ such that $H_i(K(A, n); \mathbb{Q}) = \{0\}$ holds for every $i > d$ if and only if n is odd;
- (ii) $H_*(K(\mathbb{Z}/2, n); \mathbb{Z})$ is nontrivial in infinitely many degrees;
- (iii) We have $H_i(K(A, n); \mathbb{Q}) = \{0\}$ for every $i \geq 1$ if and only if A is finite;
- (iv) We have $H_i(K(A, n); \mathbb{Z}) = \{0\}$ for every $i \geq 1$ if and only if A is trivial;
- (v) $H_i(K(A, n))$ is finitely generated for all $i \in \mathbb{Z}^{\geq 0}$.

Exercise 47. Decide which of the following Serre classes in $\mathbb{Z}\text{-Mod}$ are Serre ideals: $\mathbb{Z}\text{-Tors}_p$, $\mathbb{Z}\text{-Tors}$, $\mathbb{Z}\text{-Mod}_{\text{fg}}$, $R\text{-Mod}_{\text{fin}}$.

Exercise 48. State all the results which are presented in the script about $\pi_n(S^k)$ and the stable stems π_n^s for $n \in \mathbb{Z}^{\geq 0}$ and $k \in \mathbb{Z}^{\geq 1}$.

REFERENCES

- [1] J. F. Adams. *Stable homotopy and generalised homology*. University of Chicago Press, Chicago, Ill., 1974. Chicago Lectures in Mathematics.
- [2] M. F. Atiyah and G. B. Segal. Equivariant K -theory and completion. *J. Differential Geometry*, 3:1–18, 1969.
- [3] T. Bröcker and K. Jänich. *Einführung in die Differentialtopologie*. Springer-Verlag, Berlin, 1973. Heidelberger Taschenbücher, Band 143.
- [4] T. Bröcker and T. tom Dieck. *Kobordismentheorie*. Springer-Verlag, Berlin, 1970.
- [5] T. Bröcker and T. tom Dieck. *Representations of compact Lie groups*. Springer-Verlag, New York, 1985.
- [6] M. Clapp. Duality and transfer for parametrized spectra. *Arch. Math. (Basel)*, 37(5):462–472, 1981.
- [7] A. Dold. Erzeugende der Thomschen Algebra $\mathfrak{N}$. *Math. Z.*, 65:25–35, 1956.
- [8] A. Dold. Relations between ordinary and extraordinary homology. *Colloq. alg. topology*, Aarhus 1962, 2-9, 1962.
- [9] R. Fritsch and R. Piccinini. CW -complexes and Euclidean spaces. In *Fourth Conference on Topology (Italian) (Sorrento, 1988)*, number 24 in *Rendiconti del Circolo Matematico di Palermo Serie II. Supplemento*, pages 79–95. Springer, 1990.
- [10] R. Fritsch and R. A. Piccinini. *Cellular structures in topology*. Cambridge University Press, Cambridge, 1990.
- [11] M. W. Hirsch. *Differential topology*. Springer-Verlag, New York, 1976. Graduate Texts in Mathematics, No. 33.

- [12] D. Husemoller. *Fibre bundles*. Springer-Verlag, New York, second edition, 1975. Graduate Texts in Mathematics, No. 20.
- [13] D. C. Isaksen. Stable stems. *Mem. Amer. Math. Soc.*, 262(1269):viii+159, 2019.
- [14] D. C. Isaksen, G. Wang, and Z. Xu. Stable homotopy groups of spheres: from dimension 0 to 90. *Publ. Math. Inst. Hautes Études Sci.*, 137:107–243, 2023.
- [15] M. Land. *Introduction to infinity-categories*. Compact Textbooks in Mathematics. Birkhäuser/Springer, Cham, 2021.
- [16] W. Lück. *Transformation groups and algebraic K-theory*, volume 1408 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989.
- [17] W. Lück. *Algebraic topology. Homology and manifolds*. Vieweg Studium: Aufbaukurs Mathematik. Vieweg, 2005.
- [18] W. Lück. Rational computations of the topological K -theory of classifying spaces of discrete groups. *J. Reine Angew. Math.*, 611:163–187, 2007.
- [19] W. Lück and T. Macko. *Surgery theory – foundations*, volume 362 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, 2024. With contributions by Diarmuid Crowley.
- [20] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999.
- [21] J. Milnor and J. D. Stasheff. *Characteristic classes*. Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.
- [22] H. Miyazaki. The paracompactness of CW -complexes. *Tôhoku Math. J.*, 4:309–313, 1952.
- [23] J. R. Munkres. *Topology: a first course*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.
- [24] J.-P. Serre. Groupes d’homotopie et classes de groupes abéliens. *Ann. of Math. (2)*, 58:258–294, 1953.
- [25] N. E. Steenrod. A convenient category of topological spaces. *Michigan Math. J.*, 14:133–152, 1967.
- [26] N. P. Strickland. The category of $cgwh$ spaces. Preprint, <https://ncatlab.org/nlab/files/StricklandCGHWSpaces.pdf>, 2009.
- [27] R. M. Switzer. *Algebraic topology—homotopy and homology*. Springer-Verlag, New York, 1975. Die Grundlehren der mathematischen Wissenschaften, Band 212.
- [28] R. M. Switzer. *Algebraic topology—homotopy and homology*. Classics in Mathematics. Springer-Verlag, Berlin, 2002. Reprint of the 1975 original [Springer, New York; MR0385836 (52 #6695)].
- [29] R. Thom. Quelques propriétés globales des variétés différentiables. *Comment. Math. Helv.*, 28:17–86, 1954.
- [30] H. Toda. *Composition methods in homotopy groups of spheres*. Annals of Mathematics Studies, No. 49. Princeton University Press, Princeton, N.J., 1962.
- [31] T. tom Dieck. *Algebraic topology*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008.
- [32] C. T. C. Wall. Determination of the cobordism ring. *Ann. of Math. (2)*, 72:292–311, 1960.
- [33] G. W. Whitehead. *Elements of homotopy theory*, volume 61 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1978.

last edited on 28.10.2025

last compiled on 2025-10-28 at 19:52:06.

name of tex-file: script`AlgTop

MATHEMATICIANS INSTITUT DER UNIVERSITÄT BONN, ENDENICHER ALLEE 60, 53115 BONN, GERMANY

Email address: wolfgang.lueck@him.uni-bonn.de

URL: <http://www.him.uni-bonn.de/lueck>