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L^2 -invariants: n years later

Mathematics – Monograph (English)

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Preface

This monograph can be thought of as part II or of a continuation of the book [45]. Since n **Comment 1 (by W.):** This number has to be specified at the very end. Could it be 28? years have passed and many new interesting developments have occurred and a lot of new results have been achieved during these years, the time is ripe to give an update.

Comment 2 (by W.): Needs to be completed.

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Chapter 1

Introduction

1.1 The Organization of the Book and a User's Guide

We have written the text in a way such that one can read small units, e.g., a single chapter, independently from the rest, concentrate on certain aspects, and extract easily and quickly specific information. We hopefully have found the right mixture between definitions, theorems, examples, and remarks so that reading the book is entertaining and illuminating. We have successfully used parts of this book, sometimes a single chapter, for seminars, reading courses, and advanced lecture courses.

Each section comes with a review where we briefly recollect the basics needed from [45].

Comment 3 (by W.): Needs to be completed.

1.1.1 Supplement

The book contains a number of exercises. They are not needed for the exposition of the book, but give some illuminating insight. Moreover, the reader may test whether she or he has understood the text or improve her or his understanding by trying to solve the exercises. Hints to the solutions of the exercises are given in Chapter 12.

If one wants to find a specific topic, the extensive index of the monograph can be used to find the right spot for a specific topic. The index contains an item “Theorem”, under which all theorems with their names appearing in the book are listed, and analogously there is an item “Conjecture”.

Comment 4 (by W.): Needs to be completed.

1.1.2 Prerequisites

1.2 Notations and Conventions

We have tried to keep the notation consistent with [45].

Here is a briefing on our main conventions and notations. Details are of course discussed in the text.

- Ring will mean (not necessarily commutative) associative ring with unit unless explicitly stated otherwise;
- Module means always left module unless explicitly stated otherwise;

- Group means discrete group unless explicitly stated otherwise;
- We will always work in the category of compactly generated spaces, compare [69] and [77, I.4]. In particular every space is automatically Hausdorff;
- We use the standard symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$, and $\mathbb{Q}_p$ for the integers, the rational numbers, the real numbers, the complex numbers, the p -adic numbers, and the p -adic rationals. Given a real number $r \in \mathbb{R}$ we write $\mathbb{R}_{\geq r} = \{s \in \mathbb{R} \mid s \geq r\}$. It is now selfexplanatory what the notions $\mathbb{R}_{>r}$, $\mathbb{R}_{\leq r}$, $\mathbb{R}_{<r}$, $\mathbb{Q}_{\geq r}$, $\mathbb{Q}_{>r}$, $\mathbb{Q}_{\leq r}$, $\mathbb{Q}_{<r}$, $\mathbb{Z}_{\geq r}$, $\mathbb{Z}_{>r}$, $\mathbb{Z}_{\leq r}$, and $\mathbb{Z}_{<r}$ mean. We also write $\mathbb{N} = \mathbb{Z}_{\geq 0}$ for the set $\{0, 1, 2, \dots\}$ of natural numbers;
- We denote certain groups by:

symbol	name
$\mathbb{Z}/n$	finite cyclic group of order n
S_n	symmetric group of permutations of the set $\{1, 2, \dots, n\}$
A_n	alternating group of even permutations of the set $\{1, 2, \dots, n\}$
D_∞	infinite dihedral group
D_{2n}	dihedral group of order $2n$

1.3 Acknowledgments

1.4 Notes

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Chapter 2

The Atiyah Conjecture

2.1 Introduction

There are various versions of the Atiyah Conjecture. They all make predictions on the possible values of the L^2 -Betti numbers $b_n^{(2)}(\tilde{X})$ of the universal covering $\tilde{X}$ of a compact Riemannian manifold or of a finite CW -complex X . If $\pi_1(X)$ is torsionfree, the Atiyah Conjecture says that all these L^2 -Betti numbers $b_n^{(2)}(\tilde{X})$ are integers. All these conjectures were triggered by a question of Atiyah whether all the L^2 -Betti numbers $b_n^{(2)}(\tilde{X})$ are rational numbers.

After a short review about L^2 -Betti numbers, we state all the variants of the Atiyah Conjectures and link them. We present some of the applications of these conjectures to illustrate their importance. Finally we give also a status report for which fundamental groups or groups these conjectures are known to be true.

Comment 5 (by W.): This is only a first draft of the introduction.

2.2 Review

2.2.1 The Group von Neumann algebra

Let G be a (discrete) group. Let $l^2(G)$ be the (complex) Hilbert space of square summable formal sums $\sum_{g \in G} \lambda_g \cdot g$ for $\lambda_g \in \mathbb{C}$. The *group von Neumann algebra* $\mathcal{N}(G)$ is the $\mathbb{C}$ -algebra $\mathcal{B}(l^2(G))$ of bounded G -equivariant operators $l^2(G) \rightarrow l^2(G)$, where we let G act by left multiplication. We will consider $\mathcal{N}(G)$ just as a ring with unit and ignore the topology on it. It becomes a ring with involution $*$: $\mathcal{N}(G) \rightarrow \mathcal{N}(G)$ in the sense of Definition 3.1 by taking the adjoint of a bounded G -equivariant operator $l^2(G) \rightarrow l^2(G)$. As a consequence, $l^2(G)$ becomes canonically a left $\mathcal{N}(G)$ -module.

The so-called *von Neumann trace*

$$(2.1) \quad \text{tr}_{\mathcal{N}(G)}: \mathcal{N}(G) \rightarrow \mathbb{C}$$

sends a bounded G -equivariant operator $f: l^2(G) \rightarrow l^2(G)$ to $\langle f(e), e \rangle_{l^2(G)}$ for $e \in G$ the unit element. This extends to (n, n) -matrices by the usual formula

$$(2.2) \quad \mathrm{tr}_{\mathcal{N}(G)}: M_{n,n}(\mathcal{N}(G)) \rightarrow \mathbb{C}, \quad A \mapsto \sum_{i=1}^n \mathrm{tr}_{\mathcal{N}(G)}(a_{i,i}).$$

Given a finitely generated projective $\mathcal{N}(G)$ -module P , define its *Hattori-Stallings rank* or its *von Neumann dimension* to be

$$(2.3) \quad \dim_{\mathcal{N}(G)}(P) := \sum \mathrm{tr}_{\mathcal{N}(G)}(A) \in \mathbb{R}_{\geq 0},$$

where A is an element in $M_{n,n}(\mathcal{N}(G))$ with $A^2 = A$ such that the image of the $\mathcal{N}(G)$ -linear map $r_A: \mathcal{N}(G)^n \rightarrow \mathcal{N}(G)^n$ induced by right multiplication with A is $\mathcal{N}(G)$ -isomorphic to P . One easily checks that this definition is independent of the choice of n and A and takes values in $\mathbb{R}_{\geq 0}$.

Exercise 2.4 Show that definition (2.3) indeed is independent of the choice of n and A and takes values in $\mathbb{R}_{\geq 0}$.

Remark 2.5. The category of finitely generated projective $\mathcal{N}(G)$ -modules is equivalent to the category of finitely generated Hilbert $\mathcal{N}(G)$ -modules and under this identification the von Neumann dimension corresponds to the classical Murray-von Neumann dimension of finitely generated Hilbert $\mathcal{N}(G)$ -modules which is defined in operator theoretic terms, see [45, Theorem 6.24 on page 249].

2.2.2 Dimension Theory

Next we review how one can extend the dimension $\dim_{\mathcal{N}(G)}$ for finitely generated projective $\mathcal{N}(G)$ -modules of (2.3) to arbitrary $\mathcal{N}(G)$ -modules.

Definition 2.6 (Closure of a submodule). Let R be a ring. Consider an R -submodule M of the R -module N . Define the *closure* of M in N to be the R -submodule of N

$$\overline{M} = \{x \in N \mid f(x) = 0 \text{ for all } f \in N^* = \mathrm{hom}_R(N, R) \text{ satisfying } M \subset \ker(f)\}.$$

For an R -module M define the R -submodule $\mathbf{T}M$ and the quotient R -module $\mathbf{P}M$ by

$$\begin{aligned} \mathbf{T}M &:= \{x \in M \mid f(x) = 0 \text{ for all } f \in M^*\}; \\ \mathbf{P}M &:= M/\mathbf{T}M. \end{aligned}$$

Definition 2.7 (Extended dimension). We define for a $\mathcal{N}(G)$ -module M its *extended dimension*

$$\dim'_{\mathcal{N}(G)}(M) \in \mathbb{R}_{\geq 0} \amalg \{\infty\}$$

to be

$$\sup\{\dim_{\mathcal{N}(G)}(P) \mid P \subset M \text{ finitely generated projective submodule}\}.$$

(We will later drop the prime in $\dim'_{\mathcal{N}(G)}$, see Notation 2.9.)

Theorem 2.8 (Dimension function for arbitrary $\mathcal{N}(G)$ -modules) *Let G be a group.*

- (i) *The ring $\mathcal{N}(G)$ is semihereditary, i.e., any finitely generated submodule of a projective module is projective;*
- (ii) *For every finitely generated $\mathcal{N}(G)$ -module M there exists $n \in \mathbb{Z}_{\geq 0}$ and an $\mathcal{N}(G)$ -homomorphism $f: \mathcal{N}(G)^n \rightarrow \mathcal{N}(G)^n$ and an exact sequence of $\mathcal{N}(G)$ -modules*

$$0 \rightarrow \mathcal{N}(G)^n \xrightarrow{f^*f} \mathcal{N}(G)^n \rightarrow M \rightarrow 0;$$

- (iii) *If $K \subset M$ is a submodule of the finitely generated $\mathcal{N}(G)$ -module M , then $M/\overline{K}$ is finitely generated projective and $\overline{K}$ is a direct summand in M ;*
- (iv) *If M is a finitely generated $\mathcal{N}(G)$ -module, then $\mathbf{P}M$ is finitely generated projective and*

$$M \cong \mathbf{P}M \oplus \mathbf{T}M;$$

- (v) *The dimension $\dim'_{\mathcal{N}(G)}$ has the following properties:*

- (a) *Extension Property*

If M is a finitely generated projective $\mathcal{N}(G)$ -module, then

$$\dim'_{\mathcal{N}(G)}(M) = \dim_{\mathcal{N}(G)}(M);$$

- (b) *Additivity*

If $0 \rightarrow M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \rightarrow 0$ is an exact sequence of $\mathcal{N}(G)$ -modules, then

$$\dim'_{\mathcal{N}(G)}(M_1) = \dim_{\mathcal{N}(G)}(M_0) + \dim_{\mathcal{N}(G)}(M_2),$$

where for $r, s \in \mathbb{R}_{\geq 0} \amalg \{\infty\}$ we define $r + s$ by the ordinary sum of two real numbers if both r and s are not ∞ , and by ∞ otherwise;

- (c) *Cofinality*

Let $\{M_i \mid i \in I\}$ be a cofinal system of submodules of M , i.e., $M = \bigcup_{i \in I} M_i$ and for two indices i and j there is an index k in I satisfying $M_i, M_j \subset M_k$. Then

$$\dim'_{\mathcal{N}(G)}(M) = \sup\{\dim'_{\mathcal{N}(G)}(M_i) \mid i \in I\};$$

- (d) *Continuity*

If $K \subset M$ is a submodule of the finitely generated $\mathcal{N}(G)$ -module M , then

$$\dim'_{\mathcal{N}(G)}(K) = \dim'_{\mathcal{N}(G)}(\overline{K});$$

- (e) *If M is a finitely generated R -module, then*

$$\begin{aligned}\dim'_{\mathcal{N}(G)}(M) &= \dim_{\mathcal{N}(G)}(\mathbf{P}M); \\ \dim'_{\mathcal{N}(G)}(\mathbf{T}M) &= 0;\end{aligned}$$

(f) *Uniqueness*

The dimension $\dim'$ is uniquely determined by the Extension Property, Additivity, Cofinality, and Continuity.

Proof. See [45, Theorem 6.5 and Theorem 6.7 on page 239 and Lemma 6.28 on page 252]. $\square$

Notation 2.9. In view of Theorem 2.8 we will not distinguish between $\dim'_{\mathcal{N}(G)}$ and $\dim_{\mathcal{N}(G)}$ in the sequel.

Exercise 2.10 Let P be a finitely generated projective $\mathcal{N}(G)$ -module and let M be an arbitrary $\mathcal{N}(G)$ -module. Prove:

- (i) $P = 0 \iff \dim_{\mathcal{N}(G)}(P) = 0$;
- (ii) We have $\dim_{\mathcal{N}(G)}(M) = 0$ if and only if every finitely generated projective $\mathcal{N}(G)$ -submodule of M is trivial;

Theorem 2.11 (Dimension and colimits) *Let $\{M_i \mid i \in I\}$ be a directed system of $\mathcal{N}(G)$ -modules over the directed set I . For $i \leq j$ let $\phi_{i,j}: M_i \rightarrow M_j$ be the associated morphism of $\mathcal{N}(G)$ -modules. For $i \in I$ let $\psi_i: M_i \rightarrow \operatorname{colim}_{i \in I} M_i$ be the canonical morphism of $\mathcal{N}(G)$ -modules.*

(i) *We get for the dimension of the $\mathcal{N}(G)$ -module given by the colimit $\operatorname{colim}_{i \in I} M_i$*

$$\dim_{\mathcal{N}(G)}(\operatorname{colim}_{i \in I} M_i) = \sup\{\dim(\operatorname{im}(\psi_i)) \mid i \in I\};$$

(ii) *Suppose for each $i \in I$ that there is $i_0 \in I$ with $i \leq i_0$ such that $\dim(\operatorname{im}(\phi_{i,i_0})) < \infty$ holds. Then*

$$\begin{aligned}\dim(\operatorname{colim}_{i \in I} M_i) \\ = \sup\{\inf\{\dim(\operatorname{im}(\phi_{i,j}: M_i \rightarrow M_j)) \mid j \in I, i \leq j\} \mid i \in I\}.\end{aligned}$$

Proof. See [45, Theorem 6.13 on page 243]. $\square$

Theorem 2.12 (Dimension and inverse limits) *Let $\{M_i \mid i \in I\}$ be an inverse system of $\mathcal{N}(G)$ -modules over the directed set I . For $i \leq j$ let $\phi_{i,j}: M_j \rightarrow M_i$ be the associated morphism of $\mathcal{N}(G)$ -modules. For $i \in I$ let $\psi_i: \operatorname{invlim}_{i \in I} M_i \rightarrow M_i$ be the canonical map. Suppose that there is a countable sequence $i_1 \leq i_2 \leq \dots$ such that for each $j \in I$ there is $n \geq 0$ with $j \leq i_n$.*

(i) *We get for the dimension of the $\mathcal{N}(G)$ -module given by the inverse limit $\operatorname{invlim}_{i \in I} M_i$*

$$\dim_{\mathcal{N}(G)}(\operatorname{invlim}_{i \in I} M_i) = \sup\{\dim_{\mathcal{N}(G)}(\operatorname{im}(\psi_i: \operatorname{invlim}_{i \in I} M_i \rightarrow M_i)) \mid i \in I\};$$

(ii) Suppose that for each index $i \in I$ there is an index $i_0 \in I$ with $i \leq i_0$ and $\dim_{\mathcal{N}(G)}(\text{im}(\phi_{i,i_0})) < \infty$. Then

$$\dim_{\mathcal{N}(G)}(\text{invlim}_{i \in I} M_i) = \sup \{ \inf \{ \dim_{\mathcal{N}(G)}(\text{im}(\phi_{i,j})) \mid j \in I, i \leq j \} \mid i \in I \}.$$

Proof. See [45, Theorem 6.18 on page 244]. $\square$

Let $i: H \rightarrow G$ be an injective group homomorphism. It induces a ring homomorphism $\mathcal{N}(i): \mathcal{N}(H) \rightarrow \mathcal{N}(G)$, see [45, Definition 1.23 on page 29]. For an $\mathcal{N}(H)$ -module M let i_*M be the $\mathcal{N}(G)$ -module $\mathcal{N}(G) \otimes_{\mathcal{N}(H)} M$ obtained by induction with $\mathcal{N}(i)$.

Theorem 2.13 *Let $i: H \rightarrow G$ be an injective group homomorphism.*

- (i) *Induction with i is a faithfully flat functor from the category of $\mathcal{N}(H)$ -modules to the category of $\mathcal{N}(G)$ -modules, i.e., a sequence of $\mathcal{N}(H)$ -modules $M_0 \rightarrow M_1 \rightarrow M_2$ is exact at M_1 if and only if the induced sequence of $\mathcal{N}(G)$ -modules $i_*M_0 \rightarrow i_*M_1 \rightarrow i_*M_2$ is exact at i_*M_1 ;*
- (ii) *For any $\mathcal{N}(H)$ -module M we have:*

$$\dim_{\mathcal{N}(H)}(M) = \dim_{\mathcal{N}(G)}(i_*M).$$

Proof. See [45, Theorem 6.29 on page 253]. $\square$

2.2.3 L^2 -Betti Numbers

Definition 2.14 (L^2 -Betti numbers). Let X be a (left) G -space. Equip $\mathcal{N}(G)$ with the obvious $\mathcal{N}(G)$ - $\mathbb{Z}G$ -bimodule structure. The *singular homology* $H_n^G(X; \mathcal{N}(G))$ of X with coefficients in $\mathcal{N}(G)$ is the homology of the $\mathcal{N}(G)$ -chain complex $\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*^{\text{sing}}(X)$, where $C_*^{\text{sing}}(X)$ is the singular chain complex of X with the induced $\mathbb{Z}G$ -structure. Define the n -th L^2 -Betti number of X by

$$b_n^{(2)}(X; \mathcal{N}(G)) := \dim_{\mathcal{N}(G)}(H_n^G(X; \mathcal{N}(G))) \in \mathbb{R}_{\geq 0} \amalg \{\infty\}.$$

Define for any (discrete) group G its n -th L^2 -Betti number by

$$b_n^{(2)}(G) := b_n^{(2)}(EG; \mathcal{N}(G)) \in \mathbb{R}_{\geq 0} \amalg \{\infty\}.$$

Theorem 2.15 (Basic properties of L^2 -Betti numbers)

- (i) *Homology invariance*

Let $f: X \rightarrow Y$ be a G -map.

- (a) *Suppose for $k \in \mathbb{Z}_{\geq 0}$ that for each subgroup $H \subset G$ the induced map $f^H: X^H \rightarrow Y^H$ is $\mathbb{C}$ -homologically k -connected, i.e., the homomorphism $H_m^{\text{sing}}(f^H; \mathbb{C}): H_m^{\text{sing}}(X^H; \mathbb{C}) \rightarrow H_m^{\text{sing}}(Y^H; \mathbb{C})$ induced by f^H on*

singular homology with complex coefficients is bijective for $m < k$ and surjective for $m = k$.

Then the induced map $H_n^G(f; \mathcal{N}(G)): H_n^G(X; \mathcal{N}(G)) \longrightarrow H_n^G(Y; \mathcal{N}(G))$ is bijective for $n < k$ and surjective for $n = k$ and we get

$$\begin{aligned} b_n^{(2)}(X; \mathcal{N}(G)) &= b_n^{(2)}(Y; \mathcal{N}(G)) & \text{for } n < k; \\ b_n^{(2)}(X; \mathcal{N}(G)) &\geq b_p^{(2)}(Y; \mathcal{N}(G)) & \text{for } n = k; \end{aligned}$$

- (b) Suppose that for each subgroup $H \subset G$ the induced map $f^H: X^H \rightarrow Y^H$ is a $\mathbb{C}$ -homology equivalence, i.e., $H_m^{\text{sing}}(f^H; \mathbb{C})$ is bijective for $m \in \mathbb{Z}_{\geq 0}$. Then for all $n \in \mathbb{Z}_{\geq 0}$ the homomorphism $H_n^G(f; \mathcal{N}(G)): H_n^G(X; \mathcal{N}(G)) \rightarrow H_n^G(Y; \mathcal{N}(G))$ induced by f is bijective and we get

$$b_n^{(2)}(X; \mathcal{N}(G)) = b_n^{(2)}(Y; \mathcal{N}(G));$$

- (ii) Comparison with the Borel construction

Let X be a G -CW-complex. Suppose that for all $x \in X$ the isotropy group G_x is finite or satisfies $b_m^{(2)}(G_x) = 0$ for all $m \in \mathbb{Z}_{\geq 0}$. Then we get for all $n \in \mathbb{Z}_{\geq 0}$.

$$b_n^{(2)}(X; \mathcal{N}(G)) = b_n^{(2)}(EG \times X; \mathcal{N}(G));$$

- (iii) Invariance under non-equivariant $\mathbb{C}$ -homology equivalences

Suppose that $f: X \rightarrow Y$ is a G -equivariant map of G -CW-complexes such that the induced map $H_m^{\text{sing}}(f; \mathbb{C})$ on singular homology with complex coefficients is bijective for $m \in \mathbb{Z}_{\geq 0}$. Suppose that for all $x \in X$ the isotropy group G_x is finite or satisfies $b_m^{(2)}(G_x) = 0$ for all $m \in \mathbb{Z}_{\geq 0}$, and analogously for all $y \in Y$. Then we get for all $n \in \mathbb{Z}_{\geq 0}$

$$b_n^{(2)}(X; \mathcal{N}(G)) = b_n^{(2)}(Y; \mathcal{N}(G));$$

- (iv) Independence of equivariant cells with infinite isotropy

Let X be a G -CW-complex. Let $X[\infty]$ be the G -CW-subcomplex consisting of those points whose isotropy subgroups are infinite. Then we get for all $n \in \mathbb{Z}_{\geq 0}$

$$b_n^{(2)}(X; \mathcal{N}(G)) = b_n^{(2)}(X, X[\infty]; \mathcal{N}(G));$$

- (v) Künneth formula

Let X be a G -space and Y be an H -space.

Then $X \times Y$ is a $G \times H$ -space and we get for all $n \in \mathbb{Z}_{\geq 0}$

$$b_n^{(2)}(X \times Y) = \sum_{p+q=n} b_p^{(2)}(X) \cdot b_q^{(2)}(Y),$$

where we use the convention that $0 \cdot \infty = 0$, $r \cdot \infty = \infty$ for $r \in \mathbb{R}_{>0} \amalg \{\infty\}$ and $r + \infty = \infty$ for $r \in \mathbb{R}_{\geq 0} \amalg \{\infty\}$;

(vi) *Restriction*

Let $H \subset G$ be a subgroup of finite index $[G : H]$.

- (a) Let M be an $\mathcal{N}(G)$ -module and $\text{res}_G^H M$ be the $\mathcal{N}(H)$ -module obtained from M by restriction.

Then

$$\dim_{\mathcal{N}(H)}(\text{res}_G^H M) = [G : H] \cdot \dim_{\mathcal{N}(G)}(M),$$

where $[G : H] \cdot \infty$ is understood to be ∞ ;

- (b) Let X be a G -space and let $\text{res}_G^H X$ be the H -space obtained from X by restriction.

Then we get for all $n \in \mathbb{Z}_{\geq 0}$

$$b_n^{(2)}(\text{res}_G^H X; \mathcal{N}(H)) = [G : H] \cdot b_n^{(2)}(X; \mathcal{N}(G))$$

with the convention $[G : H] \cdot \infty = \infty$;

(vii) *Induction*

Let $i: H \rightarrow G$ be an inclusion of groups and let X be an H -space.

Then we get for all $n \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned} H_n^G(G \times_H X; \mathcal{N}(G)) &= \mathcal{N}(G) \otimes_{\mathcal{N}(H)} H_p^H(X; \mathcal{N}(H)); \\ b_n^{(2)}(G \times_H X; \mathcal{N}(G)) &= b_n^{(2)}(X; \mathcal{N}(H)); \end{aligned}$$

(viii) *Zero-th homology and L^2 -Betti number*

Let X be a path-connected G -space. Then:

- (a) There is an $\mathcal{N}(G)$ -isomorphism $H_0^G(X; \mathcal{N}(G)) \xrightarrow{\cong} \mathcal{N}(G) \otimes_{\mathbb{C}G} \mathbb{C}$;
- (b) We have $b_0^{(2)}(X; \mathcal{N}(G)) = |G|^{-1}$, where $|G|^{-1}$ is defined to be zero if the order $|G|$ of G is infinite;
- (c) $H_0^G(X; \mathcal{N}(G))$ is trivial if and only if G is non-amenable;

Proof. See [45, Theorem 6.54 on page 265]. □

For the numerous applications of L^2 -Betti numbers to problems in algebra, geometry, topology, and operator algebras we refer for instance to [45].

2.2.4 The Algebra $\mathcal{U}(G)$ of Affiliated Operators

Definition 2.16 (Properties of rings). Let R be a ring.

- (i) It is called *Noetherian* if every submodule of a finitely generated R -module is again finitely generated;
- (ii) It is called *regular* if it is Noetherian and every R -module has a projective resolution of finite dimension;

- (iii) It is called *semihereditary* if every finitely generated submodule of a projective module is projective;
- (iv) It is called *von Neumann regular* if each finitely presented R -module is projective;
- (v) It is called *semisimple* if every R -module is projective.

Recall that a ring is Noetherian if and only if any ascending sequence of ideas $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq R$ becomes stationary, i.e., there exists $n \in \mathbb{Z}_{\geq 0}$ satisfying $I_n = I_j$ for every $j \in \mathbb{Z}_{\geq n}$.

Proposition 2.17. *The following statements are equivalent for a ring R :*

- (i) *It is semisimple;*
- (ii) *Every R -module is injective;*
- (iii) *The ring R is isomorphic to $\prod_{i=1}^m M_{n_i}(D_i)$ for $m, n_1, n_2, \dots, n_m \in \mathbb{Z}_{\geq 1}$ and skew fields D_i .*
- (iv) *Every short exact sequence splits;*
- (v) *R is Noetherian and von Neumann regular.*

Proof. See [61, page 604]. □

Proposition 2.18. *The following statements are equivalent for a ring R :*

- (i) *R is von Neumann regular;*
- (ii) *For any $r \in R$ there exists $s \in R$ with $rsr = r$;*
- (iii) *Every principal ideal in R is generated by an idempotent;*
- (iv) *Every finitely generated ideal in R is generated by an idempotent;*
- (v) *Every finitely generated submodule of a finitely generated projective R -module is a direct summand;*
- (vi) *Every finitely presented R -module is projective;*
- (vii) *Every R -module is R -flat;*
- (viii) *Every finitely generated projective R -module is a direct sum of principal ideals.*

Proof. See [60, Lemma 4.15, Theorem 4.16 and Theorem 9.15], [76, Theorem 4.2.9 on page 98]. **Comment 6 (by W.):** Add the pages in the reference to [60]. Check and add a reference for assertions (iv) and (viii). Maybe add also a reference to Goodearl [21]. □

Comment 7 (by W.): Add information about von Neumann *-regular rings. See for instance [26, Section 3.3] and [7]. Meanwhile at least partially done in Section 3.2.

Comment 8 (by W.): Mention that for the surjectivity of $K_0(\mathbb{C}G) \rightarrow K_0(\mathcal{REG}_{\mathbb{C}}(G))$ one has only to show that for a projection $x \in \mathcal{REG}_F(G)$, i.e., $x^2 = x$ and $x = x^*$, the class of (x) lies in the image.

If G is torsionfree, then the Strong Atiyah Conjecture follows from the statement that all idempotents of $\mathcal{REG}_{\mathbb{C}}(G)$ are trivial.

Could it be that any projection $x \in \mathcal{REG}_{\mathbb{C}}(G)$ can be lifted to a projection in $\mathbb{C}G$ if G has property (BOF)?

Associated to the von Neumann algebra $\mathcal{N}(G)$ is the *algebra of affiliated operators* $\mathcal{U}(G)$ which contains $\mathcal{N}(G)$. Its functional analytic definition can be found in [45, Section 8.1]. For us, only the following properties will be relevant.

Proposition 2.19.

- (i) The ring $\mathcal{U}(G)$ is the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicative subset of non-zero divisors;
- (ii) $\mathcal{U}(G)$ is flat as an $\mathcal{N}(G)$ -module;
- (iii) $\mathcal{U}(G)$ is von Neumann regular;
- (iv) The inclusion $i: \mathcal{N}(G) \rightarrow \mathcal{U}(G)$ induces an isomorphism

$$K_0(i): K_0(\mathcal{N}(G)) \xrightarrow{\cong} K_0(\mathcal{U}(G)).$$

Proof. See [45, Theorem 8.2 on page 327 and Theorem 9.20 (i) on page 345]. $\square$

There is a well behaved dimension function for $\mathcal{U}(G)$ -modules: One can for an arbitrary $\mathcal{U}(G)$ -module M define

$$(2.20) \quad \dim_{\mathcal{U}(G)}(M) \in \mathbb{R}_{\geq 0} \amalg \{\infty\}.$$

This is compatible with the notion of dimension for $\mathcal{N}(G)$ -modules in the sense that for every $\mathcal{N}(G)$ -module N we have

$$(2.21) \quad \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{N}(G)} N) = \dim_{\mathcal{N}(G)}(N).$$

Moreover, $\dim_{\mathcal{U}(G)}$ shares all the good properties of $\dim_{\mathcal{N}(G)}$, such as Additivity and Cofinality. For a projective $\mathcal{U}(G)$ -module P we have

$$\dim_{\mathcal{U}(G)}(P) = 0 \iff P = \{0\}.$$

For all of these claims see [45, Definition 8.28 and Theorem 8.29 on page 330].

2.2.5 K_0 and G_0 of Rings

Definition 2.22 (Projective class group $K_0(R)$ and G -theory group $G_0(R)$). Let R be a ring. Define its *projective class group* $K_0(R)$ to be the abelian group whose generators are isomorphism classes $[P]$ of finitely generated projective R -modules P and whose relations are $[P_0] + [P_2] = [P_1]$ for any exact sequence $0 \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow 0$ of finitely generated projective R -modules.

Define $G_0(R)$ analogously but replacing finitely generated projective by finitely generated.

We denote by $\tilde{K}_0(R)$ and $\tilde{G}_0(R)$ respectively the quotient of $K_0(R)$ and $G_0(R)$ respectively by the subgroup generated by the class $[R]$ of R .

Every ring homomorphism $\varphi: R \rightarrow S$ induces a homomorphism of abelian groups $K_0(\varphi): K_0(R) \rightarrow K_0(S)$ by sending the class $[P]$ of a finitely generated projective R -module to the class $[\varphi_* P]$ of the finitely generated projective S -module $\varphi_* P = S \otimes_R P$. This is not true in general for G_0 , one needs the extra condition that the functor sending a finitely generated R -module M to the finitely generated S -module $\varphi_* M$ is exact. This condition is automatically satisfied if R is semisimple, e.g. $R = FH$ for a finite group H and a field F of characteristic zero. If G contain the free group $\mathbb{Z} * \mathbb{Z}$ as subgroup, the class of $[CG]$ in $G_0(\mathbb{C}G)$ is trivial, whereas the class of $[\mathbb{C}G]$ in $K_0(\mathbb{C}G)$ generates an infinite cyclic subgroup. The map $K_0(R) \rightarrow G_0(R)$ is in general not injective or surjective, but is a bijection if R is regular or semisimple.

If R is semisimple, it is of the form $R \cong \prod_{i=1}^k M_{l_i}(D_i)$ for skew fields D_i and one obtains explicit isomorphisms

$$(2.23) \quad K_0(R) \xrightarrow{\cong} \prod_{i=1}^k K_0(M_{l_i}(D_i)) \xrightarrow{\cong} \prod_{i=1}^k K_0(D_i) \xrightarrow{\cong} \prod_{i=1}^k \mathbb{Z}.$$

In particular $K_0(R)$ is a finitely generated free abelian group. For more information about $K_0(R)$ and $G_0(R)$ we refer for instance to [46, Chapter 2].

2.3 The Statement of the Atiyah Conjecture

2.3.1 Some Conventions Concerning Matrices

We first fix some notation concerning matrices. We use the convention that an (m, n) -matrix A has m rows and n columns and a finite matrix is an (m, n) -matrix for some $m, n \in \mathbb{Z}_{\geq 1}$. Hence an element A in $M_{m,n}(R)$ for a ring R is of the form

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-2} & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n-2} & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{m-1,1} & a_{m-1,2} & a_{m-1,3} & \cdots & a_{m-1,n-2} & a_{m-1,n-1} & a_{m-1,n} \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n-2} & a_{m,n-1} & a_{m,n} \end{pmatrix}.$$

We denote by $M_{m,n}(R)$ the set of (m, n) -matrices and by $M_{n,n}(R)$ the ring of (n, n) -matrices. Let I_n be the (n, n) -matrix, whose diagonal entries are all 1 and whose other entries are 0. Obviously I_n is the multiplicative unit in $M_{n,n}(R)$. Given $A \in M_{m,n}(R)$ and $B \in M_{n,o}(R)$, then their product $A \cdot B$ belongs to $M_{m,o}(R)$. We write elements in R^m as $(1, m)$ -matrices, i.e., as a row $(x_1, x_2, \dots, x_m)$. Given $A \in M_{m,n}(R)$, right multiplication with A yields a R -homomorphism of left R -modules

$$(2.24) \quad r_A: R^m \rightarrow R^n; \quad (x_1, x_2, \dots, x_m) \mapsto (x_1, x_2, \dots, x_m) \cdot A$$

Note for the sequel that $r_{AB} = r_B \circ r_A$ holds.

2.3.2 The Statements of Various Versions the Atiyah Conjecture

Conjecture 2.25 (Atiyah Conjecture of order Λ with coefficients in F). *Let G be a group. Let F be a field satisfying $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Let Λ be an abelian group satisfying $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{R}$.*

We say that G satisfies the Atiyah Conjecture of order Λ with coefficients in F if for any matrix $A \in M_{m,n}(FG)$ the von Neumann dimension of the kernel of the $\mathcal{N}(G)$ -homomorphism $r_A: \mathcal{N}(G)^m \rightarrow \mathcal{N}(G)^n$ satisfies

$$\dim_{\mathcal{N}(G)}(\ker(r_A: \mathcal{N}(G)^m \rightarrow \mathcal{N}(G)^n)) \in \Lambda.$$

Define the abelian group

$$(2.26) \quad \mathbb{Z} \subseteq \frac{1}{\mathcal{FLN}(G)}\mathbb{Z} \subseteq \mathbb{Q}$$

to be the additive subgroup of $\mathbb{Q}$ generated by the set $\{|H|^{-1} \mid H \subseteq G, |H| < \infty\}$. We will explain in Remark 2.37 that $\frac{1}{\mathcal{FLN}(G)}\mathbb{Z}$ is the smallest possible choice for Λ in Conjecture 2.25. So the most interesting case of Conjecture 2.25 is $\Lambda = \frac{1}{\mathcal{FLN}(G)}\mathbb{Z}$.

The case $F = \mathbb{Q}$ is the most relevant one for applications to topology as explained in Proposition 2.31, whereas for applications in algebra the case $F = \mathbb{C}$ is the desired one. If we take $F = \mathbb{C}$ and $\Lambda = \frac{1}{\mathcal{FLN}(G)}\mathbb{Z}$ in Conjecture 2.25, we obtain the so-called Strong Atiyah Conjecture.

Conjecture 2.27 (Strong Atiyah Conjecture). *A group G satisfies the strong Strong Atiyah Conjecture if for any matrix $A \in M_{m,n}(\mathbb{C}G)$ the von Neumann dimension of the kernel of $\mathcal{N}(G)$ -homomorphism $r_A: \mathcal{N}(G)^m \rightarrow \mathcal{N}(G)^n$ given by right multiplication with A satisfies*

$$\dim_{\mathcal{N}(G)}(\ker(r_A: \mathcal{N}(G)^m \rightarrow \mathcal{N}(G)^n)) \in \frac{1}{\mathcal{FLN}(G)}\mathbb{Z}.$$

In the special case that G is torsionfree, we will consider the following versions of the two conjectures above.

Conjecture 2.28 (Strong Atiyah Conjecture with coefficients in F for torsionfree groups). *Let G be a torsionfree group. Let F be a field satisfying $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$.*

We say that G satisfies the Atiyah Conjecture with coefficients in F for the torsionfree group G if for any matrix $A \in M_{m,n}(FG)$ the von Neumann dimension of the kernel of the $\mathcal{N}(G)$ -homomorphism $r_A: \mathcal{N}(G)^m \rightarrow \mathcal{N}(G)^n$ given by right multiplication with A satisfies

$$\dim_{\mathcal{N}(G)}(\ker(r_A: \mathcal{N}(G)^m \rightarrow \mathcal{N}(G)^n)) \in \mathbb{Z}.$$

Conjecture 2.29 (Strong Atiyah Conjecture for torsionfree groups). *A torsionfree group G satisfies the Strong Atiyah Conjecture if for any matrix $A \in M_{m,n}(\mathbb{C}G)$ the von Neumann dimension of the kernel of the $\mathcal{N}(G)$ -homomorphism $r_A: \mathcal{N}(G)^m \rightarrow \mathcal{N}(G)^n$ given by right multiplication with A satisfies*

$$\dim_{\mathcal{N}(G)}(\ker(r_A: \mathcal{N}(G)^m \rightarrow \mathcal{N}(G)^n)) \in \mathbb{Z}.$$

Obviously Conjecture 2.28 holds for all field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ if and only if Conjecture 2.29 is true.

Comment 9 (by T.): Shall we discard one of the many versions of the conjecture? **Comment 10 (by W.):** My preference is not to discard anything.

$$\text{source}_U(x) \subseteq \langle u_2 u_1^{-1}, \dots, u_n u_1^{-1} \rangle \subseteq U.$$

Notation 2.30. A group G has property (BOF) if there is an upper bound on the orders of the finite subgroups of G .

We will explain in Section 2.8 that there are counterexamples to Conjecture 2.25 and Conjecture 2.27 unless one makes the assumption that G has property (BOF) introduced in Notation 2.30. No counterexamples are known for Conjecture 2.29.

2.4 Some Reformulations of the Atiyah Conjecture

We can reformulate the Atiyah Conjecture 2.25 for $F = \mathbb{Q}$ in terms of L^2 -Betti numbers as follows.

Proposition 2.31. *Let G be a group. Consider any $d \in \mathbb{Z}_{\geq 2}$. Let $\mathbb{Z} \subseteq A \subseteq \mathbb{Q}$ be an additive subgroup of $\mathbb{Q}$. Then the following statements are equivalent:*

(i) *For every cocompact free proper G -manifold M and every $n \in \mathbb{Z}_{\geq 0}$, we get*

$$b_n^{(2)}(M; \mathcal{N}(G)) \in A;$$

(ii) *For every cocompact free proper G -manifold M without boundary of dimension $(2d+4)$ we get*

$$b_d^{(2)}(M; \mathcal{N}(G)) \in A;$$

(iii) *For every free G -CW-complex X of finite type and every $n \in \mathbb{Z}_{\geq 0}$, we get*

$$b_n^{(2)}(X; \mathcal{N}(G)) \in A;$$

(iv) *For every $(d+1)$ -dimensional free G -CW-complex X of finite type, we get*

$$b_d^{(2)}(X; \mathcal{N}(G)) \in A;$$

- (v) *The Atiyah Conjecture 2.25 of order Λ with coefficients in $\mathbb{Q}$ is true for G .*
 (vi) *For an arbitrary G -space X and every $n \in \mathbb{Z}_{\geq 0}$ we have*

$$b_n^{(2)}(M; \mathcal{N}(G)) \in \Lambda \amalg \{+\infty\}.$$

Proof. The proof except assertion (vi) is similar to the one of [45, Lemma 10.5 on page 371].

Assertion (vi) is proven as follows: using G -CW-approximations and Theorem 2.15 (iii) we can reduce the claim to G -CW-complexes. By Theorem 2.15 (iv) and looking at the relative cellular chain complex of $(X, X[\infty])$, we can reduce to a chain complex of a proper G -CW-complex. By Theorem 2.15 (ii) this has the same L^2 -Betti numbers as a free G -CW-complex. By a colimit argument one reduces to finite free G -CW-complexes and the claim follows. $\square$

One can rephrase the Atiyah Conjecture also in a more module-theoretic fashion.

Notation 2.32. Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Define

$$\Lambda(G, F)_{\text{fgp}} := \{\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{FG} M_1) - \dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{FG} M_0)\} \subset \mathbb{R},$$

where M_0 and M_1 run through all finitely generated projective FG -modules. The subsets $\Lambda(G, F)_{\text{fp}}$, $\Lambda(G, F)_{\text{fg}}$, or $\Lambda(G, F)_{\text{all}}$ are defined analogously, using finitely presented FG -modules, finitely generated FG -modules, or all FG -modules with $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{FG} M_i) < \infty$ for $i = 0, 1$.

Proposition 2.33. *A group G satisfies the Atiyah Conjecture of order Λ with coefficients in F if and only if $\Lambda(G, F)_{\text{fp}} \subseteq \Lambda$ holds, or, equivalently, for any finitely presented FG -module M we have*

$$\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{FG} M) \in \Lambda.$$

Proof. See [45, Lemma 10.7 on page 372]. $\square$

Proposition 2.34. *Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$.*

- (i) *We have $\frac{1}{|\mathcal{FIN}(G)|} \mathbb{Z} \subseteq \Lambda(G, F)_{\text{fgp}}$.
 If the assembly map*

$$\text{colim}_{H \subseteq G, |H| < \infty} K_0(FH) \rightarrow K_0(FG)$$

given by the collection of inclusions $H \subseteq G$ is surjective, then we have $\frac{1}{|\mathcal{FIN}(G)|} \mathbb{Z} = \Lambda(G, F)_{\text{fgp}}$;

- (ii) *We have $\Lambda(G, F)_{\text{fgp}} \subseteq \Lambda(G, F)_{\text{fp}} \subseteq \Lambda(G, F)_{\text{fg}} \subseteq \Lambda(G, F)_{\text{all}}$;*
 (iii) *$\Lambda(G, F)_{\text{fg}} \subseteq \overline{\Lambda(G, F)_{\text{fp}}}$, where the closure is taken in $\mathbb{R}$;*

(iv) We have

$$\begin{aligned}\overline{\Lambda(G, F)_{\text{fp}}} &= \overline{\Lambda(G, F)_{\text{fg}}}; \\ \Lambda(G, F)_{\text{all}} &= \overline{\Lambda(G, F)_{\text{fg}}}.\end{aligned}$$

Proof. See [45, Lemma 10.10 on page 373] □

Remark 2.35. Suppose that G satisfies property (BOF). Then the least common multiple $\text{lcm}(G)$ of the orders of finite subgroups is defined and

$$\frac{1}{|\mathcal{FIN}(G)|} \mathbb{Z} = \{r \in \mathbb{R} \mid \text{lcm}(G) \cdot r \in \mathbb{Z}\}.$$

Hence by Proposition 2.33 and Proposition 2.34 the Strong Atiyah Conjecture 2.27 is equivalent to the equality

$$\{r \in \mathbb{R} \mid \text{lcm}(G) \cdot r \in \mathbb{Z}\} = \Lambda(G, \mathbb{C})_{\text{fgp}} = \Lambda(G, \mathbb{C})_{\text{fp}} = \Lambda(G, \mathbb{C})_{\text{fg}} = \Lambda(G, \mathbb{C})_{\text{all}}.$$

In particular the Strong Atiyah Conjecture 2.29 for a torsionfree group G is equivalent to the equality

$$\mathbb{Z} = \Lambda(G, \mathbb{C})_{\text{fgp}} = \Lambda(G, \mathbb{C})_{\text{fp}} = \Lambda(G, \mathbb{C})_{\text{fg}} = \Lambda(G, \mathbb{C})_{\text{all}}.$$

Exercise 2.36 Show that the following assertions for a group G are equivalent:

- (i) The group G has property (BOF);
- (ii) The subgroup $\frac{1}{|\mathcal{FIN}(G)|} \mathbb{Z}$ of $\mathbb{R}$ is discrete;
- (iii) The subgroup $\frac{1}{|\mathcal{FIN}(G)|} \mathbb{Z}$ of $\mathbb{R}$ is closed.

Remark 2.37. If the Atiyah Conjecture 2.25 of order Λ with coefficients in F holds for G , then we must have $\frac{1}{|\mathcal{FIN}(G)|} \mathbb{Z} \subseteq \Lambda$ because of Proposition 2.34.

2.5 The K -Theoretic Atiyah Conjecture

Next we give a purely K -theoretic version of the Atiyah Conjecture. We need some notation for certain subgroups of the projective class group $K_0(\mathcal{U}(G))$ of $\mathcal{U}(G)$.

Notation 2.38. Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Define the subgroup

$$\kappa(G, F)_{\text{fgp}} := \{[\mathcal{U}(G) \otimes_{FG} M_1] - [\mathcal{U}(G) \otimes_{FG} M_0]\} \subset K_0(\mathcal{U}(G)),$$

where M_0 and M_1 run through all finitely generated projective FG -modules. Define $\kappa(G, F)_{\text{fp}}$ correspondingly, where M_0 and M_1 run through all finitely presented FG -modules.

Define the subgroup of $K_0(\mathcal{U}(G))$ to be the image of the composite

$$\kappa(G; F)_{\text{fin}} := \text{im} \left(\bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \rightarrow K_0(FG) \xrightarrow{K_0(j)} K_0(\mathcal{U}(G)) \right),$$

where the first map is induced by the various inclusions of finite subgroups $H \subseteq G$ and the second map is induced by the inclusion $j: FG \rightarrow \mathcal{U}(G)$.

The definition of $\kappa(G, F)_{\text{fp}}$ make senses, since for a finitely presented FG -module M the $\mathcal{U}(G)$ -module $\mathcal{U}(G) \otimes_{\mathcal{N}(G)} M$ is a finitely presented $\mathcal{U}(G)$ -module and hence a finitely generated projective $\mathcal{U}(G)$ -module, since $\mathcal{U}(G)$ is von Neumann regular by Proposition 2.19 (iii).

Conjecture 2.39 (K -theoretic Atiyah Conjecture). *Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Then*

$$\kappa(G, F)_{\text{fin}} = \kappa(G, F)_{\text{fp}}.$$

Proposition 2.40. *Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$.*

- (i) *We have $\kappa(G, F)_{\text{fin}} \subseteq \kappa(G, F)_{\text{fgp}} \subseteq \kappa(G, F)_{\text{fp}}$;*
- (ii) *If G satisfies the Full Farrell-Jones Conjecture, see [46, Conjecture 13.30 on page 387], then $\kappa(G, F)_{\text{fin}} = \kappa(G, F)_{\text{fgp}}$.*

Proof. (i) This is obvious since $\kappa(G, F)_{\text{fgp}}$ agrees with the image of the homomorphism $K_0(j): K_0(FG) \rightarrow K_0(\mathcal{U}(G))$.

(ii) The Full Farrell-Jones Conjecture implies that the map

$$\bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \rightarrow K_0(FG)$$

is surjective, see [46, Theorem 13.65 (Xii) on page 406]. $\square$

Remark 2.41. The Full Farrell-Jones Conjecture is known for a large class of groups, see [46, Theorem 16.1 on page 481]. For example, it contains all hyperbolic groups, finite dimensional CAT(0)-groups, lattices in path connected second countable locally compact Hausdorff groups, fundamental groups of manifolds of dimension ≤ 3 , and S -arithmetic groups. Furthermore, it is closed under taking subgroups, passing to overgroups of finite index, and passing to colimits over directed systems of groups with arbitrary structure maps. Its precise statement is not relevant for this monograph and therefore omitted.

Remark 2.42. Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. The K -theoretic Atiyah Conjecture 2.39 implies the Atiyah Conjecture 2.25 of order $\Lambda = \frac{1}{\mathcal{F}TN(G)}\mathbb{Z}$ with coefficients in F by the following argument.

The dimension function $\dim_{\mathcal{N}(G)}$ of (2.3) defines a homomorphism

$$(2.43) \quad \dim_{\mathcal{N}(G)}: K_0(\mathcal{N}(G)) \rightarrow \mathbb{R}.$$

The homomorphism $K_0(i): K_0(\mathcal{N}(G)) \xrightarrow{\cong} K_0(\mathcal{U}(G))$ is bijective by Proposition 2.19 (iv). One easily checks using (2.21) that the image of $\kappa(G, F)_{\text{fp}}$ and $\kappa(G, F)_{\text{fin}}$ under the composite $K_0(\mathcal{U}(G)) \xrightarrow{K_0(i)^{-1}} K_0(\mathcal{N}(G)) \xrightarrow{\dim_{\mathcal{N}(G)}} \mathbb{R}$ is $\Lambda(G, F)_{\text{fp}}$ and $\frac{1}{\mathcal{FIN}(G)}\mathbb{Z}$, respectively.

2.6 The Center-Valued Atiyah Conjecture

Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Using the center-valued universal trace and the associated center-valued universal dimension function, one obtains homomorphisms

$$\dim_{\mathcal{N}(G)}^u: K_0(\mathcal{N}(G)) \rightarrow \mathcal{Z}(\mathcal{N}(G))^{\mathbb{Z}/2} = \{a \in \mathcal{Z}(\mathcal{N}(G)) \mid a = a^*\},$$

and

$$\dim_{\mathcal{U}(G)}^u: K_0(\mathcal{U}(G)) \rightarrow \mathcal{Z}(\mathcal{N}(G))^{\mathbb{Z}/2} = \{a \in \mathcal{Z}(\mathcal{N}(G)) \mid a = a^*\},$$

where $\mathcal{Z}(\mathcal{N}(G))$ is the center of the von Neumann algebra and the group structure on $\mathcal{Z}(\mathcal{N}(G))^{\mathbb{Z}/2}$ comes from the addition. The composite of $\dim_{\mathcal{U}(G)}^u$ with the isomorphism $K_0(j): K_0(FG) \rightarrow K_0(\mathcal{U}(G))$ is $\dim_{\mathcal{N}(G)}^u$. The homomorphism $\dim_{\mathcal{N}(G)}^u$ is always injective. It is bijective if $\mathcal{N}(G)$ is of type II_1 . All these claims follow from [45, Theorem 9.13 (2) on page 342]. If G is finitely generated and not virtually finitely generated abelian, then $\mathcal{N}(G)$ is of type II_1 , see [45, Lemma 9.4 (3) on page 338]. If G is an ICC group – this means that the conjugacy class of each element $g \in G$ is finite – then $\mathcal{Z}(\mathcal{N}(G)) = \mathcal{Z}(\mathcal{N}(G))^{\mathbb{Z}/2} = \mathbb{R}$ holds by [45, Lemma 9.4 (4) on page 338].

Notation 2.44. Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Define

$$\begin{aligned} \Lambda^u(G, F)_{\text{fgp}} &:= \{\dim_{\mathcal{N}(G)}^u(\mathcal{N}(G) \otimes_{FG} M_1) - \dim_{\mathcal{N}(G)}^u(\mathcal{N}(G) \otimes_{FG} M_0)\} \\ &\subset \mathcal{Z}(\mathcal{N}(G))^{\mathbb{Z}/2}, \end{aligned}$$

where M_0 and M_1 run through all finitely generated projective FG -modules, and define $\Lambda^u(G, F)_{\text{fp}}$ correspondingly using finitely presented FG -modules. Let $\Lambda^u(G, F)_{\text{fin}}$ be the image of the composite

$$\text{colim}_{H \subseteq G, |H| < \infty} K_0(FH) \rightarrow K_0(FG) \rightarrow K_0(\mathcal{N}(G)) \xrightarrow{\dim_{\mathcal{N}(G)}} \mathcal{Z}(\mathcal{N}(G))^{\mathbb{Z}/2}.$$

Conjecture 2.45 (Center-Valued Atiyah Conjecture). *Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Then the center-valued Atiyah Conjecture predicts*

$$\Lambda^u(G, F)_{\text{fin}} = \Lambda^u(G, F)_{\text{fp}}.$$

Proposition 2.46. *Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$.*

Then the K -theoretic Atiyah Conjecture 2.39 and the center-valued Atiyah Conjecture 2.45 are equivalent.

Proof. This follows from the definitions and the fact that the homomorphism $\dim_{\mathcal{N}(G)}^u: K_0(\mathcal{N}(G)) \rightarrow \mathcal{Z}(\mathcal{N}(G))^{\mathbb{Z}/2}$ is injective. $\square$

2.7 Some Implications and Applications of the Atiyah Conjecture

We briefly describe a few applications of the Atiyah Conjecture. Many more will be discussed later in this monograph. **Comment 11 (by W.):** *Maybe add references later.*

2.7.1 The Zero-Divisor Conjecture of Kaplansky and the Embedding Conjecture of Malcev

We have the following prominent conjectures due to Kaplansky and Malcev.

Conjecture 2.47 (Zero-Divisor Conjecture). *Let F be a field of characteristic zero and G be a torsionfree group. Then FG contains no non-trivial zero-divisors.*

Conjecture 2.48 (Embedding Conjecture). *Let F be a field of characteristic zero and G be a torsionfree group. Then FG embeds into a skew field.*

Obviously the Embedding Conjecture 2.48 implies the Zero-Divisor Conjecture 2.47.

Remark 2.49. We conclude from Theorem 2.83 (i) that the Strong Atiyah Conjecture 2.29 for torsionfree groups implies the Embedding Conjecture 2.48 if $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ holds. Actually, there is a canonical embedding of FG in a specific ring which turns out to be a skew field if the Atiyah conjecture holds.

The Strong Atiyah Conjecture 2.29 for torsionfree groups implies the Zero-Divisor Conjecture 2.47 for all fields F of characteristic zero because of [46, Remark 2.84 on page 54].

Remark 2.50. Let F be a field and G be a group such that FG has no non-trivial zero-divisors.

Then G is amenable if and only if FG is an Ore domain, i.e., satisfies the Ore condition with respect to the multiplicative subgroup $FG \setminus \{0\}$, see [6, Theorem A.1].

Proposition 2.51. *Suppose that G be a torsionfree amenable group. Then the Strong Atiyah Conjecture 2.29 holds for G if and only if the Zero-Divisor Conjecture 2.47 holds for $\mathbb{C}G$, see [45, Lemma 10.16 on page 376]. In this case the canonical skew field of Remark 2.49 is the Ore localization of $\mathbb{C}G$ with respect to the multiplicative subgroup $\mathbb{C}G \setminus \{0\}$.*

2.7.2 Computation of L^2 -Betti Numbers

Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Given a matrix $A \in M_{m,n}(FG)$ one can consider the FG -homomorphism $r_A: \mathcal{N}(G)^m \rightarrow \mathcal{N}(G)^n$ given by right multiplication with A and ask for the computation of $\dim_{\mathcal{N}(G)}(\ker(r_A^{(2)}))$. If one has an algorithm to solve the word problem in G , one can define a monotone increasing sequence $(c_p(A))_{p \geq 0}$ of elements in $\mathbb{R}_{\geq 0}$ such that each element is computable and one has

$$\dim_{\mathcal{N}(G)}(\ker(r_A)) = \lim_{p \rightarrow \infty} c_p(A).$$

This is proved in [45, Section 3.7]. Löh and Uschold [40] investigate the computability degree of this limit. They show that the complexity class for the computation of this limiting number is bounded by the complexity class to solve the word problem in G .

Now suppose that G satisfies property (BOF) and that the Atiyah Conjecture of order $\Lambda = \frac{1}{|\mathcal{FLN}(G)|} \mathbb{Z}$ with coefficients in F , see Conjecture 2.25, holds. If there is a p such that $c_p(A) < \text{lcm}(G)^{-1}$ holds, then we conclude from Remark 2.35 that we have $\dim_{\mathcal{N}(G)}(\ker(r_A^{(2)})) = 0$, and one does not have to compute all the elements of the sequence $(c_p(A))_{p \geq 0}$.

It is clear that this can be very useful to show the vanishing of L^2 -Betti numbers, see [45, Remark 3.173 on page 195] and to compute L^2 -invariant in general.

2.7.3 Finite Generation of Projective Modules

Question 2.52 (Finite generation of projective modules over group rings and their von Neumann dimension). *Let G be a group satisfying property (BOF). Consider a ring R with $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$. Let P be a projective RG -module such that $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{RG} P) < \infty$ holds. Is then P finitely generated?*

This question is only interesting in the case where G has property (BOF).

Exercise 2.53 Consider a field F with $\mathbb{Z} \subseteq F \subseteq \mathbb{C}$. Construct an abelian group G such that for any $\epsilon \in \mathbb{R}_{>0}$ there is a projective FG -module P such that P is not finitely generated and $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{RG} P) < \epsilon$ holds.

Here is a partial answer to Question 2.52.

Proposition 2.54. *Let G be a group satisfying property (BOF). Consider a ring R and a field F with $\mathbb{Z} \subseteq R \subseteq F \subseteq \mathbb{C}$. Suppose that the Atiyah Conjecture 2.25 of order $\Lambda = \frac{1}{|\mathcal{FLN}(G)|} \mathbb{Z}$ with coefficients in F holds. Let P be a projective RG -module. Then the following assertions are equivalent:*

- (i) *The RG -module P is finitely generated;*
- (ii) *We have $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{RG} P) < \infty$;*

Proof. This is a direct consequence of the following two facts whose proof can be found in [34, Lemma 4 and Lemma 5].

Consider the situation of Proposition 2.54. Then:

- Suppose that G satisfies property (BOF) and that the Atiyah Conjecture 2.25 of order $A = \frac{1}{|\mathcal{FIN}(G)|} \mathbb{Z}$ with coefficients in F holds. Let N be an RG -module. Suppose that $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{RG} N) < \infty$. Then there exists a finitely generated RG -submodule $M \subset N$ satisfying $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{Z}G} N/M) = 0$;
- Let P be a projective RG -module such that for some finitely generated RG -submodule $M \subset P$ we have $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{RG} P/M) = 0$. Then P is finitely generated.

□

2.7.4 Applications to Groups of Low (Co-)Homological Dimension

Comment 12 (by W.): One should add a reference to this Subsection 2.7.4 in Chapter 7.

The next result is taken from [34, Theorem 2].

Theorem 2.55 (Groups of homological dimension ≤ 1) *A locally free group G has homological dimension ≤ 1 . We remind the reader that a group is locally (P) if every finitely generated subgroup has property (P).*

If G is a group of homological dimension ≤ 1 and the Strong Atiyah Conjecture 2.28 with coefficients in $F = \mathbb{Q}$ for torsionfree groups holds for G , then G is locally free.

Exercise 2.56 Find a group which is locally cyclic without being cyclic.

The next result is taken from [34, Theorem 3].

Theorem 2.57 (Elementary amenable group of cohomological dimension ≤ 2) *Let G be an elementary amenable group of cohomological dimension ≤ 2 . Then:*

- (i) *Suppose that G is finitely generated. Then G possesses a presentation of the form*

$$\langle x, y \mid yxy^{-1} = x^n \rangle.$$

In particular there is a finite 2-dimensional model for BG ;

- (ii) *Suppose that G is countable but not finitely generated. Then G is a non-cyclic subgroup of the additive group $\mathbb{Q}$.*

Recall that a group G is of type FP_n or FF_n respectively if there exists an exact sequence of $\mathbb{Z}G$ -modules $0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow \mathbb{Z}$ for $\mathbb{Z}$ equipped with the trivial G -action such that each module P_i is finitely generated projective or finitely generated free respectively. We omit the elementary proof that the next Proposition 2.58 follows from Proposition 2.54.

Proposition 2.58. *Let G be a finitely generated group of cohomological dimension ≤ 2 with $b_2^{(2)}(G) < \infty$. Suppose that G satisfies the Strong Atiyah Conjecture 2.28 with coefficients in $F = \mathbb{Q}$ for the torsionfree group G .*

Then G is of type FP_2 .

If G satisfies the Full Farrell-Jones Conjecture, then we can replace FP_2 by FF_2 in the statement above.

Theorem 2.59 (Vanishing of top L^2 -Betti numbers and passing to subgroups) *Let H be a subgroup of G . Let X be a d -dimensional proper G -CW-complex which we can also view as d -dimensional proper H -CW-complex by restriction. Suppose $b_d^{(2)}(\bar{X}; \mathcal{N}(G)) = 0$.*

Then we get $b_d^{(2)}(\bar{X}; \mathcal{N}(H)) = 0$.

Proof. For any $\mathbb{C}G$ -module M we obtain an $\mathcal{N}(H)$ -homomorphism, natural in M ,

$$(2.60) \quad T(M): \mathcal{N}(H) \otimes_{\mathbb{C}H} M \rightarrow \mathcal{N}(G) \otimes_{\mathbb{C}G} M, \quad u \otimes x \mapsto i(u) \otimes x.$$

Next we want to show that it is injective for projective $\mathbb{C}G$ -modules M . As T is compatible with direct sums over arbitrary index sets, it suffices to prove the claim for $M = \mathbb{C}G$.

Let T be a transversal of the projection $\text{pr}: G \rightarrow H \backslash G$, i.e., T is a subset of G such that $\text{pr}|_T: T \rightarrow H \backslash G$ is a bijection. We will assume $e \in T$ for the unit $e \in G$. We obtain an isomorphism of left $\mathbb{C}H$ -modules

$$\bigoplus_{t \in G/H} \mathbb{C}H \xrightarrow{\cong} \mathbb{C}G, \quad (x_t)_{t \in T} \mapsto \sum_{t \in T} x_t \cdot t.$$

It induces a bijection

$$\alpha: \bigoplus_{t \in T} \mathcal{B}(l^2(H))^H \xrightarrow{\cong} \mathcal{B}(l^2(H))^H \otimes_{\mathbb{C}H} \mathbb{C}G, \quad (f_t)_{t \in T} \mapsto \sum_{t \in T} f_t \otimes t$$

and an H -equivariant isomorphism of Hilbert spaces

$$\xi: \bigoplus_{t \in T} \overline{l^2(H)} \xrightarrow{\cong} l^2(G), \quad (u_t)_{t \in T} \mapsto \sum_{t \in T} u_t \cdot t,$$

where the source is the Hilbert space completion of the pre Hilbert space $\bigoplus_{t \in G/H} l^2(H)$. We have the isomorphism

$$\beta: \mathcal{N}(G) \otimes_{\mathbb{C}G} \mathbb{C}G \xrightarrow{\cong} \mathcal{B}(l^2(G))^G, \quad f \otimes \left(\sum_{g \in G} \lambda_g \cdot g \right) \mapsto \sum_{g \in G} \overline{\lambda_g} \cdot (f \circ r_{g^{-1}})$$

where $r_{g^{-1}}: l^2(G) \rightarrow l^2(G)$ is right multiplication with g^{-1} . Hence it remains to show that

$$\beta \circ T(\mathbb{C}G) \circ \alpha: \bigoplus_{t \in G/H} \mathcal{B}(l^2(H))^H \rightarrow \mathcal{B}(l^2(G))^G$$

is injective. Given $s \in T$ and an element f_s in the copy of $\mathcal{B}(l^2(H))^H$ belonging to s in the source of α , we get a commutative diagram of operators of Hilbert spaces

$$(2.61) \quad \begin{array}{ccc} \bigoplus_{t \in T} l^2(H) & \xrightarrow[\cong]{\xi} & l^2(G) \\ \sigma_s \downarrow & & \downarrow r_{s^{-1}} \\ \bigoplus_{t \in T} l^2(H) & \xrightarrow[\cong]{\xi} & l^2(G) \\ \oplus_{t \in T} f_s \downarrow & & \downarrow i(f_s) \\ \bigoplus_{t \in T} l^2(H) & \xrightarrow[\cong]{\xi} & l^2(G) \end{array}$$

where for $t_0, t_1 \in T$ the operator $(\sigma_s)_{t_0, t_1}: l^2(H) \rightarrow l^2(H)$ is trivial if $\text{pr}(t_0 s^{-1}) \neq \text{pr}(t_1)$ and is right multiplication with the element $t_0 s^{-1} t_1^{-1} \in H$ if $\text{pr}(t_0 s^{-1}) = \text{pr}(t_1)$.

Now suppose that the element $(f_s)_{s \in T} \in \bigoplus_{t \in G/H} \mathcal{B}(l^2(H))^H$ lies in the kernel of $\beta \circ T(\mathbb{C}G) \circ \alpha$. Then $\xi^{-1} \circ (\sum_{s \in T} i(f_s) \circ r_{s^{-1}}) \circ \xi: \bigoplus_{t \in T} l^2(H) \rightarrow \bigoplus_{t \in T} l^2(H)$ is trivial.

We get for $t \in T$ that $(\sigma_s)_{t, e}$ is zero for $s \neq t$ since $\text{pr}(ts^{-1}) = \text{pr}(e) \implies ts^{-1} \in H \implies \text{pr}(t) = \text{pr}(s) \implies s = t$. Fix $t \in T$. Let $j_t: l^2(H) \rightarrow \bigoplus_{t \in T} l^2(H)$ be the inclusion of the summand belonging to $t \in T$ and $\text{pr}_e: \bigoplus_{t \in T} l^2(H) \rightarrow l^2(H)$ be the projection onto the summand belonging to $e \in T$. Then the composite

$$\text{pr}_e \circ \xi^{-1} \circ \left(\sum_{s \in T} i(f_s) \circ r_{s^{-1}} \right) \circ \xi \circ j_t: l^2(H) \rightarrow l^2(H)$$

is zero and sends u to $f_t(u)$ because of the commutative diagram (2.61). Hence $f_t = 0$ for all $t \in T$. This finishes the proof that the natural map $T(M)$ of (2.60) is bijective for every free $\mathbb{C}G$ -module M .

If $c_d: C_*(\overline{X}) \rightarrow C_*(\overline{X})$ is the d -th differential in the cellular $\mathbb{C}G$ -chain complex $C_*(\overline{X}; \mathbb{C})$, we get a commutative diagram

$$\begin{array}{ccc} \mathcal{N}(H) \otimes_{\mathbb{C}H} C_d(\overline{X}; \mathbb{C}) & \xrightarrow{T(C_d(\overline{X}; \mathbb{C}))} & \mathcal{N}(G) \otimes_{\mathbb{C}G} C_d(\overline{X}; \mathbb{C}) \\ \text{id}_{\mathcal{N}(H)} \otimes_{\mathbb{C}H} c_d \downarrow & & \downarrow \text{id}_{\mathcal{N}(G)} \otimes_{\mathbb{C}G} c_d \\ \mathcal{N}(H) \otimes_{\mathbb{C}H} C_{d-1}(\overline{X}; \mathbb{C}) & \xrightarrow{T(C_{d-1}(\overline{X}; \mathbb{C}))} & \mathcal{N}(G) \otimes_{\mathbb{C}G} C_{d-1}(\overline{X}; \mathbb{C}) \end{array}$$

whose horizontal arrows are injective. Hence the left vertical arrow is injective if the right vertical arrow is injective. Now the claim follows from the fact that a projective $\mathcal{N}(G)$ -module P is trivial if and only if $\dim_{\mathcal{N}(G)}(P)$ vanishes, see [45, Lemma 6.28 (3) on page 252]. **Comment 13 (by T.):** [try to find a more concise argument here.](#) $\square$

Theorem 2.59 has been proved for proper G -actions on simplicial complexes using orbit equivalence in [20, Theorem 1.5].

Proposition 2.58 and Theorem 2.59 imply the next theorem which has also been proved by Jaikin-Zapirain and Linton [28, Theorem 3.7].

Theorem 2.62 (Almost coherent groups) *Let G be a group of cohomological dimension ≤ 2 with $b_2^{(2)}(G) = 0$. Suppose that G satisfies the Strong Atiyah Conjecture 2.28. with coefficients in $F = \mathbb{Q}$ for the torsionfree group G .*

Then G is almost coherent in the sense that every finitely generated subgroup $H \subseteq G$ is of type FP_2 .

If G satisfies the Full Farrell-Jones Conjecture, then we can replace FP_2 by FF_2 in the statement above.

Theorem 2.62 is interesting in connection with the following question.

Question 2.63 (Coherent groups). *Let G be a group of cohomological dimension ≤ 2 with $b_2^{(2)}(G) = 0$. Is then G coherent in the sense that any finitely generated subgroup $H \subseteq G$ has a finite 2-dimensional model for BH ?*

Exercise 2.64 Show that every elementary amenable group of cohomological dimension ≤ 2 is coherent.

Comment 14 (by W.): [Discuss here or later the relation to the existence of non-positive immersions and the vanishing of the second \$L^2\$ -Betti number. See in particular Jaikin-Zapirain and Linton \[28, Theorem 1.2\] and the preprint by Avramidi-Lück \[5\].](#)

2.7.5 Further Applications

Comment 15 (by W.): [To be filled or removed. Examples: The Strong Algebraic Eigenvalue Conjecture, The Center Conjecture, The Independence Conjecture, see \[29, Corollary 1.2\]. Existence of Hughes free epic division ring for \$EG\$ if \$G\$ is torsionfree and \$E\$ is a field of characteristic zero, see \[29, Corollary 6.7\].](#)

2.8 Counterexamples to the Atiyah Conjecture

If we drop the property (BOF) that there is an upper bound on the orders of finite subgroups, one might still ask whether $\Lambda(G; \mathbb{Q})_{\text{fp}} = \mathbb{Q}$ holds, motivated by the observation that the image of the composite

$$\bigoplus_{H \subseteq G, |H| < \infty} K_0(\mathbb{Q}H) \rightarrow K_0(\mathbb{Q}G) \rightarrow K_0(\mathcal{N}(G)) \xrightarrow{\dim_{\mathcal{U}(G)}} \mathbb{R}$$

is contained in $\mathbb{Q}$. This goes back to Atiyah's original question [3, page 72], who asked for the rationality of the L^2 -Betti numbers $b_n^{(2)}(\widetilde{M})$ of every closed manifold M . Austin [4, Corollary 1.2] gave the first example of a finitely generated group G , where for some matrix $A \in M_{m,n}(\mathbb{Q}G)$ the dimension $\dim_{\mathcal{N}(G)}(\ker(r_A))$ of the kernel of the $\mathcal{N}(G)$ -homomorphism $r_A: \mathcal{N}(G)^m \rightarrow \mathcal{N}(G)^n$ is irrational. Grabowski [22, Corollary 1.2] showed that one can choose G to be finitely presented above. Moreover, Grabowski [22, Theorem 1.3] proved, using Turing machines, that any non-negative real number can occur in this way for some finitely generated group G and some matrix A . Löh and Uschold [40] investigate the computability degree of real numbers arising as L^2 -Betti numbers or L^2 -torsion of groups, parametrised over the Turing degree of the word problem. Roughly speaking, the complexity of the computation of L^2 -invariants of a group is the same as the complexity of the word problem.

2.9 Positive Results about the Atiyah Conjectures

The notions of elementary amenable groups and amenable groups are explained for instance in [45, Subsection 6.4.1]. Examples for elementary amenable groups are virtually solvable groups. Every elementary amenable group is amenable and no amenable group contains a non-abelian free group.

Definition 2.65 (Class of groups $\mathcal{C}$). Let $\mathcal{C}$ be the smallest class of groups satisfying the following conditions:

- (i) $\mathcal{C}$ contains all free groups;
- (ii) If $\{G_i \mid i \in I\}$ is a directed system of subgroups directed by inclusion such that each G_i belongs to $\mathcal{C}$, then $G = \bigcup_{i \in I} G_i$ belongs to $\mathcal{C}$;
- (iii) Let $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ be an extension of groups such that K belongs to $\mathcal{C}$ and Q is elementary amenable, then G belongs to $\mathcal{C}$.

Definition 2.66 (Class of groups $\mathcal{A}$). Denote by $\mathcal{A}$ the collection of all groups which have property (BOF) and satisfy the Strong Atiyah Conjecture 2.27.

Definition 2.67 (Class of groups $\mathcal{D}$). Let $\mathcal{D}$ be the smallest class of groups satisfying the following conditions:

- (i) The trivial group belongs to $\mathcal{D}$;
- (ii) If $\{G_i \mid i \in I\}$ is a filtered system of groups in $\mathcal{D}$ (with arbitrary structure maps), then its colimit again belongs to $\mathcal{D}$;
- (iii) If $\{G_i \mid i \in I\}$ is a cofiltered system of groups in $\mathcal{D}$ (with arbitrary structure maps), then its limit again belongs to $\mathcal{D}$;

- (iv) If G belongs to $\mathcal{D}$ and $H \subseteq G$ is a subgroup, then $H \in \mathcal{D}$;
- (v) If $p: G \rightarrow A$ is an epimorphism of a torsionfree group G onto an elementary amenable group A and if $p^{-1}(B) \in \mathcal{D}$ for every finite group $B \subseteq A$, then $G \in \mathcal{D}$.

Note that each element in $\mathcal{D}$ is a torsionfree group, and the class $\mathcal{D}$ contains all residually (torsionfree elementary amenable) groups.

A group is called *indicable* if it is trivial or admits an epimorphism $G \rightarrow \mathbb{Z}$ onto $\mathbb{Z}$. Hence a group is called *locally indicable*, if every non-trivial finitely generated subgroup admits an epimorphism onto $\mathbb{Z}$. Locally indicable groups are torsionfree. Examples for locally indicable groups are torsionfree one-relator groups.

Theorem 2.68 (Status of the Strong Atiyah Conjecture 2.27)

- (i) If G belongs to $\mathcal{C}$ and satisfies property (BOF), then $G \in \mathcal{A}$;
- (ii) If G belongs to $\mathcal{D}$, then G is torsionfree and $G \in \mathcal{A}$;
- (iii) If $1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$ is an extension of groups, H is torsionfree and belongs to $\mathcal{A}$, and Q is locally indicable, then G is torsionfree and belongs to $\mathcal{A}$;
- (iv) If $G \in \mathcal{A}$ and $H \subseteq G$ is a subgroup with $\text{lcm}(H) = \text{lcm}(G)$, then $H \in \mathcal{A}$;
- (v) If G is the directed union $\bigcup_{i \in I} G_i$ of subgroups G_i directed by inclusion and each G_i belongs to $\mathcal{A}$, then G belongs to $\mathcal{A}$;
- (vi) The group G belongs to $\mathcal{A}$ if and only if all its finitely generated subgroups belong to $\mathcal{A}$;
- (vii) If $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ is an extension of groups such that K is finite and G belongs to $\mathcal{A}$, then Q belongs to $\mathcal{A}$;
- (viii) Let M be a connected (not necessarily compact) d -dimensional manifold (possibly with non-empty boundary) such that $d \leq 3$ and its fundamental group $\pi_1(M)$ is torsionfree, then $\pi_1(M) \in \mathcal{C}$ and hence $\pi_1(M) \in \mathcal{A}$;
- (ix) If the group G has property (BOF) and belongs to one of the following classes below, then G belongs to $\mathcal{A}$:
 - (a) Residually {torsionfree elementary amenable} groups;
 - (b) Free by elementary amenable groups;
 - (c) Braid groups;
 - (d) Right-angled Artin and Coxeter groups;
 - (e) Torsionfree p -adic analytic pro- p -groups;
 - (f) Finite extensions of fundamental groups of compact special cube complexes;
 - (g) Locally indicable groups;
 - (h) One-relator groups.

Proof. (i) This is due to Linnell, see for instance [39] or [45, Theorem 10.19 on page 378].

(ii) This follows from [26, Theorem 1.1], together with [15, Theorem 1.4].

(iii) This follows from [29, Proposition 6.5].

- (iv) This follows from [45, Theorem 6.29 (2) on page 253].
 - (v) See [45, Lemma 10.4 on page 371].
 - (vi) This follows from assertions (iv) and (v).
 - (vii) This follows from [45, Lemma 13.45 on page 473] because $\text{lcm}(G) = \text{lcm}(Q) \cdot |K|$.
 - (viii) This follows from [31, Theorem 1.1] for $d = 3$. The case $d = 2$ can be reduced to the case $d = 3$ by crossing with S^1 and assertion (iv) or just use the fact that the epimorphism the fundamental group of a connected manifold of dimension ≤ 2 onto its abelianization has a free group as kernel.
 - (ix) Items (ixa), (ixb), (ixg), and (ixh) follow from the previous assertions.
- Items (ixc), (ixd), (ixe), and (ixf) are proven in [38, Theorem 1.8], [37, Theorem 2], [18, Theorem 1.1], and [66, Theorem 1.1] for matrices over $\overline{\mathbb{Q}}[G]$. Note that [66, Theorem 1.1] states the result for matrices over $\mathbb{Q}[G]$, but the proof extends to $\overline{\mathbb{Q}}[G]$. Then apply [26, Theorem 1.1]. $\square$

Remark 2.69. The class $\mathcal{A}$ is very large by the aforementioned results. Nevertheless, we do not know whether all hyperbolic groups or all torsionfree amenable groups belong to $\mathcal{A}$. Note that a hyperbolic group automatically satisfies property (BOF).

Theorem 2.70 (Status of the K -theoretic Atiyah Conjecture 2.39)

The K -theoretic Atiyah Conjecture 2.39 holds for a group G and $F = \mathbb{C}$ if G belongs to the class $\mathcal{C}$ and has property (BOF).

Proof. This follows essentially by inspecting the proof of Linnell for the Atiyah Conjecture, see [33, Theorem 1.11]. $\square$

Remark 2.71. Note that for a torsion-free group the K -theoretic Atiyah Conjecture is equivalent to the Strong Atiyah Conjecture, compare Subsection 2.11.2.

Comment 16 (by T.): What remains interesting and open: which groups in $\mathcal{A}$ which are not torsionfree satisfy the K -theoretic Atiyah Conjecture?

2.10 Permanence Properties of the Strong Atiyah Conjecture

In contrast to the Full Farrell-Jones Conjecture one does not know much about the permanence properties of the Strong Atiyah Conjecture 2.27. It is open whether it is closed under taking finite products, subgroups, or overgroups of finite index.

There are partial results on question, whether the Strong Atiyah Conjecture 2.27 holds for a group G if it holds for a torsionfree subgroup of finite index, see for instance [38]. **Comment 17 (by T.):** I'll add a sentence making this more specific, if reasonable? **Comment 18 (by T.):** more results

in this direction, but with problems: In [75], Christian Wegner proves that the strong Atiyah conjecture for a group G implies the conjecture for any quotient group G/F if F is a finite normal subgroup of G . He also states conditions for a subgroup U the strong Atiyah conjecture for G implies the strong Atiyah conjecture for U . Unfortunately, some intermediate results used in the attempted proof are not correct, compare the Göttingen Master thesis of Steffen Schultze [67]. However, Schultze gives a different argument which implies for instance that the strong and center valued Atiyah conjecture for a product $U \times V$ of two groups is inherited by each factor U, V separately.

Sanchez Peralta [62, Theorem 1.1] showed that the Strong Atiyah Conjecture 2.27 is closed under taking amalgamations along finite groups. More generally, he shows that the Strong Atiyah Conjecture holds for the fundamental group of a graphs of groups with finite edge stabilizers, provided it holds for all vertex groups.

Exercise 2.72 Let G be a group satisfying property (BOF). Let $\{G_i \mid i \in I\}$ be a directed system of subgroups, directed by inclusion, such that G is the union of the G_i .

Show that G satisfies the Strong Atiyah Conjecture 2.27, if every group G_i satisfies the Strong Atiyah Conjecture 2.27.

Exercise 2.73 Let $\{G_i \mid i \in I\}$ be a collections of groups. Suppose that there is a natural number d such that for every $i \in I$ the order of every subgroup of G_i is bounded by d and G_i satisfies the Strong Atiyah Conjecture 2.27 and Put $G = *_{i \in I} G_i$.

Show that then G satisfies property (BOF) and the Strong Atiyah Conjecture 2.27.

2.11 Strategy for a Proof of the K -Theoretic Atiyah Conjecture

We have already explained in Section 2.8 that the Atiyah Conjecture has only a chance to be true if the group G has property (BOF), i.e., has a bound on the order of its finite subgroups. The strategy of proof of the Atiyah Conjecture for groups G which have property (BOF) and belong to the class $\mathcal{C}$ due to Linnell [39] has been elaborated on and expanded and finally carried out in details for the class $\mathcal{C}$ in [45, Chapter 10] and [59]. We summarize and record some old ideas and some new developments concerning it in this section. After we have introduced the fundamental square in Subsection 2.11.1, we give a brief summary in the torsionfree case in Subsection 2.11.2 and in the case, where G has property (BOF), in Subsection 2.11.3. Then we give more details for the interested reader in Subsection 2.11.4. Finally we give a refined strategy involving localization in Subsection 2.11.5.

2.11.1 The Fundamental Square

Let G be a group and F a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. In the sequel we will assume for convenience that F is closed under complex conjugation since then $FG \subseteq \mathcal{U}(G)$ is closed under the involution on $\mathcal{U}(G)$ coming from taking the adjoint of an operator. Recall that we have introduced $\mathcal{U}(G)$ and explained that $\dim_{\mathcal{N}(G)}$ extends to $\dim_{\mathcal{U}(G)}$ in Subsection 2.2.4.

Notation 2.74. Let S be a ring and $R \subset S$ be a subring. Denote by

$$T(R \subset S)$$

the set of all elements in R which become invertible in S . Denote by

$$\Sigma(R \subset S)$$

the set of all square matrices A over R which become invertible in S .

A subring $R \subset S$ is called *division closed* if $T(R \subset S) = R^\times$, i.e., any element in R which is invertible in S is already invertible in R . It is called *rationally closed* if $\Sigma(R \subset S) = \text{GL}(R)$, i.e., any square matrix over R which is invertible over S is already invertible over R . Notice that the intersection of division closed subrings of S is again division closed, and analogously for rationally closed subrings. Hence the following definition makes sense.

Definition 2.75 (Division and rational closure). Let S be a ring with subring $R \subset S$. The *division closure* $\mathcal{D}(R \subset S)$ or *rational closure* $\mathcal{R}(R \subset S)$ respectively is the smallest subring of S which contains R and is division closed or rationally closed respectively.

Proposition 2.76. *Given a subring R of S , the division closure $\mathcal{D}(R \subseteq S)$ can be constructed as follows.*

Put $\mathcal{D}_0(R \subseteq S) = R$. Define inductively for $n = 1, 2, \dots$ the subring $\mathcal{D}_n(R \subseteq S)$ of S to be the subring of S generated by $\mathcal{D}_{n-1}(R \subseteq S)$ and all elements $s \in S$ for which there exists $r \in \mathcal{D}_{n-1}(R \subseteq S)$ with $s = r^{-1}$ in S . Then $\mathcal{D}_{n-1}(R \subseteq S) \subseteq \mathcal{D}_n(R \subseteq S)$ and we get

$$\mathcal{D}(R \subseteq S) = \bigcup_{n \geq 0} \mathcal{D}_n(R \subseteq S).$$

Exercise 2.77 Prove Proposition 2.76.

Lemma 2.78. *Let $R \subset S \subseteq T$ be inclusions of rings.*

- (i) *We have $\mathcal{D}(R \subseteq T) = \mathcal{D}(R \subseteq \mathcal{D}(S \subseteq T)) \subseteq \mathcal{D}(S \subseteq T)$;*
- (ii) *If S is von Neumann regular, then $\mathcal{D}(R \subseteq S) = \mathcal{D}(R \subseteq T)$.*

Proof. (i) This follows from the definitions.

(ii) Consider $x \in S$ such that x is invertible in T . Then x is a non-zero-divisor

in S . Since S is von Neumann regular, we can find y in S with $xyx = x$. This implies $(xy - 1_R)x = x(yx - 1_R) = 0$. Hence we get $xy = xy = 1_R$, in other words, x is invertible in S . $\square$

Now one can consider the *division closure* $\mathcal{D}_F(G)$ of FG in $\mathcal{U}(G)$. Sometimes one views also the *rational closure* $\mathcal{R}_F(G)$ of FG in $\mathcal{U}(G)$.

Let $\mathcal{REG}_F(G)$ be the **-regular closure of FG in $\mathcal{U}(G)$* in the sense of Definition 3.8, i.e., the smallest von Neumann *-regular subring of $\mathcal{U}(G)$ containing FG .

Proposition 2.79. *Let G be a group and F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ which is closed under complex conjugation.*

- (i) *We have the inclusions $FG \subseteq \mathcal{D}_F(G) \subseteq \mathcal{R}_F(G) \subseteq \mathcal{REG}_F(G)$;*
- (ii) *If $\mathcal{D}_F(G)$ is von Neumann regular, then $\mathcal{D}_F(G) = \mathcal{R}_F(G) = \mathcal{REG}_F(G)$;*
- (iii) *If $\mathcal{REG}_F(G)$ is a skew field, then $\mathcal{D}_F(G) = \mathcal{R}_F(G) = \mathcal{REG}_F(G)$.*

Proof. (i) The inclusions $FG \subseteq \mathcal{D}_F(G) \subseteq \mathcal{R}_F(G)$ are obvious. Since $\mathcal{REG}_F(G)$ is von Neumann regular, it is not hard to check that it is division closed in $\mathcal{U}(G)$, see [58, Proposition 13.15 on page 103]. Hence $\mathcal{R}_F(G) \subseteq \mathcal{REG}_F(G)$.

(ii) If $\mathcal{D}_F(G)$ is von Neumann regular, it is *-regular and hence $\mathcal{REG}_F(G) \subseteq \mathcal{D}_F(G)$. Now the claim follows from assertion (i).

(iii) Suppose that $\mathcal{REG}_F(G)$ is a skew field. Then every element in $\mathcal{D}_F(G)$ different from 0 is a unit in $\mathcal{U}(G)$ and hence is a unit in $\mathcal{D}_F(G)$. Therefore $\mathcal{D}_F(G)$ is a skew field. Now apply assertion (ii). $\square$

The so-called *fundamental square* is given by the square of inclusions of rings

$$(2.80) \quad \begin{array}{ccc} FG & \longrightarrow & \mathcal{N}(G) \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{U}(G). \end{array}$$

where S is a von Neumann *-regular ring in the sense of Definition 3.2, e.g., $S = \mathcal{D}_F(G)$, $\mathcal{R}_F(G)$, or $\mathcal{REG}_F(G)$.

The relevance of the fundamental square was discovered by Linnell. The ring $\mathcal{D}_F(G)$ is often called the *Linnell ring*.

2.11.2 Summary in the Torsionfree Case

Let G be a torsionfree group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ which is closed under complex conjugation. Then by Theorem 2.83 and Theorem 2.84 the following assertions are equivalent:

- (i) The K -theoretic Atiyah Conjecture 2.39 holds for G and F ;

- (ii) The Strong Atiyah Conjecture 2.28 with coefficients in F for torsionfree groups holds for G .
- (iii) There is a ring S satisfying $FG \subseteq S \subseteq \mathcal{U}(G)$, which is von Neuman regular and satisfies $\tilde{K}_0(S) = \{0\}$;
- (iv) The ring $\mathcal{REG}_F(G)$ satisfies $\tilde{K}_0(\mathcal{REG}_F(G)) = \{0\}$;
- (v) The ring $\mathcal{REG}_F(G)$ is a skew field;
- (vi) The ring $\mathcal{R}_F(G)$ is a skew field;
- (vii) The ring $\mathcal{D}_F(G)$ is von Neumann regular and satisfies $\tilde{K}_0(S) = \{0\}$;
- (viii) We have $\tilde{G}_0(\mathcal{D}_F(G)) = \{0\}$;
- (ix) The ring $\mathcal{D}_F(G)$ is a skew field;

To prove the K -theoretic Atiyah Conjecture 2.39 or the equivalent Strong Atiyah Conjecture 2.28 with coefficients in F for torsionfree groups, the most promising attempt seems to be to prove assertion (iii) or (iv). Assertion (viii) looks also promising but we have explained in Subsection 2.2.5 that it is better to work with $K_0(R)$ than with $G_0(R)$. One reason is that the inclusion $FG \rightarrow \mathcal{D}_F(G)$ does *not* induce a homomorphism from $G_0(FG)$ to $G_0(\mathcal{D}_F(G))$ in general. We will also prove that the three rings $\mathcal{D}_F(G)$, $\mathcal{R}_F(G)$, and $\mathcal{REG}_F(G)$ agree provided that one of the equivalent assertions above is true.

2.11.3 Summary in the Case Where There Is a Bound on the Orders of Finite Subgroups

Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ which is closed under complex conjugation.

Notation 2.81. We say that a ring S with $FG \subseteq S \subseteq \mathcal{U}(G)$ has property (K) if the composite

$$\bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \rightarrow K_0(FG) \xrightarrow{K_0(k)} K_0(S)$$

for $k: FG \rightarrow S$ the inclusion is surjective.

Then by Theorem 2.84 the following assertions are equivalent if G has property (BOF):

- (i) The K -theoretic Atiyah Conjecture 2.39 holds for G and F
- (ii) There is a ring S satisfying $FG \subseteq S \subseteq \mathcal{U}(G)$ which is von Neuman regular and has property (K);
- (iii) The ring $\mathcal{REG}_F(G)$ has property (K);
- (iv) The ring $\mathcal{R}_F(G)$ is von Neumann regular and has property (K);
- (v) The ring $\mathcal{D}_F(G)$ is von Neumann regular and has property (K);
- (vi) The ring $\mathcal{D}_F(G)$ is semisimple and has property (K);
- (vii) The composite $\bigoplus_{H \subseteq G, |H| < \infty} G_0(FH) \rightarrow G_0(\mathcal{D}_F(G))$ induced by the various inclusions $FH \rightarrow \mathcal{D}_F(G)$ is surjective;

To prove the K -theoretic Atiyah Conjecture 2.39 the most promising attempt seems to be to prove assertion (ii) and (iii). Assertion (vii) looks also promising but we have explained in Subsection 2.2.5 that it is better to work with $K_0(R)$ than with $G_0(R)$. We will also prove that in the case that one of the equivalent assertions above is true, the three rings $\mathcal{D}_F(G)$, $\mathcal{R}_F(G)$, and $\mathcal{REG}_F(G)$ agree.

2.11.4 More Information and Details

Theorem 2.82 (Strategy for a proof of the K -theoretic Atiyah Conjecture) *Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ which is closed under complex conjugation.*

- (i) *If there is a ring S satisfying $FG \subseteq S \subseteq \mathcal{U}(G)$ which is von Neuman regular and has property (K), then the K -theoretic Atiyah Conjecture 2.39 holds for G and F ;*
- (ii) *If G satisfies the Full Farrell-Jones Conjecture and there is a ring S satisfying $FG \subseteq S \subseteq \mathcal{U}(G)$ which is von Neuman regular and for which $K_0(k): K_0(FG) \rightarrow K_0(S)$ is surjective, then the K -theoretic Atiyah Conjecture 2.39 holds for G and F ;*
- (iii) *If $\mathcal{REG}_F(G)$ has property (K), then the K -theoretic Atiyah Conjecture 2.39 holds for G and F ;*
- (iv) *If G satisfies the Full Farrell-Jones Conjecture and the homomorphism $K_0(k): K_0(FG) \rightarrow K_0(\mathcal{REG}_F(G))$ is surjective, then the K -theoretic Atiyah Conjecture 2.39 holds for G and F ;*
- (v) *If G is torsionfree, the following statements are equivalent:*
 - (a) *The K -theoretic Atiyah Conjecture 2.39 holds for G and F ;*
 - (b) *The Strong Atiyah Conjecture Conjecture 2.28 with coefficients in F for G is true;*
 - (c) *There is a skew field D with $FG \subseteq D \subseteq \mathcal{U}(G)$;*
 - (d) *$\mathcal{D}_F(G)$ is a skew field;*
 - (e) *$\mathcal{R}_F(G)$ is a skew field;*
 - (f) *$\mathcal{REG}_F(G)$ is a skew field;*
- (vi) *Let S be a ring satisfying $FG \subseteq S \subseteq \mathcal{U}(G)$ which is von Neuman regular and has property (K). Then the following assertions are equivalent:*
 - (a) *The group G has property (BOF);*
 - (b) *The ring S is semisimple;*
- (vii) *Let S be a ring satisfying $FG \subseteq S \subseteq \mathcal{U}(G)$ which is von Neuman regular and has property (K). Then the following assertions are equivalent:*
 - (a) *G is torsionfree;*
 - (b) *The ring S is a skew field;*

(viii) If G is torsionfree and there is a ring S satisfying $FG \subseteq S \subseteq \mathcal{U}(G)$ which is von Neuman regular and has property (K), then the rings $\mathcal{D}_F(G)$, $\mathcal{R}_F(G)$, and $\mathcal{REG}_F(G)$ agree and are skew fields.

Proof. (i) Let M be a finitely presented FG -module. Then $S \otimes_{FG} M$ is a finitely presented S -module. As S is von Neumann regular, $S \otimes_{FG} M$ is a finitely generated projective S -module and hence defines a class $[S \otimes_F M] \in K_0(S)$. Because of property (K) this class lies in the image of $\bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \rightarrow K_0(FG) \xrightarrow{K_0(k)} K_0(S)$. Hence $\mathcal{U}(G) \otimes_{FG} M$ is a finitely generated projective $\mathcal{U}(G)$ -module whose class lies in the image of $\bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \rightarrow K_0(FG) \xrightarrow{K_0(l)} K_0(\mathcal{U}(G))$ for the inclusion l .

(ii) This follows from assertion (i) since the Full Farrell-Jones Conjecture implies the surjectivity of $\bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \rightarrow K_0(FG)$, see [46, Theorem 13.65 (xii) on page 406].

(iii) This follows from assertion (i), since $\mathcal{REG}_F(G)$ is von Neumann regular.

(iv) This follows from assertion (ii) since $\mathcal{REG}_F(G)$ is von Neumann regular.

(v) We get the implication (va) $\implies$ (vb) from Remark 2.42. The implication (vb) $\implies$ (vd) is proved in [45, Lemma 10.39 on page 388], where only the case $F = \mathbb{C}$ is treated, but the argument carries directly over to the general case $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. The implication (vd) $\implies$ (vc) is obvious. The implication (vc) $\implies$ (va) follows from assertion (ii). Hence we have proved that the assertions (va), (vb), (vc), and (vd) are equivalent. Now assertion (v) follows from assertion (viii).

(vi) Suppose that S is semisimple. Then $K_0(S)$ is finitely generated, see (2.23). Hence the image of the composite

$$\bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \rightarrow K_0(FG) \xrightarrow{K_0(k)} K_0(S) \xrightarrow{K_0(l)} K_0(\mathcal{U}(G)) \xrightarrow{\dim_{\mathcal{U}(G)}} \mathbb{R}$$

for the inclusion $l: S \rightarrow \mathcal{U}(G)$ is finitely generated. This image is $\frac{1}{|\mathcal{FLN}(G)|} \mathbb{Z} \subseteq \Lambda$. Hence G has property (BOF).

Suppose G has property (BOF). Because of Proposition 2.17 it suffices to show that S is Noetherian.

Let $\text{lcm}(G)$ be the least common multiple of the orders of finite subgroups of G . Since S has property (K), the image of the composition

$$K_0(S) \xrightarrow{l} K_0(\mathcal{U}(G)) \xrightarrow{\dim_{\mathcal{U}(G)}} \mathbb{R}$$

lies in $\frac{1}{\text{lcm}(G)} \mathbb{Z} = \{r \in \mathbb{R} \mid \text{lcm}(G) \cdot r \in \mathbb{Z}\}$. Next we show that for any chain of ideals $\{0\} = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_r = S$ of $\mathcal{S}(G)$ with $I_i \neq I_{i+1}$ we have $r \leq \text{lcm}(G)$.

Choose $x_i \in I_i$ with $x_i \notin I_{i-1}$ for $1 \leq i \leq r-1$. Let J_i be the ideal generated by $x_1, x_2, \dots, x_i$ for $1 \leq i \leq r-1$. Then we obtain a sequence

of finitely generated ideals of the same length $\{0\} = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_r = \mathcal{S}(G)$ of $\mathcal{S}(G)$ with $J_i \neq J_{i+1}$. Since S is von Neumann regular, Proposition 2.18 implies we get a direct sum decompositions $J_i = J_{i-1} \oplus K_i$ for $i = 1, 2, \dots, r$ for finitely generated projective non-trivial S -modules $K_1, K_2, \dots, K_r$. Choose an idempotent $p_i \in M_{n_i}(S)$ representing K_i . Then p_i considered as an element $M_{n_i}(\mathcal{U}(G))$ represents $\mathcal{U}(G) \otimes_{\mathcal{S}(G)} K_i$ and is non-trivial. Hence $\mathcal{U}(G) \otimes_{\mathcal{S}(G)} K_i$ is a non-trivial finitely generated projective $\mathcal{U}(G)$ -module. We conclude from [45, Theorem 8.29 on page 330] that $\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{U}(S)} K_i) > 0$ holds for $i = 1, 2, \dots, r$. Hence we get

$$0 < \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{S}(G)} J_1) < \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{S}(G)} J_2) \\ < \cdots < \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{S}(G)} J_{r-1}) < 1.$$

Since $\text{lcm}(G) \cdot \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_S J_i)$ is an integer for $i = 1, 2, \dots, r-1$, we get $r \leq \text{lcm}(G)$.

(vii) Suppose that G is torsionfree. We have seen in the proof of assertion (vi) that S has no non-trivial ideal. This implies that S is a skew field.

For the opposite implication, suppose that S is a skew field. Then $\mathbb{Z} \rightarrow K_0(S)$ sending n to $n \cdot [S]$ is bijective. Hence the image of $K_0(S) \xrightarrow{K_0(l)} K_0(\mathcal{U}(G)) \xrightarrow{\dim_{\mathcal{U}(G)}} \mathbb{R}$ is $\mathbb{Z}$. Therefore the image of the composite

$$\bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \rightarrow K_0(FG) \xrightarrow{K_0(k)} K_0(S) \xrightarrow{K_0(l)} K_0(\mathcal{U}(G)) \xrightarrow{\dim_{\mathcal{U}(G)}} \mathbb{R}$$

is $\mathbb{Z}$. Since this image is $\frac{1}{|\mathcal{FLN}(G)|} \mathbb{Z} \subseteq \Lambda$, the group G is torsionfree.

(viii) Since $\mathcal{D}_F(G)$ is division closed in $\mathcal{U}(G)$ and hence division closed in the skew field S , it is itself a skew field. Now the equality $\mathcal{D}_F(G) = \mathcal{R}_F(G) = \mathcal{REG}_F(G)$ follows from Proposition 2.79 (ii). This finishes the proof of Theorem 2.82. $\square$

Next we record the special case where G is torsionfree.

Theorem 2.83 (The Strong Atiyah Conjecture for torsionfree groups and skew fields) *Let G be a torsionfree group. Consider a field F with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ which is closed under complex conjugation.*

(i) *The following statements are equivalent:*

- (a) *The K -theoretic Atiyah Conjecture 2.39 holds for G and F ;*
- (b) *The group G satisfies the Strong Atiyah Conjecture 2.28; with coefficients in F for torsionfree groups;*
- (c) *There exists a skew field S with $FG \subseteq S \subseteq \mathcal{U}(G)$;*
- (d) *The ring $\mathcal{D}_F(G)$ is a skew field;*
- (e) *The ring $\mathcal{R}_F(G)$ is a skew field;*
- (f) *The ring $\mathcal{REG}_F(G)$ is a skew field;*

- (g) The subrings rings $\mathcal{D}_F(G)$, $\mathcal{R}_F(G)$, and $\mathcal{REG}_F(G)$ agree and are skew fields;
- (ii) Suppose that G satisfies the Strong Atiyah Conjecture 2.28 with coefficients in F for torsionfree groups;
Then we get for every projective FG -chain complex C_* and every $n \geq 0$

$$b_n^{(2)}(\mathcal{N}(G) \otimes_{FG} C_*) = \dim_{\mathcal{D}_F(G)}(H_n(\mathcal{D}_F(G) \otimes_{\mathbb{Q}G} C_*)).$$

In particular $b_n^{(2)}(\mathcal{N}(G) \otimes_{FG} C_*)$ is an integer or ∞ .

Proof. (i) Assertions (ia) and (ib) are equivalent by Proposition 2.46 **Comment 19 (by T.):** at this point the argument has to be added . The assertions (ia) (ic), (id) (ie), and (if) are equivalent by Theorem 2.82 (v). We get the implication (id) $\implies$ (ig) from Proposition 2.79 (ii).

(ii) This follows from Proposition 2.19 (ii) and (2.21). $\square$

Consider the situation of Theorem 2.82 (vi). One can say actually more about S than just that it is semisimple. Recall that we have seen in the proof of assertion (vi) that for any chain of ideals $\{0\} = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_r = S$ of S with $I_i \neq I_{i+1}$ we have $r \leq \text{lcm}(G)$.

Theorem 2.84 (The ring $\mathcal{D}_F(G)$ is Atiyah-expected)

Let G be a group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ which is closed under complex conjugation. Suppose G has property (BOF). Then the following assertions are equivalent:

- (i) The ring $\mathcal{D}_F(G)$ is semisimple, which is Atiyah-expected in the sense of [33, Definition 3.6];
- (ii) The ring $\mathcal{D}_F(G)$ is semisimple and has property (K);
- (iii) The composite

$$\bigoplus_{H \subseteq G, |H| < \infty} G_0(FH) \rightarrow G_0(\mathcal{D}_F(G))$$

induced by the various inclusions $FH \rightarrow \mathcal{D}_F(G)$ is surjective;

- (iv) The K -theoretic Atiyah Conjecture 2.39 holds for G and F ;
- (v) The center-valued Atiyah Conjecture 2.45 holds for G and F .

Proof. The equivalence of the statements (i), (ii), (iii), and (v) is proved in [33, Theorem 3.7]. The equivalence of the statements (iv) and (v) is proved in Proposition 2.46. $\square$

The notion of Atiyah-expected in the sense of [33, Definition 3.6] makes precise predictions about the numbers k and l_i appearing in the decomposition of $\mathcal{D}_F(G)$ as product $\prod_{i=1}^k M_{l_i}(D_i)$ for appropriate division rings D_i in terms of G and F .

2.11.5 Involving Localization

Comment 20 (by W.): The following strategy is under construction. I hope that one can decide whether condition (ii) is satisfied.

Let G be a group satisfying property (BOF) and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ which is closed under complex conjugation. Next we describe a strategy for a proof of the K -Theoretic Atiyah Conjecture which is a further development of the one of Linnell presented above or in [45, Section 10.3] based on [39]. Namely, it suffices to show that the composite

$$(2.85) \quad \bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \rightarrow K_0(FG) \rightarrow K_0(\mathcal{R}\mathcal{E}\mathcal{G}_F(G))$$

is bijective, and this obviously is true if the following conditions are satisfied:

- (i) G satisfies the condition (UL) of Definition 5.92, i.e., the canonical epimorphism

$$\lambda: FG_{\Sigma(FG \subseteq \mathcal{U}(G))} \rightarrow \mathcal{R}_F(G)$$

is bijective, where $\Sigma(FG \subseteq \mathcal{U}(G))$ is the set of all square matrices over FG which become invertible in $\mathcal{U}(G)$ and $FG_{\Sigma(FG \subseteq \mathcal{U}(G))}$ denotes the associated localization, see Subsection 5.4.2

- (ii) There is the Schofield's construction of a localization sequence

$$K_1(R) \rightarrow K_1(R_{\Sigma}) \rightarrow K_0(\mathcal{T}) \rightarrow K_0(R) \rightarrow K_0(R_{\Sigma})$$

where $\mathcal{T}$ is the full subcategory of the category of the finitely presented R -modules whose objects are cokernels of elements in Σ , see [65, Theorem 5.12 on page 60].

The condition is that it can be extended in the situation $R = FG$ and $\Sigma = \Sigma(FG \subseteq \mathcal{U}(G))$ to the right

$$\begin{aligned} K_1(FG) &\rightarrow K_1(FG_{\Sigma(FG \subseteq \mathcal{U}(G))}) \rightarrow K_0(\mathcal{T}) \rightarrow K_0(FG) \\ &\rightarrow K_0(FG_{\Sigma(FG \subseteq \mathcal{U}(G))}) \rightarrow K_{-1}(\mathcal{T}) \rightarrow K_{-1}(FG) \\ &\rightarrow K_1(FG_{\Sigma(FG \subseteq \mathcal{U}(G))}) \rightarrow K_{-2}(\mathcal{T}) \rightarrow \cdots \end{aligned}$$

- (iii) $K_{-1}(\mathcal{T})$ vanishes;
 (iv) We have $\mathcal{R}_F(G) = \mathcal{R}\mathcal{E}\mathcal{G}_F(G)$, or, equivalently $\mathcal{R}_F(G)$ is von Neumann regular.
 (v) The map

$$\bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \rightarrow K_0(FG)$$

is surjective.

Suppose that G satisfies the K -theoretic Farrell-Jones Conjecture. Then condition (v) is satisfied and $K_{-1}(FG)$ is trivial. Moreover, if conditions (i)

and (iv) are true, then the map (2.85) is surjective if and only if condition (iii) holds.

To check whether this strategy makes sense, let us assume that G satisfies the K -theoretic Farrell-Jones Conjecture, satisfies property (BOF) and that the map (2.85) is surjective. Then $\mathcal{D}_F(G)$ is semisimple and $\mathcal{D}_F(G) = \mathcal{R}_F(G) = \mathcal{R}\mathcal{E}\mathcal{G}_F(G)$ holds. Hence condition (iv) and (v) are satisfied and condition (ii) implies condition (iii). Condition (i) is investigated in [36, Theorem 2.14 and 2.17].

2.12 Coefficient Change from $\overline{\mathbb{Q}}$ to $\mathbb{C}$

Comment 21 (by W.): This will be inserted by Thomas.

2.13 Notes

For more information about the Atiyah Conjecture we refer for instance to [26], [28], and [45, Chapter 10], **Comment 22 (by W.):** This list has to be completed.

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Chapter 3

Proof of the Atiyah Conjecture for Locally Indicible Groups

3.1 Introduction

In this chapter we want to present the proof of Jaikin-Zapirain and López-Álvarez [29, Theorem 1.1] that the Strong Atiyah Conjecture 2.29 for torsionfree groups holds for locally indicible groups. It uses ideas and concepts of the paper by Dicks-Herbera-Sánchez [14], where the proofs of Hughes [25] were enlightened. The material of this chapter is not needed for the remainder of the book. Nevertheless we decided to include a rather complete exposition of the proof, since it contains interesting and intriguing ideas, strategies, and techniques.

Recall that the situation of the Strong Atiyah Conjecture 2.29 simplifies if the group G is torsionfree. Let G be a torsionfree group and F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ which is closed under complex conjugation. According to Subsection 2.11.2 the K -theoretic Atiyah Conjecture 2.39 and the Strong Atiyah Conjecture 2.28 with coefficients in F for torsionfree groups are equivalent and they are true if one of (and hence all) the rings $\mathcal{D}_F(G)$, $\mathcal{R}_F(G)$, and $\mathcal{REG}_F(G)$ is a skew field. So one may try to prove the Atiyah Conjecture by studying embeddings of FG into certain rings and in particular into skew fields and use the machinery developed about epic division FG -algebras. The advantage is that no considerations about algebraic K -theory are necessary anymore, as they appear for instance in Subsection 2.11.5, since the composite $K_0(\mathbb{C}) \rightarrow K_0(\mathcal{D}_F(G))$ is obviously bijective if $\mathcal{D}_F(G)$ is a skew field.

So it suffices to show Theorem 3.174 which essentially claims for a locally indicible group G that, for a Sylvester module rank function $\dim$ on the group ring FG which is Hughes free in the sense of Definition 3.164 and comes with a positive definite $*$ -regular envelope $(\varphi: FG \rightarrow U, \dim_U)$ in the sense of Definition 3.139, the ring U is a division ring. The key ingredient is now the notion of a complexity for an element in U which allows to show by induction over the complexity that every non-zero element in U is invertible. The induction beginning says that any element in $\varphi(E^\times G)$ is invertible in U which follows directly from the fact that any element in $E^\times G$ is invertible in EG . The hard induction step is based on the theory of Hughes free EG -division rings, Sylvester rank functions, and the notion of complexity itself.

The complexity of an element in U is a rather sophisticated notion. Roughly speaking, the complexity of an element in U measures how many sums, how many products, how many brackets, and how often taking inverses are need to express an element in U in terms of elements of the shape $\varphi(eg)$ for $e \in E^\times$ and $g \in G$. This is encoded in a rooted tree associated to an ele-

ment in U and uses the fact that the set of rooted trees $\mathcal{T}$ is well-ordered. For the precise implementation of this idea one needs to study rooted trees and (universal) rational U -rings. To carry out the induction step a deep analysis of Sylvester rank functions is needed. All this will be explained in detail in the forthcoming sections of this chapter.

Comment 23 (by W.): This is only a first draft of the introduction.

3.2 Von Neumann *-Regular Rings

Definition 3.1 (Ring with involution). A *ring with involution* R is an associative ring R with unit together with an *involution of rings*

$$*: R \rightarrow R, \quad r \mapsto r^*,$$

i.e., a map satisfying $(r^*)^* = r$, $(r + s)^* = r^* + s^*$, $(r \cdot s)^* = s^* \cdot r^*$, and $1^* = 1$ for $r, s \in R$.

The involution $*$ is called *proper* if $r^*r = 0 \implies r = 0$ holds for all $r \in R$.

If R is a ring with involution, the matrix ring $M_{n,n}(R)$ inherits an involution by sending $A = (A_{i,j})$ to the matrix $A^* = (a_{j,i}^*)$.

Definition 3.2 (Von Neumann *-regular ring). A *von Neumann *-regular ring* R is a ring with proper involution in the sense of Definition 3.1 which is von Neumann regular in the sense of Definition 2.16.

We say that a von Neumann *-regular ring R is *positive definite* if $M_{n,n}(R)$ is a von Neumann *-regular ring with the induced involution for all $n \in \mathbb{Z}^{\geq 1}$.

Remark 3.3 (Explicite relative inverses). Let R be a von Neumann *-regular ring R . Then for every element $r \in R$, we can define an explicite element *called relative inverse*

$$(3.4) \quad r[-1] \in R$$

satisfying

$$(3.5) \quad rr[-1]r = r$$

see [26, Proposition 3.2]. The element $r[-1]$ is uniquely characterized by the property that $r[-1]r$ and $rr[-1]$ are projections, i.e., self-adjoint idempotents, and (3.5) holds.

Proposition 3.6. *Let R be a von Neumann *-regular ring and let I be a two-sided ideal of R . Then I is *-closed and the induced involution on R/I is proper, in other words, R/I inherits the structure of a von Neumann *-regular ring.*

Proof. See [27, Proposition 3.3] □

Proposition 3.7. *Let R be a $*$ -subring of a von Neumann $*$ -regular ring S . Then there exists a smallest $*$ -regular subring $\mathcal{REG}(R \subseteq S)$ of S containing R , namely the intersection of all von Neumann $*$ -regular rings of S containing R . Moreover, $\mathcal{REG}(R \subseteq S)$ can be constructed as follows:*

Put $\mathcal{REG}_0(R \subseteq S) = R$ which is a $*$ -subring of S . Define inductively for $n = 1, 2, \dots$ the $*$ -subring $\mathcal{REG}_n(R \subseteq S)$ of S to be the subring of S generated by $\mathcal{REG}_{n-1}(R \subseteq S)$ and all elements $r[-1]$ where r runs through the elements of $\mathcal{REG}_{n-1}(R \subseteq S)$.

Then $\mathcal{REG}_{n-1}(R \subseteq S) \subseteq \mathcal{REG}_n(R \subseteq S)$ and we put

$$\mathcal{REG}(R \subseteq S) = \bigcup_{n \geq 0} \mathcal{REG}_n(R \subseteq S)$$

Proof. See [1, Proposition 6.2]. □

Definition 3.8 ($*$ -regular closure). We call $\mathcal{REG}(R \subseteq S)$ the $*$ -regular closure of R in S .

3.3 Epic Ring Homomorphism

Definition 3.9 (Epic ring homomorphism). A homomorphism $\varphi: R \rightarrow S$ is called *epic* if it is right cancellable in the sense that for every ring T and two ring homomorphisms $\psi_1, \psi_2: S \rightarrow T$ the implication $\psi_1 \circ \varphi = \psi_2 \circ \varphi \implies \psi_1 = \psi_2$ holds.

Proposition 3.10. *Let $\varphi: R \rightarrow S$ be a ring homomorphism. Then the following statements are equivalent:*

- (i) *The ring homomorphism φ is epic;*
- (ii) *In the S -bimodule $S \otimes_R S$ we have $1_S \otimes s = s \otimes 1_S$ for all $s \in S$;*
- (iii) *The multiplication map $m: S \otimes_R S \rightarrow S$ sending $s_1 \otimes s_2$ to $s_1 s_2$ is an isomorphism of S -bimodules.*

Proof. See [12, Proposition 4.1.1]. □

If $\varphi: R \rightarrow S$ is a ring homomorphism, we say that $s \in S$ is *dominated* by φ if for any ring T and two ring homomorphisms $\alpha, \beta: S \rightarrow T$ we have the implication $\alpha \circ \varphi = \beta \circ \varphi \implies \alpha(s) = \beta(s)$. The set of elements of S dominated by φ is a subring $\text{dom}(\varphi)$ of S , called the *domination* of φ . Obviously φ is epic if and only if $\text{dom}(\varphi) = S$.

Every surjective ring homomorphism is epic. The converse is in general not true but holds for von Neumann regular rings as source.

Proposition 3.11. *Let $\varphi: R \rightarrow S$ be a ring homomorphism with a von Neumann regular ring R as source.*

- (i) *We have $\text{dom}(\varphi) = \text{im}(\varphi)$;*
- (ii) *The ring homomorphism φ is epic if and only if φ is surjective.*

Proof. See [70, Proposition 1.4 in Chapter XI on page 226]. $\square$

Exercise 3.12 Let $\varphi_1: R \rightarrow S_1$ and $\varphi_2: R \rightarrow S_2$ be two epic ring homomorphisms such that S_1 and S_2 are von Neumann regular. Let $\psi_1: S_1 \rightarrow T$ and $\psi_2: S_2 \rightarrow T$ be two ring homomorphisms with $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$.

Then $\text{im}(\psi_1) = \text{im}(\psi_2)$ holds.

Proposition 3.13. *Let $\varphi: R \rightarrow S$ be a ring homomorphism. Then*

- (i) *We have $\mathcal{D}(\text{im}(\varphi) \subseteq S) \subseteq \text{dom}(\varphi)$;*
- (ii) *If $\mathcal{D}(\text{im}(\varphi) \subseteq S) = S$ holds, then φ is epic.*

Proof. (i) Consider two ring homomorphisms $\alpha, \beta: S \rightarrow T$ with $\alpha \circ \varphi = \beta \circ \varphi$. Then $E = \{s \in S \mid \alpha(s) = \beta(s)\}$ is a subring of S containing $\text{im}(\varphi)$. If $e \in E$ is invertible in S , then e belongs already to E because of $\alpha(e^{-1}) = \alpha(e)^{-1} = \beta(e)^{-1} = \beta(e^{-1})$. Hence we get $\mathcal{D}(\text{im}(\varphi) \subseteq S) \subseteq E$. This implies $\mathcal{D}(\text{im}(\varphi) \subseteq S) \subseteq \text{dom}(\varphi)$.

(ii) This follows directly from assertion (i). $\square$

Proposition 3.14. *Let $\varphi: R \rightarrow S$ be a ring homomorphism with a division ring S as target.*

Then φ is epic if and only if $\mathcal{D}(\text{im}(\varphi) \subseteq S) = S$ holds.

Proof. The “if”-statement follows from Proposition 3.13 (ii), the “only if”-statement is proved as follows. Suppose that φ is epic. Then the inclusion $i: \mathcal{D}(\text{im}(\varphi) \subseteq S) \rightarrow S$ is epic. As S is a division ring, $\mathcal{D}(\text{im}(\varphi) \subseteq S)$ is a division ring and hence von Neumann regular. Now apply Proposition 3.11 (ii). $\square$

Proposition 3.15. *Let R be a ring with involution, S be a von Neumann *-regular ring, and $\varphi: R \rightarrow S$ be a homomorphism of rings with involution.*

*Then φ is epic if and only if S is the *-regular closure of $\varphi(R)$ in S .*

Proof. See [29, Proposition 2.4]. $\square$

Lemma 3.16. *Let $\varphi: R \rightarrow S$ be an inclusion of rings which is epic. Then we have $\mathcal{Z}(R) \subseteq \mathcal{Z}(S)$.*

Exercise 3.17 Give the proof of Lemma 3.16.

3.4 (Rooted) Trees

A graph X is a 1-dimensional CW-complex. Recall that an open n -cell e of a CW-complex X is a path component of $X_n \setminus X_{n-1}$ and the associated closed n -cell $\bar{e}$ of e is the closure of e in X . A vertex of a graph X is the same as a closed 0-cell and an edge is the same as a closed 1-cell. Recall that a CW-complex is finite, i.e., has only finitely many open cells, if and only if it is compact.

We say that two vertices x and y are *connected by the edge e* if x and y are contained in e . We call two vertices *adjacent* if they are connected by some edge.

A graph is called a *tree* if it satisfies one of the following equivalent conditions:

- (i) X is contractible;
- (ii) X is simply connected;
- (iii) X is connected and the first singular homology $H_1(X; \mathbb{Z})$ vanishes;
- (iv) X is connected and does not contain a subgraph homeomorphic to S^1 ;
- (v) X is connected and $X \setminus \{v\}$ is not connected for every vertex v ;
- (vi) For two vertices x and y there is precisely one $n \in \mathbb{Z}_{\geq 0}$ and one sequence of pairwise distinct n vertices $v_0, v_1, \dots, v_n$ such that $v_0 = x$ and $v_n = y$ holds and v_{i-1} and v_i are adjacent for $i = 0, 1, \dots, (n-1)$.

Exercise 3.18 Show that a connected finite graph X is a tree if and only if its Euler characteristic $\chi(X)$ is 1.

Let X be a tree. If we have two vertices x and y , we define their distance $d_X(x, y)$ to be the number n appearing condition (vi). One easily checks that this defines a metric d_X on the set of vertices of X . There is precisely one extension of this metric to a metric d_X on X , denoted in the same way, such that for every edge e we get an isometry $[0, 1] \xrightarrow{\cong} e$, where we equip e with the metric induced by d_X and $[0, 1]$ with the standard metric.

An isomorphism of trees is a cellular homeomorphism.

A *rooted tree* (X, x_0) is a pointed tree, i.e., a tree with a preferred choice of a vertex x_0 , called *root*. An isomorphism of rooted trees is an isomorphism of trees sending one root to the other.

The root will enable us to introduce the following notions and an addition and multiplication on the set $\mathcal{T}$ of isomorphism classes of finite rooted trees.

Let $X = (X, x_0)$ be a rooted tree. The *height* of a vertex x of X

$$(3.19) \quad \text{height}(x) \in \mathbb{Z}_{\geq 0}$$

is defined to be $d_X(x, x_0)$. The height of X

$$(3.20) \quad \text{height}(X) \in \mathbb{Z}_{\geq 0} \amalg \{\infty\}$$

is defined to be the supremum of the heights of all the vertices of X . If X is finite, we have $\text{height}(X) \in \mathbb{Z}_{\geq 0}$.

Define the *family of X* of the rooted tree $X = (X, x_0)$

$$(3.21) \quad \text{fam}(X)$$

to be the set of rooted trees given by the components of the graph obtained from X by deleting the root x_0 and every open 1-cell e for which x_0 belongs to its closure $\bar{e}$. The root of an element in $\text{fam}(X)$ is the vertex which is adjacent to the root x_0 in X .

Exercise 3.22 Let X be finite rooted tree. Show

$$\text{height}(X) = \max\{\text{height}(Y) \mid Y \in \text{fam}(X)\} + 1.$$

The *width* of a rooted tree $X = (X, x_0)$

$$(3.23) \quad \text{width}(X) \in \mathbb{Z}_{\geq 0} \amalg \{\infty\}$$

is defined to be $|\text{fam}(X)|$ if $\text{fam}(X)$ is finite, and to be ∞ otherwise. If X is finite, then $\text{width}(X) \in \mathbb{Z}_{\geq 0}$

Given a rooted tree $X = (X, x_0)$, define the *expanded rooted tree* $\text{expand}(X)$ to be the rooted tree which is obtained from X by adding a stem, i.e, we add a new vertex x_1 and one edge connecting x_1 and x_0 to X and declare x_1 to be the root of $\text{expand}(X)$.

Denote by

$$(3.24) \quad 0_{\mathcal{T}} \text{ and } 1_{\mathcal{T}}$$

the trees given by the space $\{0\}$ and by the unit interval $[0, 1]$ with $\{0, 1\}$ as 0-skeleton and 1 as root. Note that $\text{fam}(0_{\mathcal{T}})$ is the emptyset, and we have $\text{fam}(1_{\mathcal{T}}) = \{0_{\mathcal{T}}\}$, and $\text{expand}(0_{\mathcal{T}})$ is isomorphic to $1_{\mathcal{T}}$ in $\mathcal{T}$.

Remark 3.25. In the sequel we abuse notation and often do not distinguish between a finite rooted tree and its class in $\mathcal{T}$.

We define the *sum of the rooted trees* X and Y

$$(3.26) \quad X + Y$$

to be the rooted tree which is obtained from the disjoint union $X \amalg Y$ by identifying the two roots and choosing as the root in $X + Y$ the vertex represented by the root of X and Y . This sum defines an addition on $\mathcal{T}$ which is associative, commutative, and has a zero-element, namely, the one given by $0_{\mathcal{T}}$. In other words, $(\mathcal{T}, +)$ becomes a commutative additive monoid. Obviously $X + Y = 0_{\mathcal{T}}$ implies $X = Y = 0_{\mathcal{T}}$.

Next we define a product on $\mathcal{T}$. Define the *product of the two finite rooted trees* X and Y to be

$$(3.27) \quad X \cdot Y := \sum_{\substack{X' \in \text{fam}(X) \\ Y' \in \text{fam}(Y)}} \text{expand}(X' + Y').$$

Here are some explanations. This definition makes sense since the addition in $\mathcal{T}$ is commutative. We say that a *rooted tree has a stem* if there is precisely one vertex which is adjacent to the root, or, equivalently, it is an expanded tree of some finite rooted tree. If X and Y are rooted trees with stems, then their product is obtained from the disjoint union $X \amalg Y$ by identifying the

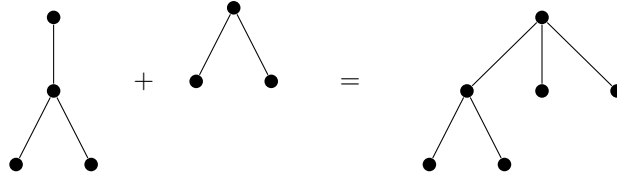
two stems. In general we can write a finite rooted tree X as the sum of finite rooted trees with stems $X = \sum_{X' \in \text{fam}(X)} \text{expand}(X')$ and now extend the formula for finite rooted trees with stems by enforcing distributivity.

The multiplication is commutative and associative and has a unit, namely, we have $X \cdot 1_{\mathcal{T}} = 1_{\mathcal{T}} \cdot X = X$ for $X \in \mathcal{T}$. By construction the multiplication is distributive over the addition. The element $0_{\mathcal{T}}$ is *absorbing*, i.e., we get for $X \in \mathcal{T}$

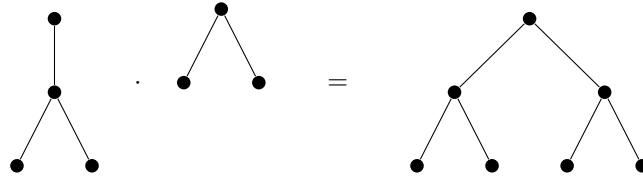
$$(3.28) \quad 0_{\mathcal{T}} \cdot X = X \cdot 0_{\mathcal{T}} = 0_{\mathcal{T}}.$$

One pictures a rooted tree by a diagram consisting of dots representing the vertices and lines representing the edges. One thing of hanging trees in the sense that on the top is the root and all vertices of height n are drawn on the horizontal line of level n below the root.

Example 3.29. We have



and



Obviously we have

$$(3.30) \quad \text{width}(X \cdot Y) = \text{width}(X) \cdot \text{width}(Y).$$

Definition 3.31 (Semiring). A *semiring* R consists of a set R and two operations

$$\begin{aligned} +: R \times R &\rightarrow R; \\ \cdot: R \times R &\rightarrow R, \end{aligned}$$

and an element $1_R \in R$ satisfying:

- (i) $(R, +)$ is an additive commutative semigroup (possibly without unit), i.e, we have $r_1 + r_2 = r_2 + r_1$ and $(r_1 + r_2) + r_3 = r_1 + (r_2 + r_3)$ for $r_1, r_2, r_3 \in R$;
- (ii) $(R, \cdot)$ is a multiplicative monoid with 1_R as unit element, i.e., we have $(r_1 \cdot r_2) \cdot r_3 = r_1 \cdot (r_2 \cdot r_3)$ and $r_1 \cdot 1_R = 1_R \cdot r_1 = r_1$ for $r_1, r_2, r_3 \in R$;

- (iii) The multiplication is left and right distributive over the addition, i.e., we have for $r_1, r_2, r_3 \in R$ the equalities $(r_1 + r_2) \cdot r_3 = (r_1 \cdot r_3) + (r_2 \cdot r_3)$ and $r_1 \cdot (r_2 + r_3) = (r_1 \cdot r_2) + (r_1 \cdot r_3)$.

A morphism $f: R \rightarrow S$ of semirings is a map satisfying $f(r_1 + r_2) = f(r_1) + f(r_2)$ and $f(r_1 \cdot r_2) = f(r_1) \cdot f(r_2)$ for $r_1, r_2 \in R$ and $f(1_R) = 1_S$.

An additive unit 0_R for a semiring is an element such that $r + 0_R = 0_R + r = r$ holds for every $r \in R$. If an additive unit 0_R exist, it is uniquely determined by this property. Note that a semiring may not have an additive unit, but is required to have a multiplicative unit.

A semiring with additive unit 0_R is a ring if and only if for every $r_0 \in R$ there exists an additive inverse s , i.e., an element $s \in R$ satisfying $r + s = 0_R$. Note that $\mathbb{Z}$ with its standard operations is a ring, $\mathbb{Z}_{\geq 0}$ with the induced standard operations is a semiring with additive unit, and $\mathbb{Z}_{\geq 1}$ is a semiring without additive unit.

Definition 3.32 (Rational semiring). A *rational semiring* is a semiring R together with a map of sets $\diamond_R: R \rightarrow R$, $r \mapsto r^\diamond$.

A morphism of semirings $f: (R_1, \diamond_{R_1}) \rightarrow (R_2, \diamond_{R_2})$ is a morphism of semirings $f: R_1 \rightarrow R_2$ satisfying $f \circ \diamond_{R_1} = \diamond_{R_2} \circ f$.

Note that the map $\diamond_R: R \rightarrow R$ is not required to be injective or surjective, to be compatible with addition or multiplication, or to respect units.

One of our main example for a rational semiring is given by the next result.

Proposition 3.33. *The set $\mathcal{T}$ becomes by the addition $+$ of (3.26), the multiplication $\cdot$ of (3.27), the element $1_{\mathcal{T}}$ of (3.24) and the map $\text{expand}^2: \mathcal{T} \rightarrow \mathcal{T}$ sending X to $\text{expand}(\text{expand}(X))$ a rational semiring.*

The underlying semiring has an additive unit, namely the element $0_{\mathcal{T}}$ of (3.24).

Next we define a total order $\leq$ on $\mathcal{T}$. For $n \in \mathbb{Z}_{\geq 0}$ let $\mathcal{T}_n \subseteq \mathcal{T}$ be the subset consisting of those elements X which have at most n edges. Then we get an ascending chain of subsets $\mathcal{T}_0 \subseteq \mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \dots \subseteq \mathcal{T}$ with $\mathcal{T} = \bigcup_{n=0}^{\infty} \mathcal{T}_n$ and $\mathcal{T}_0 = \{0_{\mathcal{T}}\}$.

Next we define inductively total orderings $\leq_n$ on $\mathcal{T}_n$ such that for $n \in \mathbb{Z}_{\geq 1}$ we have $X \leq_{n-1} Y \iff X \leq_n Y$ for $X, Y \in \mathcal{T}_{n-1}$. Since $\mathcal{T}_0$ consists of precisely one element, namely $0_{\mathcal{T}}$, there is only one total ordering on $\mathcal{T}_0$. This is the induction beginning. Next we explain the induction step from $(n-1)$ to $n \geq 1$.

Consider $X, Y \in \mathcal{T}_n$. If $X = 0_{\mathcal{T}}$, we put $X \leq Y$. If $Y = 0_{\mathcal{T}}$, we put $Y \leq X$. Hence it remains to treat the case where X and Y are different from $0_{\mathcal{T}}$. Define $\log(X)$ to be an element in $\text{fam}(X)$ such that its class in $\mathcal{T}_{n-1}$ is the largest element in the finite subset of $\mathcal{T}$ given by the classes of elements in $\text{fam}(X)$. Then $\text{expand}(\log(X))$ can be viewed as a rooted subtree of X and there is precisely one rooted subtree denoted by $X - \text{expand}(\log(X))$ of X such that

$$\text{expand}(\log(X)) + (X - \text{expand}(\log(X))) = X$$

holds, namely $X - \text{expand}(\log(X))$ is obtained from X by deleting all vertices and edges in $\log(X)$ and the edge in X connecting the root of X to the root of $\log(X)$. In particular X can be obtained from the disjoint subtrees $\log(X)$ and $X - \text{expand}(\log(X))$ of X by adding to $\log(X) \amalg (X - \text{expand}(\log(X)))$ the edge connecting the root of X and the root of $\log(X)$. Note that there might be different choices for $\log(X) \in \text{fam}(X)$, but the classes of $\log(X)$ and $(X - \text{expand}(\log(X)))$ in $\mathcal{T}$ are uniquely defined. Obviously $\log(X)$, $\log(Y)$, $(X - \text{expand}(\log(X)))$, and $(Y - \text{expand}(\log(Y)))$ belong to $\mathcal{T}_{n-1}$.

Note that precisely one of the following cases occurs:

$$\begin{aligned} \log(X) &<_{n-1} \log(Y); \\ \log(X) &= \log(Y); \\ \log(X) &>_{n-1} \log(Y). \end{aligned}$$

If $\log(X) <_{n-1} \log(Y)$ holds, we put $X <_n Y$. If $\log(X) >_{n-1} \log(Y)$ holds, we put $X >_n Y$. Suppose that $\log(X) = \log(Y)$. Then precisely one of the following cases occurs:

$$\begin{aligned} (X - \text{expand}(\log(X))) &<_{n-1} (Y - \text{expand}(\log(Y))); \\ (X - \text{expand}(\log(X))) &= (Y - \text{expand}(\log(Y))); \\ (X - \text{expand}(\log(X))) &>_{n-1} (Y - \text{expand}(\log(Y))). \end{aligned}$$

We define $X <_n Y$ if $(X - \text{expand}(\log(X))) <_{n-1} (Y - \text{expand}(\log(Y)))$ holds, we have $X = Y$ if $(X - \text{expand}(\log(X))) = (Y - \text{expand}(\log(Y)))$ holds, and we define $X >_n Y$ if $(X - \text{expand}(\log(X))) >_{n-1} (Y - \text{expand}(\log(Y)))$ holds. One easily checks that we obtain a total ordering $\leq_n$ on $\mathcal{T}_n$ such that for $n \in \mathbb{Z}_{\geq 1}$ we have $X \leq_{n-1} Y \iff X \leq_n Y$ for $X, Y \in \mathcal{T}_{n-1}$. Hence the following definition makes sense.

Definition 3.34 (Total ordering on $\mathcal{T}$). Define a total ordering $\leq$ on $\mathcal{T}$ as follows. Consider X, Y in $\mathcal{T}$. Choose $n \in \mathbb{Z}_{\geq 0}$ such that both X and Y belong to $\mathcal{T}_n$. Define $X \leq Y$ if $X \leq_n Y$.

Note that $0_{\mathcal{T}}$ is the least element in $\mathcal{T}$ and the definition of $\log(X)$ and $X - \text{expand}(\log(X))$ makes sense for all $X \in \mathcal{T} \setminus \{0_{\mathcal{T}}\}$.

One easily checks for $X, Y, Z \in \mathcal{T}$ with $X \neq 0_{\mathcal{T}}$:

$$(3.35) \quad \text{height}(\log(X)) = \text{height}(X) - 1$$

$$(3.36) \quad \text{width}(X - \text{expand}(\log(X))) = \text{width}(X) - 1;$$

$$(3.37) \quad \log(X - \text{expand}(\log(X))) \leq \log(X);$$

$$(3.38) \quad Y \leq Z \implies \text{height}(Y) \leq \text{height}(Z).$$

Theorem 3.39 ($\mathcal{T}$ is well-ordered) *The total ordering on $\mathcal{T}$ is well-ordered, i.e., every non-empty subset of $\mathcal{T}$ contains a minimal element.*

Proof. Suppose that $\leq$ is not well-ordered. Then the set S of strictly descending infinite chains $X_0 > X_1 > X_2 > \dots$ is non-empty. Choose $m \in \mathbb{Z}_{\geq 1}$ such that for every element in S we have $\text{height}(X_0) \geq m$ and there is one element with $\text{height}(X_0) = m$. Let S_m be the subset of S consisting of those element satisfying $\text{height}(X_0) = m$. The total order $\leq$ restricted to the set $\{Y \in \mathcal{T} \mid \text{height}(Y) < m\}$ must be well-ordered. Since we get $\text{height}(\log(X_0)) < \text{height}(X_0)$ from (3.35), we can choose $Z \in \mathcal{T}$ such that for every element in S_m we have $\log(X_0) \geq Z$ and there is an element in S_m with $\log(X_0) = Z$. Define S_Z to the subset of S consisting of those elements for which $\log(X_0) = Z$ holds. We conclude from (3.38) that S_Z is non-empty and any element in S satisfies $\log(X_0) \geq Z$. Now fix an element $Y_0 > Y_1 > Y_2 > \dots$ in S_Z minimizing $\text{width}(Y_0)$.

By the definition of the ordering we get $\log(Y_0) \geq \log(Y_1) \geq \log(Y_2) \geq \dots$. Next we prove

$$Z = \log(Y_0) = \log(Y_1) = \log(Y_2) = \dots$$

by contradiction. Suppose that there is $n \in \mathbb{Z}_{\geq 1}$ with $\log(Y_0) > \log(Y_n)$. Then we get the element $Y_n > Y_{n-1} > Y_{n-2} > \dots$ in S with $\log(Y_n) < \log(Y_0) = Z$, a contradiction.

By the definition of the ordering we get an element in S by

$$Y_0 - \text{expand}(\log(Y_0)) > Y_1 - \text{expand}(\log(Y_1)) > Y_2 - \text{expand}(\log(Y_2)) > \dots$$

This implies $\log(Y_0 - \text{expand}(\log(Y_0))) \geq Z = \log(Y_0)$. Hence we get

$$\log(Y_0 - \text{expand}(\log(Y_0))) = \log(Y_0) = Z$$

from (3.37). Hence $Y_0 - \text{expand}(\log(Y_0)) > Y_1 - \text{expand}(\log(Y_1)) > Y_2 - \text{expand}(\log(Y_2)) > \dots$ belongs to S_Z . Since $Y_0 > Y_1 > Y_2 > \dots$ in S_Z minimizes the width in S_Z , we get $\text{width}(Y_0) \leq \text{width}(Y_0 - \text{expand}(\log(Y_0)))$. This contradicts (3.36). $\square$

So far we have not defined $\log(0_{\mathcal{T}})$. Given the semiring $\mathcal{T}$, we define a new semiring

$$(3.40) \quad \mathcal{T} \amalg \{-\infty\}$$

to be the set consisting of $\mathcal{T}$ and a new element $-\infty$ such that the addition and multiplication on $\mathcal{T}$ is extended to $\mathcal{T} \amalg \{-\infty\}$ by putting $(-\infty) + x = -\infty = (-\infty) + x$ and $(-\infty) \cdot x = -\infty = (-\infty) \cdot x$ for every $x \in \mathcal{T} \amalg \{-\infty\}$. Note that $-\infty$ is the unit for the multiplication in $\mathcal{T} \amalg \{-\infty\}$ but there is no additive unit in $\mathcal{T} \amalg \{-\infty\}$. Then we can extend the definition of $\log$ on $\mathcal{T} \setminus \{0_{\mathcal{T}}\}$ to $\mathcal{T} \amalg \{-\infty\}$ by putting $\log(0_{\mathcal{T}}) = \log(-\infty) = -\infty$. We can extend the definition of width and height on $\mathcal{T}$ to $\mathcal{T} \amalg \{-\infty\}$ by putting $\text{width}(-\infty) = 0$ and $\text{height}(-\infty) = -1$. We can extend total ordering on $\mathcal{T}$ to a total ordering on $\mathcal{T} \amalg \{-\infty\}$ by declaring $-\infty$ to be the smallest

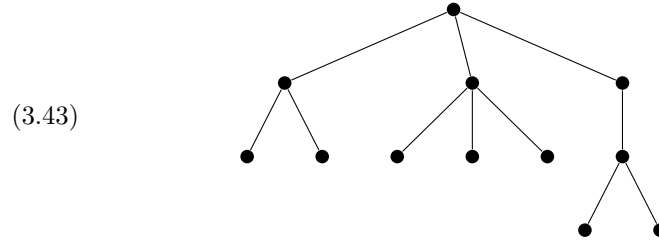
element. This total ordering on $\mathcal{T} \amalg \{-\infty\}$ is well-ordered again. One may think of $-\infty$ to be the class of the empty tree.

Next we collect some basic properties about the semiring $\mathcal{T}$, its ordering, and the notion of $\log$.

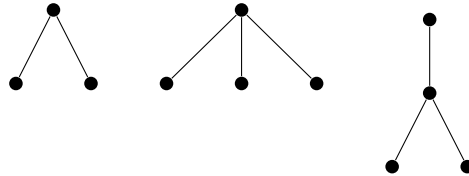
Proposition 3.41. *For $X, Y, X', Y' \in \mathcal{T}$ we get:*

- (i) *Suppose $X \leq X'$ and $Y \leq Y'$. Then we get $X + Y \leq X' + Y'$. Moreover, we have $X + Y = X' + Y'$ if and only if $X = X'$ and $Y = Y'$ hold;*
- (ii) *We have $X \leq X + Y$. We have $X = X + Y$ if and only if $Y = 0_{\mathcal{T}}$ holds;*
- (iii) *Suppose $X \leq X'$ and $Y \leq Y'$. Then we get $X \cdot Y \leq X' \cdot Y'$. Provided that X' and Y' are different from $0_{\mathcal{T}}$, we have $X \cdot Y = X' \cdot Y'$ if and only if $X = X'$ and $Y = Y'$ holds;*
- (iv) *Suppose that both X and Y are different from $0_{\mathcal{T}}$. Then we have $X \leq X \cdot Y$. Moreover, $X = X \cdot Y$ holds if and only if we have $Y = 1_{\mathcal{T}}$;*
- (v) *We have $\log(X + Y) = \max\{\log(X), \log(Y)\}$;*
- (vi) *We have $\log(X \cdot Y) = \log(X) + \log(Y)$;*
- (vii) *We have $\log^2(X \cdot Y) \leq \max\{\log^2(X), \log^2(Y)\}$. Moreover, the equality $\log^2(X \cdot Y) = \max\{\log^2(X), \log^2(Y)\}$ holds if and only if both X and Y are different from $0_{\mathcal{T}}$.*

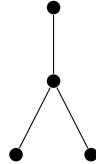
Example 3.42. Let X be the rooted tree given by



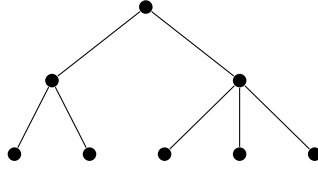
Its height and its width is 3, and $\text{fam}(X)$ consists of the following three rooted trees



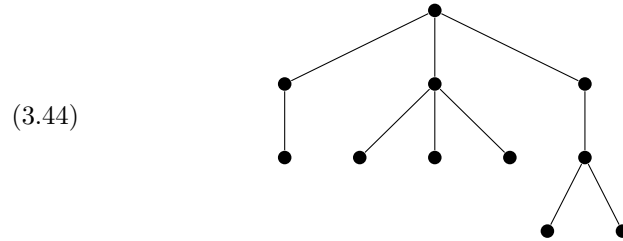
There is only one candidate for $\log(X)$, namely



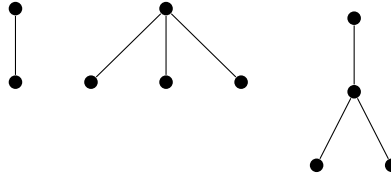
since its height is 2 and the height of the other members of $\text{fam}(X)$ is 1. The rooted tree $X - \text{expand}(\log(X))$ looks like



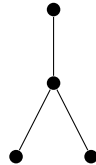
Let Y be the rooted tree given by



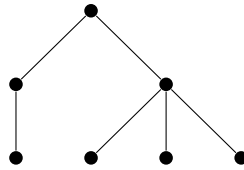
Its height and its width is 3, and its family consists of the following three rooted trees;



There is only one candidate for $\log(Y)$, namely



since its height is 2 and the height of the other members of $\text{fam}(Y)$ is 1. The rooted tree $Y - \text{expand}(\log(Y))$ looks like



Exercise 3.45 Let X and Y be the rooted trees defined in (3.43) and (3.44). Decide whether $X < Y$ or $Y < X$ holds.

3.5 Rational U -Semirings

Throughout this section U is a group, written multiplicatively.

Definition 3.46 (U -semiring). A U -semiring consists of a semiring R in the sense of Definition 3.31 together with a map $\varphi: U \rightarrow R$ such that φ is multiplicative, i.e., $\varphi(u \cdot v) = \varphi(u) \cdot \varphi(v)$ holds for $u, v \in U$, and we have $\varphi(1_U) = 1_R$.

A morphism $f: (R, \varphi) \rightarrow (R', \varphi')$ of U -semirings is a morphism $f: R \rightarrow R'$ of semirings such that $\varphi' \circ f = \varphi$ holds.

Note that R inherits a U -biset structure by $u \cdot r \cdot v := \varphi(u) \cdot r \cdot \varphi(v)$ for $u, v \in U$ and $R \in R$.

Definition 3.47 (Rational U -semiring). A *rational U -semiring* consists of a rational semiring $(R, \diamond_R)$ in the sense of Definition 3.32 together with a map $\varphi_R: U \rightarrow R$ such that (R, φ_R) is a U -semiring and $\diamond_R$ is an anti-map of U -bisets, i.e., we have $\diamond_R(\varphi_R(u) \cdot r \cdot \varphi_R(v)) = \varphi_R(v)^{-1} \cdot \diamond_R(r) \cdot \varphi_R(u)^{-1}$ for $u, v \in U$ and $r \in R$.

A morphism $f: (R, \varphi_R, \diamond_R) \rightarrow (R', \varphi_{R'}, \diamond_{R'})$ of rational U -semirings is a morphism $f: R \rightarrow R'$ of semirings such that $\varphi_{R'} = f \circ \varphi_R$ and $\diamond_{R'} \circ f = f \circ \diamond_R$ hold.

Example 3.48. Let R be a semiring. Let $R \amalg \{\infty\}$ be the semiring obtained from R by adding a new element ∞ such that R is a subsemiring and $x + \infty = \infty = \infty + x$ and $x \cdot \infty = \infty = \infty \cdot x$ hold for $x \in R \amalg \{\infty\}$. Consider a subgroup $U \subseteq R^\times$ of the multiplicative group $R^\times$ of units in R . Let $\varphi_{R \amalg \{\infty\}}: U \rightarrow R$ be the inclusion. Define a map $\diamond_{R \amalg \{\infty\}}: R \amalg \{\infty\} \rightarrow R \amalg \{\infty\}$ by sending $r \in R^\times$ to r^{-1} and any element in $R \amalg \{\infty\} \setminus R^\times$ to ∞ . Then $(R, \varphi_{R \amalg \{\infty\}}, \diamond_{R \amalg \{\infty\}})$ is a rational U -semiring.

If R is a skew field, we can take $U = R \setminus \{0_R\}$.

Example 3.49. Given any rational semiring $(R, \diamond_R)$, we can turn it for each group U into a rational U -semiring by the map $\varphi_R: U \rightarrow R$ sending $u \in U$ to 1_R . This applies for instance to the rational semiring $(\mathcal{T}, \text{expand}^2)$ of Proposition 3.33.

We have defined the semiring $\mathcal{T} \amalg \{-\infty\}$ in (3.40). Note that it corresponds to $\mathcal{T} \amalg \{\infty\}$ of Example 3.48, if we identify $-\infty$ and ∞ . We have used the notation $-\infty$ in (3.40) since $-\infty$ is the smallest element in $\mathcal{T} \amalg \{-\infty\}$ with respect to its total ordering $\leq$. We can turn $\mathcal{T} \amalg \{-\infty\}$ into a rational semiring by defining $\diamond_{\mathcal{T} \amalg \{-\infty\}}(X) = \text{expand}^2(X)$ for $X \in \mathcal{T}$ and $\diamond_{\mathcal{T} \amalg \{-\infty\}}(-\infty) = -\infty$. By the procedure above we can view it as rational U -semiring for every group U .

Definition 3.50 (Universal rational U -semiring). We call a U -semiring $\text{Rat}(U)$ *universal* if for any rational U -semiring R there is precisely one morphism of rational U -semirings from $\text{Rat}(U)$ to R .

Obviously the universal rational U -semiring is unique up to unique isomorphism of rational U -semirings if it exists. Next we show its existence.

Theorem 3.51 (Existence of the universal rational U -semiring) *For every group U the universal rational U -semiring $\text{Rat}(U)$ exists.*

Its proof needs some preparation.

Given a set M , let $\mathbb{N}[M]$ be the free commutative (additive) monoid with basis M . Elements are functions $f: M \rightarrow \mathbb{Z}_{\geq 0}$ with finite support, i.e., the complement of $f^{-1}(0)$ in M is finite. We can and will think of elements in $\mathbb{N}[M]$ as formal sums $\sum_{x \in M} n_x \cdot x$, where $n_x \in \mathbb{Z}_{\geq 0}$ and we require that the set $\{n_x \neq 0 \mid x \in M\}$ is finite. Addition is given by

$$\left(\sum_{x \in M} m_x \cdot x \right) + \left(\sum_{x \in M} n_x \cdot x \right) = \sum_{x \in M} (m_x + n_x) \cdot x.$$

The unit is given by the zero function $M \rightarrow \mathbb{Z}_{\geq 0}$, or, equivalently by $\sum_{x \in X} 0 \cdot x$. Note that $\mathbb{N}[\emptyset]$ is the trivial monoid $\{0\}$ consisting of precisely one element.

Lemma 3.52. *Let A be any commutative monoid. Then there is a bijective correspondence between the set of morphisms of sets from M to A and the set of morphisms of commutative monoids $\mathbb{N}[M]$ to A . It sends a map of sets $f: M \rightarrow A$ to the morphism of commutative monoids $\mathbb{N}[M]$ to A that maps $\sum_{x \in M} n_x \cdot x$ to $\sum_{x \in M} n_x \cdot f(x)$.*

If M is a monoid, written multiplicatively, then $\mathbb{N}[M]$ becomes a semiring by the multiplication given by convolution, i.e.,

$$\left(\sum_{x \in M} m_x \cdot x \right) \cdot \left(\sum_{x \in M} n_x \cdot x \right) = \sum_{z \in M} \left(\sum_{\substack{x, y \in M \\ z = x \cdot y}} m_x \cdot n_y \right) \cdot z.$$

The unit element for the multiplication is given by $1 \cdot 1_M$. Note that the semiring $\mathbb{N}[M]$ has an additive unit.

Let U be a group, written multiplicatively. A U -monoid consists of a monoid M together with morphism of monoids $\psi: U \rightarrow M$. If M is a U -monoid, the semiring $\mathbb{N}[M]$ with additive unit above inherits the structure of a U -semiring with additive unit by the map $U \rightarrow \mathbb{N}[M]$ sending u to the element $1 \cdot \psi(u)$.

Lemma 3.53. *Let R be a U -semiring and M be a multiplicative U -monoid. Consider R as a U -monoid with respect to its multiplicative structure.*

- (i) *There is a bijective correspondence between the set of morphism of multiplicative U -monoids from M to R and the set of morphisms of U -semirings from $\mathbb{N}[M] \setminus \{0\}$ to R . It sends a map of U -monoids $f: M \rightarrow R$ to the morphism of U -semirings $\mathbb{N}[M]$ to R that maps $\sum_{x \in M} n_x \cdot x \in \mathbb{N}[M] \setminus \{0\}$ to $\sum_{x \in M} n_x \cdot f(x)$;*

- (ii) Suppose that R has an additive unit. Then there is a bijective correspondence between the set of morphism of U -monoids M to R and the set of morphisms of U -semirings with additive unit from $\mathbb{N}[M]$ to R . It sends a map of U -monoids $f: M \rightarrow R$ to the morphism of semirings with additive unit $\mathbb{N}[M]$ to A that maps $\sum_{x \in M} n_x \cdot x \in \mathbb{N}[M]$ to $\sum_{x \in M} n_x \cdot f(x)$.

Let X be a U -biset. Define U -bisets X_U^n for $n = 0, 1, \dots$ as follows. Put $X_U^0 = U$ with the obvious U -biset structure, and $X_U^1 = X$ with the given U -biset structure. Let X_U^n for $n \in \mathbb{Z}_{\geq 2}$ be the quotient of $X^n = \prod_{i=1}^n X_i$ under the left U^{n-1} -action given by

$$(u_1, u_2, \dots, u_{n-1}) \cdot (x_1, x_2, \dots, x_n) = (x_1 u_1^{-1}, u_1 x_2 u_2^{-1}, u_2 x_3 u_3^{-1}, \dots, u_{n-1} x_n).$$

The U -biset-structure on X^n given by $u(x_1, x_2, \dots, x_n)u' = (ux_1, x_2, \dots, x_n u')$ induces a U -biset structure on X_U^n . Note that X_U^2 is just the balanced product $X \times_U X$, where the *balanced product* of a right U -set A and a left U -set B is defined to be

$$A \times_U B := A \times B / \sim$$

for the equivalence relation $\sim$ on $A \times B$ for which $(a_0, b_0) \simeq (a_1, b_1)$ holds if and only if there exists $u \in U$ with $a_1 = a_0 u^{-1}$ and $b_1 = u b_0$. Define the U -biset

$$(3.54) \quad U \sharp X := \prod_{n=0}^{\infty} X_U^n.$$

For $m, n \in \mathbb{Z}_{\geq 0}$ there is a preferred isomorphism of U -bisets

$$\mu_{m,n}: X_U^m \times_U X_U^n \xrightarrow{\cong} X_U^{m+n}$$

coming from the obvious identification $X^m \times X^n = X^{m+n}$ for $m, n \in \mathbb{Z}_{\geq 1}$, the map $U \times X^n \rightarrow X^n$ given by the obvious left U action on X^n for $n \in \mathbb{Z}_{\geq 1}$, the map $X^m \times U \rightarrow X^m$ given by the obvious right U -action on X^m for $m \in \mathbb{Z}_{\geq 1}$, and the multiplication on U . The collection of the maps $\mu_{m,n}$ induces a map

$$\mu: U \sharp X \times_U U \sharp X \rightarrow U \sharp X.$$

We claim that the composite

$$U \sharp X \times U \sharp X \xrightarrow{\text{pr}} U \sharp X \times_U U \sharp X \xrightarrow{\mu} U \sharp X.$$

for pr the canonical projection and the canonical inclusion $U \rightarrow U \sharp X$ turn $U \sharp X$ into a U -monoid. Note that $U \sharp \emptyset$ is U . We call $U \sharp X$ the *free multiplicative U -monoid* on X . This name is justified by the following result.

Lemma 3.55. *Let M be a multiplicative U -monoid and X be a U -biset. Let $f: X \rightarrow M$ be a morphism of U -bisets.*

Then there exists a morphism of U -monoids $\bar{f}: U\sharp X \rightarrow M$ which is uniquely determined by the property that $\bar{f}$ coincides on $X \subseteq U\sharp X$ with f .

By combining Lemma 3.53 and Lemma 3.55 we obtain the next result.

Lemma 3.56. *Let R be a U -semiring and X be a U -biset. Let $f: X \rightarrow R$ be a morphism of U -bisets. Then:*

- (i) *There exists a morphism of U -semirings $\bar{f}: \mathbb{N}[U\sharp X] \setminus \{0\} \rightarrow R$ which is uniquely determined by the property that $\bar{f}$ coincides on $X = X_U^1 \subseteq U\sharp X \subseteq \mathbb{N}[U\sharp X]$ with f ;*
- (ii) *Suppose that R has an additive unit. There exists a morphism of U -semirings with additive unit $\bar{f}: \mathbb{N}[U\sharp X] \rightarrow R$ which is uniquely determined by the property that $\bar{f}$ coincides on $X = X_U^1 \subseteq U\sharp X \subseteq \mathbb{N}[U\sharp X]$ with f .*

Notation 3.57 ($B^\diamond$). Let B be a U -biset. Let $B^\diamond$ be the set $B \times \{0\}$ and $\diamond_B: B^\diamond \rightarrow B$ be the bijection given by the canonical projection $\text{pr}: B \times \{0\} \rightarrow B$. Equip $B^\diamond$ with the structure of a U -biset for which $\diamond_B$ becomes an anti-map of U -bisets, i.e., put $u(b, 0)v = (v^{-1}bu^{-1}, 0)$.

The main point of Notation 3.57 is that the U -biset $B^\diamond$ is a disjoint copy of B with the anti U -biset structure and defined functorial in B .

Now we are ready to give the proof of Theorem 3.51.

Proof of Theorem 3.51. We begin with constructing an ascending chain of U -bisets $X_0 \subseteq X_1 \subseteq X_2 \cdots$ by induction over n . We put

$$\begin{aligned} X_0 &= \emptyset; \\ X_1 &= (\mathbb{N}[U] \setminus \{0\})^\diamond. \end{aligned}$$

The induction step from $n \geq 1$ to $n + 1$ is done as follows. Since $\mathbb{N}[U\sharp X_n]$ is a U -semiring, we can consider it as a U -biset. Obviously $X_{n-1} \subseteq X_n$ implies that we can consider $\mathbb{N}[U\sharp X_{n-1}]$ as a sub U -biset of $\mathbb{N}[U\sharp X_n]$. Hence we can consider the U -biset $\mathbb{N}[U\sharp X_n] \setminus \mathbb{N}[U\sharp X_{n-1}]$. Define the U -biset

$$X_{n+1} = (\mathbb{N}[U\sharp X_n] \setminus \mathbb{N}[U\sharp X_{n-1}])^\diamond \amalg X_n.$$

(If we would formally put $\mathbb{N}[U\sharp X_{-1}] = \{0\}$, the definition above would also work for $n = 0$.) Define the U -biset X to be $\text{colim}_{n \rightarrow \infty} X_n$. Roughly speaking, X is the union of the X_n -s. Now we can define a U -semiring

$$(3.58) \quad \text{Rat}(U) = \mathbb{N}[U\sharp X] \setminus \{0\}.$$

It remains to construct the map

$$\diamond_{\text{Rat}(U)}: \text{Rat}(U) \rightarrow \text{Rat}(U).$$

Since X_n is a subset of X_{n+1} and $X = \text{colim}_{n \rightarrow \infty} X_n$ as sets, $U\sharp X_n$ is a subset of $U\sharp X_{n+1}$ and have $U\sharp X = \text{colim}_{n \rightarrow \infty} U\sharp X_n$ as sets and hence also

as U -bisets. Therefore $\mathbb{N}[U\sharp X_n]$ is a subset of $\mathbb{N}[U\sharp X_{n+1}]$ and $\mathbb{N}[U\sharp X] = \text{colim}_{n \rightarrow \infty} \mathbb{N}[U\sharp X_n]$ as sets and hence also as U -bisets. Hence it suffices to define the restrictions of $\diamond_{\text{Rat}(U)}$ to $\mathbb{N}[U\sharp X_{n+1}] \setminus \mathbb{N}[U\sharp X_n]$ for $n \in \mathbb{Z}^{\geq 0}$ and to $\mathbb{N}[U\sharp X_0] \setminus \{0\}$. The restriction to $\mathbb{N}[U\sharp X_{n+1}] \setminus \mathbb{N}[U\sharp X_n]$ is given by the bijection

$$\begin{aligned} \diamond_{\mathbb{N}[U\sharp X_{n+1}] \setminus \mathbb{N}[U\sharp X_n]}^{-1} : \mathbb{N}[U\sharp X_{n+1}] \setminus \mathbb{N}[U\sharp X_n] \\ \xrightarrow{\cong} X_n \setminus X_{n-1} = (\mathbb{N}[U\sharp X_{n+1}] \setminus \mathbb{N}[U\sharp X_n])^\diamond \end{aligned}$$

and the obvious inclusion $X_n \subseteq \mathbb{N}[U\sharp X] \setminus \{0\}$. The restriction to $\mathbb{N}[U\sharp X_0] \setminus \{0\} = \mathbb{N}(U) \setminus \{0\}$ is given by

$$\diamond_{\mathbb{N}[U] \setminus \{0\}}^{-1} : \mathbb{N}[U] \setminus \{0\} \xrightarrow{\cong} X_1 = (\mathbb{N}[U] \setminus \{0\})^\diamond$$

and the obvious inclusion $X_1 \subseteq \mathbb{N}[U\sharp X] \setminus \{0\}$.

Obviously we have $\diamond_{\text{Rat}(U)}(urv) = v^{-1} \diamond_{\text{Rat}(U)}(r)u^{-1}$ for $u, v \in U$ and $r \in \text{Rat}(U)$.

Now consider any rational U -semiring $R = (R, \varphi_R, \diamond_R)$. We want to construct a morphism of rational U -semirings $f : \text{Rat}(U) \rightarrow R$. For this purpose we construct a morphism of rational U -semirings $f : \text{Rat}(U) \rightarrow R$ in a specific way and then check that it satisfies $\diamond_R \circ f = f \circ \diamond_{\text{Rat}(U)}$. It suffices to construct for $n = 0, 1, 2, \dots$ morphisms of U -semirings $f_n : \mathbb{N}[U\sharp X_n] \setminus \{0\} \rightarrow R$ such that $f_n|_{\mathbb{N}[U\sharp X_{n-1}] \setminus \{0\}} = f_{n-1}$ holds for $n \in \mathbb{Z}_{\geq 1}$.

The map

$$f_0 : \mathbb{N}[U\sharp X_0] = \mathbb{N}[U] \setminus \{0\} \rightarrow R$$

sends $\sum_{u \in U} n_U \cdot u$ to $\sum_{u \in U} n_u \cdot \varphi_R(u)$. The induction step from $n \geq 0$ to $(n+1)$ is done as follows.

By Lemma 3.56 (i) a map of U -semirings $f_{n+1} : \mathbb{N}[U\sharp X_{n+1}] \setminus \{0\} \rightarrow R$ is the same a morphisms of U -bisets $\beta_{n+1} : X_{n+1} \rightarrow R$, where we consider R as U -biset using its multiplicative structure.

If $n = 0$, the map $\beta_1 : X_1 = (\mathbb{N}[U] \setminus \{0\})^\diamond \rightarrow R$ is given by the diagram

$$(3.59) \quad \begin{array}{ccc} (\mathbb{N}[U] \setminus \{0\})^\diamond & \xrightarrow{\beta_1} & R \\ \uparrow \cong \diamond_{\mathbb{N}[X_0] \setminus \{0\}} & & \uparrow \diamond_R \\ \mathbb{N}[U] \setminus \{0\} & \xrightarrow{f_0} & R \end{array}$$

Suppose $n \in \mathbb{Z}_{\geq 1}$. Recall that X_{n+1} is $(\mathbb{N}[U\sharp X_n] \setminus \mathbb{N}[U\sharp X_{n-1}])^\diamond \amalg X_n$. We put $\beta_n|_{X_n} = f_n|_{X_n}$. It remains to explain what the restriction of β_{n+1} to $(\mathbb{N}[U\sharp X_n] \setminus \mathbb{N}[U\sharp X_{n-1}])^\diamond$ is. This is done by requiring that it makes the following diagram commutative

$$(3.60) \quad \begin{array}{ccc} (\mathbb{N}[U\sharp X_n] \setminus \mathbb{N}[U\sharp X_{n-1}])^\diamond & \xrightarrow{\beta_{n+1}|_{(\mathbb{N}[U\sharp X_n] \setminus \mathbb{N}[U\sharp X_{n-1}])^\diamond}} & R \\ \uparrow \cong & & \uparrow \diamond_R \\ \mathbb{N}[U\sharp X_n] \setminus \mathbb{N}[U\sharp X_{n-1}] & \xrightarrow{f_n|_{\mathbb{N}[U\sharp X_n] \setminus \mathbb{N}[U\sharp X_{n-1}]}} & R \end{array}$$

(If we would formally put $\mathbb{N}[U\sharp X_{-1}] = \{0\}$ and $X_{-1} = \emptyset$, the diagram (3.60) makes also sense for $n = 0$ and agrees with (3.59) but this convention could be confusing as $\mathbb{N}[U\sharp \emptyset]$ is $\mathbb{N}(U)$ and hence in general not $\{0\}$.) This finishes the construction of the morphism of U -semirings $f: \text{Rat}(U) \rightarrow R$. One easily checks that it is compatible with the rational structures. Hence we have constructed the desired morphism of rational U -semirings $f: \text{Rat}(U) \rightarrow R$. It is also easy to see that there is at most one morphism of rational U -semirings from $\text{Rat}(U)$ to R , see also Remark 3.61. $\square$

Remark 3.61. For the reader's convenience we recall the recipe how the unique map of rational U -semirings $f: \text{Rat}(U) \rightarrow R$ is defined. Since $\text{Rat}(U)$ can be thought of as the union of the ascending sequence of subsemirings $\mathbb{N}[U\sharp X_n]$, it suffices to tell what $f_n := f|_{\mathbb{N}[U\sharp X_n]}$ is. This is done by induction over $n = 0, 1, 2, \dots$

The map $f_0: \mathbb{N}[U\sharp X_0] = \mathbb{N}[U] \setminus \{0\} \rightarrow R$ sends $\sum_{u \in U} n_U \cdot u$ to $\sum_{u \in U} n_u \cdot \varphi_R(u)$.

Suppose that we have already specified f_n for $n \in \mathbb{Z}_{\geq 0}$. Then we construct f_{n+1} as follows. An element in $\mathbb{N}[U\sharp X_{n+1}]$ is a finite formal sum $\sum_{x \in U\sharp X_{n+1}} n_x \cdot x$ and we must have

$$f_{n+1} \left(\sum_{x \in U\sharp X_{n+1}} m_x \cdot x \right) = \sum_{x \in U\sharp X_{n+1}} m_x \cdot f_{n+1}(x).$$

So it suffices to specify $f_{n+1}(x)$ for $U\sharp X_{n+1}$. Since $U\sharp X_{n+1}$ is the disjoint union $\bigcup_{i=0}^{\infty} (X_{n+1})_U^i$, it suffices to specify $f_{n+1}(x)$ for $x \in (X_{n+1})_U^i$ for $i = 0, 1, 2, \dots$. The image of $f_{n+1}(u)$ for $u \in U = (X_{n+1})_U^0$ must be $\varphi_R(u)$. An element in $(X_{n+1})_U^i$ for $i = 1, 2, \dots$ is represented by an element $(x_1, x_2, \dots, x_i)$ in X_{n+1}^i and must be sent to the product $f_{n+1}(x_1)f_{n+1}(x_2) \cdots f_{n+1}(x_i)$. Hence it suffices to define $f_{n+1}(x)$ for $x \in X_{n+1}$. Recall that X_{n+1} is $(\mathbb{N}[U] \setminus \{0\})^\diamond$ for $n = 0$ and is $(\mathbb{N}[U\sharp X_n] \setminus \mathbb{N}[U\sharp X_{n-1}])^\diamond \amalg X_n$ for $n \in \mathbb{Z}_{\geq 1}$. Then $f_1(x) = f_0 \circ \diamond_{\mathbb{N}[U] \setminus \{0\}}^{-1}(x)$ for $x \in (\mathbb{N}[U] \setminus \{0\})^\diamond$, and for $n \in \mathbb{Z}_{\geq 1}$ we put $f_{n+1}(x) = f_n \circ \diamond_{\mathbb{N}[U\sharp X_n] \setminus \mathbb{N}[U\sharp X_{n-1}]}^{-1}(x)$ for $x \in (\mathbb{N}[U\sharp X_n] \setminus \mathbb{N}[U\sharp X_{n-1}])^\diamond$ and $f_{n+1}(x) = f_n(x)$ for $x \in X_n$.

We will frequently use the following direct consequence of the construction of $\diamond_{\text{Rat}(U)}$.

Lemma 3.62. *The map $\diamond_{\text{Rat}(U)}: \text{Rat}(U) \rightarrow \text{Rat}(U)$ is injective and satisfies*

$$\diamond_{\text{Rat}(U)}(\mathbb{N}[U\sharp X_n] \setminus \{0\}) = X_{n+1}$$

for $n \in \mathbb{Z}_{\geq 0}$ and

$$\text{im}(\diamond_{\text{Rat}(U)}) = \diamond_{\text{Rat}(U)}(\mathbb{N}[U\sharp X] \setminus \{0\}) = X.$$

Remark 3.63 (Functoriality of $\text{Rat}(U)$). Next we deal with the functoriality of $\text{Rat}(U)$. Let $\alpha: U \rightarrow V$ be a (not necessarily injective) group homomorphism. Given a V -biset- Y , let α^*Y be the U -biset whose underlying set is Y again and whose U -biset structure is defined by $u_1xu_2 := \alpha(u_1)x\alpha(u_2)$. Given a V -monoid $\mu: V \rightarrow M$, let α^*M be the U -monoid whose underlying monoid is M again together with the homomorphism of monoids $\mu \circ \alpha: U \rightarrow V$. Given a V -semiring $\varphi: V \rightarrow R$, let α^*R be the U -semiring whose underlying semiring is R again together with the map $\varphi \circ \alpha: U \rightarrow R$. Given a rational V -semiring consisting of a semiring R and maps $\varphi: V \rightarrow R$ and $\diamond: R \rightarrow R$, let α^*R be the rational U -semiring given by R and the maps $\varphi \circ \alpha: U \rightarrow R$ and $\diamond: R \rightarrow R$.

The universal property of the rational U -semiring $\text{Rat}(U)$, see Theorem 3.51, implies that there is precisely one morphism of rational U -semirings

$$\text{Rat}(\alpha): \text{Rat}(U) \rightarrow \alpha^* \text{Rat}(V).$$

One easily checks that for two groups homomorphisms $\alpha: U \rightarrow V$ and $\beta: V \rightarrow W$ the composite

$$\text{Rat}(U) \xrightarrow{\text{Rat}(\alpha)} \alpha^* \text{Rat}(V) \xrightarrow{\alpha^* \text{Rat}(\beta)} \alpha^* \beta^* \text{Rat}(W) = (\beta \circ \alpha)^* \text{Rat}(W)$$

agrees with $\text{Rat}(\beta \circ \alpha): \text{Rat}(U) \rightarrow (\beta \circ \alpha)^* \text{Rat}(W)$ and $\text{Rat}(\text{id}_U): \text{Rat}(U) \rightarrow \text{id}_U^* \text{Rat}(U) = \text{Rat}(U)$ is the identity on $\text{Rat}(U)$.

Using the specific model for $\text{Rat}(U)$ described in the proof of Theorem 3.51, see (3.58), one obtains a strict covariant functor from the category of groups to the category of semirings by sending U to this model $\text{Rat}(U)$ and a morphism of groups $\alpha: U \rightarrow V$ to the induced morphisms of semirings $\text{Rat}(\alpha): \text{Rat}(U) \rightarrow \text{Rat}(V)$ described in Remark 3.61 which is compatible with the structure of rational U -semirings on $\text{Rat}(U)$ and on $\alpha^* \text{Rat}(V)$.

Remark 3.64 (Universal U -semiring for rational U -semirings with absorbing additive unit). Let R be rational U -semiring. Then we can define a U -semiring with additive unit $R \amalg \{0\}$ by adding to R a new disjoint element 0 and extending the addition and multiplication of R to $R \amalg \{0\}$ by putting $0 + x = 0 = x + 0$ and $x \cdot 0 = 0 = 0 \cdot x$ for $x \in R \amalg \{0\}$. (Note that we do not specify a rational structure on $R \amalg \{0\}$, whereas on $R \amalg \{\infty\}$ there is a preferred rational structure.)

Note that a model for $\text{Rat}(U) \cup \{0\}$ is given by $\mathbb{N}[U\sharp X]$ in the notation appearing in the construction of $\text{Rat}(U)$ in (3.58).

Let R be a rational U -semiring with additive unit 0_R . Let $f: \text{Rat}(U) \rightarrow R$ be the unique morphism of rational semirings $f: \text{Rat}(U) \rightarrow R$ coming from

Theorem 3.51. It extends to a morphism of U -semirings with additive unit $\bar{f}: \text{Rat}(U) \amalg \{0\} \rightarrow R$ if and only if $0_R \in R$ is absorbing i.e., $r \cdot 0_R = 0_R = 0_R \cdot r$ for all $r \in R$.

Let U be a group. Consider the rational semiring with additive unit $\mathcal{T}$ to be a rational U -semiring using the constant map $U \rightarrow \mathcal{T}$ with value $1_{\mathcal{T}}$, see Proposition 3.33. Obviously the additive unit $0_{\mathcal{T}}$ is absorbing. From Remark 3.64 we obtain a morphism of rational U -semirings with additive unit, denoted by

$$(3.65) \quad \text{Tree}_U: \text{Rat}(U) \amalg \{0\} \rightarrow \mathcal{T}.$$

Definition 3.66 (Complexity of elements in $\text{Rat}(U)$). We call

$$\text{Tree}_U(r) \in \mathcal{T}$$

the *complexity* of the element $r \in \text{Rat}(U)$.

We record the basis properties of this notion.

Theorem 3.67 (Basic properties of complexity for elements in $\text{Rat}(U)$)

For elements $r, s, r', s' \in \text{Rat}(U) \amalg \{0\}$ the following assertions are true:

- (i) We have $\text{Tree}_U(r) = 0_{\mathcal{T}}$ if and only if $r = 0$;
- (ii) We have $\text{Tree}_U(r) = 1_{\mathcal{T}}$ if and only if $r \in U$;
- (iii) We have $\text{Tree}_U(r + s) = \text{Tree}_U(r) + \text{Tree}_U(s)$;
- (iv) If $\text{Tree}_U(r) \leq \text{Tree}_U(r')$ and $\text{Tree}_U(s) \leq \text{Tree}_U(s')$ holds, then we have $\text{Tree}_U(r + s) \leq \text{Tree}_U(r' + s')$. Moreover, $\text{Tree}_U(r + s) = \text{Tree}_U(r' + s')$ holds if and only if we have both $\text{Tree}_U(r) = \text{Tree}_U(r')$ and $\text{Tree}_U(s) = \text{Tree}_U(s')$. We have $\text{Tree}_U(r) \leq \text{Tree}(r + s)$ and $\text{Tree}_U(r) = \text{Tree}(r + s) \iff s = 0$;
- (v) We have $\text{Tree}_U(r \cdot s) = \text{Tree}_U(r) \cdot \text{Tree}_U(s)$;
- (vi) Suppose that r, r', s , and s' are different from 0. If $\text{Tree}_U(r) \leq \text{Tree}_U(r')$ and $\text{Tree}_U(s) \leq \text{Tree}_U(s')$ holds, then we have $\text{Tree}_U(r \cdot s) \leq \text{Tree}_U(r' \cdot s')$. Moreover, $\text{Tree}_U(r \cdot s) = \text{Tree}_U(r' \cdot s')$ holds if and only if we have both $\text{Tree}_U(r) = \text{Tree}_U(r')$ and $\text{Tree}_U(s) = \text{Tree}_U(s')$. We have $\text{Tree}_U(r) \leq \text{Tree}(r \cdot s)$ and $\text{Tree}_U(r) = \text{Tree}(r \cdot s) \iff s \in U$;
- (vii) We have $\log(\text{Tree}_U(r + s)) = \max\{\log(\text{Tree}_U(r)), \log(\text{Tree}_U(s))\}$;
- (viii) We have $\log(\text{Tree}_U(r \cdot s)) = \log(\text{Tree}_U(r)) + \log(\text{Tree}_U(s))$;
- (ix) We have $\log^2(\text{Tree}_U(r + s)) = \max\{\log^2(\text{Tree}_U(r)), \log^2(\text{Tree}_U(s))\}$;
- (x) We have $\log^2(\text{Tree}_U(r \cdot s)) \leq \max\{\log^2(\text{Tree}_U(r)), \log^2(\text{Tree}_U(s))\}$. Moreover if r and s are different from zero, we get $\log^2(\text{Tree}_U(r \cdot s)) = \max\{\log^2(\text{Tree}_U(r)), \log^2(\text{Tree}_U(s))\}$;
- (xi) If r is different from 0, then $\text{Tree}_U(r^\diamond) > \log^2(\text{Tree}_U(r^\diamond)) = \text{Tree}_U(r)$;
- (xii) If r is different from 0, then $\text{Tree}_U(r^\diamond) = \text{expand}^2(\text{Tree}_U(r))$;
- (xiii) If $\alpha \in U \sharp X$, then $\text{width}(\text{Tree}_U(\alpha)) = 1$.

Exercise 3.68 Let U be a group and $\text{pr}: U \rightarrow \{1\}$ be the projection onto the trivial group. Show that the map $\text{Tree}_U: \text{Rat}(U) \amalg \{0\} \rightarrow \mathcal{T}$ of (3.65) factorizes as the composite

$$\text{Tree}_U: \text{Rat}(U) \amalg \{0\} \xrightarrow{\text{Rat}(\text{pr}) \amalg \text{Id}_{\{0\}}} \text{Rat}(\{1\}) \amalg \{0\} \xrightarrow{\text{Tree}_{\{1\}}} \mathcal{T}$$

and that both maps $\text{Rat}(\text{pr})$ and $\text{Tree}_{\{1\}}$ are surjective and not injective, provided that U is non-trivial.

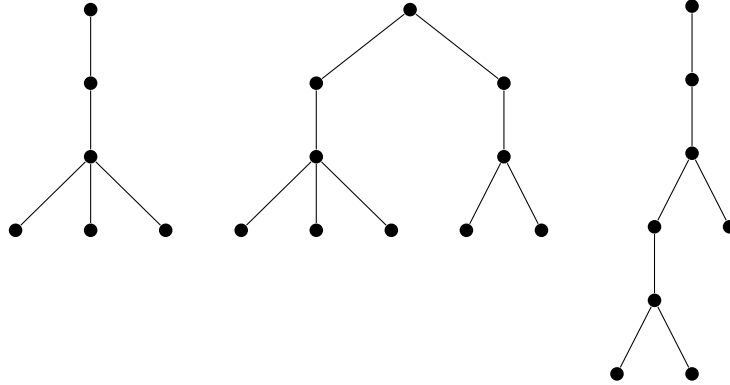
Example 3.69. Consider the following elements in $\text{Rat}(U)$.

$$\begin{aligned} x &= \diamond(\varphi(u_1) + \varphi(u_2) + \varphi(u_3)); \\ y &= \diamond(\varphi(u_4) + \varphi(u_4) + \varphi(u_6)) + \diamond(\varphi(u_7) + \varphi(u_8)); \\ z &= \diamond(\diamond(\varphi(u_9) + \varphi(u_{10})) + \diamond(\varphi(u_{11}))), \end{aligned}$$

for elements $u_i \in U$. Then we get

$$\begin{aligned} \text{Tree}_U(x) &= \text{expand}^2(1_{\mathcal{T}} + 1_{\mathcal{T}} + 1_{\mathcal{T}}); \\ \text{Tree}_U(y) &= \text{expand}^2(1_{\mathcal{T}} + 1_{\mathcal{T}} + 1_{\mathcal{T}}) + \text{expand}^2(1_{\mathcal{T}} + 1_{\mathcal{T}}); \\ \text{Tree}_U(z) &= \text{expand}^2(\text{expand}^2(1_{\mathcal{T}} + 1_{\mathcal{T}}) + 1_{\mathcal{T}}). \end{aligned}$$

Then $\text{Tree}_U(x)$, $\text{Tree}_U(y)$, and $\text{Tree}_U(z)$ look like



We get directly from the definitions or from (3.38) and Proposition 3.41(ii)

$$\text{Tree}_U(x) < \text{Tree}_U(y) < \text{Tree}_U(z).$$

Comment 24 (by W.): [Check this this example.](#)

3.6 Complexity and EG -Rings

Throughout this section we fix a division ring E . For a group G we denote by EG the group ring with coefficients in E . All the constructions below can be

extended to arbitrary crossed products $E * G$ but we will consider only group rings since this setting is simpler and this is the case we will be interested in.

Let $\varphi: EG \rightarrow R$ be a EG -ring, i.e., a ring homomorphism with EG as source. Consider the multiplicative subgroup of $(EG)^\times$.

$$E^\times G = \{eg \mid e \in E, e \neq 0, g \in G\}.$$

Then we can regard the rational $\varphi(k^\times G)$ -semiring $R \amalg \{\infty\}$, see Example 3.48. Recall that $\diamond: R \cup \{\infty\} \rightarrow R \cup \{\infty\}$ sends $r \in R \amalg \{\infty\}$ to r^{-1} if r belongs to R and its inverse r^{-1} in R exists, and to ∞ otherwise. We can consider it also as a rational $E^\times G$ -semiring by the surjection $E^\times G \rightarrow \varphi(E^\times G)$ induced by φ .

From the universal property of $\text{Rat}(E^\times G)$ we obtain a unique homomorphism of rational $E^\times G$ -rings

$$(3.70) \quad \rho = \rho_{G,R}: \text{Rat}(E^\times G) \rightarrow R \amalg \{\infty\}.$$

Proposition 3.71. *The image of ρ is $\mathcal{D}(\varphi(EG) \subseteq R) \amalg \{\infty\}$, where $\mathcal{D}(\varphi(EG) \subseteq R)$ is the division closure of $\varphi(EG)$ in R .*

Proof. In the sequel we use the notation from Section 3.5. Put

$$\begin{aligned} U &:= E^\times G; \\ \mathcal{D} &:= \mathcal{D}(\varphi(EG) \subseteq R). \end{aligned}$$

Recall that $\text{Rat}(U) = \mathbb{N}[U \sharp X] \setminus \{0\}$ is the union of subsemirings $\mathbb{N}[U \sharp X_n] \setminus \{0\}$. Moreover, we have described $\mathcal{D}$ as a union of subrings $\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n$ in Proposition 2.76. We will and it suffices show for $n = 1, 2, \dots$

$$(3.72) \quad \rho(\mathbb{N}[U \sharp X_n] \setminus \{0\}) = \mathcal{D}_n \amalg \{\infty\}.$$

Recall that $\mathcal{D}_0$ is $\varphi(EG)$. The map $\rho|_{\mathbb{N}[U \sharp X_0] \setminus \{0\}}$ is the map $\mathbb{N}[U] \setminus \{0\} \rightarrow R$ which extends the map $U \rightarrow R$ given by the map $U = E^\times G \rightarrow R$ induced by φ . This implies $\varphi(EG) \setminus \{0\} \subseteq \rho(\mathbb{N}[U \sharp X_0] \setminus \{0\}) \subseteq \varphi(EG)$. Since $1 \cdot 1_G + (-1) \cdot 1_G$ belongs to $\mathbb{N}[U] \setminus \{0\}$ and is sent to 0 under ρ , we conclude

$$(3.73) \quad \rho(\mathbb{N}[U \sharp X_0] \setminus \{0\}) = \varphi(EG) = \mathcal{D}_0.$$

Recall that $X_1 = (\mathbb{N}[U] \setminus \{0\})^\diamond$ and we have defined a map $\beta_1: X_1 \rightarrow R$ in (3.59) such that in view of (3.73)

$$\text{im}(\beta_1) = \diamond_R(EG) = \{x \in R \mid \exists y \in EG \text{ satisfying } x = y^{-1} \text{ in } R\} \amalg \{\infty\}.$$

This implies $\rho(\mathbb{N}[U \sharp X_1] \setminus \{0\}) = \mathcal{D}_1 \amalg \{\infty\}$ in view of the definition of ρ and $\mathcal{D}_1$. This finishes the induction beginning $n = 1$.

The induction step from n to $n + 1$ is done analogously. By induction hypothesis we have $\rho(\mathbb{N}[U\sharp X_n] \setminus \{0\}) = \mathcal{D}_n \amalg \{\infty\}$. Recall that $X_{n+1} = (\mathbb{N}[U\sharp X_n] \setminus \mathbb{N}[U\sharp X_{n-1}])^\diamond \amalg X_n$. By inspecting the definitions of the map β_{n+1} of (3.60), of ρ , and of D_{n+1} , we see that $\rho(X_{n+1})$ is contained in $D_{n+1} \amalg \{\infty\}$. This shows $\rho(\mathbb{N}[U\sharp X_{n+1}] \setminus \{0\}) \subseteq D_{n+1} \amalg \{\infty\}$. Consider $d \in D_n \setminus \{0\}$. Choose $x \in \mathbb{N}[U\sharp X_n] \setminus \{0\}$ with $\rho(x) = d$. Then $\rho \circ \diamond_{\text{Rat}(U)}(x) = \diamond_R \circ \rho(x) = \diamond_R(d) = d^{-1}$. Hence we get $\rho(\mathbb{N}[U\sharp X_{n+1}] \setminus \{0\}) = D_{n+1} \amalg \{\infty\}$. This finishes the proof of Proposition 3.71. $\square$

Recall that every non-empty subset of the totally ordered set $\mathcal{T}$ contains a minimal element, see Theorem 3.39, and that we have defined a map $\text{Tree}_{E \times G}: \text{Rat}(E^\times G) \amalg \{0\} \rightarrow \mathcal{T}$, see (3.65). Hence the following definition makes sense because of Proposition 3.71.

Definition 3.74 (Complexity of an element in $\mathcal{D}(\varphi(EG) \subseteq R)$). Consider an element a in $\mathcal{D}(\varphi(EG) \subseteq R)$. Define its *complexity*

$$\text{Tree}_{G,R}(a) := \min\{\text{Tree}_{E \times G}(\alpha) \mid \alpha \in \text{Rat}(E^\times G), \rho_{G,R}(\alpha) = a\} \in \mathcal{T}.$$

We say that $\alpha \in \text{Rat}(E^\times G)$ *realizes* a if $\phi_{G,R}(\alpha) = \text{Tree}_{G,R}(a)$ holds.

Exercise 3.75 Let $\varphi_i: EG \rightarrow R_i$ be a EG -ring for $i = 1, 2$. Let $\psi: (R_1, \varphi_1) \xrightarrow{\cong} (R_2, \varphi_2)$ be an isomorphism of EG -rings. Then ψ sends $\mathcal{D}(\varphi_1(G) \subseteq R_1)$ to $\mathcal{D}(\varphi_2(G) \subseteq R_2)$ and we get for $a \in \mathcal{D}(\varphi_1(G) \subseteq R_1)$

$$\text{Tree}_{G,R_2}(\psi(a)) = \text{Tree}_{G,R_1}(a).$$

Now consider a subgroup $H \subseteq G$. Then we denote by

$$(3.76) \quad \rho_{G,R}: \text{Rat}(E^\times G) \rightarrow R \amalg \{\infty\};$$

$$(3.77) \quad \rho_{H,R}: \text{Rat}(E^\times H) \rightarrow R \amalg \{\infty\},$$

the maps defined in (3.70) for $\varphi: EG \rightarrow R$ and $\varphi|_{EH}: EH \rightarrow R$ and denote by

$$(3.78) \quad \mathcal{D}_{G,R} \subseteq R;$$

$$(3.79) \quad \mathcal{D}_{H,R} \subseteq R,$$

the division closure of $\varphi(EG)$ and $\varphi(EH)$ in R . By the uniqueness part appearing in Definition 3.50, we obtain a commutative diagram

$$(3.80) \quad \begin{array}{ccc} \text{Rat}(EH) & \xrightarrow{\rho_{H,R}} & \mathcal{D}_{H,R} \\ \downarrow & & \downarrow \\ \text{Rat}(EG) & \xrightarrow{\rho_{G,R}} & \mathcal{D}_{G,R} \end{array}$$

where the vertical arrows are the inclusions induced by the inclusion $H \rightarrow G$.

One easily checks using the diagram (3.80)

Lemma 3.81. *Consider $a \in \mathcal{D}_{H,R}$. Then*

$$\text{Tree}_{G,R}(a) \leq \text{Tree}_{H,R}(a).$$

Example 3.82. Let $\varphi: EG \rightarrow \mathcal{D}$ be an epic EG -division ring. Consider elements $g_1, g_2 \in G$ with $\varphi(g_1) \neq \varphi(g_2)^{-1}$. Define elements in $\mathcal{D}$

$$\begin{aligned} x &= ((\varphi(g_1)^{-1} + (\varphi(g_2^{-1}) - \varphi(g_1))^{-1})^{-1} \\ y &= \varphi(g_1) - \varphi(g_1)\varphi(g_2)\varphi(g_1). \end{aligned}$$

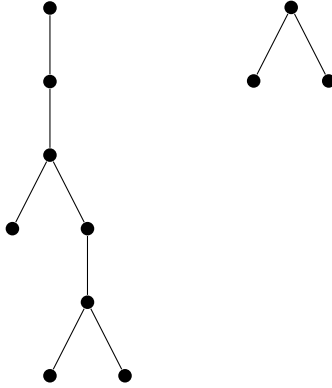
Let $(\text{Rat}(E^\times G), \varphi_{\text{Rat}}, \diamond_{\text{Rat}})$ be the universal rational $E^\times G$ -semiring. Define elements

$$\begin{aligned} x' &= \diamond_{\text{Rat}}(\varphi_{\text{Rat}}(g_1^{-1}) + \diamond_{\text{Rat}}(\varphi_{\text{Rat}}(g_2^{-1}) + \varphi_{\text{Rat}}(-g_1))); \\ y' &= \varphi_{\text{Rat}}(g_1) + \varphi_{\text{Rat}}(-g_1 g_2 g_1). \end{aligned}$$

Then the canonical map $\rho: \text{Rat}(E^\times G) \rightarrow \mathcal{D} \amalg \{\infty\}$ of (3.70) sends x' to x and y' to y . (Recall that ρ is surjective by Proposition 3.14 and Proposition 3.71.) We compute

$$\begin{aligned} \text{Tree}_{E^\times G}(x') &= \text{expand}^2(1_{\mathcal{T}} + \text{expand}^2(1_{\mathcal{T}} + 1_{\mathcal{T}})); \\ \text{Tree}_{E^\times G}(y') &= 1_{\mathcal{T}} + 1_{\mathcal{T}}. \end{aligned}$$

Hence $\text{Tree}_{E^\times G}(x')$ and $\text{Tree}_{E^\times G}(y')$ look like



We get directly from the definitions or from (3.38)

$$\text{Tree}_{E^\times G}(y') < \text{Tree}_{E^\times G}(x').$$

So one may think that the complexity of x is $\text{Tree}_{E^\times G}(x')$ and the complexity of y is $\text{Tree}_{E^\times G}(y')$. However, we get $x = y$ in $\mathcal{D}$ from the following

computation

$$\begin{aligned}
& (\varphi(g_1)^{-1} + (\varphi(g_2^{-1}) - \varphi(g_1))^{-1}) \cdot (\varphi(g_1) - \varphi(g_1)\varphi(g_2)\varphi(g_1)) \\
&= \varphi(g_1)^{-1}\varphi(g_1) + (\varphi(g_2^{-1}) - \varphi(g_1))^{-1}\varphi(g_1) - \varphi(g_1)^{-1}\varphi(g_1)\varphi(g_2)\varphi(g_1) \\
&\quad - (\varphi(g_2^{-1}) - \varphi(g_1))^{-1}\varphi(g_1)\varphi(g_2)\varphi(g_1) \\
&= 1 + (\varphi(g_2^{-1}) - \varphi(g_1))^{-1}\varphi(g_1) - \varphi(g_2)\varphi(g_1) \\
&\quad - (\varphi(g_2^{-1}) - \varphi(g_1))^{-1}\varphi(g_1)\varphi(g_2)\varphi(g_1) \\
&= 1 + (\varphi(g_2^{-1}) - \varphi(g_1))^{-1}(\varphi(g_1) - (\varphi(g_2^{-1}) - \varphi(g_1))\varphi(g_2)\varphi(g_1) \\
&\quad - \varphi(g_1)\varphi(g_2)\varphi(g_1)) \\
&= 1 + (\varphi(g_2^{-1}) - \varphi(g_1))^{-1}(\varphi(g_1) - \varphi(g_2^{-1})\varphi(g_2)\varphi(g_1) + \varphi(g_1))\varphi(g_2)\varphi(g_1) \\
&\quad - \varphi(g_1)\varphi(g_2)\varphi(g_1)) \\
&= 1 + (\varphi(g_2^{-1}) - \varphi(g_1))^{-1} \cdot 0 \\
&= 1.
\end{aligned}$$

This actually implies that the complexitiy of both x and y is represented by the tree



since x is not contained in $\varphi(E^\times G)$, the only trees which are smaller than this tree are $0_{\mathcal{T}}$ and $1_{\mathcal{T}}$ and we have Theorem 3.67 (i) and (ii). **Comment 25 (by W.):** [Check this this example.](#)

3.7 The Source Subgroup

In this section we introduce the notion of the source subgroup of an element in $\text{Rat}(U)$ for a given multiplicative group U . Recall that for any subgroup $V \subseteq U$ there are canoncial inclusions

$$\begin{aligned}
\text{Rat}(V) &\subseteq \text{Rat}(U); \\
U &\subseteq \text{Rat}(U).
\end{aligned}$$

Hence $\text{Rat}(V) \cdot U \subseteq \text{Rat}(U)$. The main result of this section is the following theorem.

Theorem 3.83 (Source subgroup) *For every $r \in \text{Rat}(U)$ the set of subgroups V of U satisfying $r \in \text{Rat}(V) \cdot U \subseteq \text{Rat}(U)$ has a unique smallest element $\text{source}(r)$.*

This subgroup $\text{source}(r)$ is finitely generated.

It leads to the following definition.

Definition 3.84 (Source subgroup and primitive elements). We call the unique subgroup $\text{source}(r) \subseteq U$ for $r \in \text{Rat}(U)$ appearing in Theorem 3.83 the *source subgroup* of r .

We call an element $r \in \text{Rat}(U)$ *primitive* if $r \in \text{Rat}(\text{source}(r))$ holds. Let $\text{Primitive}(U)$ be the subset of $\text{Rat}(U)$ consisting of primitive elements.

We will also prove the next proposition.

Proposition 3.85.

- (i) We have $\text{Primitive}(U) \cdot U = U \cdot \text{Primitive}(U) = \text{Rat}(U)$;
- (ii) If $r, r' \in \text{Rat}(U)$ and $u \in U$ satisfy $r' = r \cdot u$, then we get

$$\text{source}(r) = \text{source}(r');$$

- (iii) If the element $r \in \text{Rat}(U)$ and the subgroup $V \subseteq U$ satisfy $r \in \text{Rat}(V) \cdot U$, then we get

$$\text{source}(r) \subseteq V.$$

- (iv) The group $\text{source}_U(r)$ is finitely generated for $r \in \text{Rat}(U)$.

The rest of this section is occupied with the proof of Theorem 3.83 and Proposition 3.85 which needs some preparations.

We start with constructing for $n \in \mathbb{Z}_{\geq 0}$ subsets

$$(3.86) \quad Q_n \subseteq X_n;$$

$$(3.87) \quad P_n \subseteq \mathbb{N}[U \sharp X_n] \setminus \{0\},$$

such that

$$(3.88) \quad Q_n \subseteq Q_{n+1};$$

$$(3.89) \quad P_n \subseteq P_{n+1};$$

$$(3.90) \quad P_n = \langle Q_n \rangle + \mathbb{N}[U \sharp X_n],$$

hold for $n \in \mathbb{Z}_{\geq 0}$, where $\langle Q_n \rangle \subseteq U \sharp X_n$ is the multiplicative submonoid generated by $Q_n \subseteq X_n = (X_n)_U^1 \subseteq U \sharp X_n$ and we define

$$\langle Q_n \rangle + \mathbb{N}[U \sharp X_n] = \{x + y \mid x \in \langle Q_n \rangle, y \in \mathbb{N}[U \sharp X_n]\} \subseteq \mathbb{N}[U \sharp X_n] \setminus \{0\}$$

considering $U \sharp X_n$ and hence $\langle Q_n \rangle$ as a subset of $\mathbb{N}[U \sharp X_n] \setminus \{0\}$ in the obvious way.

For $n = 0$, put

$$Q_0 = X_0 = \emptyset.$$

The subset P_0 of $\mathbb{N}[U] \setminus \{0\}$ is given because of (3.90) by

$$P_0 = \langle \emptyset \rangle + \mathbb{N}[U \sharp \emptyset] = \{1_U\} + \mathbb{N}[U] = \left\{ \sum_{u \in U} m_u \cdot u \mid m_u \in \mathbb{Z}_{\geq 0}, m_{1_U} \geq 1 \right\}.$$

For $n = 1$ let Q_1 be the preimage of P_0 under the bijection

$$\diamond_{\mathbb{N}[U] \setminus \{0\}} : X_1 = (\mathbb{N}[U] \setminus \{0\})^\diamond \xrightarrow{\cong} \mathbb{N}[U] \setminus \{0\},$$

and define P_1 by (3.90).

The induction step from $n \geq 1$ to $n + 1$ is done as follows. Define Q_{n+1} as the image of P_n under the bijection of Lemma 3.62

$$\diamond_{\text{Rat}(U) | \mathbb{N}[U \sharp X_n] \setminus \{0\}} : \mathbb{N}[U \sharp X_n] \setminus \{0\} \xrightarrow{\cong} X_{n+1}.$$

Finally P_{n+1} is given by (3.90).

Notation 3.91 (P and Q). Define the subsets $P \subseteq \text{Rat}(U) = \mathbb{N}[U \sharp X]$ and $Q \subseteq X$ to be

$$\begin{aligned} P &= \text{colim}_{n \rightarrow \infty} P_n = \bigcup_{n=0}^{\infty} P_n; \\ Q &= \text{colim}_{n \rightarrow \infty} Q_n = \bigcup_{n=0}^{\infty} Q_n. \end{aligned}$$

We say that a subset $A \subseteq B$ of a U -biset B is *closed under U -conjugation* if $uau^{-1} \in A$ holds for all $a \in A$ and $u \in U$. The elementary proof of the next result is left to the reader.

Lemma 3.92. *We get for every $n \in \mathbb{Z}_{\geq 0}$:*

- (i) *We have $P = \langle Q \rangle + \mathbb{N}[U \sharp X]$;*
- (ii) *The image of P under the injective map $\diamond_{\text{Rat}(U)} : \text{Rat}(U) \rightarrow \text{Rat}(U)$ is Q ;*
- (iii) *The subsets $Q_n \subseteq X_n$, $\langle Q_n \rangle \subseteq U \sharp X_n$ and $P_n \subseteq \mathbb{N}[U \sharp X_n] \setminus \{0\}$ are closed under U -conjugation. The subsets $Q \subseteq X$, $\langle Q \rangle \subseteq U \sharp X$ and $P \subseteq \text{Rat}(U) = \mathbb{N}[U \sharp X] \setminus \{0\}$ are closed under U -conjugation;*
- (iv) *We have $Q_n U = U Q_n = X_n$ and $Q U = U Q = X$;*
- (v) *We have $\langle Q_n \rangle U = U \langle Q_n \rangle = U \sharp X_n$ and $\langle Q \rangle U = U \langle Q \rangle = U \sharp X$;*
- (vi) *We have $P_n U = U P_n = \mathbb{N}[U \sharp X_n] \setminus \{0\}$ and $P U = U P = \text{Rat}(U) = \mathbb{N}[U \sharp X] \setminus \{0\}$.*

Recall that the set $\mathcal{T}$ is well-ordered, see Theorem 3.39, we have constructed a map $\text{Tree}_U : \text{Rat}(U) \amalg \{0\} \rightarrow \mathcal{T}$, see (3.66), and we have defined the complexity of an element $r \in R$ to be $\text{Tree}_U(r) \in \mathcal{T}$, see Definition 3.66.

Lemma 3.93. *By transfinite induction over the complexity, we can construct for every $p \in P$ a subgroup of U , denoted by $\text{source}_U(p)$, such that the following two properties are satisfied for every $u \in U$:*

- (i) *We have $u \text{source}_U(p) u^{-1} = \text{source}_U(upu^{-1})$;*
- (ii) *If $pu \in P$, then $\text{source}_U(pu) = \text{source}_U(p)$ and $u \in \text{source}_U(p)$.*

Proof. Transfinite induction means that we have to construct for $p \in P$ the group $\text{source}_U(p)$ and prove conditions (i) and (ii), provided that the induction hypothesis holds, i.e., for every element $p' \in P$ with $\text{Tree}_U(p') <$

$\text{Tree}_U(p)$ the group $\text{source}_U(p')$ has been constructed and satisfies conditions (i) and (ii).

We can partition P into four disjoint subsets

$$(3.94) \quad \{1_{\text{Rat}(U)}\} \quad Q \quad \langle Q \rangle \setminus (Q \cup \{1_U\}) \quad \langle Q \rangle + (\mathbb{N}[U] \setminus \{0\}).$$

Case 1: $p = 1_{\text{Rat}(U)} = 1_U$

We put $\text{source}_U(\{1_{\text{Rat}(U)}\})$ to be the trivial subgroup group $\{1_U\}$ of U . Then conditions (i) and (ii) follow from the conclusion of Lemma 3.92 (i) that $P \cap U = \{1_U\} = \{1_{\text{Rat}(U)}\}$ holds.

Case 2: $p \in Q$

Let $p' \in P$ be unique the element in P which is send under the injective map $\diamond_{\text{Rat}(U)} : \text{Rat}(U) \rightarrow \text{Rat}(U)$ to p , see Lemma 3.92 (ii). Then

$$\begin{aligned} \text{height}(\text{Tree}_U(p')) &= \text{height}(\text{Tree}_U(\diamond_{\text{Rat}(U)}(p))) \\ &= \text{height}(\text{expand}^2(\text{Tree}_U(p))) < \text{height}(\text{Tree}_U(p)). \end{aligned}$$

We conclude $\text{Tree}_U(p') < \text{Tree}_U(p)$ from (3.38). Hence the induction hypothesis applies to p' , i.e., we have already constructed the subgroup $\text{source}_U(p')$ of U satisfying conditions (i) and (ii). We put

$$\text{source}_U(p) = \text{source}_U(p').$$

It remains to show that $\text{source}_U(p)$ satisfies conditions (i) and (ii).

We get $u \text{source}_U(p')u^{-1} = \text{source}_U(p')$ from the induction hypothesis for r' . This implies

$$u \text{source}_U(p)u^{-1} = \text{source}_U(p).$$

Suppose $pu \in P$. We compute

$$pu = \diamond_{\text{Rat}(U)}(p')u = \diamond_{\text{Rat}(U)}(u^{-1}p') = \diamond_{\text{Rat}(U)}(u^{-1}(p'u^{-1})u).$$

Now Lemma 3.92 (ii) implies $u^{-1}(p'u^{-1})u \in Q$. Hence $p'u^{-1} \in Q$ by Lemma 3.92 (iii). Since $\text{Tree}_U(pu) = \text{Tree}_U(p)$ and $\text{Tree}_U(p'u^{-1}) = \text{Tree}_U(p')$ hold by Theorem 3.67 (v), we get $\text{Tree}_U(pu) < \text{Tree}_U(p'u^{-1})$. Hence obtain from the definitions $\text{source}_U(pu) = \text{source}_U(p'u^{-1})$. From the induction hypothesis applied to p' we get $\text{source}_U(p') = \text{source}_U(p'u^{-1})$ and $u^{-1} \in \text{source}_U(p')$. This implies

$$\begin{aligned} \text{source}_U(pu) &= \text{source}_U(p); \\ u &\in \text{source}_U(p). \end{aligned}$$

Case 3: $p \in \langle Q \rangle \setminus (Q \cup \{1_U\})$

We can express p as a product

$$(3.95) \quad p = q \cdot r$$

for $q \in \langle Q \rangle \setminus \{1\}$ and $r \in Q \subseteq X$. Actually if $p \in X_U^n$, then $q \in X_U^{n-1}$ and $r \in X$. We conclude $\text{Tree}_U(q) < \text{Tree}_U(p)$ and $\text{Tree}_U(r) < \text{Tree}_U(p)$ from Theorem 3.67 (vi). Hence the induction hypothesis applies to both q and r , i.e., we have constructed the subgroups $\text{source}_U(q)$ and $\text{source}_U(r)$ which satisfy conditions (i) and (ii). We want to define

$$(3.96) \quad \text{source}_U(p) := \langle \text{source}_U(q) \cup \text{source}_U(r) \rangle \subset U$$

where $\langle \text{source}_U(q) \cup \text{source}_U(r) \rangle$ is the subgroup of U generated by the subset $\text{source}_U(q) \cup \text{source}_U(r)$. Note that a priori this definition depends on the decomposition (3.95).

Now consider $u \in U$ satisfying $pu \in P$. Then we have $pu \in \langle Q \rangle \setminus (Q \cup \{1_U\})$ and we can write pu as a product

$$(3.97) \quad pu = q' \cdot r'$$

for $q' \in \langle Q \rangle \setminus \{1\}$ and $r' \in Q \subseteq X$. Since $pu = qru = q'r'$, we can find $v \in U$ satisfying $q' = qv$ and $r' = v^{-1}ru = (v^{-1}rv)(v^{-1}u)$. We conclude $\text{Tree}_U(q') = \text{Tree}_U(q) < \text{Tree}_U(p)$ and $\text{Tree}_U(r') = \text{Tree}_U(r) < \text{Tree}_U(p)$ from Theorem 3.67 (vi). Hence we get from the induction hypothesis

$$\begin{aligned} \text{source}_U(q') &= \text{source}_U(q); \\ v &\in \text{source}_U(q); \\ \text{source}_U(r') &= \text{source}_U(v^{-1}rv); \\ v^{-1}u &\in \text{source}_U(v^{-1}rv) \\ \text{source}_U(v^{-1}rv) &= v^{-1} \text{source}_U(r)v. \end{aligned}$$

Suppose that $u = 1$. Then we get $\text{source}_U(q') = \text{source}_U(q)$ and $\text{source}_U(r') = \text{source}_U(v^{-1}rv) = \text{source}_U(r)$, which implies

$$\langle \text{source}_U(q) \cup \text{source}_U(r) \rangle = \langle \text{source}_U(q') \cup \text{source}_U(r') \rangle.$$

This shows that the definition (3.96) of $\text{source}_U(p)$ makes sense, i.e., is independent of the decomposition (3.95).

Next consider any $u \in U$. Then we get

$$\begin{aligned} \text{source}_U(p) &= \langle \text{source}_U(q) \cup \text{source}_U(r) \rangle = \langle \text{source}_U(q) \cup v^{-1} \text{source}_U(r)v \rangle \\ &= \langle \text{source}_U(q') \cup \text{source}_U(r') \rangle = \text{source}_U(pu), \end{aligned}$$

and $u \in \text{source}_U(p)$.

We have

$$upu^{-1} = u(qr)u^{-1} = (uqu^{-1})(uru^{-1}).$$

We conclude $uqu^{-1} \in \langle Q \rangle \setminus \{1\}$ and $ruru^{-1} \in Q$ from Lemma 3.92 (iii). Since $\text{Tree}_U(uqu^{-1}) = \text{Tree}_U(q) < \text{Tree}_U(p)$ and $\text{Tree}_U(uru^{-1}) = \text{Tree}_U(r) < \text{Tree}_U(p)$ hold by Theorem 3.67 (vi), the induction hypothesis implies

$$\begin{aligned} u \text{source}_U(q) u^{-1} &= \text{source}_U(uqu^{-1}); \\ u \text{source}_U(r) u^{-1} &= \text{source}_U(uru^{-1}). \end{aligned}$$

Hence we get

$$\begin{aligned} u \text{source}_U(p) u^{-1} &= u \langle \text{source}_U(q) \cup \text{source}_U(r) \rangle u^{-1} \\ &= \langle u \text{source}_U(q) \cup u \text{source}_U(r) \rangle u^{-1} \\ &= \langle u \text{source}_U(q) u^{-1} \cup u \text{source}_U(r) u^{-1} \rangle \\ &= \langle \text{source}_U(uqu^{-1}) \cup \text{source}_U(uru^{-1}) \rangle \\ &= \text{source}_U(upu^{-1}). \end{aligned}$$

Case 4: $p \in \langle Q \rangle + (\mathbb{N}[U\sharp X] \setminus \{0\})$

We can write

$$(3.98) \quad p = \sum_{i=1}^n f_i$$

where $n \in \mathbb{Z}_{\geq 2}$, $f_i \in U\sharp X$ for $i = 1, 2, \dots, n$, and $f_{i_0} \in \langle Q \rangle$ for some $i_0 \in \{1, 2, \dots, n\}$. Since $U\sharp X = \langle Q \rangle U$ holds by Lemma 3.92(v), we can write for $i = 1, 2, \dots, n$

$$(3.99) \quad f_i = p_i u_i,$$

where $p_i \in \langle Q \rangle$ and $u_i \in U$. Note that $\langle Q \rangle \subseteq P$ holds by Lemma 3.92 (i). We conclude $\text{Tree}_U(p_i) = \text{Tree}(p_i u_i) < \text{Tree}(p)$ for $i = 1, 2, \dots, n$ from Theorem 3.67 (iv) because of $n \geq 2$. Hence the induction hypothesis applies to $p_i \in P$ for $i = 1, 2, \dots, n$. In particular the group $\text{source}_U(p_i)$ has already been constructed. We want to define

$$(3.100) \quad \text{source}_U(p) = \langle \bigcup_{i=1}^n (\text{source}_U(p_i) \amalg \{u_i\}) \rangle$$

where $\langle \bigcup_{i=1}^n (\text{source}_U(p_i) \amalg \{u_i\}) \rangle$ is the subgroup of U generated by the subset $\bigcup_{i=1}^n (\text{source}_U(p_i) \amalg \{u_i\})$. Note that $\text{source}_U(p)$ a priori depends on the decompositions (3.98) and (3.99).

Consider $u \in U$ satisfying $pu \in P$. Then $pu \in \langle Q \rangle + (\mathbb{N}[U\sharp X] \setminus \{0\})$ and we write as above

$$(3.101) \quad pu = \sum_{i=1}^{n'} f'_i$$

where $n' \in \mathbb{Z}_{\geq 2}$, $f'_i \in U \sharp X$ for $i = 1, 2, \dots, n'$, and $f'_{i'_0} \in \langle Q \rangle$ for some $i'_0 \in \{1, 2, \dots, n'\}$ and then write

$$(3.102) \quad f'_i = p'_i u'_i,$$

where $p'_i \in \langle Q \rangle$ and $u'_i \in U$. Since $\sum_{i=1}^{n'} f'_i = \sum_{i=1}^n (f_i u)$ holds, we get $n = n'$ and there is a permutation $\sigma: \{1, 2, \dots, n\} \xrightarrow{\cong} \{1, 2, \dots, n\}$ such that $f'_i = f_{\sigma(i)} u$ holds for $i \in \{1, 2, \dots, n\}$. Since $\bigcup_{i=1}^n \text{source}_U(p_i) = \bigcup_{i=1}^n \text{source}_U(p_{\sigma(i)})$ holds, we can assume without loss of generality that $\sigma = \text{id}$ holds, in other words, $f'_i = f_i u$ holds for $i = 1, 2, \dots, n$. Then we get $p'_i u'_i = p_i u_i u$ and hence

$$p'_i = p_i u_i u (u'_i)^{-1}$$

for $i = 1, 2, \dots, n$. We conclude $\text{Tree}_U(p'_i) = \text{Tree}(p'_i u'_i) < \text{Tree}(p)$ for $i = 1, 2, \dots, n$ from Theorem 3.67 (iv) because of $n \geq 2$. Hence we get from the induction hypothesis

$$(3.103) \quad \text{source}_U(p'_i) = \text{source}_U(p_i);$$

$$(3.104) \quad u_i u (u'_i)^{-1} \in \text{source}_U(p_i),$$

for $i = 1, 2, \dots, n$.

Suppose $u = 1_U$. Then $u_i (u'_i)^{-1} \in \text{source}_U(p_i) = \text{source}_U(p'_i)$ holds and the subgroups $\langle \bigcup_{i=1}^n (\text{source}_U(p_i) \amalg \{u_i\}) \rangle$ and $\langle \bigcup_{i=1}^n (\text{source}_U(p'_i) \amalg \{u'_i\}) \rangle$ of U agree. Hence definition (3.100) makes sense, i.e., is independent of the decompositions (3.98) and (3.99).

It remains to prove that the element p satisfies conditions (i) and (ii). We can assume without loss of generality that $u_{i_0} = 1$ and $u_{i'_0} = 1$ hold for some elements $i_0, i'_0 \in \{1, 2, \dots, n\}$, since we have $p \in \langle Q \rangle + (\mathbb{N}[U \sharp X] \setminus \{0\})$ and the definition of the group $\text{source}_U(p)$ is independent of the decompositions (3.98) and (3.99). Then $f_{i_0} = p_{i_0} \in \langle Q \rangle$ and $f'_{i'_0} = p'_{i'_0} \in \langle Q \rangle$. Hence we get from (3.103) and (3.104)

$$\begin{aligned} u_{i'_0} u &\in \text{source}_U(p_{i_0}); \\ u (u'_{i_0})^{-1} &\in \text{source}_U(p'_{i'_0}). \end{aligned}$$

The first equality implies $u \in \text{source}_U(p)$, whereas the second equality implies $u \in \text{source}_U(pu)$. Since u and u_i belong to $\text{supp}(p)$ and $u_i u (u'_i)^{-1}$ belongs to $\text{supp}(p)$ by (3.104), u'_i also belongs to $\text{supp}(p)$. Since u and u'_i belong to $\text{supp}(pu)$ and $u_i u (u'_i)^{-1}$ belongs to $\text{supp}(pu)$ by (3.103) and (3.104), u_i also belongs to $\text{supp}(pu)$. This implies

$$\text{source}_U(p) = \text{source}_U(pu).$$

We have

$$\begin{aligned} upu^{-1} &= u \left(\sum_{i=1}^n f_i \right) u^{-1} = \sum_{i=1}^n (uf_i u^{-1}) \\ &= \sum_{i=1}^n (up_i u_i u^{-1}) = \sum_{i=1}^n (up_i u^{-1})(uu_i u^{-1}). \end{aligned}$$

We get $up_i u^{-1} \in P$ and $uf_i u^{-1} \in \langle Q \rangle$ from Lemma 3.92 (iii). We conclude $\text{Tree}_U(up_i u^{-1}) = \text{Tree}_U(p_i) < \text{Tree}_U(p) = \text{Tree}_U(upu^{-1})$ from Theorem 3.67 (vi). Hence we get $\text{source}_U(up_i u^{-1}) = u \text{source}_U(p) u^{-1}$ for $i = 1, 2, \dots, n$ from the induction hypothesis. We conclude

$$\text{source}_U(upu^{-1}) = u \text{source}_U(p) u^{-1}.$$

This finishes the proof of Lemma 3.93. $\square$

Consider two groups U and W . The subgroups P , Q , and X appearing in the constructions above are now denoted by P_U , Q_U , and X_U and by P_W , Q_W , and X_W to indicate whether they are understood to be with respect to U or W .

Lemma 3.105. *If p is an element in $P_U \subseteq \text{Rat}(U)$, then:*

- (i) *The group $\text{source}_U(p)$ is finitely generated;*
- (ii) *We have $p \in \text{source}_U(p)$;*
- (iii) *If U is a subgroup of W , then $p \in P_W \subseteq \text{Rat}(W)$ and $\text{source}_W(p) = \text{source}_U(p)$.*

Proof. According to the parting of (3.94) we get for U and W partitions of P_U and P_W into four disjoint subsets:

$$\{1_{\text{Rat}(U)}\} \quad Q_U \quad \langle Q_U \rangle \setminus (Q_U \cup \{1_U\}) \quad \langle Q_U \rangle + (\mathbb{N}[U] \setminus \{0\});$$

and

$$\{1_{\text{Rat}(W)}\} \quad Q_W \quad \langle Q_W \rangle \setminus (Q_W \cup \{1_W\}) \quad \langle Q_W \rangle + (\mathbb{N}[W] \setminus \{0\}).$$

Next we will prove by transfinite induction for $p \in P_U$:

- (i) The group $\text{source}_U(p)$ is finitely generated;
- (ii) We have $p \in \text{source}_U(p)$;
- (iii) If p belongs to $\{1_{\text{Rat}(U)}\}$, Q_U , $\langle Q_U \rangle \setminus (Q_U \cup \{1_U\})$, or $\langle Q_U \rangle + (\mathbb{N}[U] \setminus \{0\})$ respectively, then p belongs to $\{1_{\text{Rat}(W)}\}$, Q_W , $\langle Q_W \rangle \setminus (Q_W \cup \{1_W\})$, or $\langle Q_W \rangle + (\mathbb{N}[W] \setminus \{0\})$ respectively. Moreover, we have

$$\text{source}_W(p) = \text{source}_U(p).$$

We treat as before the following four cases.

Case 1: $p = 1_{\text{Rat}(U)} = 1_U$

This case follows directly from the definitions, as $\text{source}_U(p) = \{1_U\}$ and $\text{source}_W(p) = \{1_W\}$ hold.

Case 2: $p \in Q_U$

Let $p' \in P$ be the unique element in P which is sent under the injective map $\diamond_{\text{Rat}(U)}: \text{Rat}(U) \rightarrow \text{Rat}(U)$ to p , see Lemma 3.92 (ii). We have $\text{Tree}_U(p') < \text{Tree}_U(p)$ and hence the induction hypothesis applies to p' . We have

$$\text{source}_U(p) = \text{source}_U(p')$$

by definition. Since $\text{source}_U(p')$ is finitely generated by the induction hypothesis, $\text{source}_U(p)$ is finitely generated. We get $p' \in \text{Rat}(\text{source}_U(p'))$ from the induction hypothesis. The map $\diamond_{\text{Rat}(\text{source}_U(p))}: \text{Rat}(\text{source}_U(p')) \rightarrow \text{Rat}(\text{source}_U(p))$ sends p' to p , since $\diamond_U(p') = p$ holds and $\diamond_{\text{Rat}(U)}$ restricted to $\text{Rat}(\text{source}_U(p))$ is $\diamond_{\text{Rat}(\text{source}_U(p))}$. We conclude

$$p \in \text{Rat}(\text{source}_U(p)).$$

Since $p' \in P_W$ holds by the induction hypothesis and the map $\diamond_W: \text{Rat}(W) \rightarrow \text{Rat}(W)$ sends P_W to Q_W by Lemma 3.92 (ii), we get

$$p \in Q_W.$$

Since $\text{source}_U(p') = \text{source}_W(p')$ holds by the induction hypothesis and $\text{source}_W(p) = \text{source}_W(p')$ follows from the definitions, we get

$$\text{source}_U(p) = \text{source}_W(p).$$

Case 3: $p \in \langle Q_U \rangle \setminus (Q_U \cup \{1_U\})$

We can express p as a product $p = q \cdot r$ for $q \in \langle Q_U \rangle \setminus \{1\}$ and $r \in Q_U$, see (3.95). Since we have $\text{Tree}_U(q) < \text{Tree}_U(p)$ and $\text{Tree}_U(r) < \text{Tree}_U(p)$, the induction hypothesis applies to both q and r . Since $\text{source}_U(p)$ is by definition the subgroup generated by $\text{source}_U(q) \cup \text{source}_U(r)$ and both $\text{source}_U(q)$ and $\text{source}_U(r)$ are finitely generated by the induction hypothesis, $\text{source}_U(p)$ is finitely generated.

By the induction hypothesis we have $q \in \text{Rat}(\text{source}_U(q)) \subseteq \text{Rat}(\text{source}_U(p))$ and $r \in \text{Rat}(\text{source}_U(r)) \subseteq \text{Rat}(\text{source}_U(p))$. This implies

$$p = q \cdot r \in \text{Rat}(\text{source}_U(p)).$$

Since the induction hypothesis implies $q \in \langle Q_W \rangle \setminus \{1\}$ and $r \in Q_W$, we get

$$p = q \cdot r \in \langle Q_W \rangle \setminus (Q_W \cup \{1_W\}).$$

Since $\text{source}_W(p)$ is by definition the subgroup generated by $\text{source}_W(q) \cup \text{source}_W(R)$ and the induction hypothesis implies $\text{source}_W(q) = \text{source}_U(q)$ and $\text{source}_W(r) = \text{source}_U(r)$, we get

$$\text{source}_W(p) = \text{source}_U(p).$$

Case 4: $p \in \langle Q_U \rangle + (\mathbb{N}[U \# X] \setminus \{0\})$

We can write $p = \sum_{i=1}^n p_i u_i$ for $p_i \in \langle Q \rangle$ and $u_i \in U$, see (3.98) and (3.99). By definition $\text{source}_U(p)$ is the subgroup generated by $\bigcup_{i=1}^n (\text{source}_U(p_i) \amalg \{u_i\})$. We conclude from the induction hypothesis that $\text{source}_U(p_i)$ is finitely generated and $p_i \in \text{Rat}(\text{source}_U(p_i))$, $p_i \in \langle Q_W \rangle$, and $\text{source}_U(p_i) = \text{source}_W(p_i)$ hold for $i = 1, 2, \dots, n$. Hence $\text{source}_U(p)$ is finitely generated, $p \in \text{Rat}(\text{source}_U(p))$, $p \in \langle Q_W \rangle + (\mathbb{N}[U \# X] \setminus \{0\})$, and $\text{source}_W(p) = \text{source}_U(p)$ hold.

This finishes the proof of Lemma 3.105. $\square$

Definition 3.106 ($\text{source}(r)$). Consider the group U and an element $r \in \text{Rat}(U)$. We can write $r = pu$ for $p \in P$ and $u \in U$ by Lemma 3.92 (vi). We put

$$\text{source}(r) = \text{source}_U(p)$$

for the group $\text{source}_U(p)$ defined in Lemma 3.93.

This is independent of the decomposition $r = pu$ by Lemma 3.93 (ii).

Now we are ready to give the proof of Theorem 3.83.

Proof of Theorem 3.83. Consider an element $r \in \text{Rat}(U)$. We can write $r = pu$ for $p \in P$ and $u \in U$ by Lemma 3.92 (vi). We get $p \in \text{source}_U(p)$ by Lemma 3.105 (ii). By Definition 3.106 we have $\text{source}(r) = \text{source}_U(p)$. Hence

$$r \in \text{source}(r) \cdot U.$$

Consider any subgroup V of U satisfying $r \in \text{Rat}(V) \cdot U$. We can write $r = r_0 \cdot u$ for $r_0 \in \text{Rat}(V)$ and $u \in U$. Then we can write $r_0 = p_0 \cdot v_0$ for $p_0 \in P_V$ and $v_0 \in V$ by Lemma 3.92 (vi). Hence $p_0 \in P_U$ and $\text{source}_U(p_0) = \text{source}_V(p_0)$ by Lemma 3.105 (iii). Since $r = p_0(vu)$, we get

$$\text{source}(r) = \text{source}_U(p_0) = \text{source}_V(p_0) \subseteq V.$$

The group $\text{source}(p)$ is finitely generated by Lemma 3.105 (i). $\square$

Note that Proposition 3.85 just summarizes facts we have already proved above, namely, it follows from Theorem 3.83, Lemma 3.92 (vi), Lemma 3.105 (ii), and the fact that Definition 3.106 makes sense.

Exercise 3.107 Show that $P \subseteq \text{Primitive}(U)$ holds and we have the equivalence $P = \text{Primitive}(U) \iff U = \{1_U\}$.

Example 3.108. Consider elements $u_1, u_2, \dots, u_n$ in U . Then we get for the element $x = \sum_{i=1}^n \varphi_{\text{Rat}(U)}(u_i)$ in $\text{Rat}(U)$

$$\text{source}_U(x) = \langle u_2 u_1^{-1}, \dots, u_n u_1^{-1} \rangle \subseteq U$$

by the following argument.

Obviously $y = \sum_{i=2}^n \varphi_{\text{Rat}(U)}(u_i u_1^{-1})$ belongs to $\text{Rat}(\langle u_2 u_1^{-1}, \dots, u_n u_1^{-1} \rangle)$. Hence $x = y \cdot u_1$ belongs to $\text{Rat}(\langle u_2 u_1^{-1}, \dots, u_n u_1^{-1} \rangle) \cdot U$. This implies

$$\text{source}_U(x) \subseteq \langle u_2 u_1^{-1}, \dots, u_n u_1^{-1} \rangle.$$

Suppose that $V \subseteq U$ is a subgroup satisfying $x \in \text{Rat}(V) \cdot U$. Then $y \in \text{Rat}(V)$ holds. This implies $u_i u_1^{-1} \in V$ for $i = 2, 3, \dots, n$ and hence

$$\text{source}_U(x) \supseteq \langle u_2 u_1^{-1}, \dots, u_n u_1^{-1} \rangle.$$

3.8 An Auxiliary Result about Extensions by an Infinite Cyclic Group

Throughout this section we fix a division ring E and a group H with a normal subgroup $N \subseteq H$ such that N/H is infinite cyclic. We fix a choice of an element $t \in H$ which is mapped under the canonical projection $H \rightarrow H/N$ to a generator. Let $c_t: N \xrightarrow{\cong} N$ be the automorphism of N sending $h \in N$ to tht^{-1} . Then we obtain an isomorphism of groups

$$(3.109) \quad N \rtimes_{c_t} \mathbb{Z} \xrightarrow{\cong} H$$

which sends an element in N to the same element in H and the generator $1 \in \mathbb{Z}$ to t . Note that c_t induces an automorphism

$$(3.110) \quad \theta := E[c_t]: EN \xrightarrow{\cong} EN.$$

Given an automorphism $\tau: R \rightarrow R$ of a ring R , we denote by $R_\tau((t))$ the ring whose elements are formal sums $x = \sum_{n \in \mathbb{Z}} r_n \cdot t^n$ for $r_n \in R$ such that there exists some element $n_0 \in \mathbb{Z}$ satisfying $r_n = 0$ for $n < n_0$. (The number n_0 depends on x .) Addition and multiplication are given by

$$\begin{aligned} \left(\sum_{n \in \mathbb{Z}} r_n \cdot t^n \right) + \left(\sum_{n \in \mathbb{Z}} s_n \cdot t^n \right) &= \sum_{n \in \mathbb{Z}} (r_n + s_n) \cdot t^n; \\ \left(\sum_{n \in \mathbb{Z}} r_n \cdot t^n \right) \cdot \left(\sum_{n \in \mathbb{Z}} s_n \cdot t^n \right) &= \sum_{n \in \mathbb{Z}} \left(\sum_{k, l \in \mathbb{Z}, k+l=n} r_k \cdot \tau^k(s_l) \right) \cdot t^n. \end{aligned}$$

The additive unit is given by $\sum_{n \in \mathbb{Z}} r_n \cdot t^n$ with $r_n = 0$ for all $n \in \mathbb{Z}$ and the multiplicative unit is given by $\sum_{n \in \mathbb{Z}} r_n \cdot t^n$ with $r_n = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$ and $r_0 = 1_R$.

Define the subring $R_\tau[t]$ of $R_\tau((t))$ by considering formal sums $\sum_{n \in \mathbb{Z}} r_n \cdot t^n$ such that there exists a number $n_0 \in \mathbb{Z}_{\geq 0}$ satisfying $r_n \neq 0 \implies 0 \leq n \leq n_0$. Define the subring $R_\tau[t^{\pm 1}]$ of $R_\tau((t))$ by considering formal sums $\sum_{n \in \mathbb{Z}} r_n \cdot t^n$ such that there exists a number $n_0 \in \mathbb{Z}_{\geq 0}$ satisfying $r_n \neq 0 \implies -n_0 \leq n \leq n_0$. Define the subring $R_\tau[[t]]$ of $R_\tau((t))$ by considering formal sums $\sum_{n \in \mathbb{Z}} r_n \cdot t^n$ satisfying $r_n \neq 0 \implies 0 \leq n$. Obviously we have

$$R_\tau[t] = R_\tau[t, t^{-1}] \cap R_\tau[[t]],$$

and $R_\tau[t, t^{-1}]$ is the Ore localization of $R_\tau[t]$ with respect to the multiplicative closed subset $\{t^n \mid n \in \mathbb{Z}_{\geq 0}\}$.

One calls $R_\tau[t]$ the *ring of τ -twisted polynomials*. The ring $R_\tau[t, t^{-1}]$ is called the *ring of τ -twisted finite Laurent series* and agrees with the twisted group ring $R_\tau[\mathbb{Z}]$ associated to the group homomorphism $\mathbb{Z} \rightarrow \text{aut}(R)$ sending $n \in \mathbb{Z}$ to τ^n . The ring $R_\tau[[t]]$ is called the *ring of τ -twisted power series*. The ring $R_\tau((t))$ is called the *ring of τ -twisted Laurent series bounded from below*.

If $\tau = \text{id}_R$, we omit τ from the notation, e.g., we just write $R((t))$.

Now we assume that we have the following data:

- An EN -ring $\varphi: EN \rightarrow R$ such that R is von Neumann regular;
- A ring automorphism $\Theta: R \xrightarrow{\cong} R$ satisfying $\Theta \circ \varphi = \varphi \circ \theta$ for θ defined in (3.110);
- A ring S containing $R_\Theta((t))$ as a subring.

Then φ can be extended to a ring homomorphism

$$(3.111) \quad \bar{\varphi}: EH = EN_\theta[t^{\pm 1}] \rightarrow R_\Theta[t^{\pm 1}] \rightarrow R_\Theta((t)) \rightarrow S.$$

So we can consider the division closures $\mathcal{D}_{N,S} = \mathcal{D}(\bar{\varphi}(EN) \subseteq S)$ and $\mathcal{D}_{H,S} = \mathcal{D}(\bar{\varphi}(EH) \subseteq S)$. Obviously $\mathcal{D}_{N,S} \subseteq \mathcal{D}_{H,S} \subseteq S$. Recall that we can define a complexity $\text{Tree}_{N,S}(a)$ and $\text{Tree}_{H,S}(b)$ in $\mathcal{T}$ for elements $a \in \mathcal{D}_{N,S}$ and $b \in \mathcal{D}_{H,S}$, see Definition 3.74.

If $\mathcal{D}_{N;R}$ denotes $\mathcal{D}(\varphi(EN) \subseteq R)$, we conclude

$$(3.112) \quad \mathcal{D}_{N,R} = \mathcal{D}_{N,S}$$

from Lemma 2.78 (ii).

Note that $\Theta: R \rightarrow R$ sends $\mathcal{D}_{N,R}$ to itself and hence induces an automorphism

$$\Theta|: \mathcal{D}_{N,R} \xrightarrow{\cong} \mathcal{D}_{N,R}.$$

Hence we can consider $(\mathcal{D}_{N,S})_{\Theta|}((t))$ as a subring of $R_\Theta((t))$. Because of (3.112) we can view $\Theta|$ as an automorphism of $\mathcal{D}_{N,S}$ and get an identification

$$(3.113) \quad (\mathcal{D}_{N,R})_{\Theta|}((t)) = (\mathcal{D}_{N,S})_{\Theta|}((t))$$

and from $R_{\Theta}((t)) \subseteq S$ an inclusion

$$(3.114) \quad (\mathcal{D}_{N,R})_{\Theta|}((t)) \subseteq \mathcal{D}_{H,S}.$$

The automorphism $c_t: N \rightarrow N$ induces an automorphism of the group $\overline{c}_t: E^\times N \xrightarrow{\cong} E^\times N$ and hence an automorphism $\text{Rat}(\overline{c}_t): \text{Rat}(E^\times N) \xrightarrow{\cong} \text{Rat}(E^\times N)$. Let

$$t \in \text{Rat}(E^\times H)$$

be the image of $t \in H \subseteq E^\times H$ under the canonical inclusion $E^\times H \rightarrow \text{Rat}(E^\times H)$. Recall that we can consider $\text{Rat}(E^\times N)$ as a subset of $\text{Rat}(E^\times H)$ using Remark 3.63. Note that we get for $\alpha, \beta \in \text{Rat}(E^\times H)$ and $m, n \in \mathbb{Z}$

$$\alpha t^m \cdot \beta t^n = \alpha \cdot \text{Rat}(\overline{c}_t)^m(\beta) \cdot t^{m+n}.$$

Recall that we have defined the map

$$\text{Tree}_{E^\times H}: \text{Rat}(E^\times H) \amalg \{0\} \rightarrow \mathcal{T}$$

in (3.65), we get a map

$$\rho_{H,S}: \text{Rat}(E^\times H) \rightarrow \mathcal{D}_{H,S} \amalg \{\infty\}$$

from (3.70) and Proposition 3.71, and we have introduced the complexity $\text{Tree}_{H,S}(a) \in \mathcal{T}$ for $a \in \mathcal{D}_{H,S}$ in Definition 3.74.

Proposition 3.115. *Consider the situation described above. Fix an element $T \in \mathcal{T}$. Suppose that for every $c \in \mathcal{D}_{H,S}$ such that $\text{Tree}_{H,S}(c) < T$ and $c \neq 0$ holds, c is invertible in $\mathcal{D}_{H,S}$.*

Then for every $b \in \mathcal{D}_{H,S}$ satisfying $\text{Tree}_{H,S}(b) \leq T$ the following holds:

- (i) *The element b belongs to $(\mathcal{D}_{N,R})_{\Theta|}((t)) = (\mathcal{D}_{N,S})_{\Theta|}((t))$. In particular we can write*

$$b = \sum_{k \geq k_0} b_k \cdot t^k$$

for $b_k \in \mathcal{D}_{N,R} = \mathcal{D}_{N,S}$ and some $k_0 \in \mathbb{Z}$;

- (ii) *Suppose $b \neq 0$. Then we get for $k \in \mathbb{Z}$*

$$\text{Tree}_{H,S}(b_k t^k) \leq \text{Tree}_{H,S}(b);$$

- (iii) *Suppose that $b \neq 0$ holds and consider $n \in \mathbb{Z}$.*

Then $\text{Tree}_{H,S}(b_n t^n) = \text{Tree}_{H,S}(b)$ holds if and only if $b_k = 0$ for $k \neq n$;

- (iv) *Suppose that we have $b \neq 0$ and $\text{Tree}_{H,S}(b_n t^n) = \text{Tree}_{H,S}(b)$ holds for some $n \in \mathbb{Z}$.*

Then we the subset of $\text{Rat}(E^\times H)$

$$\{\beta \in \text{Rat}(E^\times H) \amalg \{0\} \mid \rho_{E^\times H, S}(\beta) = b \text{ and } \text{Tree}_{E^\times H}(\beta) = \text{Tree}_{H, S}(b)\}$$

is contained in $\{\alpha t^n \mid \alpha \in \text{Rat}(E^\times N)\}$.

Proof. Consider any element $b \in \mathcal{D}_{H, S}$ satisfying $\text{Tree}_{H, S}(b) \leq T$. We have to show that assertions (i) to (iv) hold for b .

If $b = 0$, this is obvious. So we can assume in the sequel $b \neq 0$.

Suppose that $\text{Tree}_{H, S}(b) \leq 1\tau$ holds. Choose $\beta \in \text{Rat}(E^\times H)$ realizing the complexity of b . Then $\beta \in E^\times H$ by Theorem 3.67 (i) and (ii). Hence we get $b \in \varphi(E^\times H)$ and therefore assertions (i) to (iv) hold for b because of (3.109). Hence we can assume $\text{Tree}_{H, S}(b) > 1\tau$ in the sequel.

By transfinite induction it suffices to show that assertions (i) to (iv) hold for b provided that assertions (i) to (iv) hold for every $c \in \mathcal{D}_{H, S}$ with $\text{Tree}_{H, S}(c) < \text{Tree}_{H, S}(b)$.

Consider any $\beta \in \text{Rat}(E^\times H)$ realizing the complexity of b .

In the sequel we abbreviate

$$U = E^\times H.$$

We can partition $\text{Rat}(E^\times H)$ into four disjoint subsets

$$U \quad X \quad U \sharp X \setminus (X \cup U) \quad \mathbb{N}[U \sharp X] \setminus U \sharp X \cup \{0\}.$$

Since $\text{Tree}_H(b) > 1\tau$ holds, we cannot have $b \in U$ by Theorem 3.67 (ii). Hence we are left with the following three cases.

Case 1: $b \in X$

Because of Lemma 3.62 we can choose $\gamma \in \text{Rat}(U) = \mathbb{N}[U \sharp X] \setminus \{0\}$ with $\beta = \gamma^\diamond$. We conclude

$$\text{Tree}_U(\gamma) < \text{Tree}_U(\beta)$$

from Theorem 3.67 (xi). Put

$$c = \rho_{H, S}(\gamma).$$

Then we get

$$b = \rho_{H, S}(\beta) = \rho_{H, S}(\gamma^\diamond) = \rho_{H, S}(\gamma)^\diamond = c^\diamond.$$

As b is different from ∞ , the inverse c^{-1} of c exists in $\mathcal{D}_{H, S}$ and we have

$$b = c^{-1}.$$

Next we show that γ realizes the complexity of c , i.e., $\text{Tree}_{H, S}(c) = \text{Tree}_U(\gamma)$. Suppose the contrary. Then we can find $\gamma' \in \text{Rat}(U)$ with $\rho_{H, S}(\gamma') = \rho_{H, S}(\gamma) = c$ and $\text{Tree}_U(\gamma') < \text{Tree}_U(\gamma)$. Since expand^2 preserves the ordering on $\mathcal{T}$, Theorem 3.67 (xii) implies

$$\text{Tree}_U((\gamma')^\diamond) < \text{Tree}_U(\gamma^\diamond) = \text{Tree}_U(\beta).$$

Since we have

$$\rho_{H,S}((\gamma')^\diamond) = (\rho_{H,S}(\gamma'))^\diamond = c^\diamond = b,$$

this contradicts the fact that β realizes the complexity of b .

Since we get

$$\text{Tree}_{H,S}(c) = \text{Tree}_U(\gamma) < \text{Tree}_U(\beta) = \text{Tree}_{H,S}(b),$$

we conclude from the induction hypothesis applied to c that c belongs to $(\mathcal{D}_{N,R})_{\Theta|}((t)) = (\mathcal{D}_{N,S})_{\Theta|}((t))$ and hence we can write

$$c = \sum_k c_k \cdot t^k$$

for $c_k \in \mathcal{D}_{N,R} = \mathcal{D}_{N,S}$ such that $\text{Tree}_{H,S}(c_k t^k) \leq \text{Tree}_{H,S}(c)$ holds for $k \in \mathbb{Z}$. We have

$$\text{Tree}_{H,S}(c_k t^k) \leq \text{Tree}_{H,S}(c) < \text{Tree}_{H,S}(b) \leq T.$$

Hence by assumption $c_k t^k$ is invertible in $\mathcal{D}_{H,S}$ for $k \in \mathbb{Z}$ if $c_k t^k$ is non-zero. This implies that for every $k \in \mathbb{Z}$ the element c_k is zero or invertible in $\mathcal{D}_{N,S} = \mathcal{D}_{H,S}$. Therefore the element c is invertible in $(\mathcal{D}_{N,S})_{\Theta|}((t))$. Since b is the inverse of c in $\mathcal{D}_{H,S}$, the element b belongs to $(\mathcal{D}_{N,S})_{\Theta|}((t))$ and is the inverse of c in $(\mathcal{D}_{N,S})_{\Theta|}((t))$.

Put $k_0 = \min\{k \in \mathbb{Z} \mid c_k \neq 0\}$. Write $b = \sum_{l \geq 0} b_l \cdot t^l$ for $b_l \in \mathcal{D}_{N,R} = \mathcal{D}_{N,S}$. Then $b_l t^l$ for $l \in \mathbb{Z}$ can be expressed using sums and products of elements $(c_{k_0} t^{k_0})^{-1}$ and $c_k t^k$ for $k \in \{k_0 + 1, k_0 + 2, \dots, 2k_0 + l\}$. Choose for $k \in \mathbb{Z}$ elements β_k, γ_k in $\text{Rat}(U) \amalg \{0\}$ satisfying

$$\begin{aligned} \text{Tree}_U(\beta_k) &= \text{Tree}_{H,S}(b_k t^k); \\ \text{Tree}_U(\gamma_k) &= \text{Tree}_{H,S}(c_k t^k). \end{aligned}$$

We conclude from Theorem 3.67 (xi)

$$(3.116) \quad \log^2(\text{Tree}_U(\gamma_{k_0}^\diamond)) = \text{Tree}_U(\gamma_{k_0}) \leq \text{Tree}_U(\gamma) = \log^2(\text{Tree}_U(\gamma^\diamond)) = \log^2(\text{Tree}_U(\beta))$$

and using (3.35) and (3.38)

$$(3.117) \quad \log^2(\text{Tree}_U(\gamma_k)) < \text{Tree}_U(\gamma_k) \leq \text{Tree}_U(\gamma) = \log^2(\text{Tree}_U(\gamma^\diamond)) = \log^2(\text{Tree}_U(\beta)).$$

Hence Theorem 3.67 (ix) and (x) imply for $l \in \mathbb{Z}$ using (3.116) and (3.117)

$$\begin{aligned} (3.118) \quad & \log^2(\text{Tree}_U(\beta_l)) \\ & \leq \max \{ \log^2(\text{Tree}_{H,S}((c_{k_0} t^{k_0})^{-1})), \max \{ \log^2(\text{Tree}_{H,S}(c_k t^k)) \mid k \in k_0 + 1, \dots, 2k_0 + l \} \} \\ & = \max \{ \log^2(\text{Tree}_U(\gamma_{k_0}^\diamond)), \max \{ \log^2(\text{Tree}_U(\gamma_k)) \mid k \in k_0 + 1, \dots, 2k_0 + l \} \} \\ & \leq \log^2(\text{Tree}_U(\beta)). \end{aligned}$$

This implies $\text{Tree}_U(\beta_l) \leq \text{Tree}_U(\beta)$ for all $l \in \mathbb{Z}$. Now suppose that for $n \in \mathbb{Z}$ we have $\text{Tree}_U(\beta_n) = \text{Tree}_U(\beta)$. Since the inequality (3.117) is strict, we conclude $\text{Tree}_U(\gamma_{k_0}) = \text{Tree}_U(\gamma)$ from (3.116) and (3.118) and hence

$$\text{Tree}_{H,S}(c_{k_0} t^{k_0}) = \text{Tree}_{H,S}(c).$$

Since we have already shown above $\text{Tree}_{H,S}(c) < \text{Tree}_{H,S}(b)$, the induction hypothesis implies that we have $c = c_{k_0} \cdot t^{k_0}$ and we can choose $\gamma' \in \text{Rat}(E^\times N)$ satisfying $\gamma = \gamma' t^{k_0}$. This implies

$$b = c^{-1} = t^{-k_0} c_{k_0}^{-1} = \Theta|^{-k_0}(c_{k_0}) t^{-k_0}$$

for $\Theta|^{-k_0}(c_{k_0}) \in \mathcal{D}_{N;R} = \mathcal{D}_{N;S}$ and

$$\beta = \gamma^\diamond = t^{-k_0} (\gamma')^\diamond = \text{Rat}(\bar{c}_t)^{-k_0} ((\gamma')^\diamond) \cdot t^{-k_0}$$

for $\text{Rat}(\bar{c}_t)^{-k_0} ((\gamma')^\diamond) \in \text{Rat}(E^\times N)$.

Case 2: $b \in U \sharp X \setminus (X \cup U)$

Choose elements $\gamma, \delta \in U \sharp X \setminus U$ satisfying $\beta = \gamma\delta$. We conclude

$$\text{Tree}_U(\gamma), \text{Tree}_U(\delta) < \text{Tree}_U(\beta)$$

for $U = E^\times H$ from Theorem 3.67 (vi). Put $c = \rho_{H,S}(\gamma)$ and $d = \rho_{H,S}(\delta)$. Then we get

$$b = cd.$$

Next we show that γ realizes the complexity of c , i.e., $\text{Tree}_{H,S}(c) = \text{Tree}_U(\gamma)$. Suppose the contrary. Then we can find $\gamma' \in \text{Rat}(U)$ with $\rho_{H,S}(\gamma') = \rho_{H,S}(\gamma) = c$ and $\text{Tree}_U(\gamma') < \text{Tree}_U(\gamma)$. We get from Proposition 3.41 (iii) and Theorem 3.67 (v)

$$\begin{aligned} \text{Tree}_U(\gamma'\delta) &= \text{Tree}_U(\gamma') \text{Tree}_U(\delta) \\ &< \text{Tree}_U(\gamma) \text{Tree}_U(\delta) = \text{Tree}_u(\gamma\delta) = \text{Tree}_U(\beta). \end{aligned}$$

Since $\rho_{H,S}(\gamma'\delta) = b$ holds, we get a contradiction to the fact that β realizes the complexity of b . Analogously one shows that δ realizes the complexity of d .

Since we have $\text{Tree}_{H,S}(c), \text{Tree}_{H,S}(d) < \text{Tree}_{H,S}(b)$, the induction hypothesis applies to c and d . Hence we can write

$$\begin{aligned} c &= \sum_{k \in \mathbb{Z}} c_k t^k; \\ d &= \sum_{k \in \mathbb{Z}} d_k t^k, \end{aligned}$$

for $c_k, d_k \in \mathcal{D}_{N,R} = \mathcal{D}_{N,S}$ such that $\text{Tree}_{H,S}(c_k t^k) \leq \text{Tree}_{H,S}(c)$ and $\text{Tree}_{H,S}(d_k t^k) \leq \text{Tree}_{H,S}(d)$ hold for $k \in \mathbb{Z}$. Since $b = cd$, we can write

$$b = \sum_{k \in \mathbb{Z}} b_k t^k$$

such that $b_k t^k = \sum_{l \in \mathbb{Z}} c_{k-l} t^{k-l} d_l t^l$ holds. Choose elements β_k, γ_k , and δ_k in $\text{Rat}(U)$ satisfying

$$\begin{aligned} \text{Tree}_{H,S}(b_k t^k) &= \text{Tree}_U(\beta_k); \\ \text{Tree}_{H,S}(c_k t^k) &= \text{Tree}_U(\gamma_k); \\ \text{Tree}_{H,S}(d_k t^k) &= \text{Tree}_U(\delta_k), \end{aligned}$$

for $k \in \mathbb{Z}$. We conclude from Theorem 3.67 (iii) and (v)

$$\text{Tree}_U(\beta_k) \leq \sum_{l \in \mathbb{Z}} \text{Tree}_U(\gamma_{k-l}) \cdot \text{Tree}_U(\delta_l).$$

Using Theorem 3.67 (vii) and (viii) we get

$$\begin{aligned} (3.119) \quad \log(\text{Tree}_U(\beta_k)) &\leq \log \left(\sum_{m \in \mathbb{Z}} \text{Tree}_U(\gamma_{k-m}) \cdot \text{Tree}_U(\delta_m) \right) \\ &= \max \{ \log(\text{Tree}_U(\gamma_{k-l}) \cdot \text{Tree}_U(\delta_l)) \mid l \in \mathbb{Z} \} \\ &= \max \{ \log(\text{Tree}_U(\gamma_{k-l})) + \log(\text{Tree}_U(\delta_l)) \mid l \in \mathbb{Z} \} \\ &\leq \log(\text{Tree}_U(\gamma)) + \log(\text{Tree}_U(\delta)) \\ &= \log(\text{Tree}_U(\gamma\delta)) \\ &= \log(\text{Tree}_U(\beta)) \\ &= \log(\text{Tree}_{H,S}(b)). \end{aligned}$$

This implies

$$\text{Tree}_U(b_k) \leq \text{Tree}_U(b)$$

for all $k \in \mathbb{Z}$. Suppose that for $n \in \mathbb{Z}$ we have $\text{Tree}_{H,S}(b_n t^n) = \text{Tree}_{H,S}(b)$, then we conclude by inspecting (3.119) that there is an integer $m \in \mathbb{Z}$ with

$$\begin{aligned} \log(\text{Tree}_U(\gamma_{n-m})) &= \log(\text{Tree}_U(\gamma)); \\ \log(\text{Tree}_U(\delta_m)) &= \log(\text{Tree}_U(\delta)). \end{aligned}$$

Since γ and δ belong to $U \sharp X$, we get $\text{width}(\gamma) = \text{width}(\delta) = 1$ from Theorem 3.67 (xiii). Since we conclude $\text{Tree}_U(\gamma) \leq \text{Tree}_U(\gamma_{n-m})$ and $\text{Tree}_U(\delta) \leq \text{Tree}_U(\delta_m)$ from (3.38), we get $\text{Tree}_U(\gamma) = \text{Tree}_U(\gamma_{n-m})$ and $\text{Tree}_U(\delta) = \text{Tree}_U(\delta_m)$ and hence

$$\begin{aligned}\text{Tree}_{H,S}(c) &= \text{Tree}_{H,S}(c_{n-m}t^{n-m}); \\ \text{Tree}_{H,S}(d) &= \text{Tree}_{H,S}(d_mt^m).\end{aligned}$$

From the induction hypothesis applied to c and d we get

$$\begin{aligned}c &= c_{n-m}t^{n-m}; \\ d &= d_mt^m,\end{aligned}$$

and there are elements γ' and δ' in $\text{Rat}(E^\times N)$ satisfying

$$\begin{aligned}\gamma &= \gamma't^{n-m}; \\ \delta &= \delta't^m.\end{aligned}$$

Hence we can write

$$b = cd = c_{n-m}t^{n-m}d_mt^m = c_{n-m}\Theta|^{n-m}(d_m)t^n$$

for $c_{n-m}\Theta|^{n-m}(d_m) \in \mathcal{D}_{N,R} = \mathcal{D}_{N,S}$ and

$$\beta = \gamma\delta = \gamma't^{n-m}\delta't^m = \gamma'\text{Rat}(\overline{c_t})^{n-m}(\delta')t^n$$

for $\gamma'\text{Rat}(\overline{c_t})^{n-m}(\delta') \in \text{Rat}(E^\times N)$.

Case 3: $b \in \mathbb{N}[U\sharp X] \setminus U\sharp X \cup \{0\}$

Choose elements $\gamma, \delta \in \mathbb{N}[U\sharp X] \setminus \{0\}$ satisfying $\beta = \gamma + \delta$. We conclude

$$\text{Tree}_U(\gamma), \text{Tree}_U(\delta) < \text{Tree}_U(\beta)$$

for $U = E^\times H$ from Theorem 3.67 (iv). Put $c = \rho_{H,S}(\gamma)$ and $d = \rho_{H,S}(\delta)$. Then we get

$$b = c + d.$$

Next we show that γ realizes the complexity of c , i.e., $\text{Tree}_{H,S}(c) = \text{Tree}_U(\gamma)$. Suppose the contrary. Then we can find $\gamma' \in \text{Rat}(U)$ with $\rho_{H,S}(\gamma') = \rho_{H,S}(\gamma) = c$ and $\text{Tree}_U(\gamma') < \text{Tree}_U(\gamma)$. We conclude from Proposition 3.41 (i) and Theorem 3.67 (iii)

$$\begin{aligned}\text{Tree}_U(\gamma' + \delta) &= \text{Tree}_U(\gamma') + \text{Tree}_U(\delta) \\ &< \text{Tree}_U(\gamma) + \text{Tree}_U(\delta) = \text{Tree}_U(\gamma + \delta) = \text{Tree}_U(\beta).\end{aligned}$$

Since $\rho_{H,S}(\gamma' + \delta) = b$, this contradicts the fact that β realizes the complexity of b . Analogously one shows that δ realizes the complexity of d .

Since we have $\text{Tree}_{H,S}(c), \text{Tree}_{H,S}(d) < \text{Tree}_{H,S}(b)$, the induction hypothesis applies to c and d . Hence we can write

$$c = \sum_{k \in \mathbb{Z}} c_k t^k;$$

$$d = \sum_{k \in \mathbb{Z}} d_k t^k,$$

for $c_k, d_k \in \mathcal{D}_{N,R} = \mathcal{D}_{N,S}$ such that $\text{Tree}_{H,S}(c_k t^k) \leq \text{Tree}_{H,S}(c)$ and $\text{Tree}_{H,S}(d_k t^k) \leq \text{Tree}_{H,S}(d)$ hold for $k \in \mathbb{Z}$. Hence we can write

$$b = \sum_{k \in \mathbb{Z}} b_k \cdot t^k$$

for $b_k = c_k + d_k$. Choose elements β_k, γ_k , and δ_k in $\text{Rat}(U)$ satisfying

$$\begin{aligned} \text{Tree}_{H,S}(b_k t^k) &= \text{Tree}_U(\beta_k); \\ \text{Tree}_{H,S}(c_k t^k) &= \text{Tree}_U(\gamma_k); \\ \text{Tree}_{H,S}(d_k t^k) &= \text{Tree}_U(\delta_k), \end{aligned}$$

for $k \in \mathbb{Z}$. We conclude from Proposition 3.41 (i) and Theorem 3.67 (iii)

(3.120)

$$\begin{aligned} \text{Tree}_{H,S}(b_k t^k) &= \text{Tree}_U(\beta_k) = \text{Tree}_U(\gamma_k + \delta_k) = \text{Tree}_U(\gamma_k) + \text{Tree}_U(\delta_k) \\ &\leq \text{Tree}_U(\gamma) + \text{Tree}_U(\delta) = \text{Tree}_U(\gamma + \delta) = \text{Tree}_U(\beta) = \text{Tree}_{H,S}(b). \end{aligned}$$

Suppose for $n \in \mathbb{Z}$ that $\text{Tree}_{H,S}(b_n) = \text{Tree}_{H,S}(b)$ holds. Since $\text{Tree}_U(\gamma_n) \leq \text{Tree}_U(\gamma)$ and $\text{Tree}_U(\delta_n) \leq \text{Tree}_U(\delta)$ hold, we conclude $\text{Tree}_U(\gamma_n) = \text{Tree}_U(\gamma)$ and $\text{Tree}_U(\delta_n) = \text{Tree}_U(\delta)$ by inspecting (3.120). This implies

$$\begin{aligned} \text{Tree}_{H,S}(c_n t^n) &= \text{Tree}_{H,S}(c); \\ \text{Tree}_{H,S}(d_n t^n) &= \text{Tree}_{H,S}(d). \end{aligned}$$

Since the induction hypothesis applies to c and d , we obtain

$$\begin{aligned} c &= c_n t^n; \\ d &= d_n t^n, \end{aligned}$$

and the existence of elements γ' and δ' in $\text{Rat}(E^\times N)$ satisfying

$$\begin{aligned} \gamma &= \gamma' t^n; \\ \delta &= \delta' t^n. \end{aligned}$$

Hence we get

$$b = c + d = (b_n + c_n) t^n$$

for $b_n + c_n \in \mathcal{D}_{N,R} = \mathcal{D}_{N,S}$ and

$$\beta = \gamma + \delta = (\gamma' + \delta') t^n$$

for $(\gamma' + \delta') \in \text{Rat}(E^\times N)$. This finishes the proof of Proposition 3.115. $\square$

3.9 Hughes-Free Epic EG -Division Rings

Let G be a locally indicible group. Consider an EG -division ring $\varphi: EG \rightarrow D$, i.e., a ring homomorphism φ with EG as source and a division ring (= skew field) D as target. Suppose that it is epic in the sense of Definition 3.9. Recall that for a subgroup $H \subseteq G$ we get an EH -division ring $\varphi|_H: EH \rightarrow D$ by restriction and that $\mathcal{D}_{H,D} \subseteq D$ is the division closure of $\varphi(EH)$ in D . Recall that φ is epic if and only if the division closure of $\text{im}(\varphi)$ in D is D itself, see Proposition 3.14. So we have $\mathcal{D}_{G,D} = D$ by assumption.

Consider any non-trivial finitely generated subgroup $H \subseteq G$. Since G is locally indicible, we can find a normal subgroup $N \subseteq H$ such that H/N is infinite cyclic. Let $t \in H$ be any generator which is mapped under the canonical projection $H \rightarrow H/N$ to a generator. Conjugation with t induces a ring automorphism $\theta: EG \rightarrow EG$, see (3.110). Since φ is epic, there is precisely one automorphism $\Theta: D \rightarrow D$ satisfying $\Theta \circ \varphi = \varphi \circ \theta$ which is given by conjugation with $\varphi(t)$. It sends $\mathcal{D}_{N,D}$ to itself and hence induces an automorphism $\Theta|: \mathcal{D}_{N,D} \rightarrow \mathcal{D}_{N,D}$. There is a canonical map $(\mathcal{D}_{N,D})_{\Theta|}[t, t^{-1}] \rightarrow \mathcal{D}_{H,D}$ which sends t to $\varphi(t)$ and extends the inclusion $\mathcal{D}_{N,D} \subseteq \mathcal{D}_{H,D}$. It induces a ring homomorphism

$$(3.121) \quad \mu_{H,N,t,D}: S^{-1}(\mathcal{D}_{N,D})_{\Theta|}[t, t^{-1}] \rightarrow \mathcal{D}_{H,D}$$

where the source is the division ring given by the Ore localization of $(\mathcal{D}_{N,D})_{\Theta|}[t, t^{-1}]$ with respect to the multiplicatively closed subset S of elements in $(\mathcal{D}_{N,D})_{\Theta|}[t, t^{-1}]$ which are different from zero.

Definition 3.122 (Hughes-free epic EG -division ring). Let G be a locally indicible group and $\varphi: EG \rightarrow D$ be an epic EG -division ring.

We call φ *Hughes-free* if for every finite generated subgroup H and normal subgroup $N \subseteq H$ and $t \in H$ such that H/N is infinite cyclic and the image of t under the canonical projection $H \rightarrow H/N$ is a generator, the ring homomorphism $\mu_{H,N,t,D}: S^{-1}(\mathcal{D}_{N,D})_{\Theta|}[t, t^{-1}] \rightarrow \mathcal{D}_{H,D}$ of (3.121) is bijective.

We leave the elementary proof of the next lemma to the reader.

Lemma 3.123. *The ring homomorphism $\mu_{H,N,t,D}$ is bijective if and only if $\{\varphi(t)^i \mid i \in \mathbb{Z}_{\geq 0}\}$ is $\mathcal{D}_{N,D}$ -linearly independent in $\mathcal{D}_{H,D}$, i.e., if we have $m \in \mathbb{Z}_{\geq 0}$ and elements $a_0, a_1, \dots, a_m \in \mathcal{D}_{N,D}$ satisfying $\sum_{i=0}^m a_i \varphi(t)^i = 0$ in $\mathcal{D}_{H,D}$, then $a_0 = a_1 = \dots = a_m$ holds.*

Note that φ is injective if φ is a Hughes-free epic EG -division ring. In this case we will identify $t = \varphi(t)$.

The following celebrated theorem is due to Hughes [25].

Theorem 3.124 (Uniqueness of Hughes-free epic EG -division rings)
Let G be a locally indicible group and $\varphi: EG \rightarrow D_i$ for $i = 1, 2$ be Hughes-free epic EG -division rings.

Then φ_1 and φ_2 are isomorphic as EG -division rings, i.e., there exists an isomorphism $\psi: D_1 \xrightarrow{\cong} D_2$ of division rings satisfying $\psi \circ \varphi_1 = \varphi_2$.

Proof. Put $D = D_1 \times D_2$ and $\varphi = \varphi_1 \times \varphi_2: EG \rightarrow D = D_1 \times D_2$. We will show by induction over the complexity that every element b in $\mathcal{D}(\varphi(EG) \subseteq D) \setminus \{0\}$ is invertible and hence that $\mathcal{D}(\varphi(EG) \subseteq D)$ is a division ring. Then pr_i induces isomorphisms of EG -division rings $\mathcal{D}(\varphi(EG) \subseteq D) \xrightarrow{\cong} D_i$ for $i = 1, 2$ because of Proposition 3.14 and hence Theorem 3.124 follows.

Next we show for every element $b \in \mathcal{D}_{G,D} = \mathcal{D}(\varphi(EG) \subseteq D)$ with $b \neq 0$ by induction over the complexity introduced in Section 3.6 that b is invertible. The induction beginning $\text{Tree}_{G,D}(b) = 1_\tau$ is obvious since then $b \in \varphi(E^\times G)$. In the induction step we have to show for $b \in \mathcal{D}_{G,D}$ with $\text{Tree}_{G,D}(b) > 1_\tau$ that b is invertible in $\mathcal{D}_{G,D}$ provided that the induction hypothesis holds, i.e., every $c \in \mathcal{D}_{G,D}$ with $c \neq 0$ and $\text{Tree}_{G,D}(c) < \text{Tree}_{G,D}(b)$ is invertible in $\mathcal{D}_{G,D}$. This is done as follows.

Let $\beta \in \text{Rat}(E^\times G)$ realize the complexity of b . By Theorem 3.83 there exists a finitely generated subgroup $\text{source}(\beta) \subseteq G$ such that β belongs to $\text{Rat}(\text{source}(\beta)) \cdot E^\times G$. Since multiplying b with an element in $\varphi(E^\times G)$ does not change its complexity and the desired conclusion for b , we can assume without loss of generality by Proposition 3.85 that β is primitive, i.e., β belongs to $\text{Rat}(\text{source}(\beta))$. Let $H \subseteq G$ be the finitely generated subgroup of G given by the image of $\text{source}(\beta)$ under the projection $E^\times G \rightarrow E^\times G / E^\times = G$. Then we have $b \in \mathcal{D}_{H,\mathcal{D}(\varphi(EH) \subseteq D)} = \mathcal{D}_{H,D}$, see Lemma 2.78 (i). We get from the commutative diagram (3.80)

$$\text{Tree}_{H,D}(b) \leq \text{Tree}_{E^\times H}(\beta) = \text{Tree}_{E^\times G}(\beta) = \text{Tree}_{G,D}(b).$$

This implies together with Lemma 3.81

$$(3.125) \quad \text{Tree}_{H,D}(b) = \text{Tree}_{G,D}(b).$$

If H is trivial, b is obviously invertible. Since G is by assumption locally indicable, we can assume without loss of generality the existence of normal subgroup $N \subseteq H$ such that H/N is infinite cyclic. Choose $t \in H$ which is sent under the canonical projection $H \rightarrow H/N$ to a generator.

Put

$$\begin{aligned} A &= \mathcal{D}_{N,D_1} \times \mathcal{D}_{N,D_2}; \\ B &= \mathcal{D}_{H,D_1} \times \mathcal{D}_{H,D_2}. \end{aligned}$$

Consider $i \in \{1, 2\}$. As φ_i is Hughes-free by assumption we have an embedding of EH -rings $\mathcal{D}_{H,D_i} \subseteq (\mathcal{D}_{N,D_i})_{\Theta_i}((t))$ where $\Theta_i: \mathcal{D}_{N,D_i} \xrightarrow{\cong} \mathcal{D}_{N,D_i}$ is the automorphism induced by conjugation with t_i . Therefore B can be embedded in $S = A_\Theta((t))$ for the automorphism $\Theta = \Theta_1 \times \Theta_2$ of $A = \mathcal{D}_{N,D_1} \times \mathcal{D}_{N,D_2}$. Since A and B are products of division rings, they are von Neumann regular. The following diagram commutes

$$\begin{array}{ccccc}
EN & \longrightarrow & EG & \longleftarrow & EH \\
\downarrow \varphi|_{EN} & & \downarrow \varphi & & \downarrow \varphi|_{EH} \\
A & \longrightarrow & D & \longleftarrow & B \\
& \searrow & & \swarrow & \\
& & S & &
\end{array}$$

where all the arrows without mark are the obvious inclusions. We conclude from Lemma 2.78

$$\begin{aligned}
\mathcal{D}_{N,A} &= \mathcal{D}_{N,D} = \mathcal{D}_{N,S}; \\
\mathcal{D}_{H,B} &= \mathcal{D}_{H,D} = \mathcal{D}_{H,S}.
\end{aligned}$$

For every $c \in \mathcal{D}_{H,D}$ with $\text{Tree}_{H,D}(c) < \text{Tree}_{H,D}(b)$ we get using (3.125) and Lemma 3.81

$$\text{Tree}_{G,D}(c) \leq \text{Tree}_{H,D}(c) < \text{Tree}_{H,D}(b) = \text{Tree}_{G,D}(b).$$

Hence c is invertible in $\mathcal{D}_{G,D}$ and hence in D by the induction hypothesis. Since $\mathcal{D}_{H,D}$ is division closed in D , c is invertible in $\mathcal{D}_{H,D}$.

Now all the conditions appearing in Proposition 3.115 are satisfied, where we take $R = A$ and $S = A_{\Theta}((t))$. Hence b belongs to $(\mathcal{D}_{N,D})_{\Theta}((t)) \subseteq \mathcal{D}_{H,D}$ and we can write $b = \sum_{k \geq k_0} b_k t^k$ for some $k_0 \in \mathbb{Z}$ and $b_k \in \mathcal{D}_{N,D}$ satisfying $b_{k_0} \neq 0$.

Next we show that there is at least one element $k_1 \in \mathbb{Z}_{\geq k_0+1}$ satisfying $b_{k_1} \neq 0$. Suppose the contrary. Then $b = b_{k_0} t^{k_0}$ and Proposition 3.115 implies $\beta = \gamma t^{k_0}$ for some $\gamma \in \text{Rat}(E^{\times} N)$ which implies $H \subseteq N$ by Proposition 3.85 (iii), a contradiction.

Hence Proposition 3.115 implies that $\text{Tree}_{H,D}(b_k t^k) < \text{Tree}_{H,D}(b)$ holds for every k and in particular for k_0 . Hence $b_{k_0} t^{k_0}$ is invertible in $\mathcal{D}_{H,D}$. As t^{k_0} is invertible in $(\mathcal{D}_{N,D})_{\Theta}((t))$ and hence in $\mathcal{D}_{H,D}$, the element b_{k_0} is invertible in $\mathcal{D}_{H,D}$ and hence in $\mathcal{D}$. As b_{k_0} belongs to $\mathcal{D}_{N,D}$, it is invertible in $\mathcal{D}_{N,D}$. This implies that b is invertible in $(\mathcal{D}_{N,D})_{\Theta}((t))$ and hence in $\mathcal{D}_{H,D} = \mathcal{D}_{H,S}$. This finishes the proof of Theorem 3.124. $\square$

3.10 Sylvester Rank Functions

3.10.1 Sylvester Matrix Rank Functions and Sylvester Module Rank Functions

The notion of a Sylvester rank function is due to Malcolmson [51].

Definition 3.126 (Sylvester module rank function). A *Sylvester module rank function* $\dim = \dim_R$ on a ring R assigns to every finitely presented R -module a non-negative real number $\dim(M) \in \mathbb{R}_{\geq 0}$ satisfying the following axioms:

- (i) We have $\dim(\{0\}) = 0$ for the zero-module $\{0\}$;
- (ii) We have $\dim(R) = 1$;
- (iii) For two finitely presented modules M and N we get

$$\dim(M \oplus N) = \dim(M) + \dim(N);$$

- (iv) If $M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is an exact sequence of finitely presented R -modules, we get

$$\dim(M_1) + \dim(M_3) \geq \dim(M_2) \geq \dim(M_3).$$

There is also a matrix-version of this notion.

Definition 3.127 (Sylvester matrix rank function). A *Sylvester matrix rank function* $\text{rk} = \text{rk}_R$ on a ring R assigns to every finite matrix A over R a non-negative real number $\text{rk}(A) \in \mathbb{R}_{\geq 0}$ satisfying the following axioms:

- (i) We have $\text{rk}(A) = 0$ if every entry in A is zero;
- (ii) We have $\text{rk}(I_1) = 1$ for the $(1, 1)$ -matrix $I_1 = (1)$ with entry the unit 1;
- (iii) For $A \in M_{l,m}(R)$ and $B \in M_{m,n}(R)$ we get for the product $AB \in M_{l,n}(R)$

$$\text{rk}(AB) \leq \min\{\text{rk}(A), \text{rk}(B)\};$$

- (iv) For $A_1 \in M_{m_1, n_1}(R)$ and $A_2 \in M_{m_2, n_2}(R)$ we get for the block sum $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \in M_{m_1+m_2, n_1+n_2}(R)$

$$\text{rk} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \text{rk}(A_1) + \text{rk}(A_2);$$

- (v) For $A_1 \in M_{m_1, n_1}(R)$, $A_2 \in M_{m_2, n_2}(R)$, and $A_3 \in M_{m_1, n_2}(R)$ we get for the matrix $\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \in M_{m_1+m_2, n_1+n_2}(R)$

$$\text{rk} \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \geq \text{rk}(A_1) + \text{rk}(A_2).$$

Proposition 3.128. *There is a one-to one-correspondence between Sylvester module rank functions and Sylvester matrix rank functions.*

Proof. Given a Sylvester module rank function $\dim$, we get a Sylvester matrix rank function rk by assigning to a matrix $A \in M_{m,n}(R)$

$$\text{rk}(A) := n - \dim(\text{coker}(r_A: R^m \rightarrow R^n)),$$

where r_A is the R -homomorphism given by right multiplication with A .

Given a Sylvester matrix rank function rk , we get a Sylvester module rank function $\dim$ by defining for a finitely presented R -module M

$$\dim(M) := n - \text{rk}(A)$$

for any choice of $m, n \in \mathbb{Z}_{\geq 1}$ and $A \in M_{m,n}(R)$, for which M is R -isomorphic to the cokernel of the R -homomorphism $r_A: R^m \rightarrow R^n$. For the elementary proof that thus we get the desired one-to-one-correspondence, we refer to [51, Section 5] or [65, Chapter 7]. $\square$

Because of Proposition 3.128 we sometimes speak of a *Sylvester rank function* without specifying whether we mean the matrix or module version and sometimes switch tacitly from Sylvester matrix rank functions to Sylvester module rank functions and vice versa.

Exercise 3.129 Let R be a ring and rk be a Sylvester matrix rank function on it. Prove:

- (i) We have $\text{rk}(A) \leq \min\{m, n\}$ for $A \in M_{m,n}(R)$;
- (ii) If $A \in M_{n,n}(R)$ is invertible, then $\text{rk}(A) = n$;
- (iii) For $A, B \in M_{m,n}(R)$ we get $\text{rk}(A + B) \leq \text{rk}(A) + \text{rk}(B)$;
- (iv) For $A, B \in M_{m,n}(R)$ with $\text{rk}(A) = 0$ we get $\text{rk}(A + B) = \text{rk}(B)$.

Remark 3.130 (Sylvester rank functions over division algebras). Let D be a division ring. Then we get a Sylvester module rank function $\dim_D$ on D by assigning to a finitely presented D -module its dimension as a finite-dimensional D -vector space. Moreover, this is the only Sylvester module rank function on D .

We conclude from Proposition 3.128 that there is precisely one Sylvester matrix rank function rk_D on D which assigns to $A \in M_{m,n}(R)$ the dimension of the finite dimensional D -vector space given by the image of the D -linear map $r_A: D^m \rightarrow D^n$. It agrees with the inner rank, i.e., the maximum size of an invertible minor, the row rank, and the column rank.

Proposition 3.131. *Let rk be a Sylvester matrix rank function on the ring R .*

- (i) *Consider $A \in M_{m,m}(R)$, $B \in M_{n,m}(R)$, and $C \in M_{n,n}(R)$. Suppose that $\text{rk}(A) = m$ or $\text{rk}(C) = n$ hold. Then we get*

$$\text{rk} \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} = \text{rk}(A) + \text{rk}(C);$$

- (ii) *Consider $A \in M_{m,m}(R)$, $B \in M_{m,n}(R)$, and $C \in M_{n,n}(R)$. Suppose that $\text{rk}(A) = m$ or $\text{rk}(C) = n$ hold. Then we get*

$$\text{rk} \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \text{rk}(A) + \text{rk}(C);$$

- (iii) *Consider $A \in M_{m,m}(R)$, $B \in M_{m,n}(R)$, and $C \in M_{n,n}(R)$. Suppose that $\text{rk}(A) = m$ and $\text{rk}(C) = n$ holds. Then we get*

$$\text{rk}(ABC) = \text{rk}(B).$$

Proof. (i) Since $\begin{pmatrix} I_m & 0 \\ B & I_n \end{pmatrix}$ is invertible, we conclude directly from the axioms of a Sylvester matrix rank functions

$$\begin{aligned} \text{rk} \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} &\geq \text{rk}(A) + \text{rk}(B); \\ \text{rk}(A) &\leq m; \\ \text{rk}(B) &\leq n; \\ \text{rk} \begin{pmatrix} I_m & 0 \\ B & I_n \end{pmatrix} &= m + n. \end{aligned}$$

Since we have

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix} \cdot \begin{pmatrix} I_m & 0 \\ B & I_n \end{pmatrix} \cdot \begin{pmatrix} I_m & 0 \\ 0 & C \end{pmatrix},$$

we get

$$\begin{aligned} \text{rk} \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} &\leq \min \left\{ \text{rk} \begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix}, \text{rk} \begin{pmatrix} I_m & 0 \\ B & I_n \end{pmatrix}, \text{rk} \begin{pmatrix} I_m & 0 \\ 0 & C \end{pmatrix} \right\} \\ &= \min \{ \text{rk}(A) + n, m + n, \text{rk}(C) + m \} \\ &= \min \{ \text{rk}(A) + n, \text{rk}(C) + m \}. \end{aligned}$$

Since $\text{rk}(A) = m$ or $\text{rk}(C) = n$ holds by assumption, this implies

$$\text{rk} \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \leq \text{rk}(A) + \text{rk}(C).$$

Hence we get

$$\text{rk} \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} = \text{rk}(A) + \text{rk}(C).$$

(ii) This follows from assertion (i) because of

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix} \cdot \begin{pmatrix} C & B \\ 0 & A \end{pmatrix} \cdot \begin{pmatrix} 0 & I_m \\ I_n & 0 \end{pmatrix}.$$

(iii) By replacing A with a block sum with the matrix I_{m-n} if $m < n$, replacing C with a block sum with the matrix I_{n-m} if $n < m$, and by extending B to a $(\max\{m, n\}, \max\{m, n\})$ -square matrix by adding 0-s, we can arrange $m = n$.

Hence it suffices to show for $A, B \in M_{n,n}(R)$

$$(3.132) \quad \text{rk}(AB) = \begin{cases} \text{rk}(A) & \text{if } \text{rk}(B) = n; \\ \text{rk}(B) & \text{if } \text{rk}(A) = n. \end{cases}$$

Since we have

$$\begin{pmatrix} AB & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \cdot \begin{pmatrix} I_n & 0 \\ -A & I_n \end{pmatrix} \cdot \begin{pmatrix} B & I_n \\ 0 & A \end{pmatrix} \cdot \begin{pmatrix} I_n & 0 \\ -B & I_n \end{pmatrix}$$

and the rank of an invertible (n, n) -matrix is n , we get

$$\text{rk}(AB) + n = \text{rk} \begin{pmatrix} B & I_n \\ 0 & A \end{pmatrix}.$$

We conclude from assertion (ii)

$$\text{rk} \begin{pmatrix} B & I_n \\ 0 & A \end{pmatrix} = \text{rk}(B) + \text{rk}(A).$$

Now (3.132) follows. This finishes the proof of Proposition 3.131. $\square$

Proposition 3.133. *Let R be a von Neumann $*$ -regular ring and rk be a Sylvester matrix rank function on R . Consider $A \in M_{m,n}(R)$. Define $A^* \in M_{n,m}(R)$ by $A^* = ((a_{j,i})^*)$. Then we get*

$$\text{rk}(A) = \text{rk}(A^*).$$

Proof. See [27, Proposition 6.11]. $\square$

3.10.2 Integer-Valued Sylvester Module Rank Functions

Let $\varphi: R \rightarrow D$ be a R -division ring, i.e., a ring homomorphism with a division ring D as target. Then we obtain a Sylvester module rank function

$$\dim_\varphi(M) := \dim_D(D \otimes_R M) \in \mathbb{Z}_{\geq 0}$$

whose values are non-negative integers.

Given a Sylvester module rank function $\dim$ on ring R taking values in $\mathbb{Z}_{\geq 0}$, there exists an epic R -division ring $\varphi: R \rightarrow D$ with $\dim = \dim_\varphi(M)$. This epic R -division ring is actually uniquely determined by this property. Namely, if $\varphi_1: R \rightarrow D_1$ and $\varphi_2: R \rightarrow D_2$ are two epic R -division rings with $\dim_{\varphi_1} = \dim_{\varphi_2}$, then there exists an isomorphism of R -division rings between them, i.e., there is precisely one isomorphism $\mu: D_1 \xrightarrow{\cong} D_2$ of division rings satisfying $\mu \circ \varphi_1 = \varphi_2$. These claims have been proved by Malcolmson [51, Theorem 2] taking Proposition 3.128 into account. Note that there is at most one isomorphism of division rings from $\varphi_1: R \rightarrow D_1$ to $\varphi_2: R \rightarrow D_2$ since φ_1 and φ_2 are epic in the sense of Definition 3.9.

3.10.3 Faithful Sylvester Rank Function

Given a Sylvester matrix rank function rk on the ring R , define the two-sided ideal of R

$$\ker(\text{rk}) = \{r \in R \mid \text{rk}(r) = 0\}.$$

We call rk *faithful* if $\ker(\text{rk}) = \{0\}$ holds.

Note that a Sylvester matrix rank function rk on R induces a faithful Sylvester matrix rank function $\overline{\text{rk}}$ on the ring $R/\ker(\text{rk})$ by

$$\overline{\text{rk}}(A) = \text{rk}(B) \quad \text{for } A = (a_{i,j}) \in M_{m,n}(R/\ker(\text{rk})),$$

where $B = (b_{i,j}) \in M_{m,n}(R)$ is any matrix satisfying $\text{pr}(b_{i,j}) = a_{i,j}$ for the canonical projection $\text{pr}: R \rightarrow R/\ker(\text{rk})$.

Exercise 3.134 Show that $\ker(\text{rk})$ is a two sided ideal of R and that $\overline{\text{rk}}$ is a well-defined faithful Sylvester matrix rank function on $R/\ker(\text{rk})$.

Exercise 3.135 Let R be a ring and rk be a Sylvester matrix rank function on R . Let $\dim_R$ be the associated Sylvester dimension function, see Proposition 3.128.

Consider the following assertions:

- (i) rk is faithful;
- (ii) We have $\text{rk}(A) = 0 \iff A = 0$ for every finite matrix A over R ;
- (iii) We have $\dim(M) = 0 \iff M = \{0\}$ for every finitely presented R -module M ;
- (iv) We have $\text{rk}(A) = n \iff A$ is invertible for $A \in M_{n,n}(R)$.

Prove:

- We have (i) $\iff$ (ii);
- We have (iii) $\implies$ (iv);
- If R is von Neumann regular, then (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv) holds.

Exercise 3.136 Let R be a principal ideal domain. Let $R^{(0)}$ be its quotient field. Show that there is precisely one integer-valued Sylvester module rank function $\dim_R$ on R , whose associated Sylvester matrix rank function is faithful, namely, the one which sends a finitely presented R -module M to $\dim_{R^{(0)}}(R^{(0)} \otimes_R M)$.

3.10.4 Envelopes

If we equip the set

$$\text{Mat}(R) = \coprod_{m,n \in \mathbb{Z}_{\geq 1}} M_{m,n}(R)$$

of finite matrices over R with the discrete topology, we have the topological space $\text{map}(\text{Mat}(R); \mathbb{R}_{\geq 0})$ of functions from $\text{Mat}(R)$ to $\mathbb{R}_{\geq 0}$. Let

$$\text{Syl}(R) \subset \text{map}(\text{Mat}(R), \mathbb{R}_{\geq 0})$$

be the subset of Sylvester matrix rank functions on R . We equip it with the subspace topology, where we use on $\text{map}(\text{Mat}(R), \mathbb{R}_{\geq 0})$ the topology coming from pointwise convergence. Then $\text{Syl}(R)$ is a compact and convex subspace of $\text{map}(\text{Mat}(R), \mathbb{R}_{\geq 0})$, see [41, Proposition 2.1.4].

A ring homomorphism $\varphi: R \rightarrow S$ induces a continuous map

$$(3.137) \quad \varphi^\# : \text{Syl}(S) \rightarrow \text{Syl}(R)$$

if for $\text{rk} \in \text{Syl}(S)$ the element $\varphi^\#(\text{rk}) \in \text{Syl}(R)$ sends a finite matrix A over R to $\text{rk}(\varphi(A))$, where $\varphi(A)$ is the finite S -matrix obtained from A by applying φ to each of its entries. On the level of Sylvester module rank functions this corresponds to the formula

$$(3.138) \quad \varphi^\#(\dim_S)(M) = \dim_S(\varphi_* M),$$

where $\varphi_* M$ is the finitely presented S -module $S \otimes_R M$ for a finitely presented R -module M .

Definition 3.139 (Envelope). Let rk_R be a Sylvester matrix rank function on a ring R . An *envelope* $(S, \varphi, \text{rk}_S)$ of rk_R consists of an epic ring homomorphism $\varphi: R \rightarrow S$ and a faithful Sylvester matrix rank function of S satisfying $\text{rk}_R = \varphi^\# \text{rk}_S$.

We call an envelope $(S, \varphi, \text{rk}_S)$ a *von Neumann regular envelope* if S is von Neumann regular and a *division envelope* if S is a division ring.

Suppose that R is a ring with involution. Then we call an envelope $(S, \varphi, \text{rk}_S)$ a *von Neumann *-regular envelope* if S is a von Neumann *-regular ring and φ is a morphism of rings with involution, and a *positive definite von Neumann *-regular envelope* if S is a positive definite von Neumann *-regular ring and φ is a morphism of rings with involution.

Remark 3.140. Let rk_R be a Sylvester rank function on a ring R . Let $\varphi: R \rightarrow S$ be a ring homomorphism and rk_S be a Sylvester rank function on S with $\text{rk}_R = \varphi^\# \text{rk}_S$. Let $\text{pr}: S \rightarrow S/\ker(\text{rk}_S)$ be the projection. Then we obtain a faithful Sylvester rank function $\overline{\text{rk}}_S$ on $S/\ker(\text{rk}_S)$ with $\text{rk}_S = \text{pr}^\# \overline{\text{rk}}_S$. Let $T := \mathcal{D}(\text{im}(\text{pr} \circ \varphi) \subseteq S/\ker(\text{rk}_S))$ be the division closure. Let rk_T be $i^\# \overline{\text{rk}}_S$ for the inclusion $i: T \rightarrow S/\ker(\text{rk}_S)$ and let $\psi: R \rightarrow T$ be the ring homomorphism induced by $\text{pr} \circ \varphi: R \rightarrow S/\ker(\text{rk}_S)$.

Then (T, ψ, rk_T) is an envelope for rk_R by Proposition 3.13 (ii).

If R is von Neumann regular or a division ring, then T is von Neumann regular or a division ring. Suppose that R is a ring with involution, S is von Neumann *-regular and φ is compatible with the involutions, then $S/\ker(\text{rk}_S)$ is a von Neumann *-regular ring by Proposition 3.6. If we replace in the construction of (T, ψ, rk_T) above the division closure $\mathcal{D}(\text{im}(\text{pr} \circ \varphi) \subseteq S/\ker(\text{rk}_S))$ by the *-regular closure $\mathcal{RE}\mathcal{G}(\text{im}(\text{pr} \circ \varphi) \subseteq S/\ker(\text{rk}_S))$ in the sense of Defini-

tion 3.8, then (T, ψ, rk_T) turns out to be a von Neumann *-regular envelope by [26, Proposition 4.1 and Proposition 6.1].

The results of Subsection 3.10.2 can be formulated in terms of envelopes as follows.

Theorem 3.141 (Existence and uniqueness of division envelopes)

Let R be a ring coming with an integer valued Sylvester matrix function rk_R .

- (i) *There exists a division envelope $(R, \varphi_1, S_1, \text{rk}_1)$ of rk_R ;*
- (ii) *Let $(R, \varphi_1, S_1, \text{rk}_1)$ and $(R, \varphi_1, S_1, \text{rk}_2)$ be two division envelopes of rk_R . Then there exists precisely one isomorphism ψ from $(R, \varphi_1, S_1, \text{rk}_1)$ to $(R, \varphi_1, S_1, \text{rk}_2)$, i.e., a ring isomorphism $\psi: S_1 \rightarrow S_2$ such that $\psi \circ \varphi_1 = \varphi_2$ and $\psi^\# \text{rk}_2 = \text{rk}_1$ hold.*

We record the following variant due to Jaikin-Zapirain [26, Theorem 6.3].

Theorem 3.142 (Uniqueness of von Neumann *-regular envelopes)

*Let R be a ring with involution coming with a Sylvester rank function rk_R . Let $(R, \varphi_1, S_1, \text{rk}_1)$ and $(R, \varphi_1, S_1, \text{rk}_2)$ be two Neumann *-regular envelopes of rk_R .*

Then there exists precisely one isomorphism ψ from $(R, \varphi_1, S_1, \text{rk}_1)$ to $(R, \varphi_1, S_1, \text{rk}_2)$, i.e., an isomorphism of rings with involution $\psi: S_1 \rightarrow S_2$ such that $\psi \circ \varphi_1 = \varphi_2$ and $\psi^\# \text{rk}_2 = \text{rk}_1$ hold.

3.10.5 Exact Sylvester Module Functions and Normalized Length Functions

Definition 3.143 (Exact Sylvester module rank function). A Sylvester module rank function $\dim$ on a ring R is called *exact* if for every surjective map $f: M \rightarrow N$ of finitely presented R -modules M and N we have

$$\dim(M) - \dim(N) = \inf\{\dim(L) \mid L \text{ a finitely presented } R\text{-module} \\ \text{possessing a surjective } R\text{-homomorphism } L \rightarrow \ker(f)\}.$$

Exercise 3.144 If R is a von Neumann regular ring, then every Sylvester module rank function is exact.

Definition 3.145 (Normalized length function). A *normalized length function* for a ring R assigns to every R -module M an element

$$l(M) \in \mathbb{R}_{\geq 0} \amalg \{\infty\}$$

satisfying

- (i) *Normalization*
We have $\dim(R) = 1$;

(ii) *Cofinality*

Let M be the union $\{M_i \mid i \in I\}$ for a directed system of R -submodules which is directed by inclusion. Then we get

$$l_{\dim}(M) = \sup\{l_{\dim}(M_i) \mid i \in I\};$$

(iii) *Exactness*

We get for every short exact sequence of R -modules $0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow 0$

$$l(M_1) = l(M_0) + l(M_2)$$

using the convention $r + \infty = \infty$ for $r \in \mathbb{R}_{\geq 0} \amalg \{\infty\}$.

Theorem 3.146 (Exact Sylvester module rank functions and normalized length functions)

There is a one-to-one-correspondence between exact Sylvester module rank functions and normalized length function on a ring R . More precisely we have:

- (i) *Let $\dim$ be an exact Sylvester module rank function. Then we obtain a normalized length function by*

$$l_{\dim}(M) := \sup\{\inf\{\dim(L) \mid L \text{ a finitely presented } R\text{-module possessing a surjective } R\text{-homomorphism } L \rightarrow N\} \mid N \subseteq M \text{ finitely generated } R\text{-submodule of } M\};$$

- (ii) *If l is a normalized length function on R , we obtain an exact Sylvester module rank $\dim_l$ by*

$$\dim_l(M) = l(M);$$

- (iii) *If l and l' are two normalized length functions with $\dim_l = \dim_{l'}$, then $l = l'$ holds.*

Its proof, which can also be found in [73], needs some preparation.

Lemma 3.147. *Let $\dim$ be an exact Sylvester module rank function and define for any R -module M an element in $\mathbb{R}_{\geq 0} \amalg \{\infty\}$ by*

$$l_{\dim}(M) := \sup\{\inf\{\dim(L) \mid L \text{ a finitely presented } R\text{-module possessing a surjective } R\text{-homomorphism } L \rightarrow N\} \mid N \subseteq M \text{ finitely generated } R\text{-submodule of } M\}.$$

Then:

- (i) *If we have the inclusion of R -modules $M_0 \subseteq M_1$, then*

$$l_{\dim}(M_0) \leq l_{\dim}(M_1);$$

- (ii) Let $0 \rightarrow M_0 \rightarrow M_1 \xrightarrow{p} M_2 \rightarrow 0$ be an exact sequence of R -modules such that M_1 is finitely generated and M_2 is finitely presented. Then M_0 is finitely generated.
- (iii) Let N be a finitely presented R -module. Then

$$\dim(N) = l_{\dim}(N);$$

- (iv) We get for any finitely generated R -module M

$$l_{\dim}(M) = \inf\{\dim(L) \mid L \text{ a finitely presented } R\text{-module} \\ \text{possessing a surjective } R\text{-homomorphism } L \rightarrow M\};$$

- (v) Let $0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow 0$ be an exact sequence of R -modules such that M_1 and M_2 are finitely presented. Then M_0 is finitely generated and we get

$$l_{\dim}(M_1) = l_{\dim}(M_0) + l_{\dim}(M_2);$$

- (vi) Let M be the union $\{M_i \mid i \in I\}$ for a directed system of R -submodules which is directed by inclusion. Then we get

$$l_{\dim}(M) = \sup\{l_{\dim}(M_i) \mid i \in I\};$$

- (vii) Let $f: M \rightarrow N$ be a surjective R -homomorphism of R -modules. Then we get

$$l_{\dim}(M) \geq l_{\dim}(N).$$

Proof. (i) This follows directly from the definitions.

(ii) Choose an exact sequence of R -modules $0 \rightarrow K \rightarrow F \xrightarrow{q} M_2 \rightarrow 0$ such that F is finitely generated free and K is finitely generated. Consider the pullback of R -modules

$$\begin{array}{ccc} N & \xrightarrow{\bar{p}} & F \\ \bar{q} \downarrow & & \downarrow q \\ M_1 & \xrightarrow{p} & M_2. \end{array}$$

Then we obtain exact sequences of R -modules $0 \rightarrow K \rightarrow N \xrightarrow{\bar{q}} M_1 \rightarrow 0$ and $0 \rightarrow M_0 \rightarrow N \xrightarrow{\bar{p}} F \rightarrow 0$. Since M_1 and K are finitely generated, N is finitely generated. Since F is free, M_0 is a quotient of N and hence finitely generated.

(iii) Let $M \subseteq N$ any finitely generated R -module. Then we obtain an exact sequence of R -modules $0 \rightarrow M \rightarrow N \xrightarrow{p} N/M \rightarrow 0$ for p the projection. Since M is finitely generated and N is finitely presented, $p: N \rightarrow N/M$ is a surjective R -homomorphism of finitely presented R -modules with M as kernel. Since $\dim$ is exact, we conclude

$$(3.148) \quad \dim(N) - \dim(N/M) = \inf\{\dim(L) \mid L \text{ a finitely presented } R\text{-module} \\ \text{possessing a surjective } R\text{-homomorphism } L \rightarrow M\}.$$

Since $\dim(N) \geq \dim(N) - \dim(M/N)$ holds, we conclude $\dim(N) \geq l_{\dim}(N)$ from (3.148) and the definition of $l_{\dim}$.

Since for a surjective R -homomorphism $f: N' \rightarrow N$ with a finitely presented R -module N' as source we conclude from the exactness of $\dim$ that $\dim(N') - \dim(N) \geq 0$ holds, we obtain $\dim(N) \leq l_{\dim}(N)$. Hence we get $\dim(N) = l_{\dim}(N)$.

(iv) Let $f: N \rightarrow M$ be any surjective R -homomorphism with a finitely presented R -module N as source and M as target. Consider any finitely generated R -submodule $M_0 \subseteq M$. Choose a finitely generated R -submodule $N_0 \subseteq N$ such that $f(N_0) = M_0$ holds. Then $0 \rightarrow N_0 \rightarrow N \rightarrow N/N_0 \rightarrow 0$ is an exact sequence of R -modules such that N and N/N_0 are finitely presented. Since $\dim$ is exact, we get

$$\dim(N) - \dim(N/N_0) = \inf\{\dim(L) \mid L \text{ a finitely presented } R\text{-module} \\ \text{possessing a surjective } R\text{-homomorphism } L \rightarrow N_0\}.$$

Since M_0 is a quotient of N_0 , this implies

$$\dim(N) \geq \inf\{\dim(L) \mid L \text{ a finitely presented } R\text{-module} \\ \text{possessing a surjective } R\text{-homomorphism } L \rightarrow M_0\}.$$

Hence we get

$$\begin{aligned} & \inf\{\dim(N) \mid N \text{ a finitely presented } R\text{-module} \\ & \quad \text{possessing a surjective } R\text{-homomorphism } N \rightarrow M\} \\ & \geq \\ & \inf\{\dim(L) \mid L \text{ a finitely presented } R\text{-module} \\ & \quad \text{possessing a surjective } R\text{-homomorphism } L \rightarrow M_0\}. \end{aligned}$$

Now assertion (iv) follows.

(v) This follows from assertions (ii) (iii), and (iv) and the exactness of $\dim$.

(vi) We conclude from assertion (i).

$$l_{\dim}(M) \geq \sup\{l_{\dim}(M_i) \mid i \in I\}.$$

Consider any $\epsilon > 0$. Then we can find by the definition of $l_{\dim}$ and assertion (iv) a finitely generated R -submodule $N \subseteq M$ satisfying $l_{\dim}(M) \leq l_{\dim}(N) + \epsilon$. Since the system $\{M_i \mid i \in I\}$ is directed, we can find an index $i \in I$ with $N \subseteq M_i$. Since we get $l_{\dim}(N) \leq l_{\dim}(M_i)$ from assertion (i), we

conclude

$$l_{\dim}(M) \leq \dim(N) + \epsilon \leq \dim(M_i) + \epsilon \leq \sup\{l_{\dim}(M_i) \mid i \in I\} + \epsilon.$$

Hence we get

$$l_{\dim}(M) \leq \sup\{l_{\dim}(M_i) \mid i \in I\}.$$

(vii) If M is finitely generated, this follows from assertion (iv). The general case follows now from (vi) applied to the directed system of finitely generated submodules of M and N since N is the directed union of all R -submodules $f(M')$, where M' runs through the finitely generated R -submodules of M .

This finishes the proof of Lemma 3.147. $\square$

Lemma 3.149. *Let $0 \rightarrow L \xrightarrow{i} M \xrightarrow{p} N \rightarrow 0$ be an exact sequence of R -modules. Then we get*

$$l_{\dim}(L) + l_{\dim}(N) \leq l_{\dim}(M).$$

Proof. **Case 1: M finitely presented**

Consider any finitely generated R -submodule L' of L . We obtain a short exact sequence $0 \rightarrow L' \rightarrow M \rightarrow M/L' \rightarrow 0$ such that M and M/L' are finitely presented. Then we get $l_{\dim}(M) = l_{\dim}(L') + l_{\dim}(M/L')$ from Lemma 3.147 (v). Since N is a quotient of M/L' , we get $l_{\dim}(N) \leq l_{\dim}(M/L')$ from Lemma 3.147 (vii). This implies

$$l_{\dim}(L') + l_{\dim}(N) \leq l_{\dim}(M).$$

Hence we conclude $l_{\dim}(L) + l_{\dim}(N) \leq l_{\dim}(M)$ from Lemma 3.147 (vi).

Case 2: M finitely generated

Consider any surjective R -homomorphism $f: M' \rightarrow M$ with some finitely presented R -module M' as source and M as target. Then Case 1 applies to the composite $p \circ f: M' \rightarrow N$ and yields $l_{\dim}(\ker(p \circ f)) + l_{\dim}(N) \leq l_{\dim}(M')$. Since $L \cong \ker(p)$ is a quotient of $\ker(p \circ f)$, we get $l_{\dim}(L) \leq l_{\dim}(\ker(p \circ f))$ from Lemma 3.147 (vii). This implies

$$l_{\dim}(L) + l_{\dim}(N) \leq l_{\dim}(M').$$

Now $l_{\dim}(L) + l_{\dim}(N) \leq l_{\dim}(M)$ follows from Lemma 3.147 (iv).

Case 3: General case

Obviously M is the union over the directed system of finitely generated submodules M' of M . Moreover, N is the union of the directed system $\{p(M')\}$ and L is the union of the direct system $\{i^{-1}(M')\}$, where M' runs over the finitely generated R -submodules of M . Hence we conclude from Lemma 3.147 (vi)

$$\begin{aligned}
l_{\dim}(L) &= \sup\{l_{\dim}(i^{-1}(M')) \mid M' \subseteq M \text{ finitely generated}\}; \\
l_{\dim}(M) &= \sup\{l_{\dim}(M') \mid M' \subseteq M \text{ finitely generated}\}; \\
l_{\dim}(N) &= \sup\{l_{\dim}(p(M')) \mid M' \subseteq M \text{ finitely generated}\}.
\end{aligned}$$

We get for every finitely generated R -submodule M' of M from Case 2

$$l_{\dim}(i^{-1}(M')) + l_{\dim}(p(M')) \leq l_{\dim}(M').$$

Now Lemma 3.149 follows using Lemma 3.147 (i). $\square$

Lemma 3.150. *Let $0 \rightarrow L \xrightarrow{i} M \xrightarrow{p} N \rightarrow 0$ be an exact sequence of R -modules. Then we get*

$$l_{\dim}(M) \leq l_{\dim}(L) + l_{\dim}(N).$$

Proof. **Case 1: The sequence splits and L and N are finitely generated**

Then we have $M \cong L \oplus N$ and M is finitely generated. Consider any $\epsilon \geq 0$. Because of Lemma 3.147 (iv) we can choose finitely presented R -modules L' and N' and surjective R -homomorphisms $f: L' \rightarrow L$ and $g: N' \rightarrow N$ satisfying

$$\begin{aligned}
\dim(L') &\leq l_{\dim}(L) - \epsilon/2; \\
\dim(N') &\leq l_{\dim}(N) - \epsilon/2.
\end{aligned}$$

Then $f \oplus g: L' \oplus N' \rightarrow L \oplus N$ is a surjective R -homomorphism with $L \oplus N$ as target and a finitely presented R -modules as source. Lemma 3.147 (iv) implies

$$\begin{aligned}
l_{\dim}(M) &= l_{\dim}(L \oplus N) \leq \dim(L' \oplus N') = \dim(L') + \dim(N') \\
&\leq l_{\dim}(L) - \epsilon/2 + l_{\dim}(N) - \epsilon/2 = l_{\dim}(L) + l_{\dim}(N) - \epsilon.
\end{aligned}$$

Since this holds for all $\epsilon > 0$, we conclude $l_{\dim}(M) \leq l_{\dim}(L) + l_{\dim}(N)$.

Case 2: M is finitely generated and N is finitely presented

Let $f: M' \rightarrow M$ be any surjective R -homomorphism with a finitely presented R -module M' as source and M as target. Consider the pullback of R -modules

$$\begin{array}{ccc}
K & \xrightarrow{\bar{i}} & M' \\
\bar{f} \downarrow & & \downarrow f \\
L & \xrightarrow{i} & M.
\end{array}$$

Then we obtain an exact sequence $0 \rightarrow K \rightarrow M' \oplus L \rightarrow M \rightarrow 0$. Lemma 3.149 implies $l_{\dim}(K) + l_{\dim}(M) \leq l_{\dim}(M' \oplus L)$. Since L is finitely generated by

Lemma 3.147 (ii), we conclude $l_{\dim}(M' \oplus L) \leq l_{\dim}(L) + l_{\dim}(M')$ from Case 1. This implies

$$l_{\dim}(K) + l_{\dim}(M) \leq l_{\dim}(L) + l_{\dim}(M').$$

Since we have the exact sequence $0 \rightarrow K \rightarrow M' \rightarrow N \rightarrow 0$ and M' and N are finitely presented, Lemma 3.147 (v) implies $l_{\dim}(K) + l_{\dim}(N) = l_{\dim}(M')$. Hence we get $l_{\dim}(M) \leq l_{\dim}(L) + l_{\dim}(N)$.

Case 3: N finitely presented

Fix $\epsilon > 0$. Because of Lemma 3.147 (vi) we can choose a finitely generated R -submodule M' of M with $l_{\dim}(M) - \epsilon \leq l_{\dim}(M')$. We can find another finitely generated R -submodule M'' of M satisfying $M' \subseteq M''$ and $p(M'') = N$. We conclude from Lemma 3.147 (i)

$$l_{\dim}(M) - \epsilon \leq l_{\dim}(M'').$$

We obtain $l_{\dim}(M'') \leq l_{\dim}(i^{-1}(M'')) + l_{\dim}(N)$ from Case 2 applied to the short exact sequence $0 \rightarrow i^{-1}(M'') \rightarrow M'' \rightarrow N \rightarrow 0$. This implies

$$l_{\dim}(M) - \epsilon \leq l_{\dim}(i^{-1}(M'')) + l_{\dim}(N).$$

Since L is the union of the directed system of R -submodules $i^{-1}(M''')$, where M''' runs through all finitely generated R -submodules of M , Lemma 3.147 (vi) implies

$$l_{\dim}(L) = \sup\{l_{\dim}(i^{-1}(M''')) \mid M''' \subseteq M \text{ finitely generated}\}.$$

Hence we get $l_{\dim}(M) - \epsilon \leq l_{\dim}(L) + l_{\dim}(N)$. Since this holds for all $\epsilon > 0$, we obtain $l_{\dim}(M) \leq l_{\dim}(L) + l_{\dim}(N)$.

Case 4: M finitely generated

Then N is finitely generated. Consider any surjective R -homomorphism $f: N' \rightarrow N$ with a finitely presented R -module N' as source and N as target. Consider the pullback of R -modules

$$\begin{array}{ccc} M' & \xrightarrow{\bar{p}} & N' \\ \bar{f} \downarrow & & \downarrow f \\ M & \xrightarrow{p} & N. \end{array}$$

Since we have the exact sequence of R -modules $0 \rightarrow L \rightarrow M' \rightarrow N' \rightarrow 0$, we conclude from Case 3 that $l_{\dim}(M') \leq l_{\dim}(L) + l_{\dim}(N')$ holds. Since f and hence $\bar{f}$ is surjective, we conclude $l_{\dim}(M) \leq l_{\dim}(M')$ from Lemma 3.147 (vii). This implies

$$l_{\dim}(M) \leq l_{\dim}(L) + l_{\dim}(N').$$

Now we get $l_{\dim}(M) \leq l_{\dim}(L) + l_{\dim}(N)$ from Lemma 3.147 (iv).

Case 5: General case

As explained in the Case 3 of the proof of Lemma 3.149 we get

$$\begin{aligned} l_{\dim}(L) &= \sup\{l_{\dim}(i^{-1}(M')) \mid M' \subseteq M \text{ finitely generated}\}; \\ l_{\dim}(M) &= \sup\{l_{\dim}(M') \mid M' \subseteq M \text{ finitely generated}\}; \\ l_{\dim}(N) &= \sup\{l_{\dim}(p(M')) \mid M' \subseteq M \text{ finitely generated}\}. \end{aligned}$$

Given any finitely presented R -submodule $M' \subseteq M$, we get an exact sequence $0 \rightarrow i^{-1}(M') \rightarrow M' \rightarrow p(M') \rightarrow 0$ and obtain from Case 4

$$l_{\dim}(M') \leq l_{\dim}(i^{-1}(M')) + l_{\dim}(p(M')).$$

Now Lemma 3.149 follows using Lemma 3.147 (i). $\square$

Now we are ready to prove Theorem 3.146.

Proof of Theorem 3.146. (i) This follows directly from Lemma 3.147 (iii) and (vi), Lemma 3.149, and Lemma 3.150.

(ii) We conclude from the property exactness of l appearing in Definition 3.145 that $\dim_l$ is a Sylvester module rank function. It remains to prove exactness of $\dim_l$. Because of Lemma 3.147 (ii) it suffices to show for any finitely generated R -module N

$$(3.151) \quad l(N) = \inf\{l(L) \mid L \text{ a finitely presented } R\text{-module} \\ \text{possessing a surjective } R\text{-homomorphism } L \rightarrow N\}.$$

Choose a finitely generated free R -module F and a surjective R -homomorphism $f: F \rightarrow N$. Since the condition Cofinality appearing in Definition 3.145 implies

$$l(\ker(f)) = \sup\{l(P) \mid P \subseteq \ker(f) \text{ finitely generated } R\text{-submodule}\}$$

and F/P is finitely presented for every finitely generated R -submodule P of $\ker(f)$, we conclude from the exactness of l

$$\begin{aligned} l(N) &= l(F) - l(\ker(f)) \\ &= l(F) - \sup\{l(P) \mid P \subseteq \ker(f) \text{ finitely generated } R\text{-submodule}\} \\ &= \inf\{l(F/P) \mid P \subseteq \ker(f) \text{ a finitely generated } R\text{-submodule}\} \\ &\geq \inf\{l(L) \mid L \text{ a finitely presented } R\text{-module} \\ &\quad \text{possessing a surjective } R\text{-homomorphism } L \rightarrow N\}. \end{aligned}$$

Additivity of l implies

$$l(N) \leq \inf\{l(L) \mid L \text{ a finitely presented } R\text{-module} \\ \text{possessing a surjective } R\text{-homomorphism } L \rightarrow N\}.$$

Hence (3.151) holds and assertion (ii) follows.

(iii) Let l be a length function. Denote by $\dim$ the exact Sylvester function obtained by restricting l to finitely presented R -modules, see assertion (ii). It suffices to prove $l = l_{\dim}$.

For every R -module M , the condition Cofinality appearing in Definition 3.145 implies

$$l(M) = \sup\{l(N) \mid N \subseteq M \text{ finitely generated } R\text{-submodule of } M\}.$$

Now we get $l = l_{\dim}$ from (3.151) and the definition of $l = l_{\dim}$. This finishes the proof of Theorem 3.146. $\square$

3.10.6 Extensions to Ore Localizations

Lemma 3.152. *Let $S \subseteq R \setminus \{0\}$ be a multiplicatively closed subset of non-zero-divisors in R satisfying the right Ore condition. Denote by $S^{-1}R$ the Ore localization. Let rk be a Sylvester module rank function on R .*

- (i) *The Sylvester matrix rank function rk on R can be extended to a Sylvester matrix rank function $S^{-1}\text{rk}$ on $S^{-1}R$ if and only if $\text{rk}(s) = 1$ holds for all $s \in S$;*
- (ii) *Two Sylvester matrix rank functions which extend the Sylvester matrix rank function rk agree.*

Proof. (i) Suppose that there is an extension $S^{-1}\text{rk}$ of rk to $S^{-1}R$. Since $s \in S^{-1}R$ is a unit, we get $\text{rk}(s) = S^{-1}\text{rk}(s) = 1$.

Suppose that $\text{rk}(s) = 1$ holds for all $s \in S$. We want to define an extension $S^{-1}\text{rk}$ of rk .

Note for the sequel the following facts. Given a matrix $B \in M_{m,n}(S^{-1}R)$, we can choose elements $s, s' \in S$ such that both $(s \cdot I_m) \cdot B$ and $B \cdot (s' \cdot I_n)$ lie in $M_{m,n}(R)$. We have $\text{rk}(s \cdot I_m) = m$ and $\text{rk}(s' \cdot I_n) = n$, and both $(s \cdot I_m)$ and $(s' \cdot I_n)$ are invertible over $S^{-1}R$.

Consider a matrix $E \in M_{n,n}(R)$ which is invertible over $S^{-1}R$. Next we show $\text{rk}(E) = n$. Choose $E' \in M_{n,n}(S^{-1}R)$ such that $E'E = I_n$ holds in $M_{n,n}(S^{-1}R)$. Choose an element $s \in S$ such that $(s \cdot I_n)E'$ lies in $M_{n,n}(R)$. Put $F = (s \cdot I_n)E'$. We have $FE = (s \cdot I_n)$. This implies

$$n = \text{rk}(s \cdot I_n) = \text{rk}(FE) \leq \min\{\text{rk}(F), \text{rk}(E)\} \leq \text{rk}(E) \leq n.$$

Hence $\text{rk}(E) = n$.

Given a matrix $B \in M_{m,n}(S^{-1}R)$, consider any matrices $A \in M_{m,m}(R)$ and $C \in M_{n,n}(R)$ such that A and C are invertible over $S^{-1}R$ and ABC lies in $M_{m,n}(R)$. We want to define

$$(3.153) \quad S^{-1}\text{rk}(B) = \text{rk}(ABC).$$

We need to show that this is independent of the choices of A and B . Let $A' \in M_{m,m}(R)$ and $C' \in M_{n,n}(R)$ be matrices such that A' and C' are invertible over $S^{-1}R$ and $A'BC' \in M_{m,n}(R)$ holds. Choose elements $s, s' \in S$ such that $(s \cdot I_m)A'A^{-1}$ lies in $M_{m,m}(R)$ and $C^{-1}C'(s' \cdot I_n)$ lies in $M_{n,n}(R)$. Put $D = (s \cdot I_m)A'A^{-1}$ and $D' = C^{-1}C'(s' \cdot I_n)$. Then $D \in M_{m,m}(R)$ and $D' \in M_{n,n}(R)$, and D and D' are invertible over $S^{-1}R$. Hence we get

$$\begin{aligned} \text{rk}(D) &= m; \\ \text{rk}(D') &= n; \\ DABCD' &= (s \cdot I_m)A'BC'(s' \cdot I_n). \end{aligned}$$

Now Proposition 3.131 (iii) implies

$$\text{rk}(ABC) = \text{rk}(DABCD') = \text{rk}((s \cdot I_m)A'BC'(s' \cdot I_n)) = \text{rk}(A'BC').$$

Hence the definition (3.153) makes sense.

It remains to show that (3.153) satisfies the axioms of a Sylvester rank function on $S^{-1}R$ appearing in Definition 3.127. This is obvious for properties (i) and (ii).

Next we deal with property (iii). Consider $A \in M_{l,m}(S^{-1}R)$ and $B \in M_{m,n}(S^{-1}R)$. Choose $C \in M_{l,l}(R)$ and $D \in M_{n,n}(R)$ such that C and D are invertible over $S^{-1}R$ and $C \cdot A \in M_{l,m}(R)$ and $B \cdot D \in M_{m,n}(R)$ hold. Then we get

$$\begin{aligned} S^{-1}\text{rk}(A \cdot B) &= \text{rk}(CABD) \leq \min\{\text{rk}(CA), \text{rk}(BD)\} \\ &= \min\{S^{-1}\text{rk}(A), S^{-1}\text{rk}(B)\}. \end{aligned}$$

Next we deal with property (iv). Consider $A_1 \in M_{m_1,n_1}(S^{-1}R)$ and $A_2 \in M_{m_2,n_2}(S^{-1}R)$. Choose $C_1 \in M_{m_1,m_1}(R)$ and $C_2 \in M_{n_1,n_1}(R)$ such that C_1 and C_2 are invertible over $S^{-1}R$ and $C_1A_1 \in M_{m_1,m_1}(R)$ and $C_2A_2 \in M_{m_2,n_2}(R)$ hold. Then we get

$$\begin{aligned} S^{-1}\text{rk} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} &= \text{rk} \left(\begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \cdot \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \right) = \text{rk} \begin{pmatrix} C_1A_1 & 0 \\ 0 & C_2A_2 \end{pmatrix} \\ &= \text{rk}(C_1A_1) + \text{rk}(C_2A_2) = S^{-1}\text{rk}(A_1) + S^{-1}\text{rk}(A_2). \end{aligned}$$

Finally we deal with property (v). Consider $A_1 \in M_{m_1,n_1}(S^{-1}R)$, $A_2 \in M_{m_2,n_2}(S^{-1}R)$, and $A_3 \in M_{m_1,n_2}(S^{-1}R)$. Choose $C_1 \in M_{m_1,m_1}(R)$ and $C_2 \in M_{n_1,n_1}(R)$ such that C_1 and C_2 are invertible over $S^{-1}R$ and $C_1A_1 \in M_{m_1,m_1}(R)$ and $C_2A_2 \in M_{m_2,n_2}(R)$ hold. Now choose $C_3 \in M_{m_1,m_1}(R)$ such that C_3 is invertible over $S^{-1}R$ and $C_3C_1A_3 \in M_{m_1,n_2}(R)$ holds. Then we get:

$$\begin{aligned}
S^{-1}\text{rk} \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} &= \text{rk} \left(\begin{pmatrix} C_3 C_1 & 0 \\ 0 & C_2 \end{pmatrix} \cdot \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \right) = \text{rk} \begin{pmatrix} C_3 C_1 A_1 & C_3 C_1 A_3 \\ 0 & C_2 A_2 \end{pmatrix} \\
&\geq \text{rk}(C_3 C_1 A_1) + \text{rk}(C_2 A_2) = S^{-1}\text{rk}(A_1) + S^{-1}\text{rk}(A_2).
\end{aligned}$$

This finishes the proof of assertion (i).

(ii) Consider any extension $S^{-1}\text{rk}$ of rk to $S^{-1}R$. Consider $A \in M_{m,n}(S^{-1}R)$. Then we can choose $C \in M_{m,m}(R)$ such that C is invertible over $S^{-1}R$ and $CA \in M_{m,n}(R)$. We conclude using Proposition 3.131 (iii) applied to the ring $S^{-1}R$ since $S^{-1}\text{rk}(C) = m$ holds

$$S^{-1}\text{rk}(A) = S^{-1}\text{rk}(CA) = \text{rk}(CA).$$

This finishes the proof of Lemma 3.152 □

3.10.7 Natural Extensions for Skew (Laurent) Polynomial Rings

Throughout this subsection we consider a ring R with a ring automorphism $\Phi: R \xrightarrow{\cong} R$ coming with a Sylvester module rank function dim . We want to construct a specific Sylvester module rank function $\widetilde{\text{dim}}$ on the twisted group ring $R_\Phi[\mathbb{Z}] = R_\Phi[t, t^{-1}]$ introduced in Section 3.8 which extends dim , i.e., $\widetilde{\text{dim}}(i_* M) = \text{dim}(M)$ holds for every finitely presented R -module M , where $i: R \rightarrow R_\Phi[\mathbb{Z}]$ is the canonical inclusion.

We call dim Φ -compatible if $\text{dim}(\Phi_* M) = \text{dim}(M)$ holds for every finitely presented R -module M .

Exercise 3.154 Let rk be the Sylvester matrix rank function associated to dim , see Proposition 3.128.

Show that dim is Φ -compatible if and only if $\text{rk} = \Phi^\#(\text{rk})$ holds.

Proposition 3.155. *Let rk be a Sylvester module rank function which extends to a Sylvester module rank function on $R_\Phi[\mathbb{Z}]$. Then rk is Φ -compatible.*

Proof. This follows from the fact that the functors i_* and $i_* \circ \Phi_*$ from $R\text{-MOD}$ to $R_\Phi[\mathbb{Z}]\text{-MOD}$ are naturally equivalent. □

Lemma 3.156. *Let R_0 and R_1 be rings. Let B be a R_1 - R_0 -bimodule such that B as a left R_1 -module is isomorphic to R_1^i for some $i \in \mathbb{Z}_{\geq 1}$. Let dim be a Sylvester module function on R_1 . Define for a finitely presented R_0 -module M*

$$\overline{\text{dim}}_i(M) := \frac{\text{dim}(B \otimes_{R_0} M)}{i} \in \mathbb{R}_{\geq 0}.$$

Then $\overline{\text{dim}}$ is a Sylvester module rank function on R_0 .

Proof. If M is a finitely presented R_0 -module, then $R_1 \otimes_{R_0} M$ is a finitely presented R' -module, since $B \otimes_{R_0} -$ is a right exact functor from $R_0\text{-MOD}$ to

R_1 -MOD and B as a R_1 -module is isomorphic to R_1^i . So $\overline{\dim}_i(M) \in \mathbb{R}^{\geq 0}$ is well-defined. It remains to prove that $\overline{\dim}_i$ satisfies all the axioms appearing in Definition 3.126. This follows from the facts that $B \otimes_{R_0} -$ is a right exact functor and the assumptions that B as a R_1 -module is isomorphic to R_1^i and that $\dim$ is a Sylvester module rank function on R_1 . $\square$

For every $i \in \mathbb{Z}_{\geq 1}$ we can define a R - $R_\Phi[t]$ -bimodule

$$(3.157) \quad B_i := R_\Phi[t]/(t^i)$$

where (t^i) is the two sided ideal in $R_\Phi[t]$ -ideal generated by t^i . The R - $R_\Phi[t]$ - R -bimodule structure is given by $r \cdot \bar{x} \cdot u = \overline{(rt^0)xu}$ for $u, x \in R_\Phi[t]$ and $r \in R$, where $\bar{x}$ denotes the class of x in $R_\Phi[t]/(t^i)$. Note that B_i as a left R -module is isomorphic to R^i . So we can define for every finitely presented $R_\Phi[t]$ -module M

$$(3.158) \quad \overline{\dim}_i(M) := \frac{\dim(B_i \otimes_{R_\Phi[t]} M)}{i} \in \mathbb{R}_{\geq 0}.$$

Note that the R -module $B_i \otimes_{R_\Phi[t]} M$ agrees with $j_i^*(\text{pr}_i)_* M$, where $\text{pr}_i: R_\Phi[t] \rightarrow R_\Phi[t]/(t^i)$ is the ring homomorphism given by the canonical projection and $j_i: R \rightarrow R_\Phi[t]/(t^i)$ is the ring homomorphism sending r to rt^0 .

We conclude from Lemma 3.156 applied to the case $R_0 = R_\Phi[t]$ and $R_1 = R$ that $\overline{\dim}_i$ is a Sylvester module rank function on $R_\Phi[t]$.

Definition 3.159 (Natural extensions). Let R be a ring coming with a ring automorphism $\Phi: R \xrightarrow{\cong} R$ and a Sylvester module rank function $\dim$.

We call a Sylvester module rank function $\widetilde{\dim}$ on $R_\Phi[t]$ *the natural extension of $\dim$ to $R_\Phi[t]$* if for every finitely presented $R_\Phi[t]$ -module M the limit $\lim_{i \rightarrow \infty} \overline{\dim}_i(M)$ exists and satisfies

$$\widetilde{\dim}(M) = \lim_{i \rightarrow \infty} \overline{\dim}_i(M).$$

We call a Sylvester module rank function $\widetilde{\dim}$ on $R_\Phi[\mathbb{Z}]$ *the natural extension of $\dim$ to $R_\Phi[\mathbb{Z}]$* if its restriction to $R_\Phi[t]$ is the natural extension of $\dim$ to $R_\Phi[t]$.

Remark 3.160. If a natural extension of $\dim$ to $R_\Phi[t]$ exists, then it is unique. Therefore we will talk about *the* natural extension of $\dim$ to $R_\Phi[t]$.

Suppose that the natural extension $\widetilde{\dim}$ of $\dim$ to $R_\Phi[t]$ exists. Let rk and $\widetilde{\text{rk}}$ be the Sylvester matrix rank functions associated to $\dim$ and $\widetilde{\dim}$. Then we get

$$(3.161) \quad \widetilde{\text{rk}}(t) = \lim_{i \rightarrow \infty} \frac{\text{rk} \begin{pmatrix} 0 & I_{i-1} \\ 0 & 0 \end{pmatrix}}{i} = \lim_{i \rightarrow \infty} \frac{i-1}{i} = 1.$$

We conclude from Lemma 3.152 that the natural extension of $\dim$ to $R_\Phi[\mathbb{Z}]$ is unique if it exists, and the natural extension of $\dim$ to $R_\Phi[\mathbb{Z}]$ exists if and only if the natural extension of $\dim$ to $R_\Phi[t]$ exists.

If the natural extension of $\dim$ to $R_\Phi[t]$ exists, then $\dim$ must be Φ -compatible by Proposition 3.155.

The natural extension of $\dim$ to $R_\Phi[t]$ exists if and only if $\lim_{i \rightarrow \infty} \overline{\dim}_i(M)$ exists for every finitely presented $R_\Phi[t]$ -module M , since then we get automatically by $\widetilde{\dim}(M) = \lim_{i \rightarrow \infty} \overline{\dim}_i(M)$ a Sylvester module rank function on $R_\Phi[t]$ as each $\overline{\dim}_i$ is a Sylvester module rank function.

In general we do not know whether the natural extension $\widetilde{\dim}$ of a Φ -compatible Sylvester module rank function $\dim$ to $R_\Phi[\mathbb{Z}]$ exists. If we assume that $\dim$ is Φ -compatible and exact in the sense of Definition 3.143, then $\widetilde{\dim}$ exists by the following theorem which follows from [72, Theorem B and Definition 4.3], where more generally amenable groups instead of $\mathbb{Z}$ are treated.

Comment 26 (by W.): If there is a rather short proof for the next result in the special case $G = \mathbb{Z}$, then we should add it for completeness; otherwise we stick to give the reference.

Theorem 3.162 (Existence of natural extensions) *Let R be a ring coming with a ring automorphism $\Phi: R \xrightarrow{\cong} R$ and a Sylvester module rank function $\dim$. Suppose that $\dim$ is exact and Φ -compatible. Let $l_{\dim}$ be the associated normalized length function on R , see Theorem 3.146.*

- (i) *Consider any $R_\Phi[\mathbb{Z}]$ -module M . Then for every R -submodule $N \subseteq M$ with $\dim(R) < \infty$ the limit*

$$E_{M,N} = \lim_{i \rightarrow \infty} \frac{l_{\dim}(T_i(N))}{i}$$

exists, where $T_i(N)$ is the R -submodule of M given by $N + tN + \cdots + t^i N$, or, equivalently, is given by

$$T_i(N) = \left\{ \sum_{k=0}^i r_k \cdot t^k \cdot n_k \mid r_k \in R, n_k \in N \right\};$$

- (ii) *If we define for any $R_\Phi[\mathbb{Z}]$ -module M*

$$\widetilde{l}_{\dim}(M) = \sup\{E_{M,N} \mid N \subseteq M \text{ finitely generated } R\text{-submodule of } M \\ \text{with } \dim_R(N) < \infty\} \in \mathbb{R}_{\geq 0} \amalg \{\infty\},$$

then $\widetilde{l}_{\dim}$ is a normalized length function on $R_\Phi[\mathbb{Z}]$ extending $l_{\dim}$;

- (iii) *The exact Sylvester rank function $\widetilde{\dim}$ on $R_\Phi[\mathbb{Z}]$ associated to $\widetilde{l}_{\dim}$, see Theorem 3.146, is the natural extension of $\dim$ to $R_\Phi[\mathbb{Z}]$ in the sense of Definition 3.159.*

Proposition 3.163. *Let R be a ring coming with a ring automorphism $\Psi: R \xrightarrow{\cong} R$. Let $\dim$ be a Sylvester module rank function on R which is integer-valued and Ψ -compatible. Let $\varphi: R \rightarrow D$ be the epic division envelope of $(R, \dim)$, see Theorem 3.141.*

- (i) *There is an automorphism $\Phi: D \xrightarrow{\cong} D$ uniquely determined by $\Phi \circ \varphi = \varphi \circ \Psi$;*
- (ii) *The Sylvester module rank function $\dim_D$ on D is Φ -compatible;*
- (iii) *Let $\bar{\varphi}: R_\Psi[\mathbb{Z}] \rightarrow D_\Phi[\mathbb{Z}]$ be the ring homomorphism sending $r \cdot t^k$ to $\varphi(r) \cdot t^k$ for $r \in R$ and $k \in \mathbb{Z}$. Then the natural extension $\widetilde{\dim_D}$ of $\dim_D$ to $D_\Phi[\mathbb{Z}]$ in the sense of Definition 3.159 exists and $\widetilde{\dim} := \bar{\varphi}^\# \widetilde{\dim_D}$ is the natural extension of $\dim$ to $R_\Psi[\mathbb{Z}]$;*
- (iv) *The Sylvester module rank function $\widetilde{\dim}$ on $R_\Psi[\mathbb{Z}]$ is integer-valued and the division envelope of $\widetilde{\dim}$ is given by the composite of $\bar{\varphi}: R_\Psi[\mathbb{Z}] \rightarrow D_\Phi[\mathbb{Z}]$ with the canonical inclusion $j: D_\Phi[\mathbb{Z}] \rightarrow S^{-1}D_\Phi[\mathbb{Z}]$ and $\dim_{S^{-1}D_\Phi[\mathbb{Z}]}$, where the division algebra $S^{-1}D_\Phi[\mathbb{Z}]$ is the Ore localization of $D_\Phi[\mathbb{Z}]$ with respect to $S = D_\Phi[\mathbb{Z}] \setminus \{0\}$.*

Proof. (i) and (ii) We get by $\varphi \circ \Psi: R \rightarrow D$ another epic division envelope of $(R, \dim)$ since rk is Ψ -compatible. We conclude from Theorem 3.141 that there exists an automorphism $\Phi: D \xrightarrow{\cong} D$ such that $\Phi \circ \varphi = \varphi \circ \Psi$ and $\Phi^\# \dim_D = \dim_D$ holds.

(iii) We conclude from Theorem 3.162 that there exists a natural extension $\widetilde{\dim_D}$ of $\dim_D$ to $D_\Phi[\mathbb{Z}]$. We have $\widetilde{\dim} = \varphi^\# \widetilde{\dim_D}$ and $\widetilde{\dim}_i = \varphi^\# (\widetilde{\dim_D})_i$ for $i \in \mathbb{Z}_{\geq 0}$, where $\widetilde{\dim}_i$ and $(\widetilde{\dim_D})_i$ have been defined in (3.158). Hence $\widetilde{\dim} := \bar{\varphi}^\# \widetilde{\dim_D}$ is the natural extension of rk to $R_\Psi[\mathbb{Z}]$.

(iv) Consider a non-trivial element $x \in D_\Phi[\mathbb{Z}]$. Then we can write $x = \sum_{i=k_0}^{k_1} d_i \cdot t^i$ where $k_0 \leq k_1$ holds and both a_{k_0} and a_{k_1} are different from 0. Then for every element $\bar{y} \in D_\Phi[\mathbb{Z}]/(D_\Phi[\mathbb{Z}]x)$ we can find a representative of the shape $y = \sum_{i=k_0}^{k_1} d'_i \cdot t^i$. This implies that $\dim_D(D_\Phi[\mathbb{Z}]/(D_\Phi[\mathbb{Z}]x)) \leq 1 + k_1 - k_0$ holds. Hence we get for every D -submodule N of $D_\Phi[\mathbb{Z}]/(D_\Phi[\mathbb{Z}]x)$ and every $i \in \mathbb{Z}_{\geq 0}$ that $\dim_D(N + t \cdot N + \cdots + t^i N) \leq 1 + k_1 - k_0$ holds and hence we have

$$\lim_{i \rightarrow \infty} \frac{\dim_D(N + t \cdot N + \cdots + t^i N)}{i} = 0.$$

This implies $\widetilde{\dim_D}(D_\Phi[\mathbb{Z}]/(D_\Phi[\mathbb{Z}]x)) = 0$ and hence that the Sylvester matrix function associated to $\widetilde{\dim_D}$ sends x to 1. We conclude from Lemma 3.152 that the Sylvester module rank function $\widetilde{\dim_D}$ on $D_\Phi[\mathbb{Z}]$ extends to a the Sylvester module rank function $S^{-1}\widetilde{\dim_D}$ on the division algebra $S^{-1}D_\Phi[\mathbb{Z}]$. By Remark 3.130 $S^{-1}\widetilde{\dim_D}$ is $\dim_{S^{-1}D_\Phi[\mathbb{Z}]}$ and in particular integer-valued. Since $\varphi: R \rightarrow D$ is epic, $\bar{\varphi}: R_\Psi[\mathbb{Z}] \rightarrow D_\Phi[\mathbb{Z}]$ is epic. Therefore $j \circ \bar{\varphi}: R_\Psi[\mathbb{Z}] \rightarrow S^{-1}D_\Phi[\mathbb{Z}]$ is epic and we have $(j \circ \bar{\varphi})^\# \dim_{S^{-1}D_\Phi[\mathbb{Z}]} = S^{-1}\widetilde{\dim_D}$. This implies

that $\widetilde{\dim}$ is integer-valued and that $j \circ \widehat{\varphi}: R_\Psi[\mathbb{Z}] \rightarrow S^{-1}D_\Phi[\mathbb{Z}]$ is a division envelope of $(R_\Phi[\mathbb{Z}], \widetilde{\dim})$. $\square$

3.10.8 Hughes-Free Sylvester Rank Functions

Next we characterize whether an epic EG -division ring is Hughes-free in terms of the associated Sylvester module rank function.

Definition 3.164 (Hughes-free Sylvester module rank function). Let $\dim$ be a Sylvester module function on EG for a group G and a division ring E . Denote for a subgroup $H \subseteq G$ by $\dim|_{EH}$ its restriction to EH . We call $\dim$ *Hughes-free* if for every finitely generated subgroup H and normal subgroup $N \subseteq H$ and $t \in H$ such that H/N is infinite cyclic and the image of t under the canonical projection $H \rightarrow H/N$ is a generator, the Sylvester module rank function $\dim|_{EH}$ is the natural extension in the sense of Definition 3.159 of $\dim|_{EN}$ to $EH = EN_\Phi[\mathbb{Z}]$, where $\Phi: EH \xrightarrow{\cong} EH$ is the ring automorphism given by conjugation with t .

Theorem 3.165 (Characterization of Hughes-free in terms of Sylvester rank functions) *An epic division ring $\varphi: EG \rightarrow D$ is Hughes-free in the sense of Definition 3.122 if and only if the associated Sylvester module rank function $\dim$ sending a finitely presented EG -module M to $\dim_D(D \otimes_{EG} M)$ is Hughes-free in the sense of Definition 3.164.*

Proof. For a subgroup $H \subseteq G$, let $\mathcal{D}_{H,D}$ be the division closure of $\varphi(EH)$ in D . Then $\varphi_{EH}: EH \rightarrow \mathcal{D}_{H,D}$ is the division envelope of $(EH, \dim|_{EH})$ as φ_{EH} is an epic division ring by Proposition 3.14 and $\dim_D$ restricted to $\mathcal{D}_{H,D}$ is $\dim_{\mathcal{D}_{H,D}}$. Recall that $\varphi: EG \rightarrow D$ is Hughes-free if for every finitely generated subgroup H and normal subgroup $N \subseteq H$ and $t \in H$ such that H/N is infinite cyclic and the image of t under the canonical projection $H \rightarrow H/N$ is a generator, the ring homomorphism $\mu_{H,N,t,D}: S^{-1}(\mathcal{D}_{N,D})_{\Theta}[t, t^{-1}] \rightarrow \mathcal{D}_{H,D}$ of (3.121) is an isomorphism. Now the claim follows from Proposition 3.163 (iv). $\square$

Our main example of a Hughes-free Sylvester rank function is described below. Let G be a group and F be a field satisfying $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. We obtain a Sylvester module rank function denoted by $\dim_{\mathcal{N}(G)}$ on FG by sending a finitely presented FG -module M to $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{FG} M)$ for the dimension function $\dim_{\mathcal{N}(G)}$ introduced in Definition 2.7.

Proposition 3.166. *If G is locally indicable, then the Sylvester module rank function $\dim_{\mathcal{N}(G)}$ on FG is Hughes-free.*

Proof. This follows from [27, Corollary 13.2]. **Comment 27 (by W.):** Is the reference sufficient? Can we give the proof here, at least for $\mathbb{Z}$, or shall we just stick to giving the reference. $\square$

3.10.9 Ultra Filters

An *ultrafilter*, denoted as ω , on the set of natural numbers $\mathbb{Z}_{\geq 0}$ is a set of nonempty subsets of $\mathbb{Z}_{\geq 0}$ satisfying:

- It is closed under finite intersections;
- It is upwards-closed, i.e., for $A \subseteq B \subseteq \mathbb{Z}_{\geq 0}$ we have the implication $A \in \omega \implies B \in \omega$
- For any subset $A \subseteq \mathbb{Z}_{\geq 0}$, we have either $A \in \omega$ or $\mathbb{Z}_{\geq 0} \setminus A \in \omega$.

An ultrafilter is called *non-principal* if it contains no finite subset of $\mathbb{Z}_{\geq 0}$. This is equivalent to the condition that there is no $i \in \mathbb{Z}_{\geq 0}$ such that ω consists of all subsets of $\mathbb{Z}_{\geq 0}$ containing i . Non-principal ultrafilters exist.

Let (X, d) be a metric space. Consider a sequence $(x_n)_{n \geq 0}$ in X . Recall that it converges to an element $x \in X$ if for every $\epsilon > 0$ there exists $N \in \mathbb{Z}_{\geq 0}$ such that $d(x, x_n) < \epsilon$ holds for $n \geq N$. If the sequence $(x_n)_{n \geq 0}$ converges to x and y in X , then $x = y$. One writes

$$x = \lim_{n \rightarrow \infty} x_n,$$

if $(x_n)_{n \geq 0}$ converges to x . Obviously a convergent sequence is bounded, i.e., there exists $R \in \mathbb{R}_{\geq 0}$ such that $d(x_m, x_n) \leq R$ holds for all $m, n \in \mathbb{Z}_{\geq 0}$. In general it is not true that every bounded sequence converges.

Given a non-principal ultrafilter ω , and a sequence $(x_n)_{n \geq 0}$ in X , we call $x \in X$ the ω -limit of $(x_n)_{n \geq 0}$, in signs

$$x = \lim_{\omega} x_n,$$

if for every $\epsilon > 0$ we have $\{n \in \mathbb{Z}_{\geq 0} \mid d(x_n, x) < \epsilon\} \in \omega$. This notion has the following useful properties:

- If x and y are ω -limits of the same sequence $(x_n)_{n \geq 0}$, then $x = y$;
- If $x = \lim_{n \rightarrow \infty} x_n$ holds, then we have $x = \lim_{\omega} x_n$;
- If X is compact, then any sequence has an ω -limit;
- Let $(x)_n$ be an bounded sequence in $\mathbb{R}$. Then its ω -limit exists.

3.10.10 Natural Extensions over von Neumann Regular Rings

The next proposition is taken from [26, Proposition 7.7].

Proposition 3.167. *Let R be a von Neumann regular ring coming with a ring automorphism $\Phi: R \xrightarrow{\cong} R$. Let $\dim$ be a faithful Φ -compatible Sylvester module rank function on R . Let $\dim'$ be a Sylvester module rank function on $R_{\Phi}[\mathbb{Z}]$ which extends $\dim$.*

Then $\dim'$ is the natural extension of $\dim$ if and only if for every $n \in \mathbb{Z}_{\geq 1}$ and $A \in M_{n,n}(R)$ we have

$$\dim'(\operatorname{coker}(r_{I_n+At}: R^n \rightarrow R^n)) = 0.$$

The explicite construction appearing in the proof of the next proposition will play a role in the proof of Theorem 3.174 and hence in the proof of the main result of this chapter Theorem 3.179.

Proposition 3.168. *Let R be a positive definite von Neumann $*$ -regular ring coming with an automorphism $\Phi: R \xrightarrow{\cong} R$ of rings with involution. Let $\dim$ be a Φ -compatible Sylvester module rank function on R .*

Then the natural extension $\widetilde{\dim}$ to $R_\Phi[\mathbb{Z}]$ of $\dim$ has a von Neumann $*$ -regular envelope.

Proof. We equip $R_\Phi[\mathbb{Z}]$ with the structure of a ring with involution $*$: $R_\Phi[\mathbb{Z}] \xrightarrow{\cong} R_\Phi[\mathbb{Z}]$ by sending $\sum_{i \in \mathbb{Z}} r_i \cdot t^i$ to $\sum_{i \in \mathbb{Z}} r_i^* \cdot t^{-i}$.

We obtain an isomorphism of rings by the composite

$$\mu_i: M_{i,i}(R) \xrightarrow{\cong} \operatorname{end}_R(R^i)^{\operatorname{op}} \xrightarrow{\cong} \operatorname{end}_R(R_\Phi[t]/(t^i))^{\operatorname{op}}$$

where the first map sends a matrix A to the R -endomorphism $r_A: R^i \rightarrow R^i$ given by right multiplication with A and the second is given by conjugation with the R -isomorphism $R^i \xrightarrow{\cong} R_\Phi[t]/(t^i)$ sending the i -th element of the standard base of R^i to the class $\overline{t^{i-1}}$ of t^{i-1} and superscript op indicates that we have passed to the opposite multiplication. Since R is a positive definite von Neumann $*$ -regular ring by assumption, $M_{i,i}(R)$ equipped with the induced involution sending $A = (a_{i,j})$ to $A^* = (a_{j,i}^*)$ is a von Neumann $*$ -regular ring. In particular $\operatorname{end}_R(R_\Phi[t]/(t^i))^{\operatorname{op}}$ is a von Neumann $*$ -regular ring and we equip $\operatorname{end}_R(R_\Phi[t]/(t^n))^{\operatorname{op}}$ with the involution for which μ_i becomes an isomorphism of rings with involution.

Let $\dim$ be a faithful Φ -compatible Sylvester module rank function on R . Note that $\dim$ is automatically exact, see Exercise 3.144, and possesses a natural extension $\widetilde{\dim}$ on $R_\Phi[\mathbb{Z}]$, see Theorem 3.162. For $i \in \mathbb{Z}_{\geq 1}$ we define the Sylvester module function $\dim_i$ on $\operatorname{end}_R(R_\Phi[t]/(t^i))^{\operatorname{op}}$ by sending a finitely presented $\operatorname{end}_R(R_\Phi[t]/(t^i))^{\operatorname{op}}$ -module M to $\frac{\dim(k_i^* M)}{i}$, where $k_i^* M$ is the finitely presented R -module obtained from M by restriction with the inclusion $k_i: R \rightarrow \operatorname{end}_R(R_\Phi[t]/(t^i))^{\operatorname{op}}$ sending r to the R -endomorphism $\overline{x} \mapsto \overline{x}r$. Fix a non-principal ultrafilter ω . Then we get a Sylvester module function $\dim_\omega$ on the von Neumann $*$ -regular ring $\prod_{i=1}^\infty \operatorname{end}_R(R_\Phi[t]/(t^i))^{\operatorname{op}}$ by putting

$$(3.169) \quad \dim_\omega(M) := \lim_\omega \dim_i((\pi_i)_* M),$$

where $\pi_i: \prod_{i=1}^\infty \operatorname{end}_R(R_\Phi[t]/(t^i))^{\operatorname{op}} \rightarrow \operatorname{end}_R(R_\Phi[t]/(t^i))^{\operatorname{op}}$ is the projection onto the i -th factor. Put

$$(3.170) \quad P_\omega^{R,\Phi} := \left(\prod_{i=1}^\infty \operatorname{end}_R(R_\Phi[t]/(t^i))^{\operatorname{op}} \right) / \ker(\operatorname{rk}_\omega),$$

where rk_ω is the Sylvester matrix function associated to $\dim_\omega$. Note that rk_ω defines a faithful Sylvester matrix function $\overline{\text{rk}_\omega}$ on $P_\omega^{R,\Phi}$. We denote by

$$(3.171) \quad \overline{\dim_\omega}$$

the associated Sylvester module rank function on $P_\omega^{R,\Phi}$.

Let

$$l'_i: R_\Phi[t] \rightarrow \text{end}_R(R_\Phi[t]/(t^i))^{\text{op}}$$

be the ring homomorphism sending rt^j to the R -endomorphism $\bar{x} \mapsto \overline{xrt^j}$. Obviously it extends to a ring homomorphism

$$l_i: R_\Phi[[t]] \rightarrow \text{end}(R_\Phi[t]/(t^i))^{\text{op}}.$$

Let the ring homomorphism

$$f_\omega: R_\Phi[[t]] \rightarrow P_\omega^{R,\Phi}$$

be the composite of $\prod_{i=0}^\infty l_i: R_\Phi[[t]] \rightarrow \prod_{i=1}^\infty \text{end}_R(R_\Phi[t]/(t^i))^{\text{op}}$ and the projection $\prod_{i=1}^\infty \text{end}_R(R_\Phi[t]/(t^i))^{\text{op}} \rightarrow (\prod_{i=1}^\infty \text{end}_R(R_\Phi[t]/(t^i))^{\text{op}}) / \ker(\text{rk}_\omega)$.

Next we show that f_ω is injective. Suppose the contrary. Then we can find $x \in R_\Phi[[t]]$ with $f_\omega(x) = 0$ such that $x = r_nt^n + r_{n+1}t^{n+1} + r_{n+2}t^{n+2} + \dots$ for some $n \in \mathbb{Z}_{\geq 0}$ and $r_n \neq 0$ holds.

We get a descending filtration of the R -module $R_\Phi[t]/(t^i)$ by R -submodules

$$R_\Phi[t]/(t^i) = F_0 \supseteq F_1 \supseteq F_2 \supseteq \dots$$

if we take F_k to be $\overline{\{s_k \cdot t^k + s_{k+1} \cdot t^{k+1} + \dots \mid s_k, s_{k+1}, \dots \in R\}}$, or, equivalently, F_k is the image of $l_i(t^k): R_\Phi[t]/(t^i) \rightarrow R_\Phi[t]/(t^i)$ for $k = 0, 1, 2, \dots$. Obviously $F_k = \{0\}$ for $k \in \mathbb{Z}_{\geq i}$ and $l_i(x)$ maps F_k to F_{k+n} for $k \in \mathbb{Z}_{\geq 0}$. Let $l_i(x)_k: F_k \rightarrow F_{k+n}$ and $\overline{l_i(x)}_{k-1}: F_{k-1}/F_k \rightarrow F_{k-1+n}/F_{k+n}$ be the maps induced by $l_i(x)$. We obtain a commutative diagram of R -modules with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_k & \longrightarrow & F_{k-1} & \longrightarrow & F_{k-1}/F_k \longrightarrow 0 \\ & & \downarrow l_i(x)_k & & \downarrow l_i(x)_{k-1} & & \downarrow \overline{l_i(x)}_{k-1} \\ 0 & \longrightarrow & F_k & \longrightarrow & F_{k-1} & \longrightarrow & F_{k-1}/F_k \longrightarrow 0 \end{array}$$

It induces an exact sequence of R -modules

$$\text{coker}(l_i(x)_k) \rightarrow \text{coker}(l_i(x)_{k-1}) \rightarrow \text{coker}(\overline{l_i(x)}_{k-1}) \rightarrow 0.$$

This implies

$$\dim(\text{coker}(l_i(x)_{k-1})) \leq \dim(\text{coker}(l_i(x)_k)) + \dim(\text{coker}(\overline{l_i(x)}_k))$$

by property (iv) appearing in Definition 3.126. Hence we get

$$\dim(\operatorname{coker}(l_i(x))) \leq \sum_{k=1}^{i-n} \dim(\operatorname{coker}(\overline{l_i(x)_{k-1}})).$$

One easily constructs a commutative diagram of R -modules with isomorphisms as vertical arrows

$$\begin{array}{ccc} R & \xrightarrow{R_{\Phi^{-n}(r_n)}} & R \\ \cong \downarrow & & \downarrow \cong \\ F_{k-1}/F_k & \xrightarrow{\overline{l_i(x)_{k-1}}} & F_{k-1}/F_k \end{array}$$

Hence we get

$$\dim(\operatorname{coker}(l_i(x))) \leq \sum_{k=1}^{i-n} \dim(\operatorname{coker}(R_{\Phi^{-n}(r_n)})) = \sum_{k=1}^{i-n} 1 - \operatorname{rk}(\Phi^{-n}(r_n)).$$

Let rk be the Sylvester matrix function on R associated to $\dim$. We have $\operatorname{rk}(r_n) = \operatorname{rk}(\Phi^{-n}(r_n))$ for $k = 1, \dots, (i-n)$, since $\dim$ is Φ -compatible by assumption. We conclude $\dim(\operatorname{coker}(l_i(x))) \leq (i-n) \cdot (1 - \operatorname{rk}(r_n))$ and hence

$$\dim_i(\operatorname{coker}(l_i(x))) \leq \frac{(i-n)}{i} \cdot (1 - \operatorname{rk}(r_n)).$$

This implies

$$\begin{aligned} \overline{\dim}_\omega(\operatorname{coker}(f_\omega(x))) &= \lim_\omega \dim_i(\operatorname{coker}(l_i(x))) \leq \lim_\omega \frac{(i-n)}{i} \cdot (1 - \operatorname{rk}(r_n)) \\ &= \lim_{i \rightarrow \infty} \frac{(i-n)}{i} \cdot (1 - \operatorname{rk}(r_n)) = 1 - \operatorname{rk}(r_n). \end{aligned}$$

Since we have $\overline{\operatorname{rk}}_\omega(f_\omega(x)) = 1 - \overline{\dim}_\omega(\operatorname{coker}(f_\omega(x)))$, we get

$$\overline{\operatorname{rk}}_\omega(f_\omega(x)) \geq \operatorname{rk}(r_n) \geq 0.$$

Since $f_\omega(x) = 0$ implies $\overline{\operatorname{rk}}_\omega(f_\omega(x)) = 0$, we obtain $\operatorname{rk}(r_n) = 0$. Since $\dim$ is faithful, we get $r_n = 0$, a contradiction. This finishes the proof that f_ω is injective.

We conclude from the definitions that for the natural extension $\widetilde{\dim}$ on $R_\Phi[\mathbb{Z}]$ of $\dim$ on R we get the equality of Sylvester module rank functions on $R_\Phi[t]$

$$\widetilde{\dim}|_{R_\Phi[t]} = f_\omega^\#(\overline{\dim}_\omega).$$

Since a product of von Neumann $*$ -regular rings is again von Neumann $*$ -regular, $\prod_{i=1}^\infty \operatorname{end}_R(R_\Phi[t]/(t^i))^{\operatorname{op}}$ is a von Neumann $*$ -regular ring. Proposition 3.6 implies that $P_\omega^{R, \Phi}$ is a von Neumann $*$ -regular ring. Note that

$\widetilde{\text{rk}}(t) = 1$ holds for the Sylvester matrix rank function $\widetilde{\text{rk}}$ associated to $\widetilde{\text{dim}}$, see (3.161). Hence we get $\overline{\text{rk}}_\omega(f_\omega(t)) = 1$. We conclude from Exercise 3.135 that $f_\omega(t)$ is invertible in $P_\omega^{R,\Phi}$. Hence the ring homomorphism f_ω extends to an injective ring homomorphism

$$(3.172) \quad \widehat{f}_\omega: R_\Phi((t)) \rightarrow P_\omega^{R,\Phi}.$$

One easily checks the equality of Sylvester module rank functions on $R_\Phi[\mathbb{Z}]$

$$\widetilde{\text{dim}} = (\widehat{f}_\omega \circ j)^\#(\overline{\text{dim}}_\omega).$$

where $j: R_\Phi[\mathbb{Z}] \rightarrow R_\Phi((t))$ is the inclusion. Obviously $\widehat{f}_\omega \circ j$ is a morphism of rings with involution. Now one passes to the $*$ -regular closure $\mathcal{REG}((\widehat{f}_\omega \circ j)(R_\Phi[\mathbb{Z}]) \subseteq P_\omega^{R,\Phi})$ to obtain a von Neumann $*$ -regular envelope of $(R_\Phi[\mathbb{Z}], \text{dim})$, see Remark 3.140. This finishes the proof of Proposition 3.168. $\square$

Proposition 3.173. *Let R be a ring with involution and $\Phi: R \xrightarrow{\cong} R$ be an automorphism of rings with involution. Let rk_R be a Sylvester matrix rank function on R which is Φ -compatible. Consider a von Neumann $*$ -regular envelope $(S, \varphi, \text{rk}_S)$ of (R, rk_R) .*

- (i) *There exists precisely one automorphism $\overline{\Phi}: S \xrightarrow{\cong} S$ of rings with involutions which extends Φ , i.e., satisfies $\overline{\Phi} \circ \varphi = \varphi \circ \Phi$;*
- (ii) *Let $\varphi_\Phi[\mathbb{Z}]: R_\Phi[\mathbb{Z}] \rightarrow S_\Phi[\mathbb{Z}]$ be the obvious extension of φ to a homomorphism of rings with involution, where we equip $R_\Phi[\mathbb{Z}]$ and $S_\Phi[\mathbb{Z}]$ with the involutions sending rt^n to r^*t^{-n} and st^n to s^*t^{-n} . Then there exists the natural extension rk_S of rk_R to $S_\Phi[\mathbb{Z}]$ and $\widetilde{\text{rk}}_R := (\varphi_\Phi[\mathbb{Z}])^\# \widetilde{\text{rk}}_S$ is the natural extension of rk_R to $R_\Phi[\mathbb{Z}]$;*
- (iii) *$(S_\Phi[\mathbb{Z}], \varphi_\Phi[\mathbb{Z}], \text{rk}_S)$ is a von Neumann $*$ -regular envelope of $(R_\Phi[\mathbb{Z}], \widetilde{\text{rk}}_R)$.*

Proof. Comment 28 (by W.): Shall we insert the proof or just refer to [29, Proposition 3.7]. It is similar to the one of 3.163. We do not need this result in this chapter, but maybe later. $\square$

3.10.11 Regular Hughes-Free Sylvester Functions and Division Envelopes

Theorem 3.174 (Regular Hughes-free Sylvester functions and division envelopes) *Let G be a locally indicible group and let F be a field satisfying $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ which is closed under complex conjugation. Let dim be a Sylvester module rank function on the group ring FG which is Hughes free in the sense of Definition 3.164 and comes with a positive definite $*$ -regular envelope $(\varphi: FG \rightarrow U, \text{dim}_U)$ in the sense of Definition 3.139.*

Then U is a division ring.

Proof. Given a subgroup $H \subseteq G$, we denote by

$$\mathcal{D}_{H,U} = \mathcal{D}(\varphi(EH) \subseteq U)$$

the division closure of $\varphi(EH)$ in U in the sense of Definition 2.75 and by

$$\mathcal{REG}_{H,U} = \mathcal{REG}(\varphi(EH) \subseteq U)$$

the $*$ -regular closure of $\varphi(EH)$ in U in the sense of Definition 3.8. Furthermore we abbreviate

$$\begin{aligned} \mathcal{D} &= \mathcal{D}_{G,U} = \mathcal{D}(\varphi(EG) \subseteq U); \\ \mathcal{D}_{H,\mathcal{D}} &= \mathcal{D}(\varphi(EH) \subseteq \mathcal{D}); \\ \mathcal{D}_{H,\mathcal{REG}_{H,U}} &= \mathcal{D}(\varphi(EH) \subseteq \mathcal{REG}_{H,U}). \end{aligned}$$

Next we show for every element $b \in \mathcal{D}$ with $b \neq 0$ by induction over the complexity introduced in Section 3.6 that b is invertible. Then Theorem 3.174 holds by the following argument. Since φ is epic and D is a division ring, the map $\mathcal{D} \rightarrow U$ is epic and we get $U = \mathcal{D}$ from Proposition 3.11 which shows that U is a division ring.

The induction beginning $\text{Tree}_{G,D}(b) = 1_{\mathcal{T}}$ is obvious since then $b \in \varphi(E^\times G)$. In the induction step we have to show for $b \in \mathcal{D}$ with $\text{Tree}_{G,D}(b) > 1_{\mathcal{T}}$ that b is invertible in $\mathcal{D}$ provided that the induction hypothesis holds, i.e., every $c \in \mathcal{D}$ with $c \neq 0$ and $\text{Tree}_{G,D}(c) < \text{Tree}_{G,D}(b)$ is invertible in $\mathcal{D}$. This is done as follows.

Let $\beta \in \text{Rat}(E^\times G)$ realize the complexity of b . By Theorem 3.83 there is a finitely generated subgroup $\text{source}(\beta) \subseteq G$ such that

$$\beta \in \text{Rat}(\text{source}(\beta)) \cdot E^\times G$$

holds. Since multiplying b with an element in $\varphi(E^\times G)$ does not change its complexity and the desired conclusion for b , we can assume without loss of generality by Proposition 3.85 that β is primitive, i.e., $\beta \in \text{Rat}(\text{source}(\beta))$. Let $H \subseteq G$ be the finitely generated subgroup of G given by the image of $\text{source}(\beta)$ under the projection $E^\times G \rightarrow E^\times G/E^\times = G$. Then we conclude from Lemma 2.78

$$(3.175) \quad \mathcal{D}_{H,\mathcal{REG}_{H,U}} = \mathcal{D}_{H,U} = \mathcal{D}_{H,\mathcal{D}},$$

and we have $b \in \mathcal{D}_{H,\mathcal{D}}$. Hence it suffices to show that b is invertible in $\mathcal{D}_{H,\mathcal{D}} \subseteq \mathcal{D}$.

If H is trivial, $\mathcal{D}_{H,\mathcal{D}}$ is isomorphic to E and b is a unit since $b \neq 0$ holds. Since G is by assumption locally indicable, we can assume without loss of generality the existence of a normal subgroup $N \subseteq H$ such that H/N is infinite cyclic. Choose $t \in H$ which is sent under the canonical projection $H \rightarrow H/N$

to a generator. Then $H = N \rtimes_{c_t} \mathbb{Z}$ if $c_t: N \xrightarrow{\cong} N$ is the automorphism given by conjugation with t .

We conclude from Proposition 3.155 and Theorem 3.142 that there is a ring automorphism $\Phi: \mathcal{REG}_{N,U} \xrightarrow{\cong} \mathcal{REG}_{N,U}$ uniquely determined by the properties that it extends the automorphism $EN \rightarrow EN$ induced by $c_t: N \xrightarrow{\cong} N$ and respects the Sylvester module function on $\mathcal{REG}_{N,U}$ obtained by restricting $\dim_U$ from U to $\mathcal{REG}_{N,U}$. Choose a non-principal ultrafilter ω on $\mathbb{Z}_{\geq 0}$. Let

$$P = P_\omega^{\mathcal{REG}_{N,U}, \Phi}$$

be the ring defined in (3.170). We have defined the injective ring homomorphism

$$\widehat{f_\omega}: (\mathcal{REG}_{N,U})_\Phi((t)) \rightarrow P$$

in (3.172) and will think of it as an inclusion. So we have inclusions of rings

$$(3.176) \quad (\mathcal{REG}_{N,U})_\Phi[\mathbb{Z}] \subseteq (\mathcal{REG}_{N,U})_\Phi((t)) \subseteq P.$$

Recall from the proof of Proposition 3.168 that $\mathcal{REG}((\mathcal{REG}_{N,U})_\Phi[\mathbb{Z}] \subseteq P)$ together with $\overline{\dim_\omega}$ defined in (3.171) is a von Neumann *-regular envelope of $((\mathcal{REG}_{N,U})_\Phi[\mathbb{Z}], \dim_U|_{\mathcal{REG}_{N,U}})$, where $\dim_U|_{\mathcal{REG}_{N,U}}$ is the natural extension of $\dim_U|_{\mathcal{REG}_{N;U}}$ from $\mathcal{REG}_{N;U}$ to $(\mathcal{REG}_{N,U})_\Phi[\mathbb{Z}]$. Since $\dim_U$ is Hughes free, $(\mathcal{REG}_{H,U}, \dim_U|_{\mathcal{REG}_{H,U}})$ is another von Neumann *-regular envelope of $((\mathcal{REG}_{N;U})_\Phi[\mathbb{Z}], \dim_U|_{\mathcal{REG}_{H;U}})$. Hence we get from Theorem 3.142 an identification of EH -rings

$$(3.177) \quad \mathcal{REG}((\mathcal{REG}_{N,U})_\Phi[\mathbb{Z}] \subseteq P) = \mathcal{REG}_{H,U}.$$

Since $\varphi(EN) \subseteq \mathcal{REG}_{N,U}$ and $\varphi(EH) \subseteq (\mathcal{REG}_{N,U})_\Phi[\mathbb{Z}]$, holds, we can define

$$\begin{aligned} \mathcal{D}_{N,P} &= \mathcal{D}(\varphi(EN) \subseteq P); \\ \mathcal{D}_{H,P} &= \mathcal{D}(\varphi(EH) \subseteq P). \end{aligned}$$

We conclude from (3.177) using Lemma 2.78

$$(3.178) \quad \mathcal{D}_{N,P} = \mathcal{D}_{N, \mathcal{REG}_{N,U}} = \mathcal{D}_{N,U} = \mathcal{D}_{N,D}.$$

For every $c \in \mathcal{D}_{H,D}$ with $\text{Tree}_{H,D}(c) < \text{Tree}_{H,D}(b)$ we get using (3.125) and Lemma 3.81

$$\text{Tree}_{G,D}(c) \leq \text{Tree}_{H,D}(c) < \text{Tree}_{H,D}(b) = \text{Tree}_{G,D}(b).$$

Hence c is invertible in $\mathcal{D}_{G,D}$ and hence in D by the induction hypothesis. Since $\mathcal{D}_{H,D}$ is division closed in D , we conclude that c is invertible in $\mathcal{D}_{H,D}$.

Recall that $\mathcal{REG}_{N,U}$ is a von Neumann regular FN -ring and P contains $(\mathcal{REG}_{N,U})_\Phi((t))$, see (3.176). Now all the conditions appearing in Proposi-

tion 3.115 are satisfied, where we take $R = \mathcal{REG}_{N,U}$, $S = U$ and $\Theta = \Phi$. In the sequel we will use tacitly the equalities (3.175) and (3.178) and the inclusion (3.176). Hence b belongs to $(\mathcal{D}_{N,D})_{\Theta}((t)) \subseteq \mathcal{D}_{H,D}$ and we can write $b = \sum_{k \geq k_0} b_k t^k$ for some $k_0 \in \mathbb{Z}$ and $b_k \in \mathcal{D}_{N,D}$ satisfying $b_{k_0} \neq 0$.

Next we show that there is at least one element $k_1 \in \mathbb{Z}_{\geq k_0+1}$ satisfying $b_{k_1} \neq 0$. Suppose the contrary. Then $b = b_{k_0} t^{k_0}$ and Proposition 3.115 implies $\beta = \gamma t^{k_0}$ for some $\gamma \in \text{Rat}(E^\times N)$ which implies $H \subseteq N$ by Proposition 3.85 (iii), a contradiction.

Hence Proposition 3.115 implies that $\text{Tree}_{H,D}(b_k t^k) < \text{Tree}_{H,D}(b)$ holds for every k and in particular for k_0 . Hence $b_{k_0} t^{k_0}$ is invertible in $\mathcal{D}_{H,D}$. This implies that b is invertible in $(\mathcal{D}_{N,D})_{\Theta}((t))$ and hence in $\mathcal{D}_{H,D}$. This finishes the proof of Theorem 3.174. $\square$

3.10.12 Finishing the Proof of the K -Theoretic Atiyah Conjecture for Locally Indicible Groups

Theorem 3.179 (The Strong Atiyah Conjecture 2.29 for torsionfree groups holds for locally indicible groups.)

Consider a locally indicible group G . Let $\mathcal{REG}_F(G)$ be the $*$ -regular closure of $\mathbb{C}G$ in $\mathcal{U}(G)$ in the sense of Definition 3.8. Then $FG \rightarrow \mathcal{REG}_F(G)$ is a Hughes free epic division FG ring.

In particular the Strong Atiyah Conjecture 2.29 for torsionfree groups holds for locally indicible groups.

Proof. The canonical map $\varphi: FG \rightarrow \mathcal{REG}_F(G)$ has a positive definite von Neumann $*$ -regular ring as target and is epic by Proposition 3.15. Since $\dim_{\mathcal{N}(G)}$ is Hughes-free by Proposition 3.166, we conclude from Theorem 3.165 that $\varphi: FG \rightarrow \mathcal{REG}_F(G)$ is a Hughes free epic division FG ring.

If we take $F = \mathbb{C}$, then $\mathcal{REG}_{\mathbb{C}}(G)$ is skew field (= division ring) and hence the Strong Atiyah Conjecture 2.29 for torsionfree groups holds for G as it explained in Subsection 2.11.2. $\square$

3.11 Notes

More details and some of the proofs which have not presented here or in [14] can be found in the PhD-thesis by Sánchez-Serdà [63].

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Chapter 4

Prime Characteristic

4.1 Introduction

4.2 Review

4.3 Title of the Section

4.4 Notes

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Chapter 5

L^2 -torsion

5.1 Introduction

Comment 29 (by W.): The introduction has to be completed.

There are various notions of classical torsion invariants, such as Reidemeister torsion, Whitehead torsion, and analytic Ray-Singer torsion, which have been intensively studied since the thirties of the last century. L^2 -torsion is the L^2 -analogue of Reidemeister torsion as L^2 -Betti numbers are the L^2 -analogues of the classical Betti numbers. We will review L^2 -torsion in Section 5.2.

Meanwhile there have been established more advanced versions of L^2 -torsion such as the ϕ -twisted L^2 -torsion functions or the universal L^2 -torsion.

5.2 Review of L^2 -torsion

5.2.1 Fuglede-Kadison Determinant

Definition 5.1 (Finitely generated Hilbert module). A *finitely generated Hilbert $\mathcal{N}(G)$ -module* V is a Hilbert space V together with a linear isometric G -action such that there exists an isometric linear G -embedding of V into $L^2(G)^r$ for some natural number r . A *morphism of Hilbert $\mathcal{N}(G)$ -modules* $f: V \rightarrow W$ is a bounded G -equivariant operator.

The additive category of finitely generated Hilbert $\mathcal{N}(G)$ -modules is equivalent to the additive category of finitely generated projective $\mathcal{N}(G)$ -modules, where $\mathcal{N}(G)$ -module is to be understood in the purely algebraic sense that we consider $\mathcal{N}(G)$ just as a ring forgetting the topology, see Remark 2.5. We denote by

$$(5.2) \quad \dim_{\mathcal{N}(G)}(V) \in \mathbb{R}_{\geq 0}$$

the *Murray-von Neumann dimension* of a finitely generated Hilbert $\mathcal{N}(G)$ -module whose precise definition can be found in [45, Subsection 1.1.3]. Under the equivalence of categories above it corresponds to the dimension of a finitely projective $\mathcal{N}(G)$ -module defined in (2.3), see Remark 2.5.

A bounded G -equivariant operator $f: V \rightarrow W$ is called a *weak isomorphism* if and only if it is injective and its image is dense. The following statements are equivalent for a morphisms $f: V \rightarrow W$ of finitely generated Hilbert $\mathcal{N}(G)$ -modules, see [45, Lemma 1.13 on page 23]:

- (i) f is a weak isomorphism;
- (ii) Its adjoint f^* is a weak isomorphism;
- (iii) The unitary part of the polar decomposition of f is an isometric G -isomorphism of finitely generated Hilbert $\mathcal{N}(G)$ -modules;
- (iv) f is injective and $\dim_{\mathcal{N}(G)}(V) = \dim_{\mathcal{N}(G)}(W)$ holds;
- (v) f has dense image and $\dim_{\mathcal{N}(G)}(V) = \dim_{\mathcal{N}(G)}(W)$ holds;
- (vi) We have $\dim_{\mathcal{N}(G)}(V) = \dim_{\mathcal{N}(G)}(W) = \dim_{\mathcal{N}(G)}(\overline{\text{im}(f)})$ for the closure $\overline{\text{im}(f)}$ of the image of f .

Definition 5.3 (Spectral density function). Let $f: V \rightarrow W$ be a morphism of finitely generated Hilbert $\mathcal{N}(G)$ -modules. Denote by $\{E_\lambda^{f^*f} \mid \lambda \in \mathbb{R}\}$ the (right-continuous) family of spectral projections of the positive operator f^*f . Define the *spectral density function of f* by

$$F_f: \mathbb{R} \rightarrow \mathbb{R}^{\geq 0} \quad \lambda \mapsto \dim_{\mathcal{N}(G)}(\text{im}(E_{\lambda^2}^{f^*f})) = \text{tr}_{\mathcal{N}(G)}(E_{\lambda^2}^{f^*f}).$$

The spectral density function is monotone non-decreasing and right-continuous. We have $F(0) = \dim_{\mathcal{N}(G)}(\ker(f))$.

Example 5.4 (Spectral density function for finite groups). Let G be finite and $f: U \rightarrow V$ be a morphism of finitely generated Hilbert $\mathcal{N}(G)$ -modules, i.e., of finite-dimensional unitary G -representations. Then $F(f)$ is the right-continuous step function whose value at λ is the sum of the complex dimensions of the eigenspaces of f^*f for eigenvalues $\mu \leq \lambda^2$ divided by the order of G , or, equivalently, the sum of the complex dimensions of the eigenspaces of $|f|$ for eigenvalues $\mu \leq \lambda$ divided by the order of G .

Example 5.5 (Spectral density function for $\mathbb{Z}^d$). Let $G = \mathbb{Z}^d$. In the sequel we use the notation and the identification $\mathcal{N}(\mathbb{Z}^d) = L^\infty(T^d)$ of [45, Example 1.4 on page 15]. For $f \in L^\infty(T^d)$ the spectral density function $F(M_f)$ of $M_f: L^2(T^d) \rightarrow L^2(T^d)$, $g \mapsto g \cdot f$ sends λ to the volume of the set $\{z \in T^d \mid |f(z)| \leq \lambda\}$.

Definition 5.6 (Fuglede-Kadison determinant). Let $f: V \rightarrow W$ be a morphism of finitely generated Hilbert $\mathcal{N}(G)$ -modules. Let F_f be the spectral density function of Definition 5.3 which is a monotone non-decreasing right-continuous function. Let dF be the unique measure on the Borel σ -algebra on $\mathbb{R}$ which satisfies $dF((a, b]) = F(b) - F(a)$ for $a < b$. Define the *Fuglede-Kadison determinant*

$$\det_{\mathcal{N}(G)}(f) \in \mathbb{R}^{\geq 0}$$

to be the positive real number

$$\det_{\mathcal{N}(G)}(f) = \exp \left(\int_{0+}^{\infty} \ln(\lambda) \, dF \right),$$

if the Lebesgue integral $\int_{0+}^{\infty} \ln(\lambda) dF$ converges to a real number, and to be 0 otherwise.

Note that in the definition above we do not require that the source and domain of f agree or that f is injective or that f is surjective. Our conventions imply that the Fuglede-Kadison determinant of the zero operator $0: V \rightarrow W$ is 1.

Example 5.7 (Fuglede-Kadison determinant for finite groups). To illustrate this definition, we look at the example where G is finite. We essentially get the classical determinant $\det_{\mathbb{C}}$. Namely, let $\lambda_1, \lambda_2, \dots, \lambda_r$ be the non-zero eigenvalues of f^*f with multiplicity μ_i . Then one obtains, if $\overline{f^*f}$ is the automorphism of the orthogonal complement of the kernel of f^*f induced by f^*f ,

$$\det_{\mathcal{N}(G)}(f) = \exp \left(\sum_{i=1}^r \frac{\mu_i}{|G|} \cdot \ln(\sqrt{\lambda_i}) \right) = \prod_{i=1}^r \lambda_i^{\frac{\mu_i}{2 \cdot |G|}} = \det_{\mathbb{C}}(\overline{f^*f})^{\frac{1}{2 \cdot |G|}},$$

where $\det_{\mathbb{C}}(\overline{f^*f})$ is put to be 1 if f is the zero operator and hence $\overline{f^*f}$ is $\text{id}: \{0\} \rightarrow \{0\}$. If $f: \mathbb{C}G^m \rightarrow \mathbb{C}G^m$ is an automorphism, we get

$$\det_{\mathcal{N}(G)}(f) = |\det_{\mathbb{C}}(f)|^{\frac{1}{|G|}}.$$

Example 5.8 (Fuglede-Kadison determinant for $\mathcal{N}(\mathbb{Z}^d)$). Let $G = \mathbb{Z}^d$. We use the identification $\mathcal{N}(\mathbb{Z}^d) = L^\infty(T^d)$ of [45, Example 1.4 on page 15]. For $f \in L^\infty(T^d)$ we conclude from Example 5.5

$$\begin{aligned} \det_{\mathcal{N}(\mathbb{Z}^d)}(M_f: L^2(T^d) \rightarrow L^2(T^d)) \\ = \exp \left(\int_{T^d} \ln(|f(z)|) \cdot \chi_{\{u \in T^d \mid f(u) \neq 0\}} d\text{vol}_z \right) \end{aligned}$$

using the convention $\exp(-\infty) = 0$.

The following theorem can be found with proof in [45, Theorem 3.14 on page 128 and Lemma 3.15 (4) on page 129].

Theorem 5.9 (Basic properties of the Fuglede-Kadison determinant)

(i) Let $f: U \rightarrow V$ and $g: V \rightarrow W$ be morphisms of finitely generated Hilbert $\mathcal{N}(G)$ -modules such that f has dense image and g is injective. Then

$$\det_{\mathcal{N}(G)}(g \circ f) = \det_{\mathcal{N}(G)}(f) \cdot \det_{\mathcal{N}(G)}(g);$$

(ii) Let $f_1: U_1 \rightarrow V_1$, $f_2: U_2 \rightarrow V_2$ and $f_3: U_2 \rightarrow V_1$ be morphisms of finitely generated Hilbert $\mathcal{N}(G)$ -modules such that f_1 has dense image and f_2 is injective. Then

$$\det_{\mathcal{N}(G)} \begin{pmatrix} f_1 & f_3 \\ 0 & f_2 \end{pmatrix} = \det_{\mathcal{N}(G)}(f_1) \cdot \det_{\mathcal{N}(G)}(f_2);$$

(iii) Let $f: U \rightarrow V$ be a morphism of finitely generated Hilbert $\mathcal{N}(G)$ -modules. Then

$$\det_{\mathcal{N}(G)}(f) = \det_{\mathcal{N}(G)}(f^*) = \sqrt{\det_{\mathcal{N}(G)}(f^*f)} = \sqrt{\det_{\mathcal{N}(G)}(ff^*)};$$

(iv) Let $i: H \rightarrow G$ be the inclusion of a subgroup of finite index $[G : H]$. Let $i^*f: i^*U \rightarrow i^*V$ be the morphism of finitely generated Hilbert $\mathcal{N}(H)$ -modules obtained from f by restriction. Then

$$\det_{\mathcal{N}(H)}(i^*f) = \det_{\mathcal{N}(G)}(f)^{[G:H]};$$

(v) Let $i: H \rightarrow G$ be an injective group homomorphism and let $f: U \rightarrow V$ be a morphism of finitely generated Hilbert $\mathcal{N}(H)$ -modules. Let i_*f be the morphism of finitely generated Hilbert $\mathcal{N}(G)$ -modules obtained by induction with i . Then

$$\det_{\mathcal{N}(G)}(i_*f) = \det_{\mathcal{N}(H)}(f).$$

5.2.2 The Determinant Conjecture

Conjecture 5.10 (Determinant Conjecture for a group G). *For any matrix $A \in M_{r,s}(\mathbb{Z}G)$, the Fuglede-Kadison determinant of the morphism of Hilbert modules $r_A^{(2)}: L^2(G)^r \rightarrow L^2(G)^s$ given by right multiplication with A satisfies*

$$\det_{\mathcal{N}(G)}^{(2)}(r_A^{(2)}) \geq 1.$$

Remark 5.11 (Status of the Determinant Conjecture). We will want to assume that the Determinant Conjecture 5.10 is true because then the condition of determinant class is automatically satisfied. This is an acceptable condition since the Determinant Conjecture 5.10 is known for a very large class of groups. Namely, the following is known, see [16, Theorem 5], [45, Section 13.2], [64, Theorem 1.21]. Let $\mathcal{DC}$ be the class of groups for which the Determinant Conjecture 5.10 is true. Then:

- (i) Amenable quotient
Let $H \subset G$ be a normal subgroup. Suppose that $H \in \mathcal{DC}$ and the quotient G/H is amenable. Then $G \in \mathcal{DC}$;
- (ii) Colimits
If $G = \operatorname{colim}_{i \in I} G_i$ is the colimit of the directed system $\{G_i \mid i \in I\}$ of groups indexed by the directed set I and each G_i belongs to $\mathcal{DC}$, then G belongs to $\mathcal{DC}$;
- (iii) Inverse limits
If $G = \operatorname{lim}_{i \in I} G_i$ is the limit of the inverse system $\{G_i \mid i \in I\}$ of groups

indexed by the directed set I and each G_i belongs to $\mathcal{DC}$, then G belongs to $\mathcal{DC}$;

(iv) Subgroups

If H is isomorphic to a subgroup of a group G with $G \in \mathcal{DC}$, then $H \in \mathcal{DC}$;

(v) Quotients with finite kernel

Let $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ be an exact sequence of groups. If K is finite and G belongs to $\mathcal{DC}$, then Q belongs to $\mathcal{DC}$;

(vi) Sofic groups belong to $\mathcal{DC}$;

(vii) The fundamental group of a 3-manifold belongs to $\mathcal{DC}$.

The class of sofic groups is very large. It is closed under direct and free products, taking subgroups, taking inverse and direct limits over directed index sets, and is closed under extensions with amenable groups as quotients and a sofic group as kernel. In particular it contains all residually amenable groups and fundamental groups of 3-manifolds.

More information about sofic groups can be found for instance in [17, 35, 57].

Comment 30 (by W.): AI has announced a counterexample to the Determinant Conjecture 5.10 see [55].

For more information about the Determinant Conjecture we refer for instance to [45, Chapter 13].

5.2.3 L^2 -Torsion of Finite Hilbert $\mathcal{N}(G)$ -Chain Complexes

Let

$$\cdots 0 \rightarrow 0 \rightarrow C_n^{(2)} \xrightarrow{c_n^{(2)}} C_{n-1}^{(2)} \xrightarrow{c_{n-1}^{(2)}} \cdots \xrightarrow{c_2^{(2)}} C_1^{(2)} \xrightarrow{c_1^{(2)}} C_0^{(2)} \rightarrow 0 \rightarrow \cdots$$

be a *finite Hilbert $\mathcal{N}(G)$ -chain complex* $(C_*^{(2)}, c_*^{(2)})$, i.e., each $C_n^{(2)}$ is a finitely generated Hilbert $\mathcal{N}(G)$ -module, each differential $c_n^{(2)}$ is a G -equivariant bounded operator, and there is $d \in \mathbb{Z}_{\geq 0}$ such that $C_n^{(2)} = 0$ holds for $n < 0$ and for $n > d$. For $n \in \mathbb{Z}_{\geq 0}$, we define the finitely generated Hilbert $\mathcal{N}(G)$ -module

$$H_n^{(2)}(C_*^{(2)}) := \ker(c_n^{(2)}) / \overline{\operatorname{im}(c_{n+1}^{(2)})}.$$

Note that we divide out the closure of the image of $c_{n+1}^{(2)}$ to ensure that we indeed obtain a finitely generated Hilbert $\mathcal{N}(G)$ -module. Denote by

$$(5.12) \quad b_n^{(2)}(C_*^{(2)}) := \dim_{\mathcal{N}(G)}(H_n^{(2)}(C_*^{(2)})) \in \mathbb{R}_{\geq 0}$$

the n -th L^2 -Betti number of $C_*^{(2)}$. We say that the complex $C_*^{(2)}$ is L^2 -acyclic if all its L^2 -Betti numbers vanish.

Define the n th *Laplace operator*

$$(5.13) \quad \Delta_n^{(2)} := c_{n+1}^{(2)} \circ (c_n^{(2)})^* + (c_{n-1}^{(2)})^* \circ c_n^{(2)} : C_n^{(2)} \rightarrow C_n^{(2)}.$$

Then $b_n^{(2)}(C_*^{(2)}) = \dim_{\mathcal{N}(G)}(\ker(\Delta_n^{(2)}))$, see [45, Lemma 1.18 on page 24]. Hence $b_n^{(2)}(C_*^{(2)})$ vanishes if and only if $\ker(\Delta_n^{(2)})$ is trivial, or, equivalently, $\Delta_n^{(2)}$ is a weak isomorphism.

We call $C_*^{(2)}$ of *determinant class* if $\det_{\mathcal{N}(G)}(c_n^{(2)}) > 0$ holds for every $n \in \mathbb{Z}_{\geq 1}$. This is equivalent to the condition that $\det_{\mathcal{N}(G)}(\Delta_n^{(2)}) > 0$ holds for every $n \in \mathbb{Z}_{\geq 0}$. We say that $C_*^{(2)}$ is $\det$ - L^2 -acyclic if it is both L^2 -acyclic and of determinant class.

If $C_*^{(2)}$ is of determinant class, then we define the L^2 -torsion of $C_*^{(2)}$ by

$$(5.14) \quad \rho^{(2)}(C_*^{(2)}) = \rho^{(2)}(C_*^{(2)}; \mathcal{N}(G)) \\ := - \sum_{n=1}^{\infty} (-1)^n \cdot \ln(\det_{\mathcal{N}(G)}(c_n^{(2)})) \in \mathbb{R}.$$

This turns out to be the same as putting

$$(5.15) \quad \rho^{(2)}(C_*^{(2)}) = -\frac{1}{2} \cdot \sum_{n=0}^{\infty} (-1)^n \cdot n \cdot \ln(\det_{\mathcal{N}(G)}(\Delta_n^{(2)})).$$

5.2.4 L^2 -Torsion of Based Finite Free Chain Complexes over Group Rings

Let G be a group and let R be one of the rings $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. Let

$$\cdots 0 \rightarrow 0 \rightarrow C_n \xrightarrow{c_n} C_{n-1} \xrightarrow{c_{n-1}} \cdots \xrightarrow{c_2} C_1 \xrightarrow{c_1} C_0 \rightarrow 0 \rightarrow \cdots$$

be a based finite free RG -chain complex (C_*, c_*) , i.e., each C_n is a finitely generated free RG -module equipped with an ordered RG -basis, each differential is an RG -homomorphism and there is $d \in \mathbb{Z}_{\geq 0}$ such that $C_n = 0$ holds for $n < 0$ and for $n > d$. The basis induces on $C_n^{(2)} := L^2(G) \otimes_{RG} C_n$ the structure of a finitely generated Hilbert $\mathcal{N}(G)$ -module in the obvious way. Note for the sequel that this structure is unchanged up to isometric G -isomorphism of finite Hilbert $\mathcal{N}(G)$ -modules if we permute the basis elements or multiply one of the basis elements with $\pm g$ for some $g \in G$. The resulting differentials $c_n^{(2)} = \text{id} \otimes c_n : C_i^{(2)} \rightarrow C_{i-1}^{(2)}$ are bounded G -equivariant operators. So we get a finite Hilbert $\mathcal{N}(G)$ -chain complex $C_*^{(2)}$. For $n \in \mathbb{Z}_{\geq 0}$, we define the finitely generated Hilbert $\mathcal{N}(G)$ -module

$$H_n^{(2)}(C_*) := H_n^{(2)}(C_*^{(2)}).$$

Denote by

$$(5.16) \quad b_n^{(2)}(C_*) = b_n^{(2)}(C_*; \mathcal{N}(G)) := b_n^{(2)}(C_*^{(2)}) \in \mathbb{R}^{\geq 0}$$

the n -th L^2 -Betti number of C_* . We say that the complex C_* is L^2 -acyclic if all its L^2 -Betti numbers vanish.

We call C_* of *determinant class* if $C_*^{(2)}$ is of determinant class. If G satisfies the Determinant Conjecture 5.10, then C_* is automatically of determinant class. We say that C_* is *det- L^2 -acyclic* if C_* is both L^2 -acyclic and of determinant class. If C_* is of determinant class, then we define the L^2 -torsion of C_* by

$$(5.17) \quad \rho^{(2)}(C_*) = \rho^{(2)}(C_*; \mathcal{N}(G)) := \rho^{(2)}(C_*^{(2)}).$$

5.2.5 L^2 -Torsion of G -Coverings of Finite CW -Complexes

Let G be a (discrete) group and X be a finite CW -complex. Let $G \rightarrow \bar{X} \rightarrow X$ be a G -principal bundle over X , or, equivalently, G -covering over X . The cellular chain complex $C_*(\bar{X})$ of $\bar{X}$ with $\mathbb{Z}$ -coefficients is a finite free $\mathbb{Z}G$ -chain complex. If we choose an ordering on the set of cells of X , an orientation for each cell in X , and a lift of each cell in X to cell in $\bar{X}$, we obtain an ordered $\mathbb{Z}G$ -basis for $C_*(X)$ and we can consider the finite Hilbert $\mathcal{N}(G)$ -chain complex $C_*^{(2)}(\bar{X}) = L^2(G) \otimes_{\mathbb{Z}G} C_*(\bar{X})$. One easily checks that $C_*^{(2)}(\bar{X})$ is up to isometric G -isomorphism and the induced the structure of a $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex on C_* are independent of the choices above. Hence we can define the n th L^2 -Betti number

$$(5.18) \quad b_n^{(2)}(\bar{X}) = b_n^{(2)}(\bar{X}; \mathcal{N}(G)) := b_n^{(2)}(C_*^{(2)}(\bar{X})).$$

This agrees with the previous Definition 2.14.

If $C_*^{(2)}(\bar{X})$ is of determinant class, we can also consider the L^2 -torsion

$$(5.19) \quad \rho^{(2)}(\bar{X}) = \rho^{(2)}(\bar{X}; \mathcal{N}(G)) := \rho^{(2)}(C_*^{(2)}(\bar{X})).$$

Let X be a finite (not necessarily connected) CW -complex. Let C be any of its path components. Let $\tilde{C} \rightarrow C$ be the universal covering of C which is a $\pi_1(C)$ -principal bundle. So $b_n^{(2)}(\tilde{C})$ is defined. We put

$$(5.20) \quad b_n^{(2)}(\tilde{X}) := \sum_{C \in \pi_0(X)} b_n^{(2)}(\tilde{C}).$$

We call X of *determinant class* if $C_*^{(2)}(\tilde{C})$ is of determinant class for every $C \in \pi_0(X)$. In this case we put

$$(5.21) \quad \rho^{(2)}(\tilde{X}) := \sum_{C \in \pi_0(X)} \rho^{(2)}(\tilde{C}).$$

We say that X is L^2 -acyclic if $b_n^{(2)}(\tilde{X})$ vanishes for all $n \in \mathbb{Z}_{\geq 0}$. We say that $\tilde{X}$ is $\det$ - L^2 -acyclic if $\tilde{X}$ is of determinant class and $b_n^{(2)}(\tilde{X})$ vanishes for all $n \in \mathbb{Z}_{\geq 0}$.

Exercise 5.22 For $n \in \mathbb{Z}_{\geq 1}$, let $X[n]$ be the finite 1-dimensional CW -complex, whose underlying topological space is $[0, 1]$ and whose 0-skeleton is $\{k \cdot n^{-1} \mid k = 0, 1, \dots, n\}$.

Show that $X[n]$ is of determinant class, but not L^2 -acyclic, and compute

$$\rho^{(2)}(\tilde{X}) = \sqrt{n+1}.$$

5.2.6 Basic Properties of L^2 -Torsion of Universal Coverings of Finite CW -Complexes

Here is a list of basic properties of L^2 -Betti numbers and L^2 -torsion of universal coverings of finite CW -complexes.

Theorem 5.23 (Basic properties of L^2 -torsion and L^2 -Betti numbers)

(i) (Simple) Homotopy invariance

Let $f: X \rightarrow Y$ be a homotopy equivalence of finite CW -complexes.

(a) Then

$$b_n^{(2)}(\tilde{X}) = b_n^{(2)}(\tilde{Y});$$

(b) Suppose that $\tilde{X}$ or $\tilde{Y}$ is $\det$ - L^2 -acyclic.

Then both $\tilde{X}$ and $\tilde{Y}$ are $\det$ - L^2 -acyclic;

(c) Suppose that f is a simple homotopy equivalence or that $\pi_1(X)$ satisfies the Determinant Conjecture 5.10. Assume that $\tilde{X}$ and $\tilde{Y}$ are $\det$ - L^2 -acyclic. Then

$$\rho^{(2)}(\tilde{Y}) = \rho^{(2)}(\tilde{X});$$

(ii) Euler-Poincaré formula

We get for a finite CW -complex X

$$\chi(X) = \sum_{n \in \mathbb{Z}_{\geq 0}} (-1)^n \cdot b_n^{(2)}(\tilde{X});$$

(iii) Sum formula

Consider the pushout of finite CW -complexes such that j_1 is an inclusion of CW -complexes, j_2 is cellular, and X inherits its CW -complex structure from X_0 , X_1 , and X_2

$$\begin{array}{ccc}
X_0 & \xrightarrow{j_1} & X_1 \\
j_2 \downarrow & & \downarrow i_1 \\
X_2 & \xrightarrow{i_2} & X.
\end{array}$$

Assume that for $k = 0, 1, 2$ the map $\pi_1(i_k, x): \pi_1(X_k, x) \rightarrow \pi_1(X, j_k(x))$ induced by the obvious map $i_k: X_k \rightarrow X$ is injective for all base points x in X_k . Then:

- (a) If $\widetilde{X}_0$, $\widetilde{X}_1$, and $\widetilde{X}_2$ are L^2 -acyclic, then $\widetilde{X}$ is L^2 -acyclic;
- (b) If $\widetilde{X}_0$, $\widetilde{X}_1$, and $\widetilde{X}_2$ are $\det$ - L^2 -acyclic, then $\widetilde{X}$ is $\det$ - L^2 -acyclic and we get

$$\rho^{(2)}(\widetilde{X}) = \rho^{(2)}(\widetilde{X}_1) + \rho^{(2)}(\widetilde{X}_2) - \rho^{(2)}(\widetilde{X}_0);$$

- (iv) Poincaré duality

Let M be a closed manifold of dimension d .

- (a) Then we get for $n \in \mathbb{Z}$.

$$b_n^{(2)}(\widetilde{M}) = b_{d-n}^{(2)}(\widetilde{M});$$

- (b) If d is even and $\widetilde{M}$ is $\det$ - L^2 -acyclic, then

$$\rho^{(2)}(\widetilde{M}) = 0;$$

- (v) Product formula

Let X and Y be finite CW-complexes.

- (a) Then

$$b_n^{(2)}(\widetilde{X \times Y}) = \sum_{i,j \in \mathbb{Z}_{\geq 0}, n=i+j} b_i^{(2)}(\widetilde{X}) \cdot b_j^{(2)}(\widetilde{Y});$$

- (b) If $\widetilde{X}$ is $\det$ - L^2 -acyclic, then $\widetilde{X \times Y}$ is $\det$ - L^2 -acyclic and

$$\rho^{(2)}(\widetilde{X \times Y}) = \chi(Y) \cdot \rho^{(2)}(\widetilde{X});$$

- (vi) Multiplicativity

Let $X \rightarrow Y$ be a finite covering of finite CW-complexes with d sheets.

- (a) Then

$$b_n(\widetilde{X}) = d \cdot b_n(\widetilde{Y});$$

- (b) Then $\widetilde{X}$ is $\det$ - L^2 -acyclic if and only if $\widetilde{Y}$ is $\det$ - L^2 -acyclic, and in this case

$$\rho^{(2)}(\widetilde{X}) = d \cdot \rho^{(2)}(\widetilde{Y});$$

- (vii) Determinant class.

- If $\pi_1(C)$ satisfies the Determinant Conjecture 5.10 for each component C of the finite CW-complex X , then $\tilde{X}$ is of determinant class;
- (viii) 0th L^2 -Betti number
- If X is a connected finite CW-complex with fundamental group π , then

$$b_0^{(2)}(\tilde{X}) = \begin{cases} \frac{1}{|\pi|} & \text{if } \pi \text{ is finite;} \\ 0 & \text{otherwise;} \end{cases}$$

(ix) Fibration formula

- (a) Let $p: E \rightarrow B$ a fibration such that B is a connected finite CW-complex and the fiber is homotopy equivalent to a finite CW-complex Z . Suppose that for every $b \in B$ and $x \in F_b := p^{-1}(b)$ the inclusion $p^{-1}(b) \rightarrow E$ induces an injection on the fundamental groups $\pi_1(F_b, x) \rightarrow \pi_1(E, x)$, and that Z is L^2 -acyclic. Then E is homotopy equivalent to a finite CW-complex X which is L^2 -acyclic;
- (b) Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a locally trivial fiber bundle of finite CW-complexes. (More generally one could consider a simple fibration $F \xrightarrow{i} E \xrightarrow{p} B$ of finite CW-complexes with a choice of a simple structure on F , see [45, page 164 and 165].) Suppose B is connected, that the map $\pi_1(F, x) \rightarrow \pi_1(E, i(x))$ is injective for every base point $x \in F$, and that $\tilde{F}$ is $\det$ - L^2 -acyclic. Then $\tilde{E}$ is $\det$ - L^2 -acyclic and

$$\rho^{(2)}(\tilde{E}) = \chi(B) \cdot \rho^{(2)}(\tilde{F});$$

(x) S^1 -actions

Let X be a connected compact S^1 -CW-complex, for instance a closed smooth manifold with smooth S^1 -action. Suppose that for one orbit S^1/H (and hence for all orbits) the inclusion into X induces a map on π_1 with infinite image. (In particular the S^1 -action has no fixed points.) Then $\tilde{X}$ is $\det$ - L^2 -acyclic and $\rho^{(2)}(\tilde{M})$ vanishes;

(xi) Aspherical spaces

- (a) Let M be an aspherical closed smooth manifold with a smooth S^1 -action. Then the conditions appearing in assertion (x) are satisfied and hence $\tilde{M}$ is $\det$ - L^2 -acyclic and $\rho^{(2)}(\tilde{X})$ vanishes;
- (b) If X is an aspherical finite CW-complex whose fundamental group contains an infinite, elementary amenable, and normal subgroup, then $\tilde{X}$ is $\det$ - L^2 -acyclic and $\rho^{(2)}(\tilde{X})$ vanishes;

(xii) Mapping tori

Let $f: X \rightarrow X$ be a self homotopy equivalence of a finite CW-complex. Denote by T_f its mapping torus.

- (a) Then $\widetilde{T}_f$ is L^2 -acyclic;
- (b) If $\widetilde{X}$ is $\det$ - L^2 -acyclic, then $\rho^{(2)}(\widetilde{T}_f)$ vanishes;

(xiii) Hyperbolic manifolds

Let M be a hyperbolic closed manifold M of dimension d .

- (a) If d is odd, $\widetilde{M}$ is $\det$ - L^2 -acyclic;
- (b) Suppose $d = 2m$ is even. Then $b_n^{(2)}(\widetilde{M})$ vanishes for $n \neq m$, and we have $(-1)^m \cdot \chi(M) = b_m^{(2)}(\widetilde{M}) > 0$;
- (c) For every number $m \in \mathbb{Z}_{\geq 0}$ there exists an explicit constant $C_m \in \mathbb{Q}_{>0}$ with the following property: If M is a hyperbolic closed manifold of dimension $(2m + 1)$ with volume $\text{vol}(M)$, then

$$\rho^{(2)}(\widetilde{M}) = (-1)^m \cdot \frac{C_m}{\pi^m} \cdot \text{vol}(M).$$

We have $C_1 = \frac{1}{6}$.

(xiv) 3-manifolds

Let M be a compact connected orientable prime 3-manifold with infinite fundamental group such that the boundary of M is empty or a disjoint union of incompressible tori. Let $M_1, M_2, \dots, M_r$ be the hyperbolic pieces in the JSJ-decomposition (They all have finite volume [54, Theorem B on page 52].)

Then $\widetilde{M}$ is $\det$ - L^2 -acyclic and

$$\rho^{(2)}(\widetilde{M}) = -\frac{1}{6\pi} \cdot \sum_{i=1}^r \text{vol}(M_i).$$

In particular, $\rho^{(2)}(\widetilde{M})$ is 0 if and only if M is $S^1 \times S^2$ or is a graph manifold.

Proof. (i) See [45, Theorem 1.35 (1) on page 37 and Theorem 3.96 (i) on page 163].

(ii) See [45, Theorem 1.35 (2) on page 37].

(iii) See [45, Theorem 3.96 (2) on page 164].

(iv) See [45, Theorem 1.35 (3) on page 37 and Theorem 3.96 (3) on page 164].

(v) See [45, Theorem 1.35 (4) on page 37 and Theorem 3.96 (4) on page 164].

(vi) See [45, Theorem 1.35 (9) on page 38 and Theorem 3.96 (3) on page 164].

(vii) This is obvious.

(viii) See [45, Theorem 1.35 (8) on page 38].

(ix) See [45, Lemma 1.41 on page 45 and Corollary 3.103 on page 166].

(x) See [45, Theorem 1.40 on page 43 and Theorem 3.105 on page 168].

(xi) See [45, Theorem 3.111 on page 171 and Theorem 3.113 on page 172].

(xii) See [45, Theorem 1.39 on page 42].

(xiii) See [23], [45, Theorem 1.39 on page 42].

(xiv) See [49, Theorem 0.6]. $\square$

Remark 5.24 (Analytic and topological version of L^2 -torsion). We have explained above the definition of L^2 -torsion in the topological setting which is due to Lück-Rothenberg [48]. The analytic setting goes back to Lott [42] and Mathai [52]. Burghleia, Friedlander, Kappeler and McDonald [8] have proved that the topological and analytic L^2 -torsion agree.

For more information about L^2 -torsion we refer for instance to [30, Chapter 6] and [45, Chapter 3].

Exercise 5.25 Show for a closed hyperbolic manifold M that every S^1 -action is trivial.

Exercise 5.26 Show for two closed hyperbolic manifolds of odd dimension with isomorphic fundamental groups that they have the same volume.

5.3 Universal L^2 -torsion

Let G be a (discrete) group.

5.3.1 Whitehead Torsion for Chain Complexes

We call two ordered $\mathbb{Z}G$ -basis B and B' of the same finitely generated free $\mathbb{Z}G$ -module M $\{\pm g\}$ -equivalent if there exists a bijection $\sigma: B \rightarrow B'$ which not necessarily respects the ordering and has the property that for every b in B we have $\sigma(b) = \epsilon(b) \cdot g(b) \cdot b$ for some $\epsilon(b) \in \{\pm 1\}$ and some $g(b) \in G$.

Definition 5.27 ($\{\pm g\}$ -based). We say that a finitely generated free $\mathbb{Z}G$ -module M is $\{\pm g\}$ -based if it is equipped with a $\{\pm g\}$ -equivalence class $[B]$ of ordered $\mathbb{Z}G$ -basis B .

A sequence $0 \rightarrow (L, [A]) \xrightarrow{i} (M, [B]) \xrightarrow{p} (L, [C]) \rightarrow 0$ of $\{\pm g\}$ -based finitely generated free $\mathbb{Z}G$ -modules is called $\{\pm g\}$ -based exact if it is exact and we can find a representative $\{b_1, b_2, \dots, b_n\}$ of $[B]$ and $m \in \{0, 1, \dots, n\}$ such that $b_i \in \text{im}(i)$ holds for $i = 1, 2, \dots, m$, $\{i^{-1}(b_1), \dots, i^{-1}(b_m)\}$ is a representative for $[A]$, and $\{p(b_{m+1}), \dots, p(b_n)\}$ is a representative for $[C]$. These notions carry over to finite free $\mathbb{Z}G$ -chain complexes degree-wise.

Let $C_* = (C_*, c_*)$ be a $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex which is contractible. Put

$$\begin{aligned} C_{\text{ev}} &= C_0 \oplus C_2 \oplus \dots; \\ C_{\text{odd}} &= C_1 \oplus C_3 \oplus \dots \end{aligned}$$

Choose a chain contraction $\gamma_*: C_* \rightarrow C_{*+1}$ for C_* . Then we get a homomorphism

$$(c + \gamma)_{\text{odd}}: C_{\text{odd}} \xrightarrow{\cong} C_{\text{ev}},$$

which turns out to be bijective, namely, an inverse is given by $(c + \gamma)_{\text{ev}}: C_{\text{ev}} \xrightarrow{\cong} C_{\text{odd}}$. Choose representatives for the $\{\pm g\}$ -equivalence class of ordered $\mathbb{Z}G$ -basis for C_{odd} and C_{ev} and let $\beta: C_{\text{ev}} \xrightarrow{\cong} C_{\text{odd}}$ be the associated ordered base preserving isomorphism. Then we get an element $[\beta \circ (c + \gamma)_{\text{odd}}] \in \text{Wh}(G)$ from the automorphism $\beta \circ (c + \gamma)_{\text{odd}}: C_{\text{odd}} \xrightarrow{\cong} C_{\text{odd}}$ of the finitely generated free $\mathbb{Z}G$ -module C_{odd} . We define the *Reidemeister torsion* of C_* to be

$$(5.28) \quad \rho(C_*) = [\beta \circ (c + \gamma)_{\text{odd}}] \in \text{Wh}(G).$$

This element turns out to be independent of the choices of γ and the representatives of the $\{\pm g\}$ -equivalence class of basis.

Let $f_*: C_* \xrightarrow{\cong} D_*$ be a $\mathbb{Z}G$ -chain homotopy equivalence of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes. Then the *mapping cone* $\text{cone}(f_*)$ is a $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex which is contractible. Its n -th-differential is

$$\begin{pmatrix} -c_{n-1} & 0 \\ f_{n-1} & d_n \end{pmatrix}: C_{n-1} \oplus D_n \rightarrow C_{n-1} \oplus D_{n-1}.$$

We define the *Whitehead torsion* of f_* to be

$$(5.29) \quad \tau(f_*) = \rho(\text{cone}(f_*)_*) \in \text{Wh}(G).$$

We call a $\mathbb{Z}G$ -chain homotopy equivalence of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes $f_*: C_* \xrightarrow{\cong} D_*$ *simple* if $\tau(f_*) = 0$ holds.

The proof of the next theorem can be found for instance in [11, Section 17], [44, Lemma 2.9], or [47, Lemma 3.18].

Theorem 5.30 (Basic properties of Whitehead torsion for chain complexes)

- (i) Let $f_*, g_*: C_* \xrightarrow{\cong} D_*$ be $\mathbb{Z}G$ -chain homotopy equivalences of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes. Suppose that f_* and g_* are chain homotopic.
Then we get $\tau(f_*) = \tau(g_*)$;
- (ii) Consider the commutative diagram of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_* & \xrightarrow{i_*} & C'_* & \xrightarrow{p_*} & C''_* \longrightarrow 0 \\ & & \downarrow f_* \simeq & & \downarrow f'_* \simeq & & \downarrow f''_* \simeq \\ 0 & \longrightarrow & D_* & \xrightarrow{j_*} & D'_* & \xrightarrow{q_*} & C'''_* \longrightarrow 0 \end{array}$$

whose rows are $\{\pm g\}$ -based exact and whose vertical arrows are $\mathbb{Z}G$ -chain homotopy equivalences.

Then we get $\tau(f_*) - \tau(f'_*) + \tau(f''_*) = 0$;

- (iii) Let $f_*: C_* \xrightarrow{\sim} D_*$ and $g_*: D_* \xrightarrow{\sim} E_*$ be $\mathbb{Z}G$ -chain homotopy equivalences of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes.

Then we get $\tau(g_* \circ f_*) = \tau(f_*) + \tau(g_*)$.

5.3.2 The Weak Whitehead Group

The universal L^2 -torsion, which we will define in Section 5.3.3, will take values in the weak Whitehead $\text{Wh}^w(G)$, which we define next.

Recall that a morphism $f: V \rightarrow W$ of finite Hilbert $\mathcal{N}(G)$ -modules is called a *weak isomorphism* if it is injective and has dense image.

Given a $\mathbb{Z}G$ -homomorphism $u: \mathbb{Z}G^m \rightarrow \mathbb{Z}G^n$, we obtain an induced morphism of finitely generated Hilbert $\mathcal{N}(G)$ -modules

$$u^{(2)} = \text{id}_{L^2(G)} \otimes_{\mathbb{Z}G} u: L^2(G)^m = L^2(G) \otimes_{\mathbb{Z}G} \mathbb{Z}G^m \rightarrow L^2(G)^n = L^2(G) \otimes_{\mathbb{Z}G} \mathbb{Z}G^n.$$

We call u a *weak isomorphism* if $u^{(2)}$ is a weak isomorphism of finitely generated Hilbert $\mathcal{N}(G)$ -modules. Let

$$(5.31) \quad u^*: \mathbb{Z}G^n \rightarrow \mathbb{Z}G^m$$

be the $\mathbb{Z}G$ -homomorphism given by the right multiplication with the (n, m) matrix $A^* = (a_{j,i})_{i,j}$ over $\mathbb{Z}G$ if u is given by right multiplication with the (m, n) -matrix $A = (\overline{a_{k,l}})_{k,l}$ over $\mathbb{Z}G$, where $\overline{\sum_{g \in G} \lambda_g \cdot g}$ is defined to be $\sum_{g \in G} \lambda_g \cdot g^{-1}$.

Definition 5.32 ($K_1^w(\mathbb{Z}G)$). Define the *weak K_1 -group*

$$K_1^w(\mathbb{Z}G)$$

to be the abelian group defined in terms of generators and relations as follows.

Generators $[f]$ are given by $\mathbb{Z}G$ -endomorphisms $f: \mathbb{Z}G^n \rightarrow \mathbb{Z}G^n$ which are weak isomorphisms.

If $f_1, f_2: \mathbb{Z}G^n \rightarrow \mathbb{Z}G^n$ are $\mathbb{Z}G$ -endomorphisms which are weak isomorphisms, then we require the relation

$$[f_2 \circ f_1] = [f_1] + [f_2].$$

If $f_0: \mathbb{Z}G^m \rightarrow \mathbb{Z}G^m$, $f_2: \mathbb{Z}G^n \rightarrow \mathbb{Z}G^n$, and $f_1: \mathbb{Z}G^n \rightarrow \mathbb{Z}G^m$ are $\mathbb{Z}G$ -maps such that f_0 and f_2 are weak isomorphisms, then we get for the $\mathbb{Z}G$ -map

$$f = \begin{pmatrix} f_0 & f_1 \\ 0 & f_2 \end{pmatrix}: \mathbb{Z}G^{m+n} = \mathbb{Z}G^m \oplus \mathbb{Z}G^n \rightarrow \mathbb{Z}G^m \oplus \mathbb{Z}G^n$$

the relation

$$[f] = [f_0] + [f_2].$$

Let

$$\tilde{K}_1^w(\mathbb{Z}G)$$

be the quotient of $K_1^w(\mathbb{Z}G)$ by the subgroup generated by the element $[-\text{id}: \mathbb{Z}G \rightarrow \mathbb{Z}G]$. This is the same as the cokernel of the obvious composite $K_1(\mathbb{Z}) \rightarrow K_1^w(\mathbb{Z}G)$.

Define the *weak Whitehead group* of G

$$\text{Wh}^w(G)$$

to be the cokernel of the homomorphism

$$\{\sigma \cdot g \mid \sigma \in \{\pm 1\}, g \in G\} \rightarrow K_1^w(\mathbb{Z}G), \quad \sigma \cdot g \mapsto [r_{\sigma \cdot g}: \mathbb{Z}G \xrightarrow{\cong} \mathbb{Z}G].$$

Definition 5.32 makes sense since the $\mathbb{Z}G$ -homomorphisms $f_2 \circ f_2$ and f appearing above are again weak isomorphisms by [45, Lemma 3.37 on page 144].

Remark 5.33. We obtain the classical notions of $K_1(\mathbb{Z}G)$, $\tilde{K}_1(\mathbb{Z}G)$, and $\text{Wh}(G)$ if we replace in Definition 5.32 everywhere the condition that f is a weak isomorphism by the stronger condition that f is an isomorphism.

Remark 5.34 (Functoriality of $\text{Wh}^w(G)$). There is an important difference for the functoriality of $\text{Wh}(G)$ and $\text{Wh}^w(G)$. Every group homomorphism $f: H \rightarrow G$ induces a homomorphism $f_*: \text{Wh}(H) \rightarrow \text{Wh}(G)$. But in order to get a group homomorphism $f_*: \text{Wh}^w(H) \rightarrow \text{Wh}^w(G)$, f has to be injective.

The problem is that for a $\mathbb{Z}H$ -homomorphism $u: \mathbb{Z}H^a \rightarrow \mathbb{Z}H^a$ which is a weak isomorphism we must ensure that $f_*u: \mathbb{Z}G^a \rightarrow \mathbb{Z}G^a$ is again a weak isomorphism. Consider for instance the case $H = \mathbb{Z}$, $G = \{e\}$, and the $\mathbb{Z}[\mathbb{Z}]$ -homomorphism $u = r_{z-1}: \mathbb{Z}[\mathbb{Z}] \rightarrow \mathbb{Z}[\mathbb{Z}]$ given by right multiplication with $(z - 1)$ for $z \in \mathbb{Z}$ a generator. Then u is a weak isomorphism, whereas $f_*u: \mathbb{Z} \rightarrow \mathbb{Z}$ is the zero map and hence not a weak isomorphism.

5.3.3 Universal L^2 -Torsion for Chain Complexes

We want to carry over the notion of Reidemeister torsion to a $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes C_* which is L^2 -acyclic.

Note that in the sequel $C_*^{(2)}$ is only defined after we have chosen an ordered $\mathbb{Z}G$ -basis for each chain module C_n representing the $\{\pm g\}$ -equivalence class, but this does not matter since the isometric G -isomorphism type of the finite $\mathcal{N}(G)$ -Hilbert chain $C_*^{(2)}$ depends only on the $\{\pm g\}$ -equivalence class. In particular the property that for a $\mathbb{Z}G$ -homomorphism $u: M \rightarrow N$ of $\{\pm g\}$ -based finite generated free $\mathbb{Z}G$ -modules $u^{(2)}$ is a weak isomorphism and the property that a $\{\pm\}$ -based finite free $\mathbb{Z}G$ -chain complex C_* is L^2 -acyclic make sense.

Recall that C_* is L^2 -acyclic if and only if

$$b_n^{(2)}(C_*) = \dim_{\mathcal{N}(G)}(H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*)) = \dim_{\mathcal{N}(G)}(H_n^{(2)}(C_*^{(2)})),$$

or, equivalently, $H_n^{(2)}(C_*^{(2)})$ vanishes for every $n \in \mathbb{Z}_{\geq 0}$.

Definition 5.35 (Weak chain contraction). Consider a $\mathbb{Z}G$ -chain complex C_* . A *weak chain contraction* (γ_*, u_*) for C_* consists of a $\mathbb{Z}G$ -chain map $u_*: C_* \rightarrow C_*$ and a $\mathbb{Z}G$ -chain homotopy $\gamma_*: u_* \simeq 0_*$ such that $u_n: C_n \rightarrow C_n$ is a weak isomorphism for all $n \in \mathbb{Z}$ and $\gamma_n \circ u_n = u_{n+1} \circ \gamma_n$ holds for all $n \in \mathbb{Z}$.

Let C_* be a based finite free $\mathbb{Z}G$ -chain complex, where based means that each chain module comes with an ordered $\mathbb{Z}G$ -basis. Denote by $\Delta_n: C_n \rightarrow C_n$ its n -th *combinatorial Laplace operator* which is the $\mathbb{Z}G$ -map

$$\Delta_n = c_{n+1} \circ c_{n+1}^* + c_n^* \circ c_n: C_n \rightarrow C_n,$$

where c_n^* has been defined in (5.31). It has the property that the morphism of finitely generated Hilbert $\mathcal{N}(G)$ -modules $\Delta_n^{(2)}: C_n^{(2)} \rightarrow C_n^{(2)}$ is the Laplace operator of the finite Hilbert $\mathcal{N}(G)$ -chain complex $C_*^{(2)}$ defined in (5.13).

Lemma 5.36. *Let C_* be a based finite free $\mathbb{Z}G$ -chain complex. Then the following assertions are equivalent:*

- (i) $\Delta_n: C_n \rightarrow C_n$ is a weak isomorphism for all $n \in \mathbb{Z}$;
- (ii) There exists a weak chain contraction (γ_*, u_*) with $\gamma_n \circ \gamma_{n-1} = 0$ for all $n \in \mathbb{Z}$;
- (iii) There exists a weak chain contraction (γ_*, u_*) ;
- (iv) C_* is L^2 -acyclic.

Proof. This follows from [19, Lemma 2.5]. □

Remark 5.37 (Homomorphisms of $\{\pm g\}$ -based finitely generated free $\mathbb{Z}G$ -modules and $\text{Wh}^w(G)$).

Let $f: M \rightarrow N$ be a $\mathbb{Z}G$ -homomorphism of $\{\pm g\}$ -based finitely generated free $\mathbb{Z}G$ -modules which is a weak isomorphism. Choose ordered $\mathbb{Z}G$ -basis for M and N representing the $\{\pm g\}$ -equivalence classes. Then we conclude from [45, Lemma 1.13 on page 23]

$$\text{rk}_{\mathbb{Z}G}(M) = \dim_{\mathcal{N}(G)}(M^{(2)}) = \dim_{\mathcal{N}(G)}(N^{(2)}) = \text{rk}_{\mathbb{Z}G}(N).$$

Hence we can choose an order preserving bijection between the two ordered $\mathbb{Z}G$ -bases for M and N . It induces a $\mathbb{Z}G$ -isomorphism $b: N \rightarrow M$. Now define an element

$$[f] := [b \circ f] \in \text{Wh}^w(G).$$

Since we are working in $\text{Wh}^w(G)$, the choice of the representatives of the $\{\pm g\}$ -equivalence classes does not matter.

Definition 5.38 (Universal L^2 -torsion for L^2 -acyclic $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes).

Let C_* be a $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex which is L^2 -acyclic. Define its *universal L^2 -torsion*

$$\rho_u^{(2)}(C_*) \in \text{Wh}^w(G)$$

by

$$\rho_u^{(2)}(C_*) = [(uc + \gamma)_{\text{odd}}] - [u_{\text{odd}}],$$

where (γ_*, u_*) is any weak chain contraction of C_* .

The proof that Definition 5.38 makes sense can be found in [19, Section 2.3].

Lemma 5.39. *Let $0 \rightarrow C_* \xrightarrow{i_*} D_* \xrightarrow{p_*} E_* \rightarrow 0$ be a $\{\pm g\}$ -based exact short sequence of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes. Suppose that two of them are L^2 -acyclic.*

Then all three are L^2 -acyclic and we get in $\text{Wh}^w(G)$

$$\rho_u^{(2)}(D_*) = \rho_u^{(2)}(C_*) + \rho_u^{(2)}(E_*).$$

Proof. This follows from [19, Lemma 2.9]. $\square$

Lemma 5.40. *Let $f: C_* \rightarrow D_*$ be a $\mathbb{Z}G$ -chain homotopy equivalence of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes. Suppose that C_* or D_* is L^2 -acyclic.*

Then both C_* and D_* are L^2 -acyclic, and we get in $\text{Wh}^w(G)$

$$\rho_u^{(2)}(D_*) - \rho_u^{(2)}(C_*) = \zeta(\tau(f_*))$$

for the canonical homomorphism $\zeta: \text{Wh}(G) \rightarrow \text{Wh}^w(G)$.

Proof. This follows from [19, Lemma 2.10]. $\square$

Note that we get an involution

$$(5.41) \quad *: \text{Wh}^w(G) \rightarrow \text{Wh}^w(G)$$

by taking the dual in the sense of (5.31).

Lemma 5.42. *Let C_* be a $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex such that $C_*^{(2)}$ is L^2 -acyclic. Choose an ordered $\mathbb{Z}G$ -basis for each C_n representing the $\{\pm g\}$ -equivalence class.*

Then for $n \in \mathbb{Z}$ the $\mathbb{Z}G$ -endomorphism given by the Laplace operator $\Delta_n: C_n \rightarrow C_n$ is a weak isomorphism, and we get in $\text{Wh}(G)$

$$\rho_u^{(2)}(C_*) + *(\rho_u^{(2)}(C_*)) = - \sum_{n \geq 0} (-1)^n \cdot n \cdot [\Delta_n].$$

Proof. This follows from [19, Lemma 2.17]. $\square$

Exercise 5.43 Suppose that the group G satisfies the Determinant Conjecture 5.10. Show:

- (i) We obtain a well-defined group homomorphism

$$\det_{\mathcal{N}(G)}: \text{Wh}^w(G) \rightarrow \mathbb{R}, \quad [f: \mathbb{Z}G^n \rightarrow \mathbb{Z}G^n] \mapsto \det_{\mathcal{N}(G)}(f^{(2)});$$

- (ii) Let X be a connected finite CW -complex with $G = \pi_1(X)$ which is L^2 -acyclic.

Then X is $\det$ - L^2 -acyclic and $\det_{\mathcal{N}(G)}: \text{Wh}^w(G) \rightarrow \mathbb{R}$ sends the universal L^2 -torsion $\rho_u^{(2)}(\tilde{X})$ to the L^2 -torsion $\rho^{(2)}(\tilde{X})$.

5.3.4 (Universal) Additive L^2 -Torsion Invariants

Next we explain the universal properties of the notions Whitehead torsion, Reidemeister torsion, and universal L^2 -torsion above. The material of this section is not needed for the rest of this chapter, but is quite illuminating.

Let M be a $\{\pm g\}$ -based finitely generated free $\mathbb{Z}G$ -module. For $n \in \mathbb{Z}_{\geq 0}$ we denote by

$$n[M]_*$$

the $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex concentrated in dimension n whose n -th chain module is M .

For $n \in \mathbb{Z}_{\geq 1}$ the *elementary $\mathbb{Z}G$ -chain complex* $\text{el}[n, M]_*$ is defined to be the contractible $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex concentrated in dimensions n and $(n-1)$ whose n -th differential is $\text{id}_M: M \rightarrow M$.

A $\mathbb{Z}G$ -homomorphism $f: M \rightarrow N$ induces $\mathbb{Z}G$ -chain maps $n[f]_*: n[M]_* \rightarrow n[N]_*$ and $\text{el}[n, f]_*: \text{el}[n, M]_* \rightarrow \text{el}[n, N]_*$ in the obvious way.

In the sequel we equip $\mathbb{Z}G$ with the $\{\pm g\}$ -equivalence class of basis represented by the $\mathbb{Z}G$ -basis $\{e\}$ for the unit $e \in G$.

Definition 5.44 ((Universal) additive L^2 -torsion invariant). An *additive L^2 -torsion invariant* $(A_u^{(2)}, a_u^{(2)})$ consists of an abelian group $A_u^{(2)}$ and an assignment which associates to every L^2 -acyclic $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex C_* an element

$$a_u^{(2)}(C_*) \in A_u^{(2)}$$

such that the following holds:

- For every short $\{\pm g\}$ -based exact sequence of such $\mathbb{Z}G$ -chain complexes $0 \rightarrow C_* \rightarrow D_* \rightarrow E_* \rightarrow 0$ we get

$$a_u^{(2)}(D_*) = a_u^{(2)}(C_*) + a_u^{(2)}(E_*);$$

- We have for every $n \in \mathbb{Z}_{\geq 1}$

$$a_u^{(2)}(\text{el}[n, \mathbb{Z}G]_*) = 0.$$

We call an additive L^2 -torsion invariant $(U_u^{(2)}, u_u^{(2)})$ *universal* if for every additive L^2 -torsion invariant $(A_u^{(2)}, a_u^{(2)})$ there is precisely one group homomorphism

$$\mu_u^{(2)}: U_u^{(2)} \rightarrow A_u^{(2)}$$

such that for every L^2 -acyclic $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex C_* we have $\mu_u^{(2)}(u_u^{(2)}(C_*)) = a_u^{(2)}(C_*)$.

Definition 5.45 ((Universal) additive Reidemeister torsion invariant). An *additive Reidemeister torsion invariant* (A^R, a^R) consists of an abelian group A^R and an assignment which associates to every contractible $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex C_* an element

$$a^R(C_*) \in A^R$$

such that the following holds:

- For every short $\{\pm g\}$ -based exact sequence of such $\mathbb{Z}G$ -chain complexes $0 \rightarrow C_* \rightarrow D_* \rightarrow E_* \rightarrow 0$ we get

$$a^R(D_*) = a^R(C_*) + a^R(E_*);$$

- We have for every $n \in \mathbb{Z}_{\geq 1}$

$$a_u^{(2)}(\text{el}[n, \mathbb{Z}G]_*) = 0.$$

We call an additive Reidemeister torsion invariant (U^R, u^R) *universal* if for every additive Reidemeister torsion invariant (A^R, a^R) there is precisely one group homomorphism

$$\mu^R: U^R \rightarrow A^R$$

such that for every contractible $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex C_* we have $\mu^R(u^R(C_*)) = a^R(C_*)$.

Definition 5.46 ((Universal) additive Whitehead torsion invariant). An *additive Whitehead torsion invariant* (A^W, a^W) consists of an abelian group A^W and an assignment which associates to every $\mathbb{Z}G$ -chain homotopy equivalence $f_*: C_* \rightarrow D_*$ of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes an element

$$a^W(f_*) \in A^W$$

such that the following holds:

- For every commutative diagram of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes

$$\begin{array}{ccccccc}
0 & \longrightarrow & C_* & \xrightarrow{i_*} & C'_* & \xrightarrow{p_*} & C''_* \longrightarrow 0 \\
& & \downarrow f_* \simeq & & \downarrow f'_* \simeq & & \downarrow f''_* \simeq \\
0 & \longrightarrow & D_* & \xrightarrow{j_*} & D'_* & \xrightarrow{q_*} & C''_* \longrightarrow 0
\end{array}$$

whose rows are $\{\pm g\}$ -based exact and whose vertical arrows are $\mathbb{Z}G$ -chain homotopy equivalences, we get

$$a^W(f_*) - a^W(f'_*) + a^W(f''_*) = 0;$$

- We have for every $n \in \mathbb{Z}_{\geq 1}$

$$a^W(0_* \rightarrow \text{el}[n, \mathbb{Z}G]_*) = 0;$$

- We have for every $n \in \mathbb{Z}_{\geq 0}$

$$a^W(\text{id}_{n[\mathbb{Z}G]_*}) = 0.$$

We call an additive Whitehead torsion invariant (U^W, u^W) *universal* if for every additive Whitehead torsion invariant (A^W, a^W) there is precisely one group homomorphism

$$\mu^W: U^W \rightarrow A^W$$

such that for every $\mathbb{Z}G$ -chain homotopy equivalence $f_*: C_* \rightarrow D_*$ of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex we have $\mu^W(u^W(C_*)) = a^W(C_*)$.

Remark 5.47 (Unique existence of universal additive torsion invariants). Obviously the universal additive L^2 -torsion invariant $(U_u^{(2)}, u_u^{(2)})$ exists and is unique up to unique isomorphism. It is construct in terms of generators and relations in the obvious way, where every L^2 -acyclic $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex C_* defines a generator $[C_*]$. The analogous statement holds for the universal additive Reidemeister torsion invariant (U^R, u^R) and the universal additive Whitehead torsion invariant (U^W, u^W) .

Exercise 5.48 Let (U^{mW}, u^{mW}) be the universal additive modified Whitehead torsion invariant, which is analogously defined as (U^W, u^W) but where the condition $a^W(\text{id}_{n[\mathbb{Z}G]_*}) = 0$ is replaced by the weaker condition that we have for every $n \in \mathbb{Z}_{\geq 0}$

$$a^W(\text{id}_{n[\mathbb{Z}G]_*}) = (-1)^n \cdot a^W(\text{id}_{n[\mathbb{Z}G]_*}).$$

Show that we get a well-defined isomorphism

$$\nu: U^{mW} \xrightarrow{\cong} U^W \times \mathbb{Z}$$

which is uniquely determined by the property that for a every $\mathbb{Z}G$ -chain homotopy equivalence $f_*: C_* \rightarrow D_*$ of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes we have $\nu(u^{mW}(f_*)) = (u^W(f_*), \chi_{\mathbb{Z}G}(C_*))$.

Exercise 5.49 Define for a $\mathbb{Z}G$ -chain homotopy equivalence $f_*: C_* \rightarrow D_*$ of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes the invariant

$$b(f_*) = \sum_{n \geq 0} \text{rk}_{\mathbb{Z}G}(D_n) - \sum_{n \geq 0} \text{rk}_{\mathbb{Z}G}(C_n) \in \mathbb{Z}.$$

Show that the pair $(\mathbb{Z}, b)$ satisfies all the axioms of an additive Whitehead torsion invariant appearing in Definition 5.46 except the one demanding $a^W(0_* \rightarrow \text{el}[n, \mathbb{Z}G]_*) = 0$ for every $n \in \mathbb{Z}_{\geq 1}$.

Lemma 5.50. *Let $(A, a) = (A^W, a^W)$ be an additive Whitehead torsion invariant.*

- (i) *Consider $\mathbb{Z}[G]$ -chain homotopy equivalences $f_*: C_* \xrightarrow{\sim} D_*$ and $f'_*: C'_* \xrightarrow{\sim} D'_*$ of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes.
Then we get $a(f_* \oplus f'_*) = a(f_*) + a(f'_*)$;*
- (ii) *Consider a $\mathbb{Z}[G]$ -chain homotopy equivalence $f_*: C_* \xrightarrow{\sim} D_*$ of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes.
Then we get $a(f_*) = a(0_* \xrightarrow{\sim} \text{cone}(f_*)_*)$;*
- (iii) *Consider a $\mathbb{Z}[G]$ -chain homotopy equivalence $f_*: C_* \xrightarrow{\sim} D_*$ of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes. Let $\Sigma f_*: \Sigma_* C_* \xrightarrow{\sim} \Sigma D_*$ be its suspension.
Then we get $a(\Sigma f_*) = -a(f_*)$;*
- (iv) *Consider $\mathbb{Z}[G]$ -chain homotopy equivalences $f_*: C_* \xrightarrow{\sim} D_*$ and $g_*: D_* \xrightarrow{\sim} E_*$ of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes.
Then we get $a(g_* \circ f_*) = a(f_*) + a(g_*)$;*
- (v) *Consider a simple $\mathbb{Z}[G]$ -chain homotopy equivalences $f_*: C_* \xrightarrow{\sim_s} D_*$ of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes.
Then we get $a(f_*) = 0$;*
- (vi) *Consider $\mathbb{Z}[G]$ -chain homotopy equivalences $f_*, g_*: C_* \xrightarrow{\sim} D_*$ of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes. Suppose that f_* and g_* are $\mathbb{Z}G$ -chain homotopic.
Then we get $a(g_*) = a(f_*)$.*

Proof. (i) This is obviously true.

(ii) We first show for every $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex C_*

$$(5.51) \quad a(\text{id}_{C_*}) = 0.$$

By induction over the dimension of C_* one shows $a(\text{id}_{C_*}) = \sum_{n=0}^{\infty} a(\text{id}_{[C_n]_*})$. Now one computes using assertion (i)

$$\begin{aligned}
a(\text{id}_{C_*}) &= \sum_{n=0}^{\infty} a(\text{id}_{n[C_n]_*}) \\
&= \sum_{n=0}^{\infty} \text{rk}_{\mathbb{Z}G}(C_n) \cdot a(\text{id}_{n[\mathbb{Z}G]_*}) \\
&= \sum_{n=0}^{\infty} \text{rk}_{\mathbb{Z}G}(C_n) \cdot 0 \\
&= 0.
\end{aligned}$$

Next we show for every contractible $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex E_*

$$(5.52) \quad a(E_* \rightarrow 0_*) = -a(0_* \rightarrow E_*).$$

This follows from (5.51) and the commutative diagram of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes with $\{\pm g\}$ -based exact rows and $\mathbb{Z}G$ -chain homotopy equivalences as vertical arrows

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0_* & \longrightarrow & E_* & \xrightarrow{\text{id}_{E_*}} & E_* \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & E_* & \xrightarrow{\text{id}_{E_*}} & E_* & \longrightarrow & 0_* \longrightarrow 0
\end{array}$$

Consider a $\mathbb{Z}[G]$ -chain homotopy equivalence $f_*: C_* \xrightarrow{\sim} D_*$ of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes. We obtain a commutative diagram of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes

$$\begin{array}{ccccccc}
0 & \longrightarrow & C_* & \longrightarrow & \text{cone}(C_*)_* & \longrightarrow & \Sigma C_* \longrightarrow 0 \\
& & f_* \downarrow \simeq & & \widetilde{f}_* \downarrow \simeq & & \Sigma \text{id}_{C_*} \downarrow \simeq \\
0 & \longrightarrow & D_* & \longrightarrow & \text{cone}(f_*)_* & \longrightarrow & \Sigma C_* \longrightarrow 0
\end{array}$$

where $\widetilde{f}_n = \text{id}_{C_{n-1}} \oplus f_n: C_{n-1} \oplus C_n \rightarrow C_{n-1} \oplus D_n$, the rows are $\{\pm g\}$ -based exact, and the vertical arrows are $\mathbb{Z}G$ -chain homotopy equivalences. We conclude

$$a(f_*) - a(\widetilde{f}_*) + a(\Sigma \text{id}_{C_*}) = 0.$$

We get $a(\Sigma \text{id}_{C_*}) = 0$ from (5.51) since $\Sigma \text{id}_{C_*} = \text{id}_{C'_*}$ for a $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex C'_* . Hence it suffices to show

$$(5.53) \quad a(\widetilde{f}_*) = a(0_* \rightarrow \text{cone}(f_*)_*).$$

For $d = -1, 0, 1, \dots, \dim(C_*)$ let $C_*|_d$ be the d -dimensional $\mathbb{Z}G$ -subchain complex of C_* whose n -th chain module is 0 for $n \geq (d+1)$ and C_n for $n \leq d$. Note that $C_*|_{-1} = 0_*$ and $C_*|_{\dim(C_*)} = C_*$ holds. Let $i_*^k: C_*|_k \rightarrow C_*$ be the inclusion. It suffices to show by induction over $d = -1, 0, 1, \dots, \dim(C_*)$

$$a(\widehat{f}_* \circ i_*^d) = a(0_* \rightarrow \text{cone}(f_*)_*) .$$

The induction beginning $d = -1$ is obvious. The induction step from $(d-1)$ to d follows from the commutative diagram of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes with $\{\pm g\}$ -based exact rows and $\mathbb{Z}G$ -chain homotopy equivalences as vertical arrows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{cone}(C_*|_{d-1})_* & \longrightarrow & \text{cone}(C_*|_d)_* & \longrightarrow & \text{el}[(d+1), c_d]_* \longrightarrow 0 \\ & & \widehat{f}_* \circ i_*^{(d-1)} \downarrow \simeq & & \widehat{f}_* \circ i_*^d \downarrow \simeq & & 0_* \downarrow \simeq \\ 0 & \longrightarrow & \text{cone}(f_*)_* & \xrightarrow{\text{id}} & \text{cone}(f_*)_* & \longrightarrow & 0_* \longrightarrow 0. \end{array}$$

Namely, we conclude using (5.52), assertion (i), and the induction hypothesis

$$\begin{aligned} a(\widehat{f}_* \circ i_*^d) &= a(\widehat{f}_* \circ i_*^{(d-1)}) + a(\text{el}[(d+1), c_d]_* \rightarrow 0_*) \\ &= a(\widehat{f}_* \circ i_*^{(d-1)}) + \text{rk}_{\mathbb{Z}G}(C_d) \cdot a(0_* \rightarrow \text{el}[(d+1), \mathbb{Z}G]_*) \\ &= 0 + \text{rk}_{\mathbb{Z}G}(C_d) \cdot 0 \\ &= 0. \end{aligned}$$

This finishes the proof of (5.53) and hence of assertion(ii).

(iii) We first show for any $\mathbb{Z}G$ -chain map $f_*: C_* \rightarrow D_*$ of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes that the induced map $\widehat{f}_*: \text{cone}(C_*)_* \rightarrow \text{cone}(D_*)_*$ satisfies

$$(5.54) \quad a(\widehat{f}_*) = 0.$$

We use induction over $d = -1, 0, 1, 2, \dots$ for which $\dim(C_*) \leq d$ and $\dim(D_*) \leq d$ holds. Note that $\dim(C_*) = -1$ means $C_* = 0_*$.

The induction beginning $d = -1$ is trivial. The induction step from $(d-1)$ to $d \geq 0$ is done as follows.

Let $C_*|_{(d-1)}$ be the $(d-1)$ -dimensional $\mathbb{Z}G$ -subchain complex of C_* which has as n th chain module $\{0\}$ for $n \geq d$ and C_k for $n \leq (d-1)$. Let $f_*|_{(d-1)}: C_*|_{(d-1)} \rightarrow D_*|_{(d-1)}$ the obvious $\mathbb{Z}G$ -chain map induced by f_* . Then we obtain a commutative diagram of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes with $\{\pm g\}$ -based exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{cone}(C_*|_{(d-1)})_* & \longrightarrow & \text{cone}(C_*)_* & \longrightarrow & \text{el}[(d+1), C_d]_* \longrightarrow 0 \\
& & \widehat{f_*|_{(d-1)}} \downarrow \simeq & & \widehat{f_*} \downarrow \simeq & & \text{el}[(d+1), f_d]_* \downarrow \simeq \\
0 & \longrightarrow & \text{cone}(D_*|_{(d-1)})_* & \longrightarrow & \text{cone}(D_*)_* & \longrightarrow & \text{el}[(d+1), D_d]_* \longrightarrow 0.
\end{array}$$

This implies $a(\widehat{f_*}) = a(\widehat{f_*|_{(d-1)}}) + a(\text{el}[(d+1), f_d]_*)$. Since $a(\widehat{f_*|_{(d-1)}}) = 0$ holds by induction hypothesis it remains to show

$$a(\text{el}[(d+1), f_d]_*) = 0.$$

From assertions (i) and (ii) and the short $\{\pm g\}$ -based exact sequence of contractible $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes

$$0 \rightarrow \text{el}[(d+1), \mathbb{Z}G^b]_* \rightarrow \text{cone}(\text{el}[(d+1), f])_* \rightarrow \text{el}[(d+2), \mathbb{Z}G^a]_* \rightarrow 0,$$

we conclude

$$\begin{aligned}
a(\text{el}[(d+1), f_d]_*) &= a(0_* \rightarrow \text{cone}(\text{el}[(d+1), f])_*) \\
&= a(0_* \rightarrow \text{el}[(d+1), \mathbb{Z}G^b]_*) + a(0_* \rightarrow \text{el}[(d+2), \mathbb{Z}G^a]_*) \\
&= b \cdot a(0_* \rightarrow \text{el}[(d+1), \mathbb{Z}G]_*) + a \cdot a(0_* \rightarrow \text{el}[(d+2), \mathbb{Z}G]_*) \\
&= b \cdot 0 + a \cdot 0 \\
&= 0.
\end{aligned}$$

This finishes the proof of (5.54).

Consider a $\mathbb{Z}[G]$ -chain homotopy equivalence $f_*: C_* \xrightarrow{\simeq} D_*$ of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes. Then we obtain a commutative diagram of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes

$$\begin{array}{ccccccc}
0 & \longrightarrow & C_* & \longrightarrow & \text{cone}(C_*)_* & \longrightarrow & \Sigma C_* \longrightarrow 0 \\
& & f_* \downarrow \simeq & & \widehat{f_*} \downarrow \simeq & & \Sigma f_* \downarrow \simeq \\
0 & \longrightarrow & D_* & \longrightarrow & \text{cone}(D_*)_* & \longrightarrow & \Sigma D_* \longrightarrow 0
\end{array}$$

where the rows are $\{\pm g\}$ -based exact and the vertical arrows are $\mathbb{Z}G$ -chain homotopy equivalences. Since this implies

$$a(f_*) - a(\widehat{f_*}) + a(\Sigma f_*) = 0,$$

assertion (iii) follows from (5.54).

(iv) Consider $\mathbb{Z}G$ -chain homotopy equivalences $f_*: C_* \xrightarrow{\simeq} D_*$ and $g_*: D_* \xrightarrow{\simeq} E_*$ of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes. Define a $\mathbb{Z}G$ -chain map

$$h_*: \Sigma^{-1} \text{cone}(v_*)_* \rightarrow \text{cone}(u_*)_*$$

by $h_n = \begin{pmatrix} 0 & 0 \\ -\text{id}_{D_n} & 0 \end{pmatrix} : D_n \oplus E_{n+1} \rightarrow C_{n-1} \oplus D_n$. Then we obtain two short $\{\pm g\}$ -based exact sequences of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes

$$0 \rightarrow \text{cone}(g_* \circ u_*)_* \rightarrow \text{cone}(h_*)_* \rightarrow \text{cone}(D_*)_* \rightarrow 0,$$

and

$$0 \rightarrow \text{cone}(f_*)_* \rightarrow \text{cone}(h_*)_* \rightarrow \text{cone}(g_*)_* \rightarrow 0.$$

They imply

$$\begin{aligned} a(0_* \xrightarrow{\cong} \text{cone}(g_* \circ u_*)_*) - a(0_* \xrightarrow{\cong} \text{cone}(h_*)_*) + a(0_* \xrightarrow{\cong} \text{cone}(D_*)_*) &= 0; \\ a(0_* \xrightarrow{\cong} \text{cone}(f_*)_*) - a(0_* \xrightarrow{\cong} \text{cone}(h_*)_*) + a(0_* \xrightarrow{\cong} \text{cone}(g_*)_*) &= 0. \end{aligned}$$

Hence we get

$$\begin{aligned} a(0_* \xrightarrow{\cong} \text{cone}(g_* \circ u_*)_*) + a(0_* \xrightarrow{\cong} \text{cone}(D_*)_*) \\ = a(0_* \xrightarrow{\cong} \text{cone}(f_*)_*) + a(0_* \xrightarrow{\cong} \text{cone}(g_*)_*). \end{aligned}$$

We conclude from assertion (ii)

$$\begin{aligned} a(0_* \xrightarrow{\cong} \text{cone}(g_* \circ u_*)_*) &= a(g_* \circ f_*) \\ a(0_* \xrightarrow{\cong} \text{cone}(D_*)_*) &= a(\text{id}_{D_*}) \stackrel{(5.51)}{=} 0; \\ a(0_* \xrightarrow{\cong} \text{cone}(f_*)_*) &= a(f_*); \\ a(0_* \xrightarrow{\cong} \text{cone}(g_*)_*) &= a(g_*). \end{aligned}$$

Now assertion (iv) follows.

(v) Because of assertion (ii) it suffices to show for a contractible $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex E_*

$$(5.55) \quad \rho(E_*) = 0 \implies a(0_* \rightarrow E_*) = 0.$$

We use induction over the dimension d of E_* . The induction beginning $d \leq 1$ is done as follows. Using assertion (iv) and the axioms one easily checks that the homomorphism

$$\text{Wh}(G) \rightarrow U^w, \quad [f : \mathbb{Z}G^a \xrightarrow{\cong} \mathbb{Z}G^a] \mapsto u^w(0[f]_* : 0[\mathbb{Z}G^a]_* \xrightarrow{\cong} 0[\mathbb{Z}G^a]_*).$$

is well-defined. It sends $\rho(E_*)$ to $u^W(0_* \rightarrow E_*)$ because of assertions (ii). This finishes the induction beginning $d \leq 1$. The induction step from $(d-1)$ to $d \geq 2$ is done as follows.

Since E_* is contractible, we can find a $\mathbb{Z}G$ -map $\gamma_{d-1} : E_{d-1} \rightarrow E_d$ satisfying $\gamma_{d-1} \circ e_d = \text{id}_{E_d}$. We obtain a $\mathbb{Z}G$ -chain complex F_* of dimension $(d-1)$ if we put $F_{d-1} = \ker(\gamma_{d-1})$, $F_n = E_n$ for $\leq (d-2)$, $f_{d-1} = e_{d-1}|_{\ker(\gamma_{d-1})}$,

and $f_n = e_n$ for $n \leq (d-2)$:

$$0 \rightarrow \ker(\gamma_{d-1}) \xrightarrow{e_{d-1}|_{\ker(\gamma_{d-1})}} E_{d-2} \xrightarrow{e_{d-2}} \cdots \xrightarrow{e_1} E_0.$$

Let $i_*: F_* \rightarrow E_*$ be the obvious inclusion of $\mathbb{Z}G$ -chain complexes. Define the $\mathbb{Z}G$ -chain map $p_*: E_* \rightarrow \text{el}[d, E_d]_*$ by $p_d = \text{id}_{E_d}$, $p_{d-1} = \gamma_{d-1}$, and $p_n = 0$ for $n \leq (d-2)$. Then we obtain a short exact sequence of $\mathbb{Z}G$ -chain complexes

$$0 \rightarrow F_* \xrightarrow{i_*} E_* \xrightarrow{p_*} \text{el}[d, E_d]_* \rightarrow 0.$$

Since E_* and $\text{el}[d, E_d]_*$ are contractible, F_* is contractible. This implies that F_{d-1} is stably free, i.e., we can find $m \in \mathbb{Z}_{\geq 0}$ such that $F_{d-1} \oplus \mathbb{Z}G^m$ is finitely generated free. By assertion (i) it suffices to show the assertion for $E_* \oplus \text{el}[d, \mathbb{Z}G^m]_*$. Hence we can assume without loss of generality that F_{d-1} is finitely generated free.

We have the exact sequence

$$0 \rightarrow F_{d-1} \xrightarrow{i_{d-1}} E_{d-1} \xrightarrow{\gamma_{d-1}} E_d \rightarrow 0$$

and E_{d-1} and E_d come with $\{\pm g\}$ -equivalences classes of $\mathbb{Z}G$ -basis. We want to equip F_{d-1} with an $\{\pm g\}$ -equivalences class of ordered $\mathbb{Z}G$ -basis such that this short exact sequence becomes $\{\pm g\}$ -based exact. This is not possible in general. But if we replace the exact sequence for large enough $m \in \mathbb{Z}_{\geq 0}$ by the short exact sequence

$$0 \rightarrow F_{d-1} \oplus \mathbb{Z}G^m \xrightarrow{i_{d-1} \oplus \text{id}_{\mathbb{Z}G^m}} E_{d-1} \oplus \mathbb{Z}G^m \xrightarrow{\gamma_{d-1} \circ \text{pr}} E_d \rightarrow 0$$

for the projection $\text{pr}: E_{d-1} \oplus \mathbb{Z}G^m \rightarrow E_{d-1}$ and use the induced $\{\pm g\}$ -equivalence classes of basis on $E_{d-1} \oplus \mathbb{Z}G^m$ and the given one on E_d , we can find a $\{\pm g\}$ -equivalence classes of basis on $F_{d-1} \oplus \mathbb{Z}G^m$ for which this sequence becomes $\{\pm g\}$ -based exact. Since we can add $\text{el}[(d-1), \mathbb{Z}G^m]_*$ to E_* , we can assume without loss of generality that the sequence $0 \rightarrow F_{d-1} \xrightarrow{i_{d-1}} E_{d-1} \xrightarrow{\gamma_{d-1}} E_d \rightarrow 0$ is $\{\pm g\}$ -based exact for an appropriate choice of $\{\pm g\}$ equivalence class of ordered $\mathbb{Z}G$ -bases on F_{d-1} . The upshot of this discussion is that we can assume without loss of generality that F_* is a $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex which is contractible and that the sequence

$$0 \rightarrow F_* \xrightarrow{i_*} E_* \xrightarrow{p_*} \text{el}[d, E_d]_* \rightarrow 0$$

is $\{\pm g\}$ -based exact.

This implies

$$\rho(F_*) = \rho(E_*) - \rho(\text{el}[d, E_d]_*) = 0 - 0 = 0,$$

and

$$a(0_* \rightarrow E_*) = a(0_* \rightarrow F_*) + a(0_* \rightarrow \text{el}[d, E_d]_*).$$

By induction hypothesis $a(0_* \rightarrow F_*) = 0$. From the axioms we get $a(0_* \rightarrow \text{el}[d, E_d]_*) = 0$. We conclude $a(0_* \rightarrow E_*) = 0$. This finishes the proof of (5.55) and hence of assertion (v).

(vi) Choose a $\mathbb{Z}G$ -chain homotopy $\gamma_*: f_* \simeq g_*$. It induces a $\mathbb{Z}G$ -chain isomorphism

$$u_*: \text{cone}(f_*)_* \xrightarrow{\cong} \text{cone}(g_*)_*$$

by $u_n = \begin{pmatrix} \text{id}_{C_{n-1}} & 0 \\ \gamma_{n-1} & \text{id}_{D_n} \end{pmatrix}: C_{n-1} \otimes D_n \rightarrow C_{n-1} \oplus D_n$. Because of assertions (ii) and (iv) it suffices to show $a(u_*) = 0$.

Since each u_n is a $\mathbb{Z}G$ -isomorphism, we conclude using assertion (i)

$$\begin{aligned} a(u_*) &= \sum_{n=0}^{\infty} a(n[u_n]_*: n[C_{n-1} \oplus D_n]_* \xrightarrow{\cong} n[C_{n-1} \oplus D_n]_*) \\ &= \sum_{n=0}^{\infty} (a(\text{id}_{n[C_{n-1}]_*}) + a(\text{id}_{n[D_n]_*})) \\ &= \sum_{n=0}^{\infty} (\text{rk}_{\mathbb{Z}G}(C_{n-1}) \cdot a(\text{id}_{n[\mathbb{Z}G]_*}) + \text{rk}_{\mathbb{Z}G}(D_n) \cdot a(\text{id}_{n[\mathbb{Z}G]_*})) \\ &= \sum_{n=0}^{\infty} (\text{rk}_{\mathbb{Z}G}(C_{n-1}) \cdot 0 + \text{rk}_{\mathbb{Z}G}(D_n) \cdot 0) \\ &= 0. \end{aligned}$$

This finishes the proof of Lemma 5.50. $\square$

Note that Lemma 5.50 shows that in Theorem 5.30 assertions (i) and (iii) are formal consequences of assertion (ii).

Notation 5.56. Let (A^W, a^W) be an additive Whitehead torsion invariant. Define the additive Reidemeister torsion invariant $(A^W, \overline{a^W})$ by putting for a contractible $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex C_*

$$\overline{a^W}(C_*) = a^W(0_* \xrightarrow{\sim} C_*).$$

Let (A^R, a^R) be an additive Reidemeister torsion invariant. Define the additive Whitehead torsion invariant $(A^R, \widetilde{a^R})$ by putting for a $\mathbb{Z}G$ -chain homotopy equivalence $f_*: C_* \rightarrow D_*$ of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes

$$\widetilde{a^R}(f_*) = a(\text{cone}(f_*)_*).$$

Remark 5.57. We leave the details of the elementary proof that $\widetilde{a^R}$ and $\overline{a^W}$ are well-defined to the reader. The only non-trivial item to check is that for

every $n \in \mathbb{Z}_{\geq 1}$ and every $\mathbb{Z}G$ -map $f: \mathbb{Z}G^a \rightarrow \mathbb{Z}G^b$ we get for the induced $\mathbb{Z}G$ -chain homotopy equivalence $\text{el}[n, f]_*: \text{el}[n, \mathbb{Z}G^a]_* \rightarrow \text{el}[n, \mathbb{Z}G^b]_*$ the equality

$$a^W(0_* \rightarrow \text{cone}(\text{el}[n, f]_*)) = 0.$$

This follows from the obvious commutative diagram of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes with $\{\pm g\}$ -based exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0_* & \longrightarrow & 0_* & \longrightarrow & 0_* & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow f'_* \simeq & & \downarrow \simeq & & \\ 0 & \longrightarrow & \text{el}[n, \mathbb{Z}G^b]_* & \longrightarrow & \text{cone}(\text{el}[n, f]_*) & \longrightarrow & \text{el}[(n+1), \mathbb{Z}G^a]_* & \longrightarrow & 0. \end{array}$$

Obviously $\widetilde{a^R} = a^R$ holds for an additive Reidemeister torsion invariant (A^R, a^R) . We conclude $\widetilde{a^W} = a^W$ for an additive Whitehead torsion invariant (A^W, a^W) from Lemma 5.50 (ii).

Hence the universal Whitehead torsion invariant (U^W, u^W) and the universal Reidemeister torsion invariant (U^R, u^R) are equivalent and carry the same information, since we can arrange $U^W = U^R$ and get $u^W = \widetilde{u^R}$ and $u^R = \overline{u^W}$.

Lemma 5.58. *Let $(A_u^{(2)}, a_u^{(2)})$ be an additive L^2 -torsion invariant. Consider two L^2 -acyclic $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes C_* and D_* .*

- (i) *We have $a_u^{(2)}(C_* \oplus D_*) = a_u^{(2)}(C_*) + a_u^{(2)}(D_*)$;*
- (ii) *We get for the suspension ΣC_* the equality $a_u^{(2)}(\Sigma C_*) = -a_u^{(2)}(C_*)$;*
- (iii) *Suppose that there is a simple $\mathbb{Z}G$ -chain homotopy equivalence $C_* \rightarrow D_*$. Then we get $a_u^{(2)}(C_*) = a_u^{(2)}(D_*)$.*

Proof. (i) This follows from the $\{\pm g\}$ -based exact short sequence $0 \rightarrow C_* \rightarrow C_* \oplus D_* \rightarrow D_* \rightarrow 0$.

(ii) The proof is analogous to the one of Lemma 5.50 (iii).

(iii) This follows from Lemma 5.50 (v) and Remark 5.57 using the composite of the canonical maps $U^R \rightarrow U_u^{(2)} \xrightarrow{\mu_u^{(2)}} A_u^{(2)}$. $\square$

Exercise 5.59 Show that any element in U^W , U^R , and $U_u^{(2)}$ respectively can be realized by $u^W(f_*)$, $u^R(C_*)$, and $u_u^{(2)}(D_*)$ respectively.

One can identify these universal invariants with Whitehead groups and torsion invariants in the following way.

Theorem 5.60 (Computing universal torsion invariants)

- (i) *The pair $(\text{Wh}^w(G), \rho_u^{(2)})$ is the additive L^2 -torsion invariant, where $\rho_u^{(2)}$ has been introduced in Definition 5.38;*

- (ii) The pair $(\text{Wh}(G), \rho)$ is the additive Reidemeister torsion invariant, where ρ has been defined in (5.28);
- (iii) The pair $(\text{Wh}(G), \tau)$ is the additive Whitehead torsion invariant, where τ has been defined in (5.29).

Proof. (i) We conclude from the universal property of $(U_u^{(2)}, u_u^{(2)})$ and Lemma 5.39 that there is precisely one homomorphism

$$(5.61) \quad \rho_u^{(2)}: U^w \rightarrow \text{Wh}^w(G)$$

such that we get for every L^2 -acyclic $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex C_*

$$\rho_u^{(2)}(u_u^{(2)}(C_*)) = \rho_u^{(2)}(C_*).$$

Next we define a homomorphism

$$(5.62) \quad e: \text{Wh}^w(G) \rightarrow U_u^{(2)}.$$

Consider a $\mathbb{Z}G$ -endomorphism $f: \mathbb{Z}G^n \rightarrow \mathbb{Z}G^n$ which is a weak isomorphism. Let $\text{el}[f]_*$ be the L^2 -acyclic 1-dimensional $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex whose first differential is f . We want to define

$$(5.63) \quad e([f]) := u_u^{(2)}(\text{el}[f]_*).$$

We have to check that the relations in $\text{Wh}^w(G)$ are satisfied. This follows for the additivity relation directly from Lemma 5.39. Next we have to show to show for $\mathbb{Z}G$ -endomorphisms $f_1, f_2: \mathbb{Z}G^n \rightarrow \mathbb{Z}G^n$ which are weak isomorphisms that we get in $\text{Wh}^w(G)$

$$(5.64) \quad u_u^{(2)}(\text{el}[f_2 \circ f_1]_*) = u_u^{(2)}(\text{el}[f_1]_*) + u_u^{(2)}(\text{el}[f_2]_*).$$

Consider the chain map

$$h_*: \Sigma^{-1} \text{el}[f_2]_* \rightarrow \text{el}[f_1]_*$$

given by $h_1 = \text{id}_{\mathbb{Z}G^n}$ and $h_i = 0$ for $i \neq 1$. We conclude from the short $\{\pm g\}$ -based exact sequence of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes $0 \rightarrow \text{el}[f_1]_* \rightarrow \text{cone}(h_*)_* \rightarrow \text{el}[f_2]_* \rightarrow 0$ that we get in $\text{Wh}^w(G)$

$$u_u^{(2)}(\text{cone}(h_*)_*) = u_u^{(2)}(\text{el}[f_1]_*) + u_u^{(2)}(\text{el}[f_2]_*).$$

There is also a $\{\pm g\}$ -based exact short sequence of L^2 -acyclic $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes $0 \rightarrow \text{el}[f_2 \circ f_1]_* \xrightarrow{j} \text{cone}(h_*)_* \rightarrow \text{el}[\text{id}_{\mathbb{Z}G^n}]_* \rightarrow 0$, which is given by

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}G^n & \xrightarrow{\begin{pmatrix} f_1 \\ -\text{id}_{\mathbb{Z}G^n} \end{pmatrix}} & \mathbb{Z}G^n \oplus \mathbb{Z}G^n & \xrightarrow{(\text{id}_{\mathbb{Z}G^n} \ f_1)} & \mathbb{Z}G^n \longrightarrow 0 \\
& & \downarrow f_2 \circ f_1 & & \downarrow \begin{pmatrix} f_2 & 0 \\ \text{id}_{\mathbb{Z}G^n} & f_1 \end{pmatrix} & & \downarrow \text{id}_{\mathbb{Z}G^n} \\
0 & \longrightarrow & \mathbb{Z}G^n & \xrightarrow{\begin{pmatrix} \text{id}_{\mathbb{Z}G^n} \\ 0 \end{pmatrix}} & \mathbb{Z}G^n \oplus \mathbb{Z}G^n & \xrightarrow{(0 \ \text{id}_{\mathbb{Z}G^n})} & \mathbb{Z}G^n \longrightarrow 0.
\end{array}$$

It yields the equation in $\text{Wh}^w(G)$

$$u_u^{(2)}(\text{cone}(h_*)_*) = u_u^{(2)}(\text{el}[f_2 \circ f_1]_*) + u_u^{(2)}(\text{el}[\text{id}_{\mathbb{Z}G^n}]_*).$$

Since $u_u^{(2)}(\text{el}[\text{id}_{\mathbb{Z}G^n}]_*) = n \cdot u_u^{(2)}(\text{el}[1, \mathbb{Z}G]) = 0$ holds by the axioms, (5.64) follows.

Since for $\epsilon \in \{-1, 1\}$ and $g \in G$ we have

$$u_u^{(2)}(\text{el}[r_{\epsilon, g}: \mathbb{Z}G \rightarrow \mathbb{Z}G]_*) = 0,$$

the homomorphism $e: \text{Wh}^w(G) \rightarrow U_u^{(2)}$ of (5.62) is well-defined.

One easily checks that $\rho_u^{(2)} \circ e$ is the identity on $\text{Wh}^w(G)$. Hence it remains to show that e is surjective. For this purpose it suffices to show for every L^2 -acyclic $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex C_* we have

$$(5.65) \quad u^{(2)}(C_*) \in \text{im}(e),$$

since U^w is generated by the elements $u^{(2)}(C_*)$. This is done by induction over the dimension d of C_* .

The induction beginning $d \leq 1$ is done as follows. If we choose representatives for the $\{\pm g\}$ -equivalence classes of basis on C_1 and C_0 , we can identify $C_1 = C_0 = \mathbb{Z}G^n$ and one gets directly from the construction of e

$$e([c_1: \mathbb{Z}G^n \rightarrow \mathbb{Z}G^n]) = u_u^{(2)}(C_*).$$

The induction step from $(d-1)$ to $d \geq 2$ is done as follows. Choose a weak chain contraction γ_* for C_* . Define a $\mathbb{Z}G$ -chain map

$$i_*: \text{el}[c_1 \circ \gamma_0: C_0 \rightarrow C_0]_* \rightarrow C_*$$

by $i_1 = \gamma_0$, $i_0 = \text{id}_{C_0}$ and $i_k = 0$ for $k \neq 0, 1$. We conclude from Lemma 5.39 applied to the $\{\pm g\}$ -based exact short sequence

$$0 \rightarrow C_* \rightarrow \text{cone}(i_*) \rightarrow \Sigma \text{el}[c_1 \circ \gamma_0: C_0 \rightarrow C_0]_* \rightarrow 0$$

that $\text{cone}(i_*)$ is an L^2 -acyclic $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complex and we get in $U_u^{(2)}$

$$u_u^{(2)}(\text{cone}(i_*)_*) = u_u^{(2)}(C_*) - u_u^{(2)}(\text{el}[c_1 \circ \gamma_0: C_0 \rightarrow C_0]_*).$$

Define a $\mathbb{Z}G$ -chain map

$$j_*: \text{el}[\text{id}_{C_0}]_* \rightarrow \text{cone}(i_*)_*$$

by $j_0 = \text{id}_{C_0}: C_0 \rightarrow C_0$, $j_1 = \begin{pmatrix} \text{id}_{C_0} \\ 0 \end{pmatrix}: C_0 \rightarrow C_0 \oplus C_1$. Then we can equip $\text{coker}(j_*)$ with an obvious $\{\pm g\}$ -equivalence class of ordered $\mathbb{Z}G$ -basis such that we obtain a $\{\pm g\}$ -based exact short sequence of L^2 -acyclic $\{\pm g\}$ -based free $\mathbb{Z}G$ -chain complexes

$$0 \rightarrow \text{el}[\text{id}_{C_0}]_* \xrightarrow{j_*} \text{cone}(i_*)_* \rightarrow \text{coker}(j_*) \rightarrow 0.$$

We conclude from Lemma 5.39 that $\text{coker}(j_*)$ is L^2 -acyclic and we get in $U_u^{(2)}$

$$u_u^{(2)}(\text{cone}(i_*)_*) = u_u^{(2)}(\text{el}[\text{id}_{C_0}]_*) + u_u^{(2)}(\text{coker}(j_*)).$$

Hence we get

$$u_u^{(2)}(C_*) = u_u^{(2)}(\text{el}[c_1 \circ \gamma_0: C_0 \rightarrow C_0]_*) + u_u^{(2)}(\text{el}[\text{id}_{C_0}]_*) + u_u^{(2)}(\text{coker}(j_*)).$$

Since $u_u^{(2)}(\text{el}[\text{id}_{C_0}]_*)$ and $u_u^{(2)}(\text{el}[c_1 \circ \gamma_0: C_0 \rightarrow C_0]_*)$ lie in the image of e , it suffices to show that $u_u^{(2)}(\text{coker}(j_*))$ lies in the image of e . Since $\text{coker}(j_*)$ has dimension $\leq d$ and its zeroth-chain module is trivial, it can be written as ΣD_* for an L^2 -acyclic $\{\pm g\}$ -based free $\mathbb{Z}G$ -chain complex D_* of dimension $(d-1)$. We get from Lemma 5.58 (ii)

$$u_u^{(2)}(\text{coker}(j_*)) = u^{(2)}(\Sigma D_*) = -u^{(2)}(D_*)$$

Since $u^{(2)}(D_*)$ lies in the image of el by induction hypothesis, the proof of (5.65) is finished. Hence assertion (i) follows.

(ii) The proof is analogous to the one of assertion (i), just replace weak chain contractions by chain contractions.

(iii) This follows from assertion (ii) and Remark 5.57. This finishes the proof of Theorem 5.60. $\square$

5.3.5 Whitehead Torsion for CW -Complexes

Define the *Whitehead group* of a (not necessarily connected) finite CW -complex X by

$$(5.66) \quad \text{Wh}(\pi_1(X)) := \bigoplus_{C \in \pi_0(X)} \text{Wh}(\pi_1(C)).$$

Obviously a map $g: X_1 \rightarrow X_2$ induces a homomorphism of abelian groups

$$g_*: \text{Wh}(\pi_1(X_1)) \rightarrow \text{Wh}(\pi_1(X_2))$$

by the homomorphisms between the various fundamental groups of the path components induced by g .

Let $f: X \xrightarrow{\sim} Y$ be a cellular homotopy equivalence of finite CW -complexes. Define the *Whitehead torsion* of f

$$(5.67) \quad \tau(f) \in \text{Wh}(\pi_1(Y)) = \bigoplus_{D \in \pi_0(X)} \text{Wh}(\pi_1(D))$$

by the collection of the Whitehead torsions $\tau(C_*(\widetilde{f_D})) \in \text{Wh}(\pi_1(D))$ defined in (5.29) for the $\mathbb{Z}[\pi_1(D)]$ -chain homotopy equivalences $C_*(\widetilde{f_D}): C_*(\widetilde{C}) \rightarrow C_*(\widetilde{D})$ between the cellular chain complexes associated to the homotopy equivalences $f_C: C \xrightarrow{\sim} D$ for $C = \pi_0(f)^{-1}(D)$.

We call f a *simple homotopy equivalence* if $\tau(f)$ vanishes.

The basic properties of this invariant are summarized in the following theorem, whose proof can be found for instance in [11, (22.1), (22.4), (23.1), and (23.2)], [44, Theorem 2.1], and [47, Theorem 3.1].

Theorem 5.68 (Basic properties of Whitehead torsion for CW -complexes)

(i) Sum formula

Let the following two diagrams be pushouts of finite CW -complexes

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ i_2 \downarrow & & \downarrow j_1 \\ X_2 & \xrightarrow{j_2} & X \end{array} \qquad \begin{array}{ccc} Y_0 & \xrightarrow{k_1} & Y_1 \\ k_2 \downarrow & & \downarrow l_1 \\ Y_2 & \xrightarrow{l_2} & Y \end{array}$$

where the left vertical arrows are inclusions of CW -complexes, the upper horizontal maps are cellular, and X and Y are equipped with the induced CW -structure. Let $f_i: X_i \rightarrow Y_i$ be cellular homotopy equivalences for $i = 0, 1, 2$ satisfying $f_1 \circ i_1 = k_1 \circ f_0$ and $f_2 \circ i_2 = k_2 \circ f_0$. Put $l_0 = l_1 \circ k_1 = l_2 \circ k_2$. Denote by $f: X \rightarrow Y$ the map induced by f_0, f_1 , and f_2 and the pushout property.

Then f is a cellular homotopy equivalence and

$$\tau(f) = (l_1)_* \tau(f_1) + (l_2)_* \tau(f_2) - (l_0)_* \tau(f_0);$$

(ii) Homotopy invariance

Let $f, g: X \rightarrow Y$ be cellular maps of finite CW -complexes. Suppose that they are cellularly homotopic.

Then the homomorphisms $f_*, g_*: \text{Wh}(\pi_1(X)) \rightarrow \text{Wh}(\pi_1(Y))$ agree. If additionally f and g are cellular homotopy equivalences, then we obtain

$$\tau(g) = \tau(f);$$

(iii) Composition formula

Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be cellular homotopy equivalences of finite CW -complexes.

Then we get

$$\tau(g \circ f) = g_*\tau(f) + \tau(g);$$

(iv) Product formula

Let $f: X' \rightarrow X$ and $g: Y' \rightarrow Y$ be cellular homotopy equivalences of connected finite CW -complexes.

Then we get

$$\tau(f \times g) = \chi(X) \cdot j_*\tau(g) + \chi(Y) \cdot i_*\tau(f)$$

where $\chi(X), \chi(Y) \in \mathbb{Z}$ denote the Euler characteristics, $j_*: \text{Wh}(\pi_1(Y)) \rightarrow \text{Wh}(\pi_1(X \times Y))$ is the homomorphism induced by $j: Y \rightarrow X \times Y, y \mapsto (y, x_0)$ for some base point $x_0 \in X$ and i_* is defined analogously.

Remark 5.69 (Dropping cellular). If $f: X \rightarrow Y$ is a (not necessarily cellular) homotopy equivalence of finite CW -complex, we can define in view of the Cellular Approximation Theorem and Theorem 5.68 (ii)

$$\tau(f) := \tau(f') \in \text{Wh}(Y)$$

for any cellular homotopy equivalence $f': X \rightarrow Y$ of finite CW -complex which is homotopic to f . Moreover, in Theorem 5.68 we can drop the word “cellular” everywhere.

Let X be a finite simplicial complex. Let X' be its barycentric subdivision. Then one can show $\tau(f) = 0$ for the map $f: X \rightarrow X'$ whose underlying map of spaces is the identity. However, if X and Y are two finite CW -complexes with the same underlying topological space Z , then the map $\text{id}_Z: X \rightarrow Y$ is not necessarily cellular and it is not at all clear that $\tau(f) = 0$ holds for any cellular map $f: X \rightarrow Y$ which is homotopic to id_Z . This problem is solved by the following (in comparison with the other statements above much deeper) result due to Chapman [9], [10], see also [11, Appendix] and [56, Section 5].

Theorem 5.70 (Topological invariance of Whitehead torsion) *The Whitehead torsion of a homeomorphism $f: X \rightarrow Y$ of finite CW -complexes vanishes.*

5.3.6 Universal L^2 -Torsion for finite free G - CW -Complexes

Definition 5.71 (Universal L^2 -torsion for finite free G - CW -complexes). Let G be a (discrete) group. Let X be a finite free G - CW -complex, or, equivalently, $X \rightarrow X/G$ is a finite G -covering over a finite CW -complex X/G .

Suppose that X is L^2 -acyclic, i.e., its n -th L^2 -Betti number $b_n^{(2)}(X; \mathcal{N}(G))$ vanishes for all $n \geq 0$.

Then we define its *universal L^2 -torsion*

$$\rho_u^{(2)}(X; \mathcal{N}(G)) \in \text{Wh}^w(G)$$

to be the element $\rho_u^{(2)}(C_*(X); \mathcal{N}(G))$ introduced in Definition 5.38 for the cellular $\mathbb{Z}G$ -chain complex $C_*(X)$.

Example 5.72 ($G = \mathbb{Z}$). The quotient field $\mathbb{Q}(z, z^{-1})$ of the integral domain $\mathbb{Z}[z, z^{-1}] = \mathbb{Z}[\mathbb{Z}]$ consists of rational functions with coefficients in $\mathbb{Q}$ in one variable z . Given a matrix $A \in M_{n,n}(\mathbb{Z}[\mathbb{Z}])$, the operator $r_A: \mathbb{Z}[\mathbb{Z}]^n \rightarrow \mathbb{Z}[\mathbb{Z}]^n$ is a weak isomorphism if and only if $\det_{\mathbb{Z}[\mathbb{Z}]}(A) \in \mathbb{Z}[\mathbb{Z}]$ is non-zero. This follows from [45, Lemma 1.34 on page 35]. Hence we obtain a well-defined homomorphism

$$\det_{\mathbb{Z}[\mathbb{Z}]}: K_1^w(\mathbb{Z}[\mathbb{Z}]) \rightarrow \mathbb{Q}(z, z^{-1})^\times, \quad [r_A: \mathbb{Z}[\mathbb{Z}]^n \rightarrow \mathbb{Z}[\mathbb{Z}]^n] \mapsto \det_{\mathbb{Z}[\mathbb{Z}]}(A).$$

Define

$$i: \mathbb{Q}(z, z^{-1})^\times \rightarrow K_1^w(\mathbb{Z}[\mathbb{Z}])$$

by sending an element $x \in \mathbb{Q}(z, z^{-1})^\times$ to $[r_p: \mathbb{Z}[\mathbb{Z}] \rightarrow \mathbb{Z}[\mathbb{Z}]] - [r_q: \mathbb{Z}[\mathbb{Z}] \rightarrow \mathbb{Z}[\mathbb{Z}]]$ for any two elements $p, q \in \mathbb{Z}[\mathbb{Z}]$ with $p \neq 0, q \neq 0$ satisfying $x = p \cdot q^{-1}$ in $\mathbb{Q}(z, z^{-1})^\times$. One easily checks that i is well-defined and $\det_{\mathbb{Z}[\mathbb{Z}]} \circ i = \text{id}_{\mathbb{Q}(z, z^{-1})^\times}$. It is not hard to prove using the standard Euclidean algorithm on the polynomial ring $\mathbb{Q}[z]$ that i is surjective. Hence $\det_{\mathbb{Z}[\mathbb{Z}]}$ and i are inverses of each other. We obtain an induced isomorphism

$$(5.73) \quad \text{Wh}^w(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Q}(z, z^{-1})^\times / \{\sigma \cdot z^n \mid \sigma \in \{\pm 1\}, n \in \mathbb{Z}\}.$$

Consider $\mathbb{R}$ with the standard $\mathbb{Z}$ -action given by translation. This is the universal covering of S^1 with the standard action of $\pi_1(S^1) = \mathbb{Z}$. The cellular $\mathbb{Z}[\mathbb{Z}]$ -chain complex of $\mathbb{R}$ is 1-dimensional and has as first differential $r_{z-1}: \mathbb{Z}[\mathbb{Z}] \rightarrow \mathbb{Z}[\mathbb{Z}]$. Hence the isomorphism (5.73) sends $\rho_u^{(2)}(\mathbb{R}; \mathcal{N}(\mathbb{Z})) \in \text{Wh}^w(\mathbb{Z})$ to the class of the element $(z - 1)$.

More generally, let X be any finite free $\mathbb{Z}$ -CW-complex. Then X is L^2 -acyclic if and only if all of the $\mathbb{Q}[\mathbb{Z}]$ -modules $H_n(X; \mathbb{Q})$ are torsion $\mathbb{Q}[\mathbb{Z}]$ -modules, see [45, Lemma 1.34 on page 35]. It is now straightforward to see that under the isomorphism (5.73) the universal L^2 -torsion of X corresponds to the Milnor-Turaev torsion [53, 71] of X .

Theorem 5.74 (Basic properties of universal L^2 -torsion for finite free G -CW-complexes)

(i) G -homotopy invariance

Let $f: X \rightarrow Y$ be a G -homotopy equivalence of finite free G -CW-complexes. Suppose that X or Y is L^2 -acyclic.

Then both X and Y are L^2 -acyclic and we get

$$\rho_u^{(2)}(Y) - \rho_u^{(2)}(X) = \zeta(\tau(f)),$$

where $\tau(f) \in \text{Wh}(G)$ is the Whitehead torsion of f and $\zeta: \text{Wh}(G) \rightarrow \text{Wh}^w(G)$ is the obvious homomorphism;

(ii) Sum formula

Consider a G -pushout of finite free G -CW-complexes

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array}$$

where the upper horizontal arrow is cellular, the left vertical arrow is an inclusion of G -CW-complexes, and X has the obvious G -CW-structure coming from the ones on X_0 , X_1 , and X_2 . Suppose that X_0 , X_1 , and X_2 are L^2 -acyclic.

Then X is L^2 -acyclic and we get

$$\rho_u^{(2)}(X; \mathcal{N}(G)) = \rho_u^{(2)}(X_1; \mathcal{N}(G)) + \rho_u^{(2)}(X_2; \mathcal{N}(G)) - \rho_u^{(2)}(X_0; \mathcal{N}(G));$$

(iii) Induction

Let $j: H \rightarrow G$ be an inclusion of groups. Denote by $j_*: \text{Wh}^w(H) \rightarrow \text{Wh}^w(G)$ the induced homomorphism. Let X be an L^2 -acyclic finite free H -CW-complex.

Then $j_*X = G \times_H X$ is an L^2 -acyclic finite free G -CW-complex and we get

$$\rho_u^{(2)}(j_*X; \mathcal{N}(G)) = j_*(\rho_u^{(2)}(X; \mathcal{N}(H)));$$

(iv) Restriction

Let $j: H \rightarrow G$ be an inclusion of groups such that the index $[G : H]$ is finite. Denote by $j^*: \text{Wh}^w(G) \rightarrow \text{Wh}^w(H)$ the homomorphism given by restriction. Let X be a finite free G -CW-complex. Let j^*X be the restriction to H by j .

Then j^*X is a finite free H -CW-complex, X is L^2 -acyclic if and only if j^*X is L^2 -acyclic, and, if this is the case, we get

$$\rho_u^{(2)}(j^*X; \mathcal{N}(H)) = j^*(\rho_u^{(2)}(X; \mathcal{N}(G)));$$

(v) Product formula

Let G_0 and G_1 be groups and denote by $j: G_0 \rightarrow G_0 \times G_1$ the obvious inclusion. Denote by $j_*: \text{Wh}^w(G_0) \rightarrow \text{Wh}^w(G_0 \times G_1)$ the induced homomorphism. Let X_i be a finite free G_i -CW-complex for $i = 0, 1$. Suppose that X_0 is L^2 -acyclic.

Then $X_0 \times X_1$ is an L^2 -acyclic finite free $G_0 \times G_1$ -CW-complex, and we get

$$\rho_u^{(2)}(X_0 \times X_1; \mathcal{N}(G_0 \times G_1)) = \chi(X_1/G_1) \cdot j_* (\rho_u^{(2)}(X_0; \mathcal{N}(G_0)));$$

(vi) Fibrations

Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. Suppose that F and B are finite CW-complexes. Let $q: \bar{E} \rightarrow E$ be a G -covering. Let $\bar{F} \rightarrow F$ be the G -covering obtained from q by the pullback construction with i . Suppose that $\bar{F}$ is L^2 -acyclic. Assume that $\text{Wh}(G)$ vanishes.

Then there is an L^2 -acyclic free finite CW-complex $\bar{E}'$ which is G -homotopy equivalent to $\bar{E}$, and we can define $\rho_u^{(2)}(\bar{E}; \mathcal{N}(G)) = \rho_u^{(2)}(\bar{E}'; \mathcal{N}(G))$ for any such choice of $\bar{E}'$ by assertion (i). Moreover, we get

$$\rho_u^{(2)}(\bar{E}; \mathcal{N}(G)) = \chi(B) \cdot \rho_u^{(2)}(\bar{F}; \mathcal{N}(G));$$

(vii) S^1 -actions

Let X be a connected finite S^1 -CW-complex. Fix a base point $x \in X$. Let $\mu: \pi_1(X, x) \rightarrow G$ be a group homomorphism. Let $\bar{X}$ be the finite free G -CW-complex associated to μ with $\bar{X}/G = X$. Suppose that the composite

$$\pi_1(S^1, 1) \xrightarrow{\pi_1(\text{ev}_x, 1)} \pi_1(X, x) \xrightarrow{\mu} G$$

is injective, where $\text{ev}_x: S^1 \rightarrow X$ sends z to $z \cdot x$. Let I_n be the set of open n -dimensional S^1 -cells of X . For each $e \in I_n$ choose a point $x(e)$ in its interior and a path $w(e)$ from $x(e)$ to x . Denote by $S_{x(e)}^1 \subseteq S^1$ the isotropy group of $x(e)$ which must be finite because of the assumptions above. Let $\bar{\text{ev}}_e: S^1/S_{x(e)}^1 \rightarrow X$ be the injective map which sends $z \cdot S_{x(e)}^1$ to $z \cdot x(e)$. Choose a homeomorphism $f_e: S^1 \xrightarrow{\cong} S^1/S_{x(e)}^1$ with $f_e(1) = 1 \cdot S_{x(e)}^1$. Identify $\mathbb{Z} = \pi_1(S^1, 1)$. Let $j(e): \mathbb{Z} \rightarrow G$ be the injective homomorphism given by the composite

$$\begin{aligned} j(e): \mathbb{Z} = \pi_1(S^1, 1) &\xrightarrow{\pi_1(f_e, 1)} \pi_1(S^1/S_{x(e)}^1, 1 \cdot S_{x(e)}^1) \xrightarrow{\pi_1(\bar{\text{ev}}_e, 1 \cdot S_{x(e)}^1)} \\ &\pi_1(X, x(e)) \xrightarrow{c_{w(e)}} \pi_1(X, x) \xrightarrow{\mu} G, \end{aligned}$$

where $c_{w(e)}$ is given by conjugation with the path $w(e)$. Denote by

$$j(e)_*: \text{Wh}^w(\mathbb{Z}) \rightarrow \text{Wh}^w(G)$$

the homomorphism induced by $j(e)$. Equip $\mathbb{R}$ with the $\mathbb{Z}$ -action given by translation.

Then $\bar{X}$ is L^2 -acyclic and we get

$$\rho_u^{(2)}(\overline{X}; \mathcal{N}(G)) = \sum_{n \geq 0} (-1)^n \cdot \sum_{e \in I_n} j(e)_* (\rho_u^{(2)}(\mathbb{R}, \mathcal{N}(\mathbb{Z})));$$

(viii) Poincaré duality

Let M be an orientable n -dimensional manifold with free proper G -action and boundary ∂M . Let $w: G \rightarrow \{\pm 1\}$ be the orientation homomorphism sending $g \in G$ to 1, if multiplication with g respects the orientation and to -1 otherwise. Equip $\text{Wh}^w(G)$ with the involution $*$ coming from the w -twisted involution on $\mathbb{Z}G$ sending $\sum_{g \in G} r_g \cdot g$ to $\sum_{g \in G} r_g \cdot w(g) \cdot g^{-1}$. Then we get

$$\rho_u^w(M, \partial M; \mathcal{N}(G)) = (-1)^{n+1} \cdot *(\rho_u^w(M; \mathcal{N}(G)))$$

using the obvious definition of $\rho_u^w(M, \partial M; \mathcal{N}(G))$.

Proof. See [19, Theorem 3.5]. □

Remark 5.75 (Universal L^2 -torsion for manifolds). Every compact topological manifold has a preferred simple homotopy type, see [32, IV]. We can therefore extend the definition of the universal L^2 -torsion from finite CW-complexes to closed topological manifolds in the usual way.

Remark 5.76 (Universal L^2 -torsion in terms of the combinatorial Laplace operator). Let M be an orientable n -dimensional manifold with free proper G -action and empty boundary such that the G action is orientation preserving. Suppose that the dimension of M is odd. Let $\Delta_n: C_*(M) \rightarrow C_*(M)$ be its combinatorial Laplace operator. Then we conclude from Lemma 5.42 and Theorem 5.74 (viii) that we get in $\text{Wh}^w(G)$

$$2 \cdot \rho_u^{(2)}(M; \mathcal{N}(G)) = - \sum_{n \in \mathbb{Z}} (-1)^n \cdot n \cdot [\Delta_n].$$

The advantage of the formula above is that one can derive the right hand side directly from the differentials without having to find an explicit weak chain contraction.

Note that in the case, where the dimension of M is even, we get no interesting information about the universal L^2 -torsion $\rho_u^{(2)}(M; \mathcal{N}(G))$, namely, we just obtain

$$0 = - \sum_{n \in \mathbb{Z}} (-1)^n \cdot n \cdot [\Delta_n].$$

whereas $\rho_u^{(2)}(M; \mathcal{N}(G))$ is not necessarily trivial.

Example 5.77 (Torus T^n). Let T^n be the n -dimensional torus for $n \geq 2$. Let G be a torsionfree group. Let $\mu: \pi_1(T^n) \rightarrow G$ be a non-trivial group homomorphism. Let $\overline{T^n} \rightarrow T^n$ be the G -covering associated to μ . Then $\overline{T^n}$ is L^2 -acyclic and we get

$$\rho_u^{(2)}(\overline{T^n}; \mathcal{N}(G)) = 0$$

by the following argument.

We can choose an integer k with $k \geq 1$, an isomorphism $\nu: \pi_1(T^n) \xrightarrow{\cong} \mathbb{Z}^k \times \mathbb{Z}^{n-k}$ and an injection $i: \mathbb{Z}^k \rightarrow G$ such that $\mu = i \circ \text{pr} \circ \nu$ for $\text{pr}: \mathbb{Z}^k \times \mathbb{Z}^{n-k} \rightarrow \mathbb{Z}^k$ the projection. We can find a homeomorphism $f: T^n \xrightarrow{\cong} T^k \times T^{n-k}$ such that $\pi_1(f) = \nu$. Hence we obtain a G -homeomorphism

$$i_*(\widetilde{T^k} \times T^{n-k}) = G \times_{\mathbb{Z}^k} (\widetilde{T^k} \times T^{n-k}) \xrightarrow{\cong} \overline{T^n},$$

where the $\mathbb{Z}^k$ -action on $\widetilde{T^k} \times T^{n-k}$ is given by the standard $\mathbb{Z}^k$ -action on $\widetilde{T^k}$ and the trivial $\mathbb{Z}^k$ -action on T^{n-k} . We conclude from Theorem 5.74 (iii) and (v) that $\overline{T^n}$ is L^2 -acyclic and

$$\rho_u^{(2)}(\overline{T^n}; \mathcal{N}(G)) = i_*(\rho_u^{(2)}(\widetilde{T^k} \times T^{n-k}; \mathcal{N}(\mathbb{Z}^k))) = \chi(T^{n-k}) \cdot i_*(\rho_u^{(2)}(\widetilde{T^k}; \mathcal{N}(\mathbb{Z}^k))).$$

If $k \neq n$, then the claim follows from $\chi(T^{n-k}) = 0$. Suppose that $k = n$. Then we have to show $\rho_u^{(2)}(\widetilde{T^n}; \mathcal{N}(\mathbb{Z}^n)) = 0$ for $n \geq 2$. This follows from Theorem 5.74 (v) applied to $\widetilde{S^1} \times \widetilde{T^{n-1}}$ using $\chi(T^{n-1}) = 0$.

Lemma 5.78. *Let G be a group such that there exists a connected L^2 -acyclic finite free G -CW-complex X . Consider any element $\omega \in \text{Wh}^w(G)$.*

Then there exists a connected L^2 -acyclic finite free G -CW-complex Y obtained from X by attaching trivially equivariant 2-cells and attaching equivariant 3-cells with $\rho_u^{(2)}(Y; \mathcal{N}(G)) = \omega$.

Proof. See [19, Lemma 3.8]. □

Lemma 5.79. *Let G be a group such that there exists a connected L^2 -acyclic finite free G -CW-complex X . Consider any element $\omega \in \text{Wh}^w(G)$ and any integer $d \geq 2 \cdot \max\{\dim(X), 3\} + 1$.*

Then there exists a connected orientable smooth d -dimensional manifold M without boundary and orientation preserving cocompact free proper smooth G -action such that M is L^2 -acyclic and we get

$$\rho_u^{(2)}(M; \mathcal{N}(G)) = (-1)^{d+1} \cdot *(\omega) + \omega.$$

for the involution $*$ defined in (5.41). Furthermore, if X is simply-connected, then M can also be chosen to be simply-connected.

Proof. See [19, Lemma 3.9]. □

5.3.7 The universal L^2 -torsion for universal coverings

The most natural and interesting case is the one of the universal covering. For the reader's convenience we record the basic properties of the universal L^2 -torsion in this setting.

Definition 5.80 (Universal L^2 -torsion for universal coverings). Let X be a finite connected CW -complex which is L^2 -acyclic, i.e., $b_n^{(2)}(\tilde{X})$ vanishes for all $n \in \mathbb{Z}_{\geq 0}$. We define its *universal L^2 -torsion*

$$\rho_u^{(2)}(\tilde{X}) \in \text{Wh}^w(\pi_1(X))$$

by $\rho_u^{(2)}(\tilde{X}; \mathcal{N}(\pi_1(X)))$ as introduced in Definition 5.71.

If X is an L^2 -acyclic (not necessarily connected) finite CW -complex, we define

$$\begin{aligned} \text{Wh}^w(\pi_1(X)) &:= \bigoplus_{C \in \pi_0(X)} \text{Wh}^w(\pi_1(C)); \\ \rho_u^{(2)}(\tilde{X}) &:= (\rho_u^{(2)}(\tilde{C}))_{C \in \pi_0(X)} \in \text{Wh}^w(\pi_1(X)). \end{aligned}$$

For a map $f: X \rightarrow Y$ of finite C -complexes such that $\pi_1(f, x): \pi_1(X, x) \rightarrow \pi_1(Y, f(x))$ is injective for all $x \in X$, we get a homomorphism

$$f_*: \text{Wh}^w(\pi(X)) \rightarrow \text{Wh}^w(\pi(Y))$$

by the collection of maps $(f|_C)_*: \text{Wh}^w(\pi_1(C)) \rightarrow \text{Wh}^w(\pi_1(D))$ for $C \in \pi_0(X)$ and $D \in \pi_1(Y)$ with $f(C) \subseteq D$.

It is evident that Theorem 5.74 gives rise to statements about the universal L^2 -torsions for universal coverings. Most statements of Theorem 5.74 specialize in an obvious way. Therefore in the next theorem we spell out only four properties.

Theorem 5.81 (Basic properties of the universal L^2 -torsion for universal coverings)

(i) Simple homotopy invariance

Let $f: X \rightarrow Y$ be a homotopy equivalence of finite CW -complexes. Suppose that X or Y is L^2 -acyclic.

Then both X and Y are L^2 -acyclic and we get

$$\rho_u^{(2)}(\tilde{Y}) - \rho_u^{(2)}(\tilde{X}) = \zeta(\tau(f)),$$

where $\tau(f) \in \text{Wh}(\pi_1(Y))$ is the Whitehead torsion of f and $\zeta: \text{Wh}(\pi_1(Y)) \rightarrow \text{Wh}^w(\pi_1(Y))$ is the obvious homomorphism;

(ii) Sum formula

Consider a pushout of finite CW -complexes

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ & \searrow j_0 & \downarrow j_1 \\ X_2 & \longrightarrow & X \end{array}$$

where the upper horizontal arrow is cellular, the left vertical arrow is an inclusion of CW-complexes, and X has the obvious CW-structure coming from the ones on X_0 , X_1 , and X_2 . Suppose that X_0 , X_1 , and X_2 are L^2 -acyclic and that for $i = 0, 1, 2$ and any point $x_i \in X$ the homomorphism $\pi_1(j_i, x_i): \pi_1(X_i, x_i) \rightarrow \pi_1(X, j_i(x_i))$ is injective. Then X is L^2 -acyclic and we get

$$\rho_u^{(2)}(\tilde{X}) = (j_1)_*(\rho_u^{(2)}(\tilde{X}_1)) + (j_2)_*(\rho_u^{(2)}(\tilde{X}_2)) - (j_0)_*(\rho_u^{(2)}(\tilde{X}_0));$$

(iii) Fibrations

Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. Suppose that F and B are connected finite CW-complexes. Assume that $\pi_1(i): \pi_1(F) \rightarrow \pi_1(E)$ is injective and that F is L^2 -acyclic. Suppose that $\text{Wh}(\pi_1(E))$ vanishes.

Then there is an L^2 -acyclic connected finite CW-complex E' which is homotopy equivalent to E , and we can define $\rho_u^{(2)}(\tilde{E}) = \rho_u^{(2)}(\tilde{E}')$ for any such choice of E' by assertion (i). Moreover, we get

$$\rho_u^{(2)}(\tilde{E}) = \chi(B) \cdot i_*(\rho_u^{(2)}(\tilde{F}));$$

(iv) S^1 -actions

Let X be a connected finite S^1 -CW-complex. If we use the notation and make the assumptions of Theorem 5.74 (vii) in the special case $G = \pi_1(X)$ and $\mu = \text{id}_{\pi_1(X)}$, then X is L^2 -acyclic and we get

$$\rho_u^{(2)}(\tilde{X}) = \sum_{n \geq 0} (-1)^n \cdot \sum_{e \in I_n} j(e)_*(\rho_u^{(2)}(\tilde{S}^1)).$$

Remark 5.82 (Realizability of the universal L^2 -torsion for universal coverings). Let π be group such that there exists a connected finite CW-complex X with $\pi = \pi_1(X)$ which is L^2 -acyclic. Consider any element $\omega \in \text{Wh}^w(\pi)$. As a special case of Lemma 5.78 we get that there is a connected finite CW-complex Y with $\pi = \pi_1(Y)$ such that Y is L^2 -acyclic and $\rho_u^{(2)}(\tilde{Y}) = \omega$.

Also Lemma 5.79 has an obvious analogue for $\rho_u^{(2)}(\tilde{M})$ for a closed manifold M .

Example 5.83 ($\rho_u^{(2)}(\tilde{T}^n)$). If $n \geq 2$, we conclude $\rho_u^{(2)}(\tilde{T}^n) = 0$ from Example 5.77. We have computed $\rho_u^{(2)}(\tilde{S}^1)$ in Example 5.72.

Definition 5.84 (Admissible 3-manifold). A 3-manifold is called *admissible* if it is compact, connected, orientable, and irreducible, its boundary is empty or a disjoint union of tori, and its fundamental group is infinite.

Theorem 5.85 (Universal L^2 -torsion of admissible 3-manifolds) Let M be an admissible 3-manifold and let $M_1, M_2, \dots, M_r$ be its pieces in the Jaco-Shalen-Johannson decomposition. Let $j_i: \pi_1(M_i) \rightarrow \pi_1(M)$ be the injection induced by the inclusion $M_i \rightarrow M$.

Then each M_i and M are L^2 -acyclic and we have

$$\rho_u^{(2)}(\widetilde{M}) = \sum_{i=1}^r (j_i)_* (\rho_u^{(2)}(\widetilde{M}_i)).$$

Proof. Each piece M_i is L^2 -acyclic by [43, Theorem 0,1]. Now the claim follows from Example 5.77 and Theorem 5.81 (2). $\square$

Remark 5.86 (Graph manifolds). We recall that a *Seifert manifold* is loosely speaking a singular S^1 -bundle over a surface. A *graph manifold* is a 3-manifold that is obtained by gluing Seifert manifolds along boundary tori. We refer to [2] for precise definitions. There is an obvious analogue of Theorem 5.81 (iv) for Seifert manifolds M with infinite fundamental groups where the role of S^1 is played by the regular fiber whose inclusion to M always induces an injection on the fundamental groups, and the cells corresponds to tubular neighborhoods of fibers. So we get again a formula for $\rho_u^{(2)}(\widetilde{M})$ which essentially reduces the computation to the one of $\rho_u^{(2)}(\widetilde{S^1})$. In view of Theorem 5.85 this extends to graph manifolds. The hyperbolic pieces in the Jaco-Shalen-Johannson decomposition are much harder to deal with.

5.4 The Weak Whitehead Group and the Whitehead Group of the Linnell Ring

5.4.1 The Construction of the Homomorphisms Θ from the Weak Whitehead Group to the Whitehead Group of the Linnell Ring

Recall that we have defined the K -groups $K_1^w(\mathbb{Z}G)$, $\tilde{K}_1^w(\mathbb{Z}G)$, and $\text{Wh}^w(G)$ in Definition 5.32 and the divison closure $\mathcal{D}_{\mathbb{Q}}(G) = \mathcal{D}(\mathbb{Q}G \subseteq \mathcal{U}(G))$ and the rational division closure $\mathcal{R}_{\mathbb{Q}}(G) = \mathcal{R}(\mathbb{Q}G \subseteq \mathcal{U}(G))$ in Definition 2.75. The K -groups $K_1(\mathcal{D}_{\mathbb{Q}}(G))$, $K_1(\mathcal{R}_{\mathbb{Q}}(G))$, $\tilde{K}_1(\mathcal{D}_{\mathbb{Q}}(G))$, and $\tilde{K}_1(\mathcal{R}_{\mathbb{Q}}(G))$ are defined in the standard way, and we define

$$(5.87) \quad \text{Wh}(\mathcal{D}_{\mathbb{Q}}(G)) \quad \text{and} \quad \text{Wh}(\mathcal{R}_{\mathbb{Q}}(G))$$

to be the cokernel of the map from $\{\sigma \cdot g \mid \sigma \in \{\pm 1\}, g \in G\}$ to $K_1(\mathcal{D}_{\mathbb{Q}}(G))$ or $K_1(\mathcal{R}_{\mathbb{Q}}(G))$ respectively which sends $\sigma \cdot g$ to $[r_{\sigma \cdot g} : \mathcal{D}_{\mathbb{Q}}(G) \xrightarrow{\cong} \mathcal{D}_{\mathbb{Q}}(G)]$.

Next we want to define homomorphisms

$$(5.88) \quad \Theta : K_1^w(\mathbb{Z}G) \rightarrow K_1(\mathcal{R}_{\mathbb{Q}}(G));$$

$$(5.89) \quad \tilde{\Theta} : \tilde{K}_1^w(\mathbb{Z}G) \rightarrow \tilde{K}_1(\mathcal{R}_{\mathbb{Q}}(G));$$

$$(5.90) \quad \Theta^w : \text{Wh}^w(G) \rightarrow \text{Wh}^w(\mathcal{R}_{\mathbb{Q}}(G)).$$

We give the construction only for Θ , the other homomorphisms are then induced by Θ in the obvious way.

Consider a weak isomorphism $f: \mathbb{Z}G^n \rightarrow \mathbb{Z}G^n$. We want to define

$$\Theta([f]) = [\text{id}_{\mathcal{D}_{\mathbb{Q}}(G)} \otimes_{\mathbb{Z}G} f: \mathcal{R}_{\mathbb{Q}}(G) \otimes_{\mathbb{Z}G} \mathbb{Z}G^n \rightarrow \mathcal{R}_{\mathbb{Q}}(G) \otimes_{\mathbb{Z}G} \mathbb{Z}G^n].$$

This definition makes sense if we can show that $\text{id}_{\mathcal{R}_{\mathbb{Q}}(G)} \otimes_{\mathbb{Z}G} f$ is an $\mathcal{R}_{\mathbb{Q}}(G)$ -isomorphism. By definition $\text{id}_{\mathcal{R}_{\mathbb{Q}}(G)} \otimes_{\mathbb{Z}G} f$ is an $\mathcal{R}_{\mathbb{Q}}(G)$ -isomorphism if and only if $\text{id}_{\mathcal{U}(G)} \otimes_{\mathbb{Z}G} f$ is a $\mathcal{U}(G)$ -isomorphism. We conclude from [45, Theorem 6.24 on page 249 and Theorem 8.22 (5) on page 327] that $f^{(2)}: L^2(G)^n \rightarrow L^2(G)^n$ is a weak isomorphism if and only if $\text{id}_{\mathcal{U}(G)} \otimes_{\mathbb{Z}G} f$ is a $\mathcal{U}(G)$ -isomorphism. Hence the definitions of Θ , $\tilde{\Theta}$, and Θ^w make sense.

5.4.2 Review of Universal Localization

Let R be a (associative) ring (with unit) and let Σ be a set of homomorphisms between finitely generated projective (left) R -modules. A ring homomorphism $f: R \rightarrow S$ is called Σ -*inverting* if for every element $\alpha: M \rightarrow N$ of Σ the induced map $S \otimes_R \alpha: S \otimes_R M \rightarrow S \otimes_R N$ is an isomorphism. A Σ -inverting ring homomorphism $i: R \rightarrow R_{\Sigma}$ is called *universal Σ -inverting* if for any Σ -inverting ring homomorphism $f: R \rightarrow S$ there is precisely one ring homomorphism $f_{\Sigma}: R_{\Sigma} \rightarrow S$ satisfying $f_{\Sigma} \circ i = f$. If $f: R \rightarrow R_{\Sigma}$ and $f': R \rightarrow R'_{\Sigma}$ are two universal Σ -inverting homomorphisms, then by the universal property there is precisely one isomorphism $g: R_{\Sigma} \rightarrow R'_{\Sigma}$ satisfying $g \circ f = f'$. This shows the uniqueness of the universal Σ -inverting homomorphism. The universal Σ -inverting ring homomorphism exists, see [65, Section 4]. If Σ is a set of matrices, a model for R_{Σ} is given by considering the free R -ring generated by the set of symbols $\{\bar{a}_{i,j} \mid A = (a_{i,j}) \in \Sigma\}$ and dividing out the relations given in matrix form by $\bar{A}\bar{A} = \bar{A}\bar{A} = 1$, where $\bar{A}$ stands for $(\bar{a}_{i,j})$ for $A = (a_{i,j})$. The map $i: R \rightarrow R_{\Sigma}$ does not need to be injective and the functor $R_{\Sigma} \otimes_R -$ does not need to be exact in general.

A special case of a universal localization is the Ore localization $S^{-1}R$ of a ring R for a multiplicative closed subset $S \subseteq R$ which satisfies the Ore condition, namely take Σ to be the set of R -homomorphisms $r_s: R \rightarrow R$, $r \mapsto rs$, where s runs through S . For the Ore localization the functor $S^{-1}R \otimes_R -$ is exact and the kernel of the canonical map $R \rightarrow S^{-1}R$ is $\{r \in R \mid \exists s \in S \text{ with } rs = 0\}$.

5.4.3 Bijectivity Results for Θ

Definition 5.91 (Class of groups $\mathcal{X}$). Let $\mathcal{X}$ be the smallest class of groups which contains all finitely generated free groups and all amenable groups and is closed under directed unions and group extensions.

Note that $\mathcal{X}$ is subgroup closed, see [36, Lemma 2.5], and contains the class $\mathcal{C}$ of Definition 2.65.

Definition 5.92 (Property (UL)). We say that a group G has the property (UL), if the canonical epimorphism

$$\lambda: \mathbb{Q}G_{\Sigma(\mathbb{Z}G \subseteq \mathcal{U}(G))} \rightarrow \mathcal{R}(G; \mathbb{Q})$$

is bijective, where $\Sigma(\mathbb{Z}G \subseteq \mathcal{U}(G))$ is the set of all $\mathbb{Z}G$ -endomorphisms $f: \mathbb{Z}G^n \rightarrow \mathbb{Z}G^n$ for which $\text{id}_{\mathcal{U}(G)} \otimes_{\mathbb{Z}G} f: \mathcal{U}(G) \otimes_{\mathbb{Z}G} \mathbb{Z}G^n \rightarrow \mathcal{U}(G) \otimes_{\mathbb{Z}G} \mathbb{Z}G^n$ is bijective.

Recall that $\text{id}_{\mathcal{U}(G)} \otimes_{\mathbb{Z}G} f: \mathcal{U}(G) \otimes_{\mathbb{Z}G} \mathbb{Z}G^n \rightarrow \mathcal{U}(G) \otimes_{\mathbb{Z}G} \mathbb{Z}G^n$ is bijective if and only if $f: \mathbb{Z}G^n \rightarrow \mathbb{Z}G^n$ is a weak isomorphism, see [45, Theorem 6.24 on page 249 and Theorem 8.22 (5) on page 327]. Note that

$$\mathbb{Q}G_{\Sigma(\mathbb{Z}G \subseteq \mathcal{U}(G))} = \mathbb{Q}G_{\Sigma(\mathbb{Q}G \subseteq \mathcal{U}(G))}$$

holds if $\Sigma(\mathbb{Q}G \subseteq \mathcal{U}(G))$ is the set of all $\mathbb{Q}G$ -endomorphisms $f: \mathbb{Q}G^n \rightarrow \mathbb{Q}G^n$ for which $\text{id}_{\mathcal{U}(G)} \otimes_{\mathbb{Q}G} f: \mathcal{U}(G) \otimes_{\mathbb{Q}G} \mathbb{Q}G^n \rightarrow \mathcal{U}(G) \otimes_{\mathbb{Q}G} \mathbb{Q}G^n$ is bijective.

Definition 5.93 (Property (ULA)). We say that a torsionfree group G has the property (ULA) if the canonical epimorphism

$$\lambda: \mathbb{Q}G_{\Sigma(\mathbb{Z}G \subseteq \mathcal{U}(G))} \rightarrow \mathcal{R}_{\mathbb{Q}}(G)$$

is bijective, and $\mathcal{D}_{\mathbb{Q}}(G)$ is a skew field.

Remark 5.94. For a torsionfree group G , we conclude from Theorem 2.83 and Proposition 2.79 (ii) that $\mathcal{D}_{\mathbb{Q}}(G)$ is a skew field if and only if G satisfies the Strong Atiyah Conjecture 2.28 with coefficients in $\mathbb{Q}$ for torsionfree groups and that we have $\mathcal{D}_{\mathbb{Q}}(G) = \mathcal{R}_{\mathbb{Q}}(G)$, provided that $\mathcal{D}_{\mathbb{Q}}(G)$ is a skew field. So G satisfies condition (ULA) if and only if G satisfies both condition (UL) and the Strong Atiyah Conjecture 2.28 with coefficients in $\mathbb{Q}$ for torsionfree groups

Theorem 5.95 (Bijectivity of Θ)

- (i) Suppose that the torsionfree group G satisfies condition (ULA). Then $\mathcal{D}_{\mathbb{Q}}(G)$ is a skew field, $\mathcal{D}_{\mathbb{Q}}(G) = \mathcal{R}_{\mathbb{Q}}(G)$, and the map

$$\Theta: K_1^w(\mathbb{Z}G) \rightarrow K_1(\mathcal{D}_{\mathbb{Q}}(G))$$

appearing in (5.88) is bijective. This is also true for the maps $\tilde{\Theta}$ of (5.89) and Θ^w of (5.90);

- (ii) Let G be a group which contains a torsionfree subgroup of finite index such that H satisfies the Strong Atiyah Conjecture 2.28 with coefficients in $\mathbb{Q}$ for torsionfree groups and H belongs to $\mathcal{X}$. Then G has property (UL);
- (iii) Let G be a torsionfree group G such that G satisfies the Strong Atiyah Conjecture 2.28 with coefficients in $\mathbb{Q}$ for torsionfree groups and G belongs to $\mathcal{X}$, then G has property (ULA);

(iv) Each torsionfree group belonging to the class $\mathcal{C}$ has property (ULA).

Proof. (i) See [36, Theorem 3.5].

(ii) This follows Remark 5.94 and [36, Theorem 2.14].

(iii) This follows from assertion (ii).

(iv) This follows from assertion (iii), since $\mathcal{C} \subseteq \mathcal{X}$ holds and we have Theorem 2.68 (i). $\square$

5.4.4 Dieudonné Determinant

There is a *Dieudonné determinant* for invertible matrices over a skew field D which takes values in the abelianization $D_{\text{ab}}^\times$ of the multiplicative group of units $D^\times$ and induces an isomorphism, see [68, Corollary 4.3 in page 133]

$$(5.96) \quad \det_D: K_1(D) \xrightarrow{\cong} D_{\text{ab}}^\times.$$

The inverse

$$(5.97) \quad J_D: D_{\text{ab}}^\times \xrightarrow{\cong} K_1(D)$$

sends the class of a unit in D to the class of the corresponding $(1, 1)$ -matrix.

Theorem 5.98 (Computing $K_1^w(\mathbb{Z}G)$)

Let G be a torsionfree group satisfying one of the following conditions:

- G satisfies the Strong Atiyah Conjecture 2.28 with coefficients in $\mathbb{Q}$ for torsionfree groups and G belongs to the class of groups $\mathcal{X}$;
- G belongs to the class of groups $\mathcal{C}$.

Then the map of (5.88)

$$\Theta: K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(\mathcal{D}_{\mathbb{Q}}(G))$$

is bijective, $\mathcal{D}_{\mathbb{Q}}(G)$ is a skew field, and the Dieudonné determinant induces an isomorphism

$$\det_{\mathcal{D}_{\mathbb{Q}}(G)}: K_1(\mathcal{D}_{\mathbb{Q}}(G)) \xrightarrow{\cong} \mathcal{D}_{\mathbb{Q}}(G)_{\text{ab}}^\times.$$

Proof. This follows from Theorem 5.95 and (5.96). $\square$

Note that the isomorphism (5.73) is the special case $G = \mathbb{Z}$ of Theorem 5.98.

Remark 5.99. All constructions and results of this Section 5.4 carry over in the obvious way to the more general case, where we replace $\mathbb{Z}$ by any ring R with $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$, the field $\mathbb{Q}$ by the quotient field F of R which automatically satisfies $F \subseteq \mathbb{C}$, and define the new versions of $\text{Wh}^w(G)$ and $\text{Wh}(\mathcal{D}_F(G))$ to be $K_1^w(RG)/\{\pm g\}$ and $K_1(\mathcal{D}_F(G))/\{\pm g\}$.

5.5 Polytopes

5.5.1 Basics on Polytopes

5.5.2 The Polytope Group

5.5.3 The Polytope Homomorphism

5.5.4 The L^2 -Polytope

5.5.5 Survey on the Thurston Norm and the Thurston Polytope

5.5.6 Identifying the Thurston Polytope and the L^2 -Polytope

5.5.7 The Thurston Norm and L^2 -Euler Characteristics

5.6 Twisting with Finite Dimensional Representations

5.7 Notes

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Chapter 6

Approximation

6.1 Introduction

6.2 Review

6.3 Title of the Section

6.4 Notes

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Chapter 7

Applications to Group Theory

7.1 Introduction

7.2 Review

7.3 Title of the Section

7.4 Notes

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Chapter 8

Measurable Group Theory

8.1 Introduction

8.2 Review

8.3 Title of the Section

8.4 Notes

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Chapter 9

Applications to Geometry

9.1 Introduction

9.2 Review

9.3 Title of the Section

9.4 Notes

We might want to cover among others the following topics:

- The (old) article of Davaux [13] proves a relation between the bottom of the scalar Laplacian and lower bounds on scalar curvature. She uses a refined Kato inequality and to be able to apply this she uses Atiyah's L^2 -index theorem (to provide non-trivial harmonic L^2 -spinors), as well as Ramachandran's L^2 -version of the APS index theorem and spectral flow.
- Wang [74] proves a far-reaching generalization of Atiyah's L^2 -index theorem to proper cocompact actions of unimodular locally compact groups (using the heat kernel method).
This is further generalized to a corresponding version of the Atiyah-Patodi-Singer index theorem in [24].

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Chapter 10

Fibering Over S^1 and $\mathbb{Z}$

10.1 Introduction

10.2 Farrell's PHD-Thesis

10.3 L^2 -Betti Numbers as Obstructions

10.4 Novikov Homology

10.5 Aspherical Counterexamples

10.6 Twisted L^2 -Torsion Function as a Secondary Obstruction

10.7 Universal L^2 -Torsion as a Secondary Obstruction

10.8 Notes

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Chapter 11

Further Topics

11.1 Introduction

11.2 Title of the Section

11.3 Errata to Part I

Comment 31 (by W.): Add a list Errata correcting misprints and wrong statement appearing in [45].

11.4 Notes

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Chapter 12

Solutions of the Exercises

Chapter 2

2.4. Let R be a ring coming with a trace $\text{tr}_R: R \rightarrow G$ with values in an abelian group G . This trace extends to matrices by the usual formula $\text{tr}_R(A) = \sum_{i=1}^n \text{tr}_R(a_{i,i})$ for $A \in M_n(R)$. Then the Hattori-Stallings rank $\dim_R(P) \in G$ of a finitely generated projective R -module P is defined to be $\text{tr}_R(A)$ where A is an element in $M_n(R)$ with $A^2 = A$ such that the image of the R -linear map $r_A: R^n \rightarrow R^n$ induced by right multiplication with A is R -isomorphic to P . Let $B \in M_m(R)$ be another such matrix. By taking the block sum with trivial square matrices one can arrange $m = n$. Then one can find $U \in \text{GL}_n(R)$ with $UAU^{-1} = B$. Now the trace property implies $\text{tr}_R(B) = \text{tr}_R(UAU^{-1}) = \text{tr}_R(AU^{-1}U) = \text{tr}_R(A)$.

If we apply this to $R = \mathcal{N}(G)$ and $\text{tr}_R = \text{tr}_{\mathcal{N}(G)}$, we obtain a well-defined element $\dim_{\mathcal{N}(G)}(P) \in \mathbb{C}$. It remains to prove that it actually lies in $\mathbb{R}_{\geq 0}$. This follows from the fact that we can always find $A \in M_n(\mathcal{N}(G))$ satisfying $A = A^2$, $\text{im}(r_A) \cong_{\mathcal{N}(G)} P$, and $A = B^*B$ for some $B \in M_n(\mathcal{N}(G))$. This follows from Remark 2.5 and taking orthogonal projections. This implies $\dim_{\mathcal{N}(G)}(P) = \text{tr}_{\mathcal{N}(G)}(B^*B) \in \mathbb{R}_{\geq 0}$.

2.10. The first assertion follows from [45, Theorem 1.9 (3) on page on page 18] and obviously implies the second assertion.

2.36. The implication (i) $\implies$ (ii) follows from Remark 2.35. The implication (ii) $\implies$ (iii) is obvious. The implication (iii) $\implies$ (i) is proved as follows. Let $r = \inf\{q \in \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z} \mid q > 0\}$. As the subgroup $\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$ is countable and closed in $\mathbb{R}$, we must have $r \in \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$ and $r > 0$. This implies $r \leq |H|^{-1}$ for every finite subgroup $H \subseteq G$. Hence r^{-1} is an upper bound on the orders of the finite subgroups of G .

2.53. Take $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2^k$. Choose $n \in \mathbb{Z}_{\geq 1}$ such that $2^{1-n} \leq \epsilon$ holds. Put $P = \bigoplus_{k=n}^{\infty} F[G/\mathbb{Z}_{2^k}]$, where $F[G/\mathbb{Z}_{2^k}]$ is the permutation FG -module given by the FG -set $G/\mathbb{Z}_{2^k}$ obtained from G by dividing out the summand $\mathbb{Z}/2^k$. Then P is projective but not finitely generated and we conclude from Theorem 2.11

$$\begin{aligned} \dim_{\mathcal{N}(G)}(P) &= \sum_{k=n}^{\infty} \dim_{\mathcal{N}(G)}(F[G/\mathbb{Z}/2^k]) = \sum_{k=n}^{\infty} \dim_{\mathcal{N}(\mathbb{Z}/2^k)}(F) \\ &= \sum_{k=n}^{\infty} 2^{-k} = 2^{1-n} < \epsilon. \end{aligned}$$

2.56. A possible answer is $\mathbb{Q}$.

2.64. This follows from Theorem 2.57 (i).

2.72. This is Theorem 2.68 (v) in the special case $F = \mathbb{C}$.

2.73. Any finite subgroup H of G is conjugate to a subgroup of G_i for some element $i \in I$ by the Kurosh Subgroup Theorem, see for instance [50, Theorem 1.10 on page 178]. Hence the order of every finite subgroup of G is bounded by d and hence G satisfies condition (B).

It remains to show that G satisfies the strong Strong Atiyah Conjecture 2.27. Because Theorem 2.68 (v) we can assume without loss of generality that I is finite. By induction over the cardinality of I we can reduce the claim to the case where I contains two elements. This case is already taking care of by Sanchez Peralta [62, Theorem 3.4].

2.77. Consider $x \in \bigcup_{n \geq 0} \mathcal{D}_n(R \subseteq S)$ such that the inverse of x exists in S . Choose $n \in \mathbb{Z}_{\geq 0}$ with $x \in \mathcal{D}_n(R \subseteq S)$. Then $x \in \mathcal{D}_{n+1}(R \subseteq S)$ holds by construction and hence we get $x \in \bigcup_{n \geq 0} \mathcal{D}_n(R \subseteq S)$. This implies $\mathcal{D}(R \subseteq S) \subseteq \bigcup_{n \geq 0} \mathcal{D}_n(R \subseteq S)$.

Consider any division closed subring $T \subseteq S$ with $R \subseteq T$. One easily checks by induction that $\mathcal{D}_n(R \subseteq S) \subseteq T$ holds for $n = 0, 1, 2, \dots$. This implies $\mathcal{D}(R \subseteq S) \subseteq \bigcup_{n \geq 0} \mathcal{D}_n(R \subseteq S)$.

Chapter 3

3.12. Since $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$ holds it suffices to show $\text{dom}(\psi_1 \circ \varphi_1) = \text{im}(\psi_1)$.

Consider two ring homomorphisms $\alpha, \beta: S \rightarrow U$ satisfying $\alpha \circ \psi_1 \circ \varphi_1 = \beta \circ \psi_1 \circ \varphi_1$. Since φ_1 is epic, this implies $\alpha \circ \psi_1 = \beta \circ \psi_1$. Hence we get $\alpha(\psi_1(s_1)) = \beta(\psi_1(s_1))$ for every $s_1 \in S_1$. This shows $\text{im}(\psi_1) \subseteq \text{dom}(\psi_1 \circ \varphi_1)$.

Consider $s_1 \in \text{dom}(\psi_1 \circ \varphi_1)$. Proposition 3.11 (i) implies $s_1 \in \text{im}(\psi_1 \circ \varphi_1) \subseteq \text{im}(\psi_1)$. Hence we get $\text{dom}(\psi_1 \circ \varphi_1) \subseteq \text{im}(\psi_1)$.

3.17. For every $r \in \mathcal{Z}(R)$ the map $S \times S \rightarrow S \otimes_R S$ sending $s_1 \otimes s_2$ to $s_1 r s_2$ is R -bilinear and hence defines a homomorphism $\varphi: S \otimes_R S \rightarrow S \otimes_R S$ satisfying $\varphi(s_1 \otimes s_2) = s_1 \otimes r s_2$. If $m: S \otimes_R S \rightarrow S$ is the multiplication map sending $s_1 \otimes s_2$ to $s_1 s_2$, then Proposition 3.10 implies for all $s \in S$

$$sr = m \circ \varphi(s \otimes 1_S) = m \circ \varphi(1_S \otimes s) = rs$$

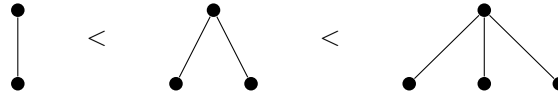
and hence $r \in \mathcal{Z}(S)$.

3.18. Since X is connected, finite, and 1-dimensional, $H_0(X; \mathbb{Z}) \cong \mathbb{Z}$ and $H_1(X; \mathbb{Z}) \cong \mathbb{Z}^r$ for some $r \in \mathbb{Z}_{\geq 0}$. We have $\chi(X) = 1 - r$. Hence $H_1(X; \mathbb{Z})$ vanishes if and only if $\chi(X) = 1$.

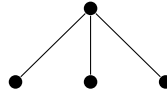
3.22. Let x be a vertex in X different from the root x_0 . Let Y be the unique element in $\text{fam}(X)$ containing y . Let y_0 be the root of Y . Then $d_Y(y_0, y) = d_X(y, y_0) = d_X(y, x_0) - 1$.

3.45. Since $\log(X)$ and $\log(Y)$ agree, we have to decide for $A = X - \text{expand}(\log(X))$ and $B = Y - \text{expand}(\log(Y))$ whether $A < B$ or $B < A$ holds.

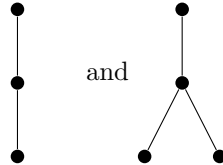
We have



Hence $\log(A)$ and $\log(B)$ agree, they are both given by



So we have to decide whether $A - \text{expand}(\log(A)) < B - \text{expand}(\log(B))$ or $A - \text{expand}(\log(A)) > B - \text{expand}(\log(A))$ holds. One easily checks that $A - \text{expand}(\log(A))$ and $B - \text{expand}(\log(B))$ look like



Since $\log(B - \text{expand}(\log(B)))$ and $\log(A - \text{expand}(\log(A)))$ look like



we get $\log(B - \text{expand}(\log(B))) < \log(A - \text{expand}(\log(A)))$. Hence $Y < X$.

3.68. The factorization is a direct consequence of Theorem 3.51.

Let $X_{\mathcal{U}}$ be the U -biset occurring in the construction of $\text{Rat}(U)$ and let X be the corresponding $\{1\}$ -biset occurring in the construction of $\text{Rat}(\{1\})$. Now the claim for $\text{Rat}(\text{pr})$ follows directly from Remark 3.63, since $\text{Rat}(\text{pr})$

comes from maps $(X_U)_n \rightarrow X_n$ which are surjective but not injective for $n \in \mathbb{Z}_{\geq 1}$. For instance for $n = 1$ it is given by the map $(X_U)_1 = \mathbb{N}[U] \setminus \{0\} \rightarrow X_1 = \mathbb{N}[\{1\}] \setminus \{0\} = \mathbb{Z}_{\geq 1}$ sending $\sum_{u \in U} n_u \cdot u$ to $\sum_{u \in U} n_u$.

Next we show that the map $\text{Tree}_{\{1\}}$ is not injective. This follows by inspecting the construction of $\text{Tree}_{\{1\}}$ described in Remark 3.63. Namely, we have

$$\begin{aligned} X_0 &= \emptyset; \\ X_1 &= (\mathbb{N}[\{1\}] \setminus \{0\})^\diamond = \mathbb{Z}_{\geq 1}; \\ X_2 &= (\mathbb{N}[\{1\} \# \mathbb{Z}_{\geq 1}] \setminus \mathbb{N}[\{1\} \# \emptyset])^\diamond \amalg X_1. \end{aligned}$$

The map $\text{Tree}_{\{1\}}$ sends an element m in $X_1 = \mathbb{Z}_{\geq 1}$ to $m \cdot 1_{\mathcal{T}}$.

An element in $\mathbb{N}[\{1\} \# \mathbb{Z}_{\geq 1}] \setminus \mathbb{N}[\{1\} \# \emptyset]$ is a finite formal sum $\sum_{\underline{m}} n_{\underline{m}} \cdot \underline{m}$ where $\underline{m}$ runs through all finite sequences $(m_1, m_2, \dots, m_l)$ for $l \in \mathbb{Z}_{\geq 1}$ and $m_i \in \mathbb{Z}_{\geq 0}$, we have $n_{\underline{m}} \in \mathbb{Z}_{\geq 0}$, and at least one of the elements $n_{\underline{m}}$ is different from zero. The map $\text{Tree}_{\{1\}}$ restricted to $(\mathbb{N}[\{1\} \# \mathbb{Z}_{\geq 1}] \setminus \mathbb{N}[\{1\} \# \emptyset])^\diamond \subseteq X_2 \subseteq \text{Rat}(\{1\})$ sends $\sum_{\underline{m}} n_{\underline{m}} \cdot \underline{m}$ to the element $\sum_{\underline{m}} n_{\underline{m}} \cdot \text{expand}^2((\prod_{i=1}^l m_i) \cdot 1_{\mathcal{T}})$ in $\mathcal{T}$ and is obviously not injective.

Finally we explain why $\text{Rat}_{\{1\}}$ is surjective. We will show by induction over $n = 0, 1, 2, \dots$ that T lies in the image of $\text{Tree}_{\{1\}}$ for any rooted tree T with $\text{height}(T) = n$. If $n = 0$, this is obvious. If the height of T is 1, this follows from the description of $\text{Tree}_{\{1\}}$ restricted to X_1 above since T looks like $m \cdot 1_{\mathbb{R}}$ for $m \in \mathbb{Z}_{\geq 1}$. The induction step from $(n-1)$ to $n \geq 2$ is done as follows.

Consider a tree T of height n . We can write $T = \sum_{i=1}^m \text{expand}(T_i)$ for appropriate elements $T_i \in \mathcal{T}$ satisfying $\text{height}(T_i) \leq n-1$. For every $i \in \{1, 2, \dots, m\}$ we can write $\text{expand}(T_i)$ as a finite product $\prod_{j=1}^n \text{expand}^2(S_j)$ for appropriate elements $S_j \in \mathcal{T}$ with $\text{height}(S) \leq n-2$. Hence it suffices to show that $\text{expand}^2(S)$ lies in the image of $\text{Tree}_{\{1\}}$ for every rooted tree S with $\text{height}(S) \leq (n-2)$. By induction hypothesis there is an element $r \in \text{Rat}_{\{1\}} \amalg \{0\}$ satisfying $\text{Tree}_{\{1\}}(r) = S$. Now the claim follows from

$$\text{expand}^2(S) = \diamond_{\mathcal{T}}(S) = \diamond_{\mathcal{T}} \circ \text{Tree}_{\{1\}}(r) = \text{Tree}_{\{1\}} \circ \diamond_R(r).$$

3.75. Obviously ψ sends $\mathcal{D}(\varphi_1(G) \subseteq R_1)$ to $\mathcal{D}(\varphi_2(G) \subseteq R_2)$. We conclude $(\psi \amalg \text{id}_{\infty}) \circ \Phi_{G, D_1} = \Phi_{G, D_2}$ from the uniqueness part appearing in Definition 3.50.

3.107. We get $P \subseteq \text{Primitive}(U)$ from Lemma 3.105 (ii). If $U = \{1\}$, we get $P = \text{Rat}(U)$ from Lemma 3.92 (vi), and hence $P = \text{Primitive}(U)$.

Suppose that there exists $p \in P$ with non-trivial $\text{source}_U(p)$. Choose $u \in \text{source}_U(p)$ with $u \neq 1_U$. Put $r = pu$. Then r does not belong to P since Lemma 3.92 (i) together with $Q \subseteq X$ and $X \cap U = \emptyset$ implies $P \cap U =$

$\{1\}$. From $p \in \text{Rat}(\text{source}_U(p))$ and $u \in \text{source}_U(p) \subseteq \text{Rat}(\text{source}_U(p))$, we conclude $r \in \text{Rat}(\text{source}_U(p)) = \text{Rat}(\text{source}(r))$ and hence $P \neq \text{Rat}(U)$.

Suppose that $\text{source}_U(p) = \{1\}$ holds for all $p \in P$. Since $PU = \text{Rat}(U)$ holds by Lemma 3.92 (vi), we conclude $\text{source}(p) = \{1_U\}$ for every $r \in \text{Rat}(U)$. Hence $r \in \text{Rat}(\{1_U\}) \cdot U$ for every $r \in \text{Rat}(U)$. This implies $\text{Rat}(U) = \text{Rat}(\{1_U\}) \cdot U$ and hence $U = \{1_U\}$.

3.129. (i) This follows from $\text{rk}(A) = \text{rk}(I_m \cdot A) \leq \min\{\text{rk}(I_m), \text{rk}(A)\} = \min\{m, \text{rk}(A)\}$ and $\text{rk}(A) = \text{rk}(A \cdot I_n) \leq \min\{\text{rk}(A), \text{rk}(I_n)\} = \min\{\text{rk}(A), n\}$ coming from property (iii) appearing in Definition 3.127.

(ii) This follows from $n = \text{rk}(I_n) = \text{rk}(A \cdot A^{-1}) \leq \min\{\text{rk}(A), \text{rk}(A^{-1})\} \leq \text{rk}(A)$ and assertion (ii).

(iii) We have

$$\begin{pmatrix} 0 & 0 \\ 0 & A+B \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ I_m & I_m \end{pmatrix} \cdot \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \cdot \begin{pmatrix} 0 & I_n \\ 0 & I_n \end{pmatrix}.$$

We conclude from properties (i), (iv), and (iii) appearing in Definition 3.127

$$\begin{aligned} \text{rk}(A+B) = \text{rk} \begin{pmatrix} 0 & 0 \\ 0 & A+B \end{pmatrix} &\leq \min \left\{ \text{rk} \begin{pmatrix} 0 & 0 \\ I_m & I_m \end{pmatrix}, \text{rk} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \text{rk} \begin{pmatrix} 0 & I_n \\ 0 & I_n \end{pmatrix} \right\} \\ &\leq \text{rk} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \text{rk}(A) + \text{rk}(B). \end{aligned}$$

(iv) In particular we get from assertion (iii) the implication $\text{rk}(A) = 0 \implies \text{rk}(A+B) \leq \text{rk}(B)$. Since $\text{rk}(A) = 0 \implies \text{rk}(-A) = 0$ by property (iii) appearing in Definition 3.127, we obtain the implication $\text{rk}(A) = 0 \implies \text{rk}(A+B) = \text{rk}(B)$.

3.134. Given A, B in $M_{m,n}(R)$, we get from Exercise 3.129 (iv) the implication $\text{rk}(A) = 0 \implies \text{rk}(A+B) = \text{rk}(B)$.

Given $r, s \in \ker(\text{rk})$, we conclude $r+s \in \ker(\text{rk})$. If $r \in R$ and $s \in \ker(\text{rk})$, then $rs \in \ker(\text{rk})$ follows directly from the property (iii) appearing in Definition 3.127. Hence $\ker(\text{rk})$ is an ideal in R .

One easily checks that $\overline{\text{rk}}$ is a well-defined faithful Sylvester matrix rank function.

3.135. Obviously (ii) $\implies$ (i) holds. Next we show (i) $\implies$ (ii). Consider $A \in M_{m,n}(R)$. Fix $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$. Let $E \in M_{1,m}(R)$ be the matrix all whose entries are zero except the entry at $(1, i)$ which is 1. Let $F \in M_{n,1}(R)$ be the matrix all whose entries are zero except the entry at $(j, 1)$ which is 1. Then EAF is the $(1, 1)$ -matrix $(a_{i,j})$. We conclude from property (iii) appearing in Definition 3.127

$$\mathrm{rk}(a_{i,j}) = \mathrm{rk}(EAF) \leq \mathrm{rk}(A) = 0.$$

Hence $a_{i,j} = 0$. This implies $A = 0$.

Suppose that (iii) holds. Consider $A \in M_{n,n}(R)$ with $\mathrm{rk}(A) = n$. Then $\dim(\mathrm{coker}(r_A)) = 0$ and hence $r_A: R^n \rightarrow R^n$ is surjective. Then $\ker(r_A)$ is finitely generated projective and we have $\mathrm{rk}(r_A) \oplus R^n \cong R^n$. This implies $\mathrm{rk}(r_A) + n = n$ and hence $\mathrm{rk}(\ker(r_A)) = 0$. We conclude that $\ker(r_A)$ is trivial. Hence A is invertible. We have $\mathrm{rk}(A) = n$ if A is invertible, see Exercise 3.129 (ii). Hence (iii) $\iff$ (iv) holds.

Now suppose that R is von Neumann regular. Next we show (i) $\implies$ (iii). Consider a finitely presented R -module M with $\dim(M) = 0$. We can find finitely many idempotents $x_1, x_2, \dots, x_n$ with $M = \bigoplus_{i=1}^n (Rx_i)$ by Proposition 2.18. Fix $i \in \{1, 2, \dots, n\}$. We conclude $\dim(Rx_i) = 0$ from $\dim(M) = \sum_{i=1}^n \dim(Rx_i)$. This implies $\mathrm{rk}(Rx_i) = 0$. Hence we get $\mathrm{rk}(R(1 - x_i)) = 1$. We have $Rx_i = 0 \iff (1 - x_i) \in R^\times$. Hence it suffices to show for $x \in R$ with $\mathrm{rk}(x) = 1$ that x is invertible. Choose $y \in R$ with $xyx = x$. Then we conclude from Proposition 3.131 (iii)

$$\mathrm{rk}(1 - yx) = \mathrm{rk}(x(1 - yx)) = \mathrm{rk}(x - xyx) = \mathrm{rk}(0) = 0.$$

Since rk is faithful by assumption, we get $yx = 1$. This implies

$$1 = \mathrm{rk}(yx) \leq \min\{\mathrm{rk}(y), \mathrm{rk}(x)\} \leq \mathrm{rk}(y) \leq 1.$$

and hence $\mathrm{rk}(y) = 1$. By the argument above now applied to y instead of x we get $z \in R$ with $zy = 1$. This implies $x \in R^\times$. This finishes the proof of the implication (i) $\implies$ (iii).

We have already shown (iii) $\implies$ (iv)

Finally we show (iv) $\implies$ (i). Consider $r \in R$ with $\mathrm{rk}(r) = 0$. Since R is von Neumann regular, there exists an idempotent $x \in R$ with $Rr = Rx$ by Proposition 2.18. Hence $\mathrm{rk}(x) = 1 - \dim(R/Rx) = 1 - \dim(R/Rr) = \mathrm{rk}(r) = 0$. Since $r = 0$ holds if and only if $Rr = Rx$ vanishes, it suffices to show for an idempotent $x \in R$ with $\mathrm{rk}(x) = 0$ that $x = 0$ holds. Since $1 = \mathrm{rk}(1) = \mathrm{rk}((1 - x) + x) = \mathrm{rk}(1 - x)$ holds by Exercise 3.129 (iv), $(1 - x)$ is a unit in R . Since $(1 - x) \cdot x = x - x^2 = x - x = 0$ holds, we get $x = 0$.

3.136. Since the functor $R^{(0)} \otimes_R -$ is right exact, $\dim_R$ is a faithful integer-valued Sylvester module rank function on R whose associated Sylvester rank function is faithful.

Let $\dim$ be any integer-valued Sylvester module rank function on R whose associated Sylvester rank function rk is faithful. We want to show $\dim = \dim_R$. Consider any finitely presented R -module M . Since M is the direct sum $R^m \oplus_{i=1}^n R/(r_i)$ for appropriate $m, n \in \mathbb{Z}_{\geq 0}$ and non-zero-elements $r_1, r_2, \dots, r_n$ and we have $\dim(M) = m + \sum_{i=1}^n \dim(R/(r_i))$ and $\dim_R(M) = m$, it suffices to show for any $r \in R$ that $\dim(R/(r)) \neq 0 \implies r = 0$ holds.

Since $\dim$ is integer-valued, $\dim(R/(r)) \neq 0$ implies $\dim(R/(r)) = 1$ and hence $\operatorname{rk}(r) = 0$. Since rk is faithful, we get $r = 0$.

3.144. Since every finitely presented R -module is projective for a von Neumann regular ring R , every short exact sequence of finitely presented R -modules is split exact.

3.154. Consider $A \in M_{m,n}(R)$. Let M be the finitely presented R -module given by the cokernel of the R -homomorphism $r_A: R^m \rightarrow R^n$. One easily checks that Φ_*M is isomorphic to the cokernel of the R -homomorphism $r_{\Phi(A)}: R^m \rightarrow R^n$. Hence $\dim$ is Φ -compatible if and only if $\operatorname{rk}(A) = \operatorname{rk}(\Phi(A)) = \Phi^\#(\operatorname{rk})(A)$ holds for every $A \in M_{m,n}(R)$.

Chapter 4

5.22. Since X is simply connected, we have $\pi_1(X) = \{1\}$ and $\tilde{X} = X$. The cellular $\mathbb{Z}$ -chain complex $C_*(X[n])$ is concentrated in dimensions 0 and 1 and its first differential $c_1: \mathbb{Z}^n \rightarrow \mathbb{Z}^{n+1}$ is given by the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & +1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & +1 \end{pmatrix}$$

The kernel of $c_1^{(2)} = \operatorname{id}_{\mathbb{C}} \otimes_{\mathbb{Z}} c_1: \mathbb{C}^n \rightarrow \mathbb{C}^{n+1}$ is $H_1([0, 1]; \mathbb{C})$ and hence is trivial, whereas the cokernel is $H_0([0, 1]; \mathbb{C})$ and hence $\mathbb{C}$. Therefore X is not L^2 -acyclic.

The operator $(c_1^{(2)})^* \circ c_1^{(2)}$ is given by the (n, n) -matrix

$$A[n] = \begin{pmatrix} +2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & +2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & +2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & +2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & +2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & +2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix}$$

We compute for $n \in \mathbb{Z}_{\geq 2}$

$$\det(A[n+1]) = 2 \cdot \det(A[n]) - \det(A[n-1]).$$

Since $\det(A[1]) = 2$ and $\det(A[2]) = 3$ holds, we get by induction for $n \in \mathbb{Z}_{\geq 1}$

$$\det(A[n]) = n + 1$$

Hence X is of determinant class and we get from Example 5.8

$$\rho^{(2)}(\widetilde{X}) = \sqrt{\det(A[n])} = \sqrt{n+1}.$$

5.25. Suppose that M carries a non-trivial S^1 -action. Since M is aspherical, we conclude from Theorem 5.23 (xi)

that all L^2 -Betti numbers $b_n^{(2)}(\widetilde{M})$ and the L^2 -torsion $\rho^{(2)}(\widetilde{M})$ vanish. This contradicts from Theorem 5.23 (xiii).

5.26. Since hyperbolic closed manifolds are aspherical, two hyperbolic closed manifolds with the isomorphic fundamental groups are homotopy equivalent. Since the fundamental group of a hyperbolic closed manifolds is residually finite and hence satisfies the Determinant Conjecture 5.10, two closed hyperbolic manifolds with isomorphic fundamental groups have the same L^2 -torsion by Theorem 5.23 (i). Now the claim follows from Theorem 5.23 (xiii).

5.43. Assertion (i) follows from Theorem 5.9., whereas assertion (ii) follows from the definitions.

5.48. One easily checks that ν is well-defined using the universal property of (U^{mW}, u^{mW}) . The inverse of ν^{-1} of ν is uniquely determined by the property that for every $\mathbb{Z}G$ -chain homotopy equivalence $f_*: C_* \rightarrow D_*$ of $\{\pm g\}$ -based finite free $\mathbb{Z}G$ -chain complexes and every $n \in \mathbb{Z}$ we have

$$\nu^{-1}(\tau^W(f_*), n) = u^{mW}(f_*) + (n - \chi_{\mathbb{Z}G}(C_*)) \cdot u^{mW}(\text{id}_{0[\mathbb{Z}G]_*}).$$

5.49. This follows immediately from the additivity of $\text{rk}_{\mathbb{Z}G}$ for short exact sequence of finitely generated free $\mathbb{Z}G$ -modules.

5.59. This follows from Lemma 5.50 (i) and (iii), from Remark 5.57, and from Lemma 5.58 (i) and (ii), since the elements $u^W(f_*)$, $u^R(C_*)$, and $u^{(2)}(D_*)$ respectively generate U^W , U^R , and $U_u^{(2)}$, see Remark 5.47.

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Chapter 9

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property

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