

L^2 -TORSION OF AUTOMORPHISMS

SAM HUGHES AND WOLFGANG LÜCK

ABSTRACT. We develop the theory of L^2 -torsion of an automorphism of a group and compute it for every automorphism of a group which is hyperbolic and one-ended relative to a finite collection of virtually polycyclic groups. We also prove a combination formula for the L^2 -torsion of a group in terms of the L^2 -torsion of its stabilisers of a sufficiently nice action on a contractible space. We apply it to compute the L^2 -torsion of a selection of CAT(0) lattices, of many relatively hyperbolic groups and their automorphisms, of higher dimensional graph manifolds, and of handlebody groups.

1. INTRODUCTION

The paper deals with L^2 -torsion and computations of it, in particular for groups and group automorphisms. Before we are describing our results, we give a brief survey about the relevance of L^2 -torsion.

The L^2 -torsion $\rho^{(2)}$ is an invariant of groups and spaces which is defined for a large class of groups and spaces with vanishing L^2 -homology and can be defined analytically and topologically. When defined, the invariant is a real number which behaves similarly to an Euler characteristic in the sense that it is multiplicative through finite covers. It seems plausible that it should behave like a hyperbolic volume. Outside of the world of closed locally symmetric spaces [41] and 3-manifolds [38, 33], computing L^2 -torsion remains a formidable challenge and there are very few examples, e.g., [13, 48]. For the definition of L^2 -torsion and a discussion about the Determinant Conjecture, which ensures the L^2 -torsion is defined, we refer the reader to Section 2. For a comprehensive introduction to L^2 -invariants the reader is referred to [33].

One should think of L^2 -torsion as a generalization of the notion of volume. Namely, if M is a compact manifold of odd dimension n whose interior is a complete hyperbolic manifold of finite volume, then its volume is up to a dimension constant $C_n \neq 0$ proportional to the L^2 -torsion, see Lück–Schick [38]. There are other notions generalizing the notion of the volume for hyperbolic manifolds of finite volume such as its minimal volume entropy $\mathcal{E}_{\min}(M)$ and its simplicial volume $\|M\|$. Work of Gromov [18], Pieroni [42], Soma [46], and Thurston [47] shows that $\text{Vol}(M)$, $\mathcal{E}_{\min}(M)^3$, and $\|M\|$ are proportional for hyperbolic orientable closed 3-manifolds and that $\text{Vol}(M)$ and $\|M\|$ are proportional for hyperbolic orientable closed manifolds up to a dimension constant. Moreover, for a closed, orientable n -manifold Gromov [18, page 37] showed that $\mathcal{E}_{\min}(M)^n \geq c_n \|M\|$ where c_n only depends on $n = \dim M$. We remark that there are examples of closed aspherical manifolds M , where $\rho^{(2)}(\widetilde{M}) = 0$ and $\|M\| \neq 0$ holds, see [33, Example 4.18 on page 498]. There is the question whether for an aspherical closed manifold M with $\|M\| = 0$ the universal covering is det- L^2 -acyclic and satisfies $\rho^{(2)}(\widetilde{M}) = 0$, see [33,

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Question 14.39 on page 488]. Moreover, there are examples where $\rho^{(2)}(G) = 0$, but $\mathcal{E}_{\min}(G) \neq 0$, compare [11, Theorem 1.1] with [13].

The following conjecture is taken from [34, Conjecture 1.12 (2)]. For locally symmetric spaces it reduces to the conjecture of Bergeron and Venkatesh [9, Conjecture 1.3].

Conjecture 1.1 (Homological torsion growth and L^2 -torsion). *Let M be an aspherical closed manifold. Consider a descending chain of subgroups $\pi_1(M) = G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$ such that G_i is normal in G , the index $[G : G_i]$ is finite, and $\bigcap_{i \geq 0} G_i = \{1\}$. Let $p: \widetilde{M} \rightarrow M$ be the universal covering. Put $M[i] := G_i \backslash \widetilde{M}$. We obtain a $[G : G_i]$ -sheeted covering $p[i]: M[i] \rightarrow M$.*

Then we get for any natural number n with $2n + 1 \neq \dim(M)$

$$\lim_{i \rightarrow \infty} \frac{\ln(|\text{tors}(H_n(M[i]; \mathbb{Z}))|)}{[G : G_i]} = 0.$$

If the dimension $\dim(M) = 2m + 1$ is odd, then $\widetilde{M}$ is $\det$ - L^2 -acyclic and we get

$$\lim_{i \rightarrow \infty} \frac{\ln(|\text{tors}(H_m(M[i]; \mathbb{Z}))|)}{[G : G_i]} = (-1)^m \cdot \rho^{(2)}(\widetilde{M}).$$

Considerations concerning the Singer Conjecture due to Avramidi-Okun-Schreue before Theorem 4 appearing in [6] lead to the following modification whose conclusion is weaker and appears in [28, Section 6.7] for X an aspherical closed manifold of odd dimension and in [35, Conjecture 8.9] in general.

Conjecture 1.2 (Modified Homological torsion growth and L^2 -torsion). *Let X be a connected finite CW-complex which is $\det$ - L^2 -acyclic. Put*

$$\rho^{\mathbb{Z}}(X[i]) = \sum_{n=0}^{2m+1} (-1)^n \cdot \frac{\ln(|\text{tors}(H_n(X[i]; \mathbb{Z}))|)}{[G : G_i]}.$$

Then the limit $\lim_{i \rightarrow \infty} \rho^{\mathbb{Z}}(X[i])$ exists and is given by

$$\lim_{i \rightarrow \infty} \rho^{\mathbb{Z}}(X[i]) = \rho^{(2)}(\widetilde{X}).$$

If M is a closed hyperbolic 3-manifold, the conjecture of Bergeron and Venkatesh, Conjecture 1.1, and Conjecture 1.2 are equivalent and reduce to the assertion

$$\lim_{i \rightarrow \infty} \frac{\ln(|\text{tors}(H_1(M[i]; \mathbb{Z}))|)}{[G : G_i]} = \frac{1}{6\pi} \cdot \text{Vol}(M).$$

Note that there are groups G where $\mathcal{E}_{\min}(BG) \neq 0$ but the torsion homology growth vanishes in every degree [5, Corollary C].

Our first result is a ‘sum’ or ‘combination’ formula for the L^2 -torsion of a group in terms of the L^2 -torsion of its stabilisers of a sufficiently nice action on a contractible space. One should compare this to the cheap α -rebuilding property of Abert, Bergeron, Fraczyk, and Gaboriau [1] and its algebraic analogue due to Li, Löh, Moraschini, Sauer, and Uschold [31]. We note that an exposition of a version of the following theorem also appeared in [31, §4.2.4].

Theorem 3.7. *Let G be a group acting cocompactly on a contractible CW-complex X such that the fixed point sets of finite subgroups of G are contractible. Suppose that each cell stabiliser H_σ of the action of G is L^2 -acyclic and admits a finite model for $\underline{E}H_\sigma$. If G satisfies the Determinant Conjecture, then*

$$\rho^{(2)}(G) = \sum_{n \geq 0} \sum_{i \in I_n} (-1)^n \cdot \rho^{(2)}(H_{i_n}^n).$$

Here $\underline{EG}$ is the *classifying space for proper actions* of G . That is, the classifying space for the family of finite subgroups. In the case where G is torsion-free we are simply asking that the stabilisers admit a finite classifying space. The additional complexity in the torsion case is because traditionally L^2 -torsion is not defined for groups which are not virtually torsion-free. We circumvent this issue by developing the theory using $\underline{EG}$ in Section 2.

Using Theorem 3.7 we make a number of computations in Section 7. For example we give a vanishing criterion for a CAT(0) lattice in a product of locally compact groups (Proposition 7.2), prove vanishing for Leary–Minasyan groups (Example 7.4) and we compute the L^2 -torsion of a generalised graph manifold (Theorem 7.5). We now highlight one more computation here in the introduction. Handlebody groups are mapping class groups of 3-dimensional handlebodies and are an important class of groups arising in low-dimensional topology, see [4] for more information. It is known [4, Theorem 6.1] that the L^2 -Betti numbers and homology torsion growth of the handlebody groups vanishes. Here we show that L^2 -torsion vanishes as well, verifying Conjecture 1.1 for the groups and solving [4, Problem 28].

Theorem 7.20. *Let $g \geq 2$ and let V_g denote the genus g handlebody. Then,*

$$\rho^{(2)}(\text{Mod}(V_g)) = 0.$$

We also prove a theorem about the vanishing of L^2 -torsion for a polynomially growing automorphisms of many families of groups. An automorphism Φ is *polynomially growing of degree* at most d if for each $g \in G$ there is a constant C such that $|\Phi^n(g)| < Cn^d + C$ for all $n \in \mathbb{N}$. The following theorem answers [3, Question 1.2] and when combined with [3, Theorem A] resolves Conjecture 1.1 for the relevant groups.

Theorem 7.19. *Let Γ be a group isomorphic to one of*

- $G \rtimes_{\Phi} \mathbb{Z}$ with G residually finite and hyperbolic;
- $G \rtimes_{\Phi} \mathbb{Z}$ with G residually finite and hyperbolic relative to a finite collection of virtually polycyclic groups;
- $A_L \rtimes_{\Phi} \mathbb{Z}$ where A_L is a right-angled Artin group and $\Phi \in \text{Aut}(A_L)$ is untwisted; or
- $W_L \rtimes_{\Phi} \mathbb{Z}$ where W_L is a right-angled Coxeter group.

If Φ is polynomially growing, then $\rho^{(2)}(\Gamma) = 0$.

Whilst the previous theorem is specifically about groups we actually work much more generally and study the L^2 -torsion $\rho^{(2)}(\Phi)$ of an automorphism $\Phi: G \rightarrow G$. This generalisation allows us to side-step assumptions about the Determinant Conjecture and only assume it for certain subgroups of G . Moreover, when the Determinant Conjecture holds for $\Gamma = G \rtimes_{\Phi} \mathbb{Z}$ we have $\rho^{(2)}(\Phi) = \rho^{(2)}(\Gamma)$. We take up this task in Sections 4 and 5. Our main application is that we compute the L^2 -torsion of any automorphism of a one-ended hyperbolic group in terms of its canonical JSJ decomposition. For background on JSJ decompositions see [23]. More generally we prove:

Theorem 7.10. *Let G be a group hyperbolic and one-ended relative to a finite collection $\mathcal{P}$ of virtually polycyclic groups, let $\Phi \in \mathcal{K}(\mathcal{T}_G)$, and let $\Gamma = G \rtimes_{\Phi} \mathbb{Z}$. Then*

$$\rho^{(2)}(\Phi) = \sum_{v \in \text{Flex}(G)} \rho^{(2)}(G_v \rtimes_{\Phi|_{G_v}} \mathbb{Z}),$$

where $\text{Flex}(G)$ is the set of flexible vertices in a JSJ decomposition of G .

In the previous theorem, $\mathcal{K}(\mathcal{T}_G)$ is a certain finite index subgroup of $\text{Aut}(G, \mathcal{P})$, see Section 7.4.

As an application of the previous theorem we obtain vanishing for all polynomially growing automorphisms of all one-ended groups hyperbolic relative to a finite collection of virtually polycyclic groups — see Theorem 7.15. We refer the interested reader to Section 7.5 for further results. Motivated by our results we raise the following conjecture:

Conjecture 1.3 (Vanishing of the L^2 -torsion of an automorphism of subexponential growth). *Let $\Phi: G \rightarrow G$ be an automorphism of a det-finite group G which has subexponential growth. Then $\rho^{(2)}(\Phi) = 0$.*

Remark 1.4. We remark that all of our results and formulas here can be adapted to the setting of twisted L^2 -torsion as developed in [36].

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2. BASICS ABOUT L^2 -TORSION

There are several notions of volume for a finite volume hyperbolic 3-manifold M . Its hyperbolic volume $\text{Vol}(M)$, its L^2 -torsion $\rho^{(2)}(M)$, and its minimal volume entropy $\mathcal{E}_{\min}(M)$. Work of Lück and Schick shows that $\text{Vol}(M)$ and $\rho^{(2)}(M)$ are proportional [38]. Work of Soma [46], Gromov [18], and Thurston [47] shows that $\text{Vol}(M)$ and $\mathcal{E}_{\min}(M)^3$ are proportional.

We collect some basic facts about L^2 -torsion. Most of it is described in [33, Section 3.4] provided we consider finite free G -CW-complexes. In this section we want to explain that all of this carries over to proper finite G -CW-complexes. Recall that a G -CW-complex is proper if and only if all its isotropy groups are finite, and is finite if and only if it is cocompact. The motivation is that thus the notion of L^2 -torsion makes sense for groups which are not torsionfree but have a finite model for the classifying space $\underline{E}G = E_{\mathcal{FIN}}(G)$ for proper G -actions, for instance for all hyperbolic groups, provided that one such model is det- L^2 -acyclic.

Given a proper finite G -CW-complex, we denote by $C_*^c(X)$ its cellular $\mathbb{Z}G$ -chain complex. Suppose that we have chosen cellular G -pushouts for every $n \in \mathbb{Z}^{\geq 0}$

$$(2.1) \quad \begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \longrightarrow & X_n. \end{array}$$

Note that then I_n can be identified with the set of open n -cells of the finite CW-complex X/G . They induce preferred $\mathbb{Z}[G]$ -isomorphisms

$$(2.2) \quad \varphi_n: \bigoplus_{i \in I_n} \mathbb{Z}[G/H_i] \xrightarrow{\cong} C_n^c(X),$$

and hence preferred $\mathbb{C}G$ -isomorphisms

$$(2.3) \quad \varphi_n^{(2)}: \bigoplus_{i \in I_n} L^2(G) \otimes_{\mathbb{Z}G} \mathbb{Z}[G/H_i] \xrightarrow{\cong} \bigoplus_{i \in I_n} L^2(G) \otimes_{\mathbb{Z}G} C_n^c(X).$$

Note that $L^2(G) \otimes_{\mathbb{Z}G} \mathbb{Z}[G/K]$ is for any finite subgroup $K \subseteq G$ a finitely generated Hilbert $\mathcal{N}(G)$ -module, as it embeds isometrically and G -linearly into $L^2(G)$, namely, by sending $x \otimes gK$ to $\frac{1}{|K|} \cdot \sum_{k \in K} xgk$. Hence we get from the isomorphism (2.3) the structure of a finite Hilbert $\mathcal{N}(G)$ -chain complex on $L^2(G) \otimes_{\mathbb{Z}G} C_*^c(G)$. Note that the choice of the G -pushouts (2.1) is not part of the G -CW-structure, only their existence is required. So we have to show that the structure of a finite Hilbert $\mathcal{N}(G)$ -chain complex on $L^2(G) \otimes_{\mathbb{Z}G} C_*^c(G)$ is independent of the choice of the G -pushouts (2.3). If we make a different choices, then this changes the isomorphism φ_n of (2.2) by an automorphism of the shape

$$\nu := \bigoplus_{i \in I_n} \epsilon_i \cdot \nu_i: \bigoplus_{i \in I_n} \mathbb{Z}[G/H_i] \xrightarrow{\cong} \bigoplus_{i \in I_n} \mathbb{Z}[G/H'_i]$$

where $\epsilon_i \in \{\pm 1\}$ and ν_i is induced by a bijective G -map $G/H_i \xrightarrow{\cong} G/H'_i$. Since the map

$$\text{id}_{L^2(G)} \otimes \nu: \bigoplus_{i \in I_n} L^2(G) \otimes_{\mathbb{Z}G} \mathbb{Z}[G/H_i] \xrightarrow{\cong} \bigoplus_{i \in I_n} L^2(G) \otimes_{\mathbb{Z}G} \mathbb{Z}[G/H'_i]$$

is an isometric G -linear automorphism, the structure of a finite Hilbert $\mathcal{N}(G)$ -chain complex on $L^2(G) \otimes_{\mathbb{Z}G} C_*^c(G)$ is independent of the choice of the G -pushouts (2.1). Hence the finite $\mathcal{N}(G)$ -Hilbert chain complex $L^2(G) \otimes_{\mathbb{Z}G} C_*^2(X)$ depends only the G -CW-structure of the finite proper G -CW-complex X .

Now for a proper finite G -CW-complex X , the notions of L^2 -Betti numbers $b_n^{(2)}(X; \mathcal{N}(G))$, of determinant class, and of being $\det$ - L^2 -acyclic are defined, and we also have the notion of L^2 -torsion $\rho^{(2)}(X; \mathcal{N}(G)) \in \mathbb{R}$, provided that X is $\det$ - L^2 acyclic. (It is not necessary to understand the precise definition of these notions to read this paper.) Note that the class of groups which satisfies the Determinant Conjecture, see [33, Conjecture 13.2 on page 454], is quite large. It includes all sofic groups. If G satisfies the Determinant Conjecture, the condition $\det$ - L^2 -acyclic on the proper finite G -CW-complex X reduces to the condition that the L^2 -Betti number $b_n^{(2)}(X; \mathcal{N}(G))$ vanishes for all $n \in \mathbb{Z}^{\geq 0}$.

A *Farrell-Jones group* G is a group G satisfying the *Full Farrell-Jones Conjecture*. Explanations about the Full Farrell-Jones Conjecture and informations about the class of Farrell-Jones groups, which is rather large and contains for instance all hyperbolic groups, can be found in [37].

For a group L define the group homomorphism

$$(2.4) \quad \det_L^{(2)} : \text{Wh}(L) \rightarrow \mathbb{R}^{>0}$$

by sending the class $[A]$ of an invertible (m, m) -matrix $A \in \text{GL}_m(\mathbb{Z}L)$ to the Fuglede-Kadison determinant $\det^{(2)}(r_A^{(2)}) \in \mathbb{R}^{>0}$ of the automorphism of a finite Hilbert $\mathcal{N}(G)$ -module $r_A^{(2)} : L^2(L)^m \rightarrow L^2(L)^m$ given by A .

Lemma 2.5. *Suppose that the group L satisfies the Determinant Conjecture or is a Farrell-Jones group. Then the homomorphism*

$$\det_L^{(2)} : \text{Wh}(L) \rightarrow \mathbb{R}^{>0}$$

defined in (2.4) is trivial.

Proof. Suppose that L satisfies the Determinant Conjecture. Given an invertible (m, m) -matrix $A \in \text{GL}_m(\mathbb{Z}L)$, we conclude $\det^{(2)}(r_A^{(2)}) \geq 1$ and $\det^{(2)}(r_{A^{-1}}^{(2)}) \geq 1$ from the Determinant Conjecture. Since we have

$$\det^{(2)}(r_A^{(2)}) \cdot \det^{(2)}(r_{A^{-1}}^{(2)}) = \det^{(2)}(r_A^{(2)} \cdot r_{A^{-1}}^{(2)}) = \det^{(2)}(\text{id}_{L^2(G)^m}) = 1.$$

we get $\det^{(2)}(r_A^{(2)}) = 1$.

Suppose that L is a Farrell-Jones group. If L is torsionfree, then $\text{Wh}(L)$ vanish and the claim is obviously true. If L is not torsionfree, $\text{Wh}(L)$ is in general non-trivial. Nevertheless, the arguments in the proof of [36, Theorem 6.7 (2)] reduce to a proof that $\det_L^{(2)} : \text{Wh}(L) \rightarrow \mathbb{R}$ is trivial, take in [36, Theorem 6.7 (2)] V to be the trivial 1-dimensional L -representation. $\square$

Now we collect some properties of L^2 -torsion. The proof of the following theorem can be found in the case, where G is torsionfree, in [33, Theorem 3.93 on page 161]. Here we want to drop the condition torsionfree.

Definition 2.6 (Condition (DFJ)). The group G satisfies condition (DFJ) if it satisfies one of the following conditions:

- (i) The group G satisfies the Determinant Conjecture;
- (ii) For any finite subgroup $H \subseteq G$ the Weyl group $W_G H$ is a Farrell-Jones group;
- (iii) The group G contains a torsionfree group H of finite index such that H satisfies the Determinant Conjecture or is a Farrell-Jones Group.

If G satisfies condition (DFJ), then every subgroup of G does, since every subgroup of a group which satisfies the Determinant Conjecture or is a Farrell-Jones group respectively, satisfies the Determinant Conjecture or is a Farrell-Jones group respectively, see [33, Theorem 3.14 (6) on page 129 and Lemma 13.45 (7) on page 473] and [37, Theorem 16.5 (iia)].

Theorem 2.7 (Basic properties of L^2 -torsion).

- (i) Homotopy invariance

Let X and Y be proper finite G -CW-complexes which are G -homotopy equivalent. Suppose that X is $\det$ - L^2 -acyclic. Then:

- (a) *Y is $\det$ - L^2 -acyclic.*
- (b) *If X and Y are simple G -homotopy equivalent or if G satisfies condition (DFJ), we get*

$$\rho^{(2)}(X; \mathcal{N}(G)) = \rho^{(2)}(Y; \mathcal{N}(G));$$

(ii) Sum formula

Consider the G -pushout of proper finite G -CW-complexes

$$\begin{array}{ccc} X_0 & \xrightarrow{i} & X_1 \\ f \downarrow & & \downarrow \bar{f} \\ X_2 & \xrightarrow{\underline{i}} & X \end{array}$$

where i is an inclusion of proper G -CW-complexes, f is a cellular G -map of G -CW-complexes, and the G -CW-structure on X has as n -skeleton $X_n = \bar{f}((X_2)_n) \cup \underline{i}((X_1)_n)$. Suppose that X_0 , X_1 , and X_2 are proper finite G -CW-complexes which are $\det$ - L^2 -acyclic.

Then X is a proper finite G -CW-complex which is $\det$ - L^2 -acyclic, and we get

$$\rho^{(2)}(X; \mathcal{N}(G)) = \rho^{(2)}(X_1; \mathcal{N}(G)) + \rho^{(2)}(X_2; \mathcal{N}(G)) - \rho^{(2)}(X_0; \mathcal{N}(G));$$

(iii) Product formula

Let X be a proper finite G -CW-complex which is $\det$ - L^2 -acyclic. Let Z be a finite proper H -CW-complex. Denote by $\chi^{(2)}(Z; \mathcal{N}(H))$ the L^2 -Euler characteristic of Z .

Then the $G \times H$ -space $X \times Z$ is a proper $G \times H$ -CW-complex which $\det$ - L^2 -acyclic, and we get

$$\rho^{(2)}(X \times Z; \mathcal{N}(G \times H)) = \chi^{(2)}(Z; \mathcal{N}(H)) \cdot \rho^{(2)}(X; \mathcal{N}(G));$$

(iv) Restriction

Let H be a subgroup of G of finite index $[G : H]$. Let X be a proper finite G -CW-complex. Let $X|_H$ be the H -space obtained from X by restriction with i .

Then $X|_H$ is a proper finite H -CW-complex. It is $\det$ - L^2 -acyclic if and only if X is $\det$ - L^2 -acyclic. If X is $\det$ - L^2 -acyclic, then we get

$$\rho^{(2)}(X|_H; \mathcal{N}(H)) = [G : H] \cdot \rho^{(2)}(X; \mathcal{N}(G));$$

(v) Induction

Let H be a subgroup of G . Let X be a proper finite H -CW-complex.

Then $G \times_H X$ is a finite proper G -CW-complex. It is $\det$ - L^2 -acyclic if and only if X is $\det$ - L^2 -acyclic. If X is $\det$ - L^2 -acyclic, then we get

$$\rho^{(2)}(G \times_H X; \mathcal{N}(G)) = \rho^{(2)}(X; \mathcal{N}(H));$$

(vi) Finite kernels

Let $1 \rightarrow K \rightarrow G \xrightarrow{p} Q \rightarrow 1$ be an extension of groups with finite K . Let X be a proper finite Q -CW-complex. Let p^*X be the G -space obtained from X by restriction with p .

Then p^*X is a finite proper G -CW-complex. If X or p^*Y is $\det$ - L^2 -acyclic, then both X and p^*Y are $\det$ - L^2 -acyclic and get

$$\rho^{(2)}(X; \mathcal{N}(G)) = \frac{\rho^{(2)}(X; \mathcal{N}(Q))}{|K|};$$

(vii) Poincaré duality

If M is a proper cocompact smooth G -manifold without boundary. Then M inherits from a smooth triangulation the structure of a proper finite

G-CW-complex. Suppose that M is $\det$ - L^2 -acyclic and its dimension is even. Then we get

$$\rho^{(2)}(M; \mathcal{N}(G)) = 0.$$

Proof. (i) We only give a brief sketch of the proof. One key fact is that one has the notion of equivariant Whitehead torsion $\tau^G(f)$ for a G -homotopy equivalence $f: X \rightarrow Y$ of proper finite G -CW-complexes which takes values in $\bigoplus_{(H)} \text{Wh}(W_G H)$, where (H) runs through the conjugacy classes of finite subgroup $H \subseteq G$ and $W_G H$ is the Weyl group $N_G H/H$, see [32, Chapters 4 and 12]. Then the homomorphism

$$\alpha: \bigoplus_{(H)} \frac{1}{|H|} \cdot \ln \circ \det_{W_G H}^{(2)}: \bigoplus_{(H)} \text{Wh}(W_G H) \rightarrow \mathbb{R}$$

sends $\tau^G(f)$ to $\rho^{(2)}(X; \mathcal{N}(G)) - \rho^{(2)}(Y; \mathcal{N}(G))$, where $\det_{W_G H}^{(2)}$ has been defined in (2.4). If f is a simple G -homotopy equivalence, then $\tau^G(f)$ vanishes. It remains to show that α is trivial if G satisfies condition (DFJ).

If $W_G H$ satisfies the Determinant Conjecture or is a Farrell-Jones group for every finite subgroup $H \subseteq G$, then the vanishing of α follows from Lemma 2.5. If G satisfies the Determinant Conjecture, then $W_G H$ satisfies the Determinant Conjecture for every finite subgroup $H \subseteq G$ by [33, Theorem 3.14 (6) on page 129 and Lemma 13.45 (7) on page 473]. If G is a Farrell-Jones group and $H \subseteq G$ is a finite subgroup, we know that $N_G H$ is a Farrell-Jones group but we do not know in general that $W_G H$ is a Farrell-Jones-group. Now suppose that K is a torsionfree subgroup of G of finite index. If K satisfies the Determinant Conjecture, then G satisfies the Determinant Conjecture by [33, Theorem 3.14 (5) on page 128]. If K is a Farrell-Jones group, then $K \cap N_G H \subseteq N_G H$ is a Farrell-Jones group and is isomorphic to a subgroup of $W_G H$ of finite index which implies that $W_G H$ is a Farrell-Jones group, see [37, Theorem 16.5 (ia) and (iif) on page 503].

(ii) (iv) and (v) The proofs of [33, Theorem 3.93 (2), (5), and (6) on page 161] carry over.

(iii) We can replace $\chi^{(2)}(Z; \mathcal{N}(H))$ by the orbifold Euler characteristic

$$\chi_{\text{orb}}(Z) = \sum_e (-1)^{\dim(e)} \cdot \frac{1}{|H_e|},$$

where e runs through the equivariant cells of Z , because of $\chi^{(2)}(Z; \mathcal{N}(H)) = \chi_{\text{orb}}(Z)$, see [33, Subsection 6.6.1]. Using assertions (i) and (ii) one can reduce the claim to the special case $Z = H/L$ for any finite subgroup $L \subseteq H$ by induction over the dimension of Z and subinduction over the number of top-dimensional equivalent cells of Z . We get an isomorphism of proper finite $G \times H$ -CW-complexes $G \times H \times_{G \times L} \text{pr}_L^* X \xrightarrow{\cong} Y \times H/L$ by sending $((g, h), x)$ to (gx, hL) , where $\text{pr}_L^* X$ is the proper finite $G \times L$ -CW-space obtained from X by restriction with the projection $\text{pr}_L: G \times L \rightarrow G$. We conclude from assertion (v) that it suffices to show the claim for the finite proper $G \times L$ -CW-complex $\text{pr}_L^* X$. This follows from assertion (iv) applied to $G \subseteq G \times L$.

(vi) This follows from [33, Lemma 13.45 on page 473].

(vii) This is proved in [33, Theorem 3.93 (3) on page 161] under the assumption that the G -action is free. If G contains a subgroup of finite index which acts freely on M then the claim follows from this special case and assertion (iv). The general case requires some additional arguments which we do not present here. $\square$

For a group G define its n th L^2 -Betti number $b_n^{(2)}(G)$ to be $b_n^{(2)}(\underline{EG}; \mathcal{N}(G))$ for any G -CW-model for $\underline{EG}$. This is the same as $b_n^{(2)}(EG; \mathcal{N}(G))$ for any G -CW-model of EG , see [33, Theorem 6.54 (1) and (2) on page 265].

Definition 2.8 (L^2 -acyclic group). A group G is called L^2 -acyclic if $b_p^{(2)}(G)$ vanishes for all $n \in \mathbb{Z}^{\geq 0}$.

Definition 2.9 ($\mathcal{FIN}$ -finite group). A group G is called $\mathcal{FIN}$ -finite if it has a finite G -CW-model for the classifying space of proper G -actions $\underline{EG} = E_{\mathcal{FIN}}(G)$.

Definition 2.10 (det- L^2 -acyclic group). A $\mathcal{FIN}$ -finite group G is called det-finite if one (and hence every) finite G -CW-model for $\underline{EG}$ is of determinant class. It is called det- L^2 -acyclic if its both L^2 -acyclic and det-finite, or, equivalently, one (and hence every) finite G -CW-model for $\underline{EG}$ is det- L^2 -acyclic.

Definition 2.11 (L^2 -torsion of groups). Suppose that the $\mathcal{FIN}$ -finite group G is det- L^2 -acyclic and satisfies condition (DFJ) appearing in Definition 2.6

Define its L^2 -torsion

$$\rho^{(2)}(G) = \rho^{(2)}(X; \mathcal{N}(G))$$

for any finite G -CW-model X for $\underline{EG}$.

This is independent of the choice of the finite G -CW-model for $\underline{EG}$ by Theorem 2.7 (i).

3. BLOWING UP ORBITS BY CLASSIFYING SPACES OF FAMILIES

Consider a family of subgroups $\mathcal{F}$. Let X be a G -CW-complex with skeletal filtration

$$X_{-1} = \emptyset \subseteq X_0 \subseteq X_2 \subseteq \cdots \subseteq X = \operatorname{colim}_{n \rightarrow \infty} X_n.$$

Let I_n be the set of open n -cells of X/G . Choose for every $n \in \mathbb{Z}^{\geq 0}$ a cellular G -pushout

$$(3.1) \quad \begin{array}{ccc} \coprod_{i_n \in I_n} G/H_{i_n}^n \times S^{n-1} & \xrightarrow{\coprod_{i_n \in I_n} q_{i_n}^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i_n \in I_n} G/H_{i_n}^n \times D^n & \xrightarrow{\coprod_{i_n \in I_n} Q_{i_n}^n} & X_n. \end{array}$$

For each $n \in \mathbb{Z}^{\geq 0}$ and $i_n \in I_n$ let $E_{i_n}^n$ be an $H_{i_n}^n$ -CW-complex which is a model for $E_{\mathcal{F}|_{H_{i_n}^n}}(H_{i_n}^n)$, where $\mathcal{F}|_{H_{i_n}^n}$ is the family of subgroups of $H_{i_n}^n$ given by $\{K \cap H_{i_n}^n \mid K \in \mathcal{F}\}$. Fix a model Z for $E_{\mathcal{F}}(G)$. We get a filtration by G -cofibrations

$$(3.2) \quad Z \times X_{-1} = \emptyset \subseteq Z \times X_0 \subseteq Z \times X_2 \subseteq \cdots \subseteq Z \times X = \operatorname{colim}_{n \rightarrow \infty} (Z \times X_n).$$

and G -pushouts

$$(3.3) \quad \begin{array}{ccc} \coprod_{i_n \in I_n} Z \times G/H_{i_n}^n \times S^{n-1} & \xrightarrow{\coprod_{i_n \in I_n} \operatorname{id}_Z \times q_{i_n}^n} & Z \times X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i_n \in I_n} Z \times G/H_{i_n}^n \times D^n & \xrightarrow{\coprod_{i_n \in I_n} \operatorname{id}_Z \times Q_{i_n}^n} & Z \times X_n, \end{array}$$

where here and in the sequel we use on a product of two G -spaces the diagonal G -action. Let $Z|_{H_{i_n}^n}$ be the restriction of the G -CW-complex Z to $H_{i_n}^n \subseteq G$. We have the G -homeomorphism

$$G \times_{H_{i_n}^n} Z|_{H_{i_n}^n} \xrightarrow{\cong} Z \times G/H_{i_n}^n, \quad (g, z) \mapsto (gz, gH_{i_n}^n).$$

Since $Z|_{H_{i_n}^n}$ is a model for $E_{\mathcal{F}|_{H_{i_n}^n}}(H_{i_n}^n)$, we can choose a $H_{i_n}^n$ -homotopy equivalence $E_{i_n}^n \xrightarrow{\cong} Z|_{H_{i_n}^n}$. Thus we get a G -homotopy equivalence

$$\varphi_{i_n}^n : G \times_{H_{i_n}^n} E_{i_n}^n \xrightarrow{\cong} Z \times G/H_{i_n}^n.$$

Next we construct a sequence of inclusions of G -CW-complexes

$$Y_{-1} = \emptyset \subseteq Y_0 \subseteq Y_2 \subseteq Y_3 \subseteq \dots$$

and cellular G -homotopy equivalences $f_n: Y_n \rightarrow Z \times X_n$ satisfying $f_{n+1}|_{Z \times X_n} = f_n$ by induction over $n = -1, 0, 1, 2, \dots$. The induction beginning is trivial as $Y_{-1} = Z \times X_{-1} = \emptyset$ holds. The induction step from $(n-1)$ to $n \geq 0$ is done as follows. Since $f_{n-1}: Y_{n-1} \rightarrow Z \times X_{n-1}$ is a G -homotopy equivalence, we can find a cellular G -map

$$\mu^n: \coprod_{i_n \in I_n} G \times_{H_{i_n}^n} E_{i_n}^n \times S^{n-1} \rightarrow Y_{n-1}$$

and a G -homotopy

$$h^n: \left(\coprod_{i_n \in I_n} G \times_{H_{i_n}^n} E_{i_n}^n \times S^{n-1} \right) \times I \rightarrow Z \times X_{n-1}$$

satisfying $h_0^n = f_{n-1} \circ \mu_{n-1}$ and $h_1^n = \coprod_{i_n \in I_n} (\text{id}_Z \times q_{i_n}^n) \circ (\varphi_{i_n}^n \times \text{id}_{S^{n-1}})$. Define the G -CW-complex Y_n by the G -pushout

$$(3.4) \quad \begin{array}{ccc} \coprod_{i \in I_n} G \times_{H_{i_n}^n} E_{i_n}^n \times S^{n-1} & \xrightarrow{\mu^n} & Y_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G \times_{H_{i_n}^n} E_{i_n}^n \times D^n & \longrightarrow & Y_n. \end{array}$$

Consider the following commutative diagram

$$\begin{array}{ccccc} \coprod_{i \in I_n} G \times_{H_{i_n}^n} E_{i_n}^n \times D^n & \longleftarrow & \coprod_{i \in I_n} G \times_{H_{i_n}^n} E_{i_n}^n \times S^{n-1} & \xrightarrow{\mu^n} & Y_{n-1} \\ J_0 \downarrow & & j_0 \downarrow & & f_{n-1} \downarrow \\ \left(\coprod_{i \in I_n} G \times_{H_{i_n}^n} E_{i_n}^n \times D^n \right) \times I & \longleftarrow & \left(\coprod_{i \in I_n} G \times_{H_{i_n}^n} E_{i_n}^n \times S^{n-1} \right) \times I & \xrightarrow{h^n} & Z \times X_{n-1} \\ J_1 \uparrow & & j_1 \uparrow & & \text{id}_{Z \times X_{n-1}} \uparrow \\ \coprod_{i \in I_n} G \times_{H_{i_n}^n} E_{i_n}^n \times D^n & \longleftarrow & \coprod_{i \in I_n} G \times_{H_{i_n}^n} E_{i_n}^n \times S^{n-1} & \xrightarrow{h_1^n} & Z \times X_{n-1} \\ \coprod_{i \in I_n} \varphi_{i_n}^n \times \text{id}_{D^n} \downarrow & & \coprod_{i \in I_n} \varphi_{i_n}^n \times \text{id}_{S^{n-1}} \downarrow & & \text{id}_{Z \times X_{n-1}} \downarrow \\ \coprod_{i \in I_n} Z \times G/H_{i_n}^n \times D^n & \longleftarrow & \coprod_{i \in I_n} Z \times G/H_{i_n}^n \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} \text{id}_Z \times q_{i_n}^n} & Z \times X_{n-1} \end{array}$$

where j_k and J_k come from the inclusion $\{\bullet\} \rightarrow I = [0, 1]$ with image $\{k\}$ for $k = 0, 1$. The pushout of the uppermost row is Y_n , see (3.4), whereas the pushout of the lowermost row is $Z \times X_n$, see (3.3). Let T_2 be the pushout of the second row and T_3 be the pushout of the third row. Then we have pairs $(T_2, Z \times X_{n-1})$ and $(T_3, Z \times X_{n-1})$. The diagram above yields G -maps of pairs

$$\begin{aligned} (a, f_{n-1}): (Y_n, Y_{n-1}) &\rightarrow (T_2, Z \times X_{n-1}); \\ (b, \text{id}_{Z \times X_{n-1}}): (T_3, Z \times X_{n-1}) &\rightarrow (T_2, Z \times X_{n-1}); \\ (c, \text{id}_{Z \times X_{n-1}}): (T_3, Z \times X_{n-1}) &\rightarrow (Z \times X_n, Z \times X_{n-1}). \end{aligned}$$

Since in the diagram above all vertical arrows are G -homotopy equivalences and the left horizontal arrow in each row is a G -cofibration, these three G -maps are G -homotopy equivalences of pairs and we can find a G -homotopy equivalence of pairs

$$(b', \text{id}_{Z \times X_{n-1}}): (T_2, Z \times X_{n-1}) \rightarrow (T_3, Z \times X_{n-1}).$$

Now define the desired G -homotopy equivalence of pairs

$$(f_n, f_{n-1}): (Y_n, Y_{n-1}) \rightarrow (Z \times X_n, Z \times X_{n-1})$$

by the composite $(c, \text{id}_{Z \times X_{n-1}}) \circ (b', \text{id}_{Z \times X_{n-1}}) \circ (a, f_{n-1})$. If we put $Y = \text{colim}_{n \rightarrow \infty} Y_n$, we obtain a G -CW-complex Y and a G -homotopy equivalence

$$(3.5) \quad f = \text{colim}_{n \rightarrow \infty} f_n: Y \xrightarrow{\simeq_G} Z \times X.$$

Next we collect the basic properties of this construction.

Theorem 3.6 (Blowing up).

- (i) Consider a number $l \in \mathbb{Z}^{\geq 0}$. Suppose for $n \in \mathbb{Z}^{\geq 0}$ and $i_n \in I_n$ that the $H_{i_n}^n$ -CW-complex $E_{i_n}^n$ has finite $(l - n)$ skeleton and X has a finite l -skeleton.
Then the G -CW-complex Y has a finite l -skeleton;
- (ii) Suppose for $n \in \mathbb{Z}^{\geq 0}$ and $i_n \in I_n$ that the $H_{i_n}^n$ -CW-complex $E_{i_n}^n$ is of finite type and the G -CW-complex X is of finite type.
Then the G -CW-complex Y is of finite type;
- (iii) Consider a number $d \in \mathbb{Z}^{\geq 0}$. Suppose for $n \in \mathbb{Z}^{\geq 0}$ and $i_n \in I_n$ that the $H_{i_n}^n$ -CW-complex $E_{i_n}^n$ has dimension $\leq (l - d)$ and X has a dimension $\leq d$.
Then the G -CW-complex Y has dimension $\leq d$.
- (iv) If for all $n \in \mathbb{Z}^{\geq 0}$ and $i_n \in I_n$ the $H_{i_n}^n$ -CW-complex $E_{i_n}^n$ is finite and the G -CW-complex X is finite, then the G -CW-complex Y is finite;
- (v) If each element in $\mathcal{F}$ is finite, then Y and $Z \times X$ are proper.
- (vi) If each of the $H_{i_n}^n$ -CW-complexes $E_{i_n}^n$ is L^2 -acyclic, then the G -CW-complex $Z \times X$ is L^2 -acyclic;
- (vii) Suppose that the following conditions are satisfied:
 - (a) G satisfies condition (DFJ), see Definition 2.6;
 - (b) Every element in $\mathcal{F}$ is finite;
 - (c) For all $n \in \mathbb{Z}^{\geq 0}$ and $i_n \in I_n$ the $H_{i_n}^n$ -CW-complex $E_{i_n}^n$ is finite and $\det$ - L^2 -acyclic;
 - (d) X is a finite G -CW-complex.

Then the G -CW-complex Y is finite, proper, and $\det$ - L^2 -acyclic, and we get for its L^2 -torsion

$$\rho^{(2)}(Y; \mathcal{N}(G)) = \sum_{n \geq 0} \sum_{i \in I_n} (-1)^n \cdot \rho^{(2)}(E_{i_n}^n; \mathcal{N}(H_{i_n}^n)).$$

Suppose additionally that X^H is contractible for any finite subgroup $H \subseteq G$. Then each group $H_{i_n}^n$ satisfies condition (DFJ), there is a finite G -CW-model for $\underline{E}G$ which is $\det$ - L^2 -acyclic, and we get

$$\rho^{(2)}(G) = \sum_{n \geq 0} \sum_{i \in I_n} (-1)^n \cdot \rho^{(2)}(H_{i_n}^n).$$

Proof. Assertions (i), (ii), (iii), (iv) and (v) follow directly from the construction of Y by inspecting the G -pushouts (3.4).

Assertion (vi) follows from [33, Theorem 6.54 on page 265].

Assertion (vii) follows from Theorem 2.7 and the fact that the projection $\underline{E}G \times X \rightarrow \underline{E}G$ is a G -homotopy equivalence if X^H is contractible for every finite subgroup H . $\square$

We extract out a version of (viii) which appeared in the introduction.

Theorem 3.7. *Let G be a group acting cocompactly on a contractible CW-complex X such that the fixed point sets of finite subgroups of G are contractible. Suppose*

that each cell stabiliser H_σ of the action of G is L^2 -acyclic and admits a finite model for $\underline{E}H_\sigma$. If G satisfies the Determinant Conjecture, then

$$\rho^{(2)}(G) = \sum_{n \geq 0} \sum_{i \in I_n} (-1)^n \cdot \rho^{(2)}(H_{i_n}^n).$$

Example 3.8 (Group extensions). Suppose we can write G as an extension $1 \rightarrow K \rightarrow G \xrightarrow{p} Q \rightarrow 1$, and there is a finite model for $\underline{E}Q$. Then we can consider the G -CW-complex $X = p^* \underline{E}Q$ obtained from the finite Q -CW-complex $\underline{E}Q$ by restriction with p . Obviously $X^H = \underline{E}Q^{p(H)}$ is contractible for any finite subgroup $H \subseteq G$. There is a bijective correspondence between the open equivariant cells of X and the open equivariant cells of $\underline{E}Q$ given by $c \mapsto p(c)$. We have $\dim(c) = \dim(p(c))$ and $G_c = p^{-1}(Q_{p(c)})$. Hence G_c contains K as a subgroup of finite index $[G_c : K] = |Q_{p(c)}|$. Suppose that G_c has a finite G_c -model for $\underline{E}G_c$ which is $\det$ - L^2 -acyclic. Assume that G satisfies condition (DFJ), see Definition 2.6. We conclude from Theorem 2.7 (iv) that $|Q_{p(c)}| \cdot \rho^{(2)}(G_c) = \rho^{(2)}(K)$. Define the orbifold Euler characteristic of $\underline{E}Q$ to be

$$(3.9) \quad \chi_{\text{orb}}(\underline{E}Q) = \sum_c (-1)^{\dim(e)} \cdot \frac{1}{|Q_{p(c)}|}.$$

If $\chi^{(2)}(\underline{E}Q; \mathcal{N}(Q))$ is the L^2 -Euler characteristic, we get, see [33, Subsection 6.6.1].

$$(3.10) \quad \chi_{\text{orb}}(\underline{E}Q) = \chi^{(2)}(\underline{E}Q; \mathcal{N}(Q)).$$

Theorem 3.6 (vii) implies that both K and G have a $\det$ - L^2 -acyclic finite proper model for their classifying space of proper actions and we have

$$\rho^{(2)}(G) = \chi_{\text{orb}}(\underline{E}Q) \cdot \rho^{(2)}(K) = \chi^{(2)}(\underline{E}Q; \mathcal{N}(Q)) \cdot \rho^{(2)}(K).$$

In particular we get $\rho^{(2)}(G) = 0$ if $\chi^{(2)}(\underline{E}Q; \mathcal{N}(Q))$ vanishes.

Example 3.11 (Graph of groups). Let Y be a connected non-empty graph in the sense of [44, Definition 1 in Section 2.1 on page 13]. Let (G, Y) be a graph of groups in the sense of [44, Definition 8 in Section 4.4. on page 37]. (In the sequel we use the notation of [44]). Denote by $\text{edge}(Y)$ the quotient of $\text{edge}(Y)$ under the involution $y \mapsto \bar{y}$. Note that y considered as a CW-complex has the set $\text{vert}(T)$ as set of 0-cells and $\text{edge}(Y)$ as set of 1-cells. Since by definition $G_y = G_{\bar{y}}$ holds, we can define for $\bar{y} \in \text{edge}(Y)$ the group $G_{\bar{y}}$ to be G_y for a representative $y \in \text{edge}(Y)$ of $\bar{y}$.

Let P_0 be an element in $\text{vert}(Y)$ and let T be a maximal tree of Y . Let $\pi_1(G, Y, P_0)$ and $\pi_1(G, Y, T)$ be the fundamental groups in the sense of [44, page 42]. Note that $\pi_1(G, Y, P_0)$ and $\pi_1(G, Y, T)$ are isomorphic, see [44, Proposition 20 in Section 5.1. on page 44]. Then there exists

- A graph $\tilde{X} = \tilde{X}(G, Y, T)$;
- An action of $\pi = \pi_1(G, Y, T)$ on $\tilde{X}$;
- A morphism $p: \tilde{X} \rightarrow X$ inducing an isomorphism $\pi \backslash \tilde{X} \rightarrow Y$,

such that the following is true:

- $\tilde{X}$ is a tree;
- $\tilde{X}^H$ is contractible for every finite subgroup $H \subseteq \pi$;

- $\tilde{X}$ is a 1-dimensional π -CW-complex for which there exists π -pushout

$$\begin{array}{ccc} \coprod_{P \in \text{vert}(T)} \pi/\pi_P \times S^0 & \longrightarrow & \coprod_{\bar{y} \in \overline{\text{edge}(Y)}} \pi/\pi_{\bar{y}} \\ \downarrow & & \downarrow \\ \coprod_{P \in \text{vert}(T)} \pi/\pi_P \times D^1 & \longrightarrow & \tilde{X} \end{array}$$

such that $\pi_P \cong G_P$ for $P \in \text{vert}(T)$ and $\pi_{\bar{y}} = G_{\bar{y}}$ for $\bar{y} \in \overline{\text{edge}(Y)}$ holds.

All these claims follows from [44, Section 5.3 and Theorem 15 in Section 6.1 on page 58].

Now suppose that Y is finite, each of the groups G_P for $P \in \text{vert}(T)$ and $G_{\bar{y}}$ for $\bar{y} \in \overline{\text{edge}(Y)}$ has a finite model for its classifying space for proper actions which is $\det$ - L^2 -acyclic, and π satisfies condition (DFJ).

Then we conclude from Theorem 5.7 that the groups G_P for $P \in \text{vert}(T)$ and $G_{\bar{y}}$ for $\bar{y} \in \overline{\text{edge}(Y)}$ satisfy condition (DFJ), there is a finite π -CW-model for $\underline{E}\pi$ which is $\det$ - L^2 -acyclic, and we get

$$\rho^{(2)}(\pi) = \sum_{P \in \text{vert}(T)} \rho^{(2)}(G_P) - \sum_{\bar{y} \in \overline{\text{edge}(Y)}} \rho^{(2)}(G_{\bar{y}}).$$

Example 3.12 (Amalgamated Products). Let G_0 be a common subgroup of the group G_1 and the group G_2 . Denote by $G = G_1 *_{G_0} G_2$ the amalgamated product. Suppose that there is a $\det$ - L^2 -acyclic finite G_i -CW-model for $\underline{E}G_i$ for $i = 0, 1, 2$ and that G satisfies condition (DFJ). Then G_i satisfies condition (DFJ) for $i = 0, 1, 2$, there is a $\det$ - L^2 -acyclic finite G -CW-module for $\underline{E}G$, and we get

$$\rho^{(2)}(G) = \rho^{(2)}(G_1) + \rho^{(2)}(G_2) - \rho^{(2)}(G_0).$$

This follows from Example 3.11 applied to the graph of groups associated to $G_1 *_{G_0} G_2$, see [44, page 43].

4. L^2 -TORSION OF A SELFHOMOTOPY EQUIVALENCE

For the remainder of this section we fix a group automorphism $\Phi: G \xrightarrow{\cong} G$.

Let $G \rtimes_{\Phi} \mathbb{Z}$ be the associated semidirect product. In the sequel $t \in \mathbb{Z}$ is a fixed generator of $\mathbb{Z}$. Then we can write every element in $G \rtimes_{\Phi} \mathbb{Z}$ uniquely as gt^n for $g \in G$ and $n \in \mathbb{Z}$ and the multiplication in $G \rtimes_{\Phi} \mathbb{Z}$ is given by $g_0 t^{n_0} g_1 t^{n_1} = g_0 \Phi^{n_0}(g_1) t^{n_0+n_1}$.

Given two G -spaces X and Y , a Φ -map $f: X \rightarrow Y$ is a map f from X to Y satisfying $f(gx) = \Phi(g)f(x)$ for $g \in G$ and $x \in X$. We call f a Φ -homotopy equivalence if there exists a Φ^{-1} -map $f': Y \rightarrow X$ such that $f' \circ f$ is G -homotopic to id_X and $f \circ f'$ is G -homotopic to id_Y . We call f a weak Φ -homotopy equivalence if $f^H: X^H \rightarrow Y^{\Phi(H)}$ is a weak homotopy equivalence for every subgroup $H \subseteq G$.

A G -CW-approximation (X, a) of a G -space Y is a G -CW-complex X together with a weak G -homotopy equivalence $a: X \rightarrow Y$. Every G -space Y admits a G -CW-approximation, see [32, Proposition 2.3 on page 35]. Given two G -CW-approximations (X, a) and (X', a') there exists a cellular G -homotopy equivalence $s: X \rightarrow X'$ which is uniquely characterized up to G -homotopy by the property that $a' \circ s$ and a are G -homotopic. This follows from the Equivariant Approximation Theorem, see [32, Theorem 2.1 on page 32], and the Equivariant Whitehead Theorem, see [32, Theorem 2.4 on page 36].

Definition 4.1 ($\mathcal{FIN}$ -finite G -space). A G -space Y is called $\mathcal{FIN}$ -finite, if it possesses a G -CW-approximation $a: X \rightarrow Y$ with a finite proper G -CW-complex as source.

Obviously the property $\mathcal{FIN}$ -finite depends only on the G -homotopy type of Y , actually, only on the weak G -homotopy type. Note that a group G is $\mathcal{FIN}$ -finite in the sense of Definition 2.9 if and only if the G -space $\underline{EG}$ is $\mathcal{FIN}$ -finite in the sense of Definition 4.1.

Recall that a finite proper G -CW-complex X is called of *determinant class*, if the associated Hilbert $\mathcal{N}(G)$ -chain complex $L^2(G) \otimes_{\mathbb{Z}G} C_*^c(X)$ is of determinant class in the sense of [33, Definition 3.20 on page 140]. Note that we are not demanding that X is L^2 -acyclic and the property being of determinant class depends only on the G -homotopy type of X . This follows from [33, Theorem 3.35 (1) on page 142], since the mapping cone of a $\mathbb{Q}G$ -chain homotopy equivalence of finite projective $\mathbb{Q}G$ -chain complexes is a contractible finite projective $\mathbb{Q}G$ -chain complexes and hence of determinant class by [33, Lemma 2.18 on page 83 and Lemma 3.30 on page 140]. If G satisfies the Determinant Conjecture, then every finite G -CW-complex X is of determinant class.

Definition 4.2 (Det-finite G -space). We call a G -space Y *det-finite* if there exists a G -CW-approximation (X, a) with a finite proper G -CW-complex X as source which is of determinant class.

Consider a weak Φ -homotopy equivalence $f: Y \rightarrow Y$ of a $\mathcal{FIN}$ -finite G -space Y . Choose a G -CW-approximation (X, a) with a finite proper G -CW-complex X as source, and a Φ -homotopy equivalence $\hat{f}: X \rightarrow X$ such that $a \circ \hat{f}$ and $f \circ a$ are Φ -homotopic. We assign to $\hat{f}$ a finite proper $G \rtimes_{\Phi} \mathbb{Z}$ -CW-complex $T_{\hat{f}; \Phi}$ by the $G \rtimes_{\Phi} \mathbb{Z}$ -pushout

$$(4.3) \quad \begin{array}{ccc} (G \rtimes_{\Phi} \mathbb{Z}) \times_G X \times \{0, 1\} & \xrightarrow{q} & (G \rtimes_{\Phi} \mathbb{Z}) \times_G X \\ \downarrow i & & \downarrow \\ (G \rtimes_{\Phi} \mathbb{Z}) \times_G X \times [0, 1] & \longrightarrow & T_{\hat{f}; \Phi} \end{array}$$

where i is the obvious inclusion and q sends (gt^n, x, k) to (gt^n, x) if $k = 0$ and to $(gt^{n-1}, \hat{f}(x))$ if $k = 1$. Note that $T_{\hat{f}; \Phi}$ is a kind of to both sides infinite mapping telescope. If $\Phi = \text{id}_G$, the quotient space $T_{\hat{f}; \Phi}/\mathbb{Z}$ is the mapping torus of $\hat{f}$. The quotient space $T_{\hat{f}; \Phi}/(G \rtimes_{\Phi} \mathbb{Z})$ is the mapping torus of $\hat{f}/G$.

Lemma 4.4. (i) *The finite $G \rtimes_{\Phi} \mathbb{Z}$ -CW-complex $T_{\hat{f}; \Phi}$ is L^2 -acyclic;*
(ii) *If X is of determinant class, then $T_{\hat{f}; \Phi}$ is det- L^2 -acyclic;*
(iii) *Suppose that there is $n \in \mathbb{Z}^{\geq 1}$ such that $\Phi^n = \text{id}_G$ and f^n is G -homotopic to the identity. Then $T_{\hat{f}; \Phi}$ is det- L^2 -acyclic.*

Proof. (i) Consider $d \in \mathbb{Z}^{\geq 1}$. Let $\text{pr}: G \rtimes_{\Phi} \mathbb{Z} \rightarrow \mathbb{Z}$ be the projection. Then $\text{pr}^{-1}(d \cdot \mathbb{Z})$ has index d in $G \rtimes_{\Phi} \mathbb{Z}$ and can be identified with $G \rtimes_{\Phi^d} \mathbb{Z}$. The restriction of $T_{\hat{f}; \Phi}$ to $\text{pr}^{-1}(d \cdot \mathbb{Z}) = G \rtimes_{\Phi^d} \mathbb{Z}$ is $G \rtimes_{\Phi^d} \mathbb{Z}$ -homotopy equivalent to $T_{\hat{f}^d; \Phi^d}$. We get from Theorem 2.7 (i) and (iv)

$$b_m^{(2)}(T_{\hat{f}; \Phi}; \mathcal{N}(G \rtimes_{\Phi} \mathbb{Z})) = \frac{1}{[d]} \cdot b_m^{(2)}(T_{\hat{f}^d; \Phi^d}; \mathcal{N}(G \rtimes_{\Phi^d} \mathbb{Z})).$$

If C is the number of cells of X/G , then the number of cells of $T_{\hat{f}^d; \Phi^d}/(G \rtimes_{\Phi^d} \mathbb{Z})$ is bounded by $2C$. This implies

$$\begin{aligned} b_m^{(2)}(T_{\hat{f}^d; \Phi^d}; \mathcal{N}(G \rtimes_{\Phi^d} \mathbb{Z})) \\ \leq \dim_{\mathcal{N}(G \rtimes_{\Phi^d} \mathbb{Z})}(L^2(G \rtimes_{\Phi^d} \mathbb{Z}) \otimes_{\mathbb{Z}[G \rtimes_{\Phi^d} \mathbb{Z}]} C_m(T_{\hat{f}^d; \Phi^d})) \leq 2C. \end{aligned}$$

Hence we get for every $d \in \mathbb{Z}^{\geq 1}$

$$b_m^{(2)}(T_{\hat{f};\Phi}; \mathcal{N}(G \rtimes_{\Phi} \mathbb{Z})) \leq \frac{2C}{d}.$$

Therefore $b_m^{(2)}(T_{\hat{f};\Phi}; \mathcal{N}(G \rtimes_{\Phi} \mathbb{Z}))$ vanishes for every $m \in \mathbb{Z}^{\geq 0}$.

(ii) The $G \times_{\Phi} \mathbb{Z}$ -CW-complex $(G \rtimes_{\Phi} \mathbb{Z}) \times_G X$ is of determinant class, since the G -CW-complex X is of determinant class, see [33, Theorem 3.14 (6) on page 129]. Hence $T_{\hat{f};\Phi}$ is of determinant class by [33, Theorem 3.35 (1) on page 142].

(iii) By inspecting the proof of assertion (i), we can show the following: There is an inclusion of groups $G \times \mathbb{Z} \rightarrow G \rtimes_{\Phi} \mathbb{Z}$ of finite index n such that the restriction $i^*T_{\hat{f};\Phi}$ of the proper finite $G \rtimes_{\Phi} \mathbb{Z}$ -CW-complex $T_{\hat{f};\Phi}$ to $G \times \mathbb{Z}$ with i is $G \times \mathbb{Z}$ -homotopy equivalent to $X \times \mathbb{R}$ with the obvious $G \times \mathbb{Z}$ -action coming from the given G -action on X and the $\mathbb{Z}$ -action on $\mathbb{R}$ given by translation. We conclude from [33, Theorem 3.14 (5) on page 128] that it suffices to show that the proper finite G -CW-complex $i^*T_{\hat{f};\Phi}$ is of determinant class. We have explained already above that the property being of determinant class depends only on the homotopy type of a finite Hilbert $\mathcal{N}(G \times \mathbb{Z})$ -chain complex. Therefore it suffices to show that the proper finite $G \times \mathbb{Z}$ -CW-complex $X \times \mathbb{R}$ is of determinant class. Using [33, Theorem 3.35 (1) on page 142] one reduces the claim to the case $X = G/H$. Since the proper finite G -CW-complex $G/H \times \mathbb{R}$ is G -homeomorphic to $(G \times \mathbb{Z}) \times_{H \times \mathbb{Z}} \mathbb{R}$ with respect to the $H \times \mathbb{Z}$ -action on $\mathbb{R}$ given by the projection $H \times \mathbb{Z} \rightarrow \mathbb{Z}$, it suffices to show by [33, Theorem 3.14 (6) on page 129] that the proper finite $H \times \mathbb{Z}$ -complex $\mathbb{R}$ is of determinant class. Because of [33, Theorem 3.14 (5) on page 128] it suffices to show that the proper finite $\mathbb{Z}$ -CW-complex $\mathbb{R}$ is of determinant class. This follows by a direct inspection or the fact that the Determinant Conjecture holds for the group $\mathbb{Z}$. $\square$

Definition 4.5 (Det-finite Φ -self-homotopy equivalence). We call a weak Φ -self-homotopy equivalence $f: Y \rightarrow Y$ of a $\mathcal{FLN}$ -finite G -space Y det-finite if the following holds. For every G -CW-approximation (X, a) with a finite proper G -CW-complex X as source and every Φ -homotopy equivalence $\hat{f}: X \rightarrow X$ such that $a \circ \hat{f}$ and $f \circ a$ are Φ -homotopic, the finite $G \rtimes_{\Phi} \mathbb{Z}$ -CW-complex $T_{\hat{f};\Phi}$ is of determinant class.

Note that for a det-finite Φ -automorphism the finite $G \rtimes_{\Phi} \mathbb{Z}$ -CW-complex $T_{\hat{f};\Phi}$ is det- L^2 -acyclic by Lemma 4.4 (i).

Definition 4.6 (L^2 -torsion of a selfhomotopy equivalence). Consider a weak Φ -self homotopy equivalence $f: Y \rightarrow Y$ of the $\mathcal{FLN}$ -finite G -space Y which is det-finite. Choose a G -CW-approximation (X, a) with a finite proper G -CW-complex X as source, and a Φ -homotopy equivalence $\hat{f}: X \rightarrow X$ such that $a \circ \hat{f}$ and $f \circ a$ are Φ -homotopic.

Then $T_{\hat{f};\Phi}$ is det- L^2 -acyclic and we define the L^2 -torsion of $(f; \Phi)$ to be

$$\rho^{(2)}(f; \Phi) := \rho^{(2)}(T_{\hat{f};\Phi}; \mathcal{N}(G \rtimes_{\Phi} \mathbb{Z})) \in \mathbb{R}.$$

In the remainder of this section will show that the notion of $\rho^{(2)}(f; \Phi)$ appearing in Definition 4.6 is well-defined and collect its main properties, namely we will prove the following Lemma 4.7 and the following Theorem 4.8.

Lemma 4.7. Consider a Φ -self-homotopy equivalence $f: Y \rightarrow Y$ of a $\mathcal{FLN}$ -finite G -space Y .

- (i) Then $(f; \Phi)$ is det-finite if and only there is a G -CW-approximation (X, a) with a finite proper G -CW-complex X as source and a Φ -homotopy equivalence $\hat{f}: X \rightarrow X$ such that $a \circ \hat{f}$ and $f \circ a$ are Φ -homotopic and the finite $G \rtimes_{\Phi} \mathbb{Z}$ -CW-complex $T_{\hat{f}; \Phi}$ is of determinant class;
- (ii) The number $\rho^{(2)}(f; \Phi)$ appearing in Definition 4.6 is independent of the choices of (X, a) and $\hat{f}$.

Proof. Suppose that (X_l, a_l) for $l = 0, 1$ is a G -CW-approximation of Y with a finite proper G -CW-complex X_l as source and we have a Φ -homotopy equivalence $\hat{f}_l: X_l \rightarrow X_l$ such that $a_l \circ \hat{f}_l$ and $f \circ a_l$ are Φ -homotopic. Then we can choose a cellular G -homotopy equivalence $s': X_0 \rightarrow X_1$ such that $a_1 \circ s'$ and a_0 are cellularly G -homotopic. We obtain a diagram of finite G -CW-complexes

$$\begin{array}{ccc} X_0 & \xrightarrow{\hat{f}_0} & X_0 \\ s' \downarrow & & \downarrow s' \\ X_1 & \xrightarrow{\hat{f}_1} & X_1 \end{array}$$

where all arrows are cellular and which commutes up to cellular Φ -homotopy. In the sequel we often abbreviate

$$G_{\Phi} = G \times_{\Phi} \mathbb{Z}.$$

Let $s: G_{\Phi} \times_G X_0 \rightarrow G_{\Phi} \times_G X_1$ be the cellular G_{Φ} -homotopy equivalence $s = \text{id}_{G_{\Phi}} \times_G s'$. Let $s^{-1}: G_{\Phi} \times_G X_1 \rightarrow G_{\Phi} \times_G X_0$ be some cellular G_{Φ} -homotopy inverse of s . Define

$$q_k: G_{\Phi} \times_G X_k \times \{0, 1\} \rightarrow G_{\Phi} \times_G X_k$$

by sending (gt^n, x, k) to (gt^n, x) , if $k = 0$, and to $(gt^{n-1}, \hat{f}_k(x))$, if $k = 1$. Choose a cellular G_{Φ} -homotopy $h: G_{\Phi} \times_G X_0 \times \{0, 1\} \times [0, 1] \rightarrow G_{\Phi} \times_G X_1$ satisfying $h_0 = s \circ q_0$ and $h_1 = q_1 \circ (s \times \text{id}_{\{0, 1\}})$.

Consider the following commutative diagram of finite G_{Φ} -CW-complexes

$$\begin{array}{ccccc} G_{\Phi} \times_G X_0 \times [0, 1] & \longleftarrow & G_{\Phi} \times_G X_0 \times \{0, 1\} & \xrightarrow{q_0} & G_{\Phi} \times_G X_0 \\ \downarrow \text{id} & & \downarrow \text{id} & & \downarrow s \\ G_{\Phi} \times_G X_0 \times [0, 1] & \longleftarrow & G_{\Phi} \times_G X_0 \times \{0, 1\} & \xrightarrow{s \circ q_0} & G_{\Phi} \times_G X_1 \\ \downarrow l_0 & & \downarrow k_0 & & \downarrow \text{id} \\ G_{\Phi} \times_G X_0 \times [0, 1] \times [0, 1] & \longleftarrow & G_{\Phi} \times_G X_0 \times \{0, 1\} \times [0, 1] & \xrightarrow{h} & G_{\Phi} \times_G X_1 \\ \uparrow l_1 & & \uparrow k_1 & & \uparrow \text{id} \\ G_{\Phi} \times_G X_0 \times [0, 1] & \longleftarrow & G_{\Phi} \times_G X_0 \times \{0, 1\} & \xrightarrow{q_1 \circ (s \times \text{id}_{\{0, 1\}})} & G_{\Phi} \times_G X_1 \\ \downarrow s \times \text{id}_{[0, 1]} & & \downarrow s \times \text{id}_{\{0, 1\}} & & \downarrow \text{id} \\ G_{\Phi} \times_G X_1 \times [0, 1] & \longleftarrow & G_{\Phi} \times_G X_1 \times \{0, 1\} & \xrightarrow{q_1} & G_{\Phi} \times_G X_1 \end{array}$$

where the maps k_m and l_m are induced by the inclusion $\{\bullet\} \rightarrow [0, 1]$ with image $\{m\}$ for $m = 0, 1$.

Recall that $T_{\hat{f}_0; \Phi}$ is the G_{Φ} -pushout of the uppermost row and $T_{\hat{f}_1; \Phi}$ is the G_{Φ} -pushout of the lowermost row. Let Z_m be the G -pushout of the m -th row for

$m = 2, 3, 4$. Then the diagram above induced a zigzag of G_Φ -homotopy equivalences of finite proper G_Φ -CW-complexes

$$T_{\hat{f}_0; \Phi} \xrightarrow{u_1} Z_2 \xrightarrow{u_2} Z_3 \xleftarrow{u_3} Z_4 \xrightarrow{u_4} T_{\hat{f}_1; \Phi}.$$

Let u_3^{-1} be a G_Φ -homotopy inverse of u_3 . Define the G_Φ -homotopy equivalence

$$u = u_4 \circ u_3^{-1} \circ u_2 \circ u_1 : T_{\hat{f}_0; \Phi} \rightarrow T_{\hat{f}_1; \Phi}.$$

The equivariant Whitehead torsion $\tau^{G_\Phi}(u)$ vanishes by the following computation based on [32, Theorem 4.8 on page 62]

$$\begin{aligned} \tau^{G_\Phi}(u) &= \tau^{G_\Phi}(u_1) + \tau^{G_\Phi}(u_2) + \tau^{G_\Phi}(u_3^{-1}) + \tau^{G_\Phi}(u_4) \\ &= \tau^{G_\Phi}(u_1) + \tau^{G_\Phi}(u_2) - \tau^{G_\Phi}(u_3) + \tau^{G_\Phi}(u_4) \\ &= \tau^{G_\Phi}(s) + 0 - 0 + (\tau^{G_\Phi}(s \times \text{id}_{[0,1]}) - \tau^{G_\Phi}(s \times \text{id}_{\{0,1\}})) \\ &= \tau^{G_\Phi}(s) + 0 - 0 + (\tau^{G_\Phi}(s) - 2 \cdot \tau^{G_\Phi}(s)) \\ &= 0. \end{aligned}$$

Hence $T_{\hat{f}_0; \Phi}$ and $T_{\hat{f}_1; \Phi}$ are simple G_Φ -homotopy equivalent. Now Lemma 4.7 follows from Theorem 2.7 (i). $\square$

Theorem 4.8 (Main properties of the L^2 -torsion of a self homotopy equivalence).

Let $\Phi : G \xrightarrow{\cong} G$ be a group automorphism.

(i) Equivariant homotopy invariance

Let $f : Y \rightarrow Y$ and $f' : Y' \rightarrow Y'$ be weak Φ -homotopy equivalences of $\mathcal{FLN}$ -finite G -spaces Y and Y' and $u : Y \rightarrow Y'$ be a weak G -homotopy equivalence such that $u \circ f$ and $f' \circ u$ are Φ -homotopy equivalent. Suppose that $(f; \Phi)$ or $(f'; \Phi)$ is det-finite.

Then both $(f; \Phi)$ and $(f'; \Phi)$ are det-finite and we get

$$\rho^{(2)}(f; \Phi) = \rho^{(2)}(f'; \Phi).$$

In particular the property det-finite and the number $\rho^{(2)}(f; \Phi)$ depends only on the Φ -homotopy class of f ;

(ii) Trace formula

Let $\Psi : G \xrightarrow{\cong} G'$ and $\Psi' : G' \rightarrow G$ be group isomorphisms. Consider a $\mathcal{FLN}$ -finite G -space Y and a $\mathcal{FLN}$ -finite G' -space Y' .

Let $f : Y \rightarrow Y'$ be a weak Ψ -homotopy equivalence and $f' : Y' \rightarrow Y$ be a weak Ψ' -homotopy equivalence. Suppose that $(f' \circ f; \Psi' \circ \Psi)$ or $(f \circ f'; \Psi \circ \Psi')$ is det-finite.

Then both $(f' \circ f; \Psi' \circ \Psi)$ and $(f \circ f'; \Psi \circ \Psi')$ are det-finite and we get

$$\rho^{(2)}(f' \circ f; \Psi' \circ \Psi) = \rho^{(2)}(f \circ f'; \Psi \circ \Psi');$$

(iii) Multiplicativity

Let $f : Y \rightarrow Y$ be weak Φ -homotopy equivalence for the $\mathcal{FLN}$ -finite G -space Y . Consider $n \in \mathbb{Z}^{\geq 1}$. Suppose that $(f^n; \Phi^n)$ or $(f; \Phi)$ is det-finite.

Then both $(f^n; \Phi^n)$ and $(f; \Phi)$ are det-finite and we get

$$\rho^{(2)}(f^n; \Phi^n) = n \cdot \rho^{(2)}(f; \Phi);$$

(iv) Restriction

Let $f : Y \rightarrow Y$ be weak Φ -homotopy equivalence for the $\mathcal{FLN}$ -finite G -space Y . Let $H \subseteq G$ be a subgroup of G of finite index $[G : H]$ satisfying $\Phi(H) = H$.

Then the restriction $Y|_H$ of Y to an H -space is $\mathcal{FIN}$ -finite. If we additionally assume that $(f|_H; \Phi|_H)$ or $(f; \Phi)$ is det-finite, then both $(f|_H; \Phi|_H)$ and $(f; \Phi)$ are det-finite and we get

$$\rho^{(2)}(f|_H; \Phi|_H) = [G : H] \cdot \rho^{(2)}(f; \Phi);$$

(v) Induction

Let $H \subseteq G$ be a subgroup satisfying $\Phi(H) = H$. Let $f: Y \rightarrow Y$ be a weak $\Phi|_H: H \rightarrow H$ -homotopy equivalence for the $\mathcal{FIN}$ -finite H -space Y .

Then the $G \times_H Y$ is a $\mathcal{FIN}$ -finite G -space and we get a weak Φ -homotopy equivalence $F: G \times_H Y \rightarrow G \times_H Y$ by sending (g, x) to $(\Phi(g), f(x))$. If we additionally assume that $(f; \Phi|_H)$ or $(F; \Phi)$ is det-finite, then both $(f; \Phi|_H)$ and $(F; \Phi)$ are det-finite and we have

$$\rho^{(2)}(F; \Phi) = \rho^{(2)}(f; \Phi|_H);$$

(vi) Finite quotients

Let $1 \rightarrow K \rightarrow G \xrightarrow{p} Q \rightarrow 1$ be an extension of groups with finite K . Let $\tilde{\Phi}: G \xrightarrow{\cong} G$ and $\Phi: Q \xrightarrow{\cong} Q$ be group automorphisms satisfying $p \circ \tilde{\Phi} = \Phi \circ p$. Let $f: Y \rightarrow Y$ be a weak Φ -homotopy equivalence for the $\mathcal{FIN}$ -finite Q -space Y . Let p^*X be the G -space obtained from X by restriction with p . Then p^*Y is $\mathcal{FIN}$ -finite and $p^*f: p^*Y \rightarrow p^*Y$ is a weak $\tilde{\Phi}$ -homotopy equivalence.

If (f, Φ) or $(p^*f, \tilde{\Phi})$ is det-finite, then both (f, Φ) and $(p^*f, \tilde{\Phi})$ are det-finite and we get

$$\rho^{(2)}(p^*f, \tilde{\Phi}) = \frac{\rho^{(2)}(f; \Phi)}{|K|};$$

(vii) Inner automorphisms

Let $f: Y \rightarrow Y$ be weak Φ -homotopy equivalence for the $\mathcal{FIN}$ -finite G -space Y . Consider $g \in G$. Let $l_g: Y \rightarrow Y$ be the map given by multiplication with g and $c_g: G \rightarrow G$ be the inner automorphism sending to g' to $gg'g^{-1}$. Suppose that $(l_g \circ f; c_g \circ \Phi)$ or $(f; \Phi)$ is det-finite.

Then both $(l_g \circ f; c_g \circ \Phi)$ and $(f; \Phi)$ are det-finite and we get

$$\rho^{(2)}(l_g \circ f; c_g \circ \Phi) = \rho^{(2)}(f; \Phi);$$

(viii) Sum formula

Consider the G -pushout

$$\begin{array}{ccc} Y_0 & \xrightarrow{j_2} & Y_2 \\ j_1 \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Y \end{array}$$

where j_1 is a G -fibration, and the commutative diagram

$$\begin{array}{ccccc} Y_1 & \xleftarrow{j_1} & Y_0 & \xrightarrow{j_2} & Y_2 \\ \downarrow f_1 & & \downarrow f_0 & & \downarrow f_2 \\ Y_1 & \xleftarrow{j_1} & Y_0 & \xrightarrow{j_2} & Y_2 \end{array}$$

where the vertical arrows are weak Φ -homotopy equivalences. Let $f: Y \rightarrow Y$ be the map given by f_0, f_1 , and f_2 and the G -pushout property. Suppose that Y_0, Y_1 , and Y_2 are $\mathcal{FIN}$ -finite and that $(f_0; \Phi)$, $(f_1; \Phi)$, and $(f_2; \Phi)$ are det-finite.

Then Y is $\mathcal{FLN}$ -finite, $f: Y \rightarrow Y$ is a weak Φ -homotopy equivalence, $(f; \Phi)$ is det-finite, and we get

$$\rho^{(2)}(f; \Phi) = \rho^{(2)}(f_1; \Phi) + \rho^{(2)}(f_2; \Phi) - \rho^{(2)}(f_0; \Phi);$$

(ix) Product formula

Let $f: Y \rightarrow Y$ be a weak Φ -homotopy equivalence for the $\mathcal{FLN}$ -finite G -space Y and let H be a group. Let Z be a $\mathcal{FLN}$ -finite H -space. Denote by $\chi^{(2)}(Z; \mathcal{N}(H))$ the L^2 -Euler characteristic of Z . Suppose that f is det-finite.

Then $f \times \text{id}_Z: Y \times Z \rightarrow Y \times Z$ is a weak $\Phi \times \text{id}_H: G \times H \rightarrow G \times H$ -homotopy equivalence for the $\mathcal{FLN}$ -finite $G \times H$ -space $Y \times Z$, is det-finite, and we get

$$\rho^{(2)}(f \times \text{id}_Z; \Phi \times \text{id}_H) = \chi^{(2)}(Z; \mathcal{N}(H)) \cdot \rho^{(2)}(f; \Phi);$$

(x) L^2 -acyclic space

Let $f: Y \rightarrow Y$ be weak Φ -homotopy equivalence for the $\mathcal{FLN}$ -finite G -space Y . Suppose that Y is det- L^2 -acyclic in the sense that there is a det- L^2 -acyclic finite proper G -CW-complex X together with a weak homotopy equivalence $X \rightarrow Y$.

Then $(f; \Phi)$ is det-finite and we get

$$\rho^{(2)}(f; \Phi) = 0.$$

(xi) Periodic self equivalence

Let $f: Y \rightarrow Y$ be a weak Φ -homotopy equivalence for a G -space Y . Suppose that there exists $n \in \mathbb{Z}^{\geq 1}$ such that $\Phi^n = \text{id}_G$ and f^n is G -homotopic to the identity id_Y .

Then $(f; \Phi)$ is det-finite and we get

$$\rho^{(2)}(f; \Phi) = 0.$$

Proof. (i) Choose a cellular G -approximation (X, a) of Y for a finite G -CW-complex X of determinant class and a Φ -homotopy equivalence $\hat{f}: X \rightarrow X$ such that $a \circ \hat{f}$ and $f \circ a$ are Φ -homotopic. Then $a' := u \circ a: X \rightarrow Y'$ is a cellular G -approximation and $a' \circ \hat{f}$ and $f' \circ a'$ are G -homotopic. Now we conclude

$$\rho^{(2)}(f; \Phi) = \rho^{(2)}(T_{\hat{f}; \Phi}; \mathcal{N}(G \rtimes_{\Phi} \mathbb{Z})) = \rho^{(2)}(f'; \Phi).$$

from Definition 4.6.

(ii) Because of Definition 4.6 it suffices to show for finite proper G -CW-complexes X and X' which are of determinant class, a Ψ -homotopy equivalence $f: X \rightarrow X'$, and a Ψ' -homotopy equivalence $f': X' \rightarrow X$

$$(4.9) \quad \rho^{(2)}(T_{f' \circ f, \Psi' \circ \Psi}; \mathcal{N}(G \rtimes_{\Psi' \circ \Psi} \mathbb{Z})) = \rho^{(2)}(T_{f \circ f', \Psi \circ \Psi'}; \mathcal{N}(G' \rtimes_{\Psi \circ \Psi'} \mathbb{Z}))$$

holds. We obtain an isomorphism of groups $\mu: G \rtimes_{\Psi' \circ \Psi} \mathbb{Z} \xrightarrow{\cong} G' \rtimes_{\Psi \circ \Psi'} \mathbb{Z}$ by sending gt^n to $\psi(g)t^n$. It is not hard to check that there is a simple μ -homotopy equivalence $T_{f' \circ f, \Psi' \circ \Psi} \rightarrow T_{f \circ f', \Psi \circ \Psi'}$. Now (4.9) follows from Theorem 2.7 (i).

(iii) Let $\text{pr}: G \rtimes_{\Phi} \mathbb{Z} \rightarrow \mathbb{Z}$ be the projection. Then $\text{pr}^{-1}(n \cdot \mathbb{Z})$ has index n in $G \rtimes_{\Phi} \mathbb{Z}$ and can be identified with $G \rtimes_{\Phi^n} \mathbb{Z}$. The restriction of $T_{\hat{f}; \Phi}$ to $\text{pr}^{-1}(n \cdot \mathbb{Z}) = G \rtimes_{\Phi^n} \mathbb{Z}$ is simple $G \rtimes_{\Phi^n} \mathbb{Z}$ -homotopy equivalent to $T_{\hat{f}_n; \Phi^n}$. Now the claim follows from Theorem 2.7 (i) and (iv).

(iv) We can view $H \rtimes_{\Phi_H} \mathbb{Z}$ as a subgroup of finite index in $G \rtimes_{\Phi} \mathbb{Z}$. The restriction of the $G \rtimes_{\Phi} \mathbb{Z}$ -space $T_{\hat{f}; \Phi}$ to $G \rtimes_{\Phi} \mathbb{Z}$ can be identified with the $H \rtimes_{\Phi|_H} \mathbb{Z}$ -space $T_{\hat{f}|_H; \Phi|_H}$, where $\hat{f}|_H$ is the $\Phi|_H$ -homotopy equivalence obtained from $\hat{f}$ by restriction. Now the claim follows from Theorem 2.7 (iv).

(v) We can view $H \rtimes_{\Phi|_H} \mathbb{Z}$ as a subgroup of $G \rtimes_{\Phi} \mathbb{Z}$. The $G \rtimes_{\Phi} \mathbb{Z}$ -space $T_{\widehat{f};\Phi}$ can be identified with $(G \rtimes_{\Phi} \mathbb{Z}) \times_{H \rtimes_{\Phi|_H} \mathbb{Z}} T_{\widehat{f};\Phi|_H}$. Now the claim follows from Theorem 2.7 (v).

(vi) Note that we obtain an exact sequence $1 \rightarrow K \rightarrow G \rtimes_{\widehat{\Phi}} \mathbb{Z} \xrightarrow{\widehat{p}} Q \rtimes_{\Phi} \mathbb{Z} \rightarrow 1$, where $\widehat{p}$ sends gt^n to $p(g)t^n$. Now assertion (vi) follows from Theorem 2.7 (vi) since $T_{p^*\widehat{f};\widehat{\Phi}}$ is $\widehat{p}^*T_{\widehat{f};\widehat{\Phi}}$.

(vii) The map $l_g: X \xrightarrow{\cong} X$ is a c_g -homeomorphism. Let $\mu: G \rtimes_{\Phi} \mathbb{Z} \xrightarrow{\cong} G \rtimes_{c_g \circ \Phi} \mathbb{Z}$ be the group isomorphism sending $g't^n$ to $g'(g^{-1}t)^n$. Then we obtain a cellular μ -homeomorphism $T_{\widehat{f};\Phi} \xrightarrow{\cong} T_{l_g \circ \widehat{f}; c_g \circ \Phi}$ since the following diagram commutes

$$\begin{array}{ccc} (G \rtimes_{\Phi} \mathbb{Z}) \times_G X & \xrightarrow{q_{\widehat{f}}} & (G \rtimes_{\Phi} \mathbb{Z}) \times_G X \\ \mu \times_G \text{id}_X \downarrow & & \downarrow \mu \times_G \text{id}_X \\ (G \rtimes_{\Phi} \mathbb{Z}) \times_G X & \xrightarrow{q_{l_g \circ \widehat{f}}} & (G \rtimes_{\Phi} \mathbb{Z}) \times_G X \end{array}$$

where $q_{\widehat{f}}$ sends $(g't^n, x)$ to $(g't^{n-1}, \widehat{f}(x))$ and $q_{l_g \circ \widehat{f}}$ sends $(g't^n, x)$ to $(g't^{n-1}, l_g \circ \widehat{f}(x))$. This implies

$$\begin{aligned} \rho^{(2)}(l_g \circ \widehat{f}; c_g \circ \Phi) &= \rho^{(2)}(T_{l_g \circ \widehat{f}; c_g \circ \Phi}; \mathcal{N}(G \times_{c_g \circ \Phi} \mathbb{Z})) \\ &= \rho^{(2)}(T_{\widehat{f};\Phi}; \mathcal{N}(G \times_{\Phi} \mathbb{Z})) = \rho^{(2)}(\widehat{f}; \Phi). \end{aligned}$$

(viii) We can assume because of assertion (i) without loss of generality that both inclusions $j_l: Y_0 \rightarrow Y_l$ for $l = 1, 2$ are G -cofibrations, otherwise replace j_2 by the inclusion of Y_0 into the mapping cylinder of j_2 .

By assumption the G -space Y_l has the weak G -homotopy type of a proper finite G -CW-complex for $l = 0, 1, 2$. Then there is a commutative diagram of G -spaces

$$\begin{array}{ccccc} X_1 & \xleftarrow{i_1} & X_0 & \xrightarrow{i_2} & X_2 \\ \downarrow a_1 & & \downarrow a_0 & & \downarrow a_2 \\ Y_1 & \xleftarrow{j_1} & Y_0 & \xrightarrow{j_2} & Y_2 \end{array}$$

such that $i_l: X_0 \rightarrow X_l$ is an inclusion of proper finite G -CW-complexes for $l = 1, 2$, see [32, 4.31 on page 76]. We obtain a commutative diagram of G -spaces

$$\begin{array}{ccccc} X_1 & \xleftarrow{i_1} & X_0 & \xrightarrow{i_2} & X_2 \\ \downarrow b_1 & & \downarrow b_0 & & \downarrow b_2 \\ \text{cyl}(a_1) & \xleftarrow{k_1} & \text{cyl}(a_0) & \xrightarrow{k_2} & \text{cyl}(a_2) \\ \uparrow c_1 & & \uparrow c_0 & & \uparrow c_0 \\ Y_1 & \xleftarrow{j_1} & Y_0 & \xrightarrow{j_2} & Y_2 \end{array}$$

where the vertical maps are the canonical inclusions into the mapping cylinders, and k_l is the obvious map induced by i_l and j_l for $l = 0, 1$. Note a_l is a weak G -homotopy equivalence and b_l is a G -homotopy equivalence for $l = 0, 1, 2$. One

also obtains a commutative diagram of G -spaces

$$\begin{array}{ccccc} \text{cyl}(a_1) & \xleftarrow{k_1} & \text{cyl}(a_0) & \xrightarrow{k_2} & \text{cyl}(a_2) \\ \downarrow p_1 & & \downarrow p_0 & & \downarrow p_2 \\ Y_1 & \xleftarrow{j_1} & Y_0 & \xrightarrow{j_2} & Y_2 \end{array}$$

where the vertical maps are the canonical projections. Note that p_l is a G -homotopy equivalence and $p_l \circ b_l = a_l$ holds for $l = 0, 1, 2$. For the proof we need the following lemma.

Lemma 4.10. *In the situation above we can construct Φ -homotopy equivalences $\widehat{f}_l: X_l \rightarrow X_l$ and $v_l: \text{cyl}(a_l) \rightarrow \text{cyl}(a_l)$ for $l = 0, 1, 2$ such that*

$$\begin{aligned} \widehat{f}_l \circ i_l &= i_l \circ \widehat{f}_0 & \text{for } l = 1, 2; \\ v_l \circ k_l &= k_l \circ v_0 & \text{for } l = 1, 2; \\ v_l \circ b_l &= b_l \circ \widehat{f}_l & \text{for } l = 0, 1, 2; \\ v_l \circ c_l &= c_l \circ f_l & \text{for } l = 0, 1, 2, \end{aligned}$$

holds.

Proof. Define $v_l: \text{cyl}(a_l) \rightarrow \text{cyl}(a_l)$ to be $c_l \circ f_l \circ p_l$ for $l = 0, 1, 2$. Then we get $v_l \circ c_l = c_l \circ f_l$ for $l = 0, 1, 2$ and $v_l \circ k_l = k_l \circ v_0$ for $l = 1, 2$.

We can find for $l = 0, 1, 2$ a Φ -homotopy equivalence $\widehat{f}_l: X_l \rightarrow X_l$ such that $v_l \circ b_l$ and $b_l \circ \widehat{f}_l$ are G -homotopic by the Equivariant Whitehead Theorem, see [32, Theorem 2.4 on page 36]. Note that then also $f_l \circ p_l$ and $p_l \circ \widehat{f}_l$ are G -homotopic for $l = 0, 1, 2$. Since $b_0 \amalg c_0: X_0 \amalg Y_0 \rightarrow \text{cyl}(a_0)$ is a G -cofibration, we can change $v_0: \text{cyl}(a_0) \rightarrow \text{cyl}(a_0)$ up to Φ -homotopy such that $v_0 \circ b_0 = b_0 \circ \widehat{f}_0$ holds and we keep $v_0 \circ c_0 = c_0 \circ f_0$. Fix $l \in \{1, 2\}$. Since $i_l: X_0 \rightarrow X_l$ is a G -cofibration, we can change $\widehat{f}_l$ up to G -homotopy such that $\widehat{f}_l \circ i_l = i_l \circ \widehat{f}_0$ holds. Since the inclusion the subspace of $\text{cyl}(a_l)$ given by $b_l(X_l) \cup c_l(Y_l) \cup k_l(\text{cyl}(f_0))$ into $\text{cyl}(f_l)$ is a cofibration, we can change v_l up to Φ -homotopy such that $v_l \circ k_l = k_l \circ v_0$ and $v_l \circ b_l = b_l \circ \widehat{f}_l$ hold and we keep $v_l \circ c_l = c_l \circ f_l$ and $v_l \circ k_l = k_l \circ v_0$. $\square$

Consider the three G -pushouts

$$\begin{array}{ccc} X_0 \xrightarrow{i_1} X_1 & Y_0 \xrightarrow{j_1} Y_1 & \text{cyl}(a_0) \xrightarrow{k_1} \text{cyl}(a_1) \\ i_2 \downarrow & j_2 \downarrow & k_2 \downarrow \\ X_2 \longrightarrow X & Y_2 \longrightarrow Y & \text{cyl}(a_2) \longrightarrow Z. \end{array}$$

From the Φ -homotopy equivalences $\widehat{f}_l$ for $l = 0, 1, 2$, we obtain by the G -pushout property a Φ -homotopy equivalence $\widehat{f}: X \rightarrow X$. From the Φ -homotopy equivalences v_l for $l = 0, 1, 2$ we obtain by the G -pushout property a Φ -homotopy equivalence $v: Z \rightarrow Z$. From the Φ -homotopy equivalences f_l for $l = 0, 1, 2$ we obtain by the G -pushout property a Φ -homotopy equivalence $f: Y \rightarrow Y$. From the weak G -homotopy equivalences b_l for $l = 0, 1, 2$ we obtain by the G -pushout property a weak G -homotopy equivalence $b: X \rightarrow Z$. From the G -homotopy equivalences c_l for $l = 0, 1, 2$ we obtain by the G -pushout property a G -homotopy equivalence

$c: X \rightarrow Z$. One easily checks that following diagram comutes

$$\begin{array}{ccc} X & \xrightarrow{\hat{f}} & X \\ \downarrow b & & \downarrow b \\ Z & \xrightarrow{v} & Z \\ \uparrow c & & \uparrow c \\ Y & \xrightarrow{f} & Y. \end{array}$$

Hence the following diagram for $l = 0, 1, 2$

$$\begin{array}{ccc} X_l & \xrightarrow{\hat{f}_l} & X_l \\ \downarrow a_l & & \downarrow a_l \\ Y_l & \xrightarrow{f_l} & Y_l \end{array}$$

and the diagram

$$\begin{array}{ccc} X & \xrightarrow{\hat{f}} & X \\ \downarrow a & & \downarrow a \\ Y & \xrightarrow{f} & Y \end{array}$$

commute up to G -homotopy and have weak G -homotopy equivalences as vertical arrows, and Φ -homotopy equivalences as horizontal arrows. Since X_l for $l = 0, 1, 2$ is a proper finite G -CW-complexes of determinant class, X is a proper finite G -CW-complexes of determinant class by Theorem 4.8 (viii). We get

$$\begin{aligned} \rho^{(2)}(f_l; \Phi) &= \rho^{(2)}(T_{\hat{f}_l; \Phi}; \mathcal{N}(G \rtimes_{\Phi} \mathbb{Z})) \quad \text{for } l = 0, 1, 2; \\ \rho^{(2)}(f; \Phi) &= \rho^{(2)}(T_{\hat{f}; \Phi}; \mathcal{N}(G \rtimes_{\Phi} \mathbb{Z})). \end{aligned}$$

from the definitions. We obtain a G -pushout of finite proper $G \rtimes_{\Phi} \mathbb{Z}$ -CW-complexes

$$\begin{array}{ccc} T_{\hat{f}_0; \Phi} & \longrightarrow & T_{\hat{f}_1; \Phi} \\ \downarrow & & \downarrow \\ T_{\hat{f}_2; \Phi} & \longrightarrow & T_{\hat{f}; \Phi} \end{array}$$

where all maps are inclusions of $G \rtimes_{\Phi} \mathbb{Z}$ -CW-complexes. Theorem 2.7 (ii) implies

$$\begin{aligned} &\rho^{(2)}(T_{\hat{f}; \Phi}; \mathcal{N}(G \rtimes_{\Phi} \mathbb{Z})) \\ &= \rho^{(2)}(T_{\hat{f}_1; \Phi}; \mathcal{N}(G \rtimes_{\Phi} \mathbb{Z})) + \rho^{(2)}(T_{\hat{f}_2; \Phi}; \mathcal{N}(G \rtimes_{\Phi} \mathbb{Z})) - \rho^{(2)}(T_{\hat{f}_0; \Phi}; \mathcal{N}(G \rtimes_{\Phi} \mathbb{Z})). \end{aligned}$$

This finishes the proof of assertion (viii).

(ix). Obviously we can assume without loss of generality that Z itself is a finite proper H -CW-complex. Moreover we can replace $\chi^{(2)}(Z; \mathcal{N}(H))$ by the orbifold Euler characteristic

$$\chi_{\text{orb}}(Z) = \sum_e (-1)^{\dim(e)} \cdot \frac{1}{|H_e|},$$

where e runs through the equivariant cells of Z , because of $\chi^{(2)}(Z; \mathcal{N}(H)) = \chi_{\text{orb}}(Z)$, see [33, Subsection 6.6.1]. Using assertions (i) and (viii) one can reduce the claim to the special case $Z = H/L$ for any finite subgroup $L \subseteq H$

by induction over the dimension of Z and subinduction over the number of top-dimensional equivalent cells of Z . We get an isomorphism of proper finite $G \times H$ - CW -complexes $G \times H \times_{G \times L} \text{pr}_L^* Y \xrightarrow{\cong} Y \times H/L$ by sending $((g, h), y)$ to (gy, hL) , where $\text{pr}_L^* Y$ is the $G \times L$ - CW -space obtained from Y by restriction with the projection $\text{pr}_L: G \times L \rightarrow G$. Under this identification the $\Phi \times \text{id}_H$ -map $f \times \text{id}_{H/L}$ becomes the induction from $G \times L$ to $G \times H$ of the $\Phi \times \text{id}_L$ -map $\text{pr}_L^* f: \text{pr}_L^* Y \rightarrow \text{pr}_L^* Y$. We conclude from assertion (v) that suffices to show the claim for the $\Phi \times \text{id}_L$ -map $\text{pr}_L^* f: \text{pr}_L^* Y \rightarrow \text{pr}_L^* Y$. This follows from assertion (iv) applied to $G \subseteq G \times L$.

(x) This follows from Theorem 2.7 (ii) and assertion (ix) applied to the $G \rtimes_{\Phi} \mathbb{Z}$ -pushout (4.3).

(xi) Lemma 4.4 (iii) implies that (f, Φ) is det-finite. Now $\rho^{(2)}(f; \Phi) = 0$ follows from assertions (i) and (iii).

This finishes the proof of Theorem 4.8. $\square$

5. L^2 -TORSION OF A GROUP AUTOMORPHISM

Consider a group automorphism $\Phi: G \xrightarrow{\cong} G$. Then there is up to Φ -homotopy precisely one Φ -homotopy equivalence $f_{\Phi}: \underline{EG} \rightarrow \underline{EG}$. Now suppose that G is $\mathcal{FIN}$ -finite. In Definition 4.5 we have defined when we call f_{Φ} to be det-finite. This notion depends only on the Φ -homotopy type of f_{Φ} by Theorem 4.8 (i). Hence the following definition makes sense, i.e., is independent of the choice of f_{Φ} .

Definition 5.1 (Det-finite group automorphism). A group automorphism $\Phi: G \xrightarrow{\cong} G$ of a $\mathcal{FIN}$ -finite group G is called *det-finite*, if f_{Φ} is det-finite for one (and hence every) choice of a Φ -homotopy equivalence $f_{\Phi}: \underline{EG} \rightarrow \underline{EG}$.

Again by Theorem 4.8 (i) the following definition makes sense, i.e., is independent of the choice of f_{Φ} .

Definition 5.2 (L^2 -torsion of a group automorphism). Let $\Phi: G \xrightarrow{\cong} G$ be an automorphism of the $\mathcal{FIN}$ -finite group G . Suppose that Φ is det-finite.

Then we define its L^2 -torsion

$$\rho^{(2)}(\Phi) := \rho^{(2)}(f_{\Phi}; \Phi) \in \mathbb{R}$$

for any choice of a Φ -homotopy equivalence $f_{\Phi}: \underline{EG} \rightarrow \underline{EG}$, where $\rho^{(2)}(f_{\Phi}; \Phi)$ has been introduced in Definition 4.6.

Next we collect the basic properties of this invariant.

Theorem 5.3 (Main properties of the L^2 -torsion of a group automorphism).

(i) Trace formula

Let $\Psi: G \xrightarrow{\cong} G'$ and $\Psi': G' \rightarrow G$ be group isomorphisms. Suppose that $\Psi' \circ \Psi$ or $\Psi \circ \Psi'$ is det-finite. Then both $\Psi' \circ \Psi$ and $\Psi \circ \Psi'$ are det-finite and we get

$$\rho^{(2)}(\Psi' \circ \Psi) = \rho^{(2)}(\Psi \circ \Psi');$$

(ii) Multiplicativity

Let $\Phi: G \xrightarrow{\cong} G$ a group automorphism of the $\mathcal{FIN}$ -finite group G . Consider $n \in \mathbb{Z}^{\geq 1}$. Suppose that Φ or Φ^n is det-finite.

Then both Φ and Φ^n are det-finite and we get

$$\rho^{(2)}(\Phi^n) = n \cdot \rho^{(2)}(\Phi);$$

(iii) Periodic automorphism

Let $\Phi: G \xrightarrow{\cong} G$ a group automorphism of the $\mathcal{FIN}$ -finite group G . Consider $n \in \mathbb{Z}^{\geq 1}$. Suppose that $\Phi^n = \text{id}_G$.

Then Φ is det-finite and we get

$$\rho^{(2)}(\Phi) = 0;$$

(iv) Product formula

Let $\Phi: G \xrightarrow{\cong} G$ a group automorphism of the $\mathcal{FIN}$ -finite group G . Suppose that Φ is det-finite. Let H be a $\mathcal{FIN}$ -finite group.

Then $G \times H$ is a $\mathcal{FIN}$ -finite group, the group automorphism $\Phi \times \text{id}_H: G \times H \xrightarrow{\cong} G \times H$ is det-finite, and we get

$$\rho^{(2)}(\Phi \times \text{id}_H) = \chi^{(2)}(H) \cdot \rho^{(2)}(\Phi);$$

(v) Restriction

Let $H \subseteq G$ be a subgroup of G of finite index $[G : H]$. Let $\Phi: G \xrightarrow{\cong} G$ be a group automorphism satisfying $\Phi(H) = H$. Suppose that one of the following two conditions holds:

- (a) G is $\mathcal{FIN}$ -finite and Φ is det-finite;
- (b) H is $\mathcal{FIN}$ -finite and $\Phi|_H$ is det-finite.

Then both conditions are satisfied and we get

$$\rho^{(2)}(\Phi|_H) = [G : H] \cdot \rho^{(2)}(\Phi);$$

(vi) Finite quotients

Let $1 \rightarrow K \rightarrow G \xrightarrow{p} Q \rightarrow 1$ be an extension of groups with finite K . Let $\tilde{\Phi}: G \xrightarrow{\cong} G$ and $\Phi: Q \xrightarrow{\cong} Q$ be group automorphisms satisfying $p \circ \tilde{\Phi} = \Phi \circ p$. If Φ or $\tilde{\Phi}$ is det-finite, then both Φ and $\tilde{\Phi}$ are det-finite and we get

$$\rho^{(2)}(\tilde{\Phi}) = \frac{\rho^{(2)}(\Phi)}{|K|};$$

(vii) Conjugation invariance

Let $\Phi: G \xrightarrow{\cong} G$ a group automorphism of the $\mathcal{FIN}$ -finite group G . Consider $g \in G$. Let $c_g: G \rightarrow G$ be associated the inner automorphisms sending g' to $gg'g^{-1}$. Suppose that Φ or $c_g \circ \Phi$ is det-finite.

Then both Φ and $c_g \circ \Phi$ are det-finite and we get

(viii) L^2 -acyclic group

Let $\Phi: G \xrightarrow{\cong} G$ be an automorphism of the $\mathcal{FIN}$ -finite group G . Suppose that one and (hence every) finite G -CW-model for $\underline{EG}$ is det- L^2 -acyclic. Then Φ is det-finite and we get

$$\rho^{(2)}(f) = 0.$$

Proof. (i) This follows from This follows from Theorem 4.8 (ii)

(ii) This follows from Theorem 4.8 (iii).

(iii) This follows from Theorem 4.8 (xi).

(iv) This follows from Theorem 4.8 (ix).

(v) This follows from Theorem 4.8 (iv).

(vi) This follows from Theorem 4.8 (vi).

(vii) This follows from Theorem 4.8 (vii).

(viii) This follows from Theorem 4.8 (x) □

Remark 5.4 (No composition formula). We mention that there is *no* composition formula for the L^2 -torsion of group automorphisms. In other words, the formula $\rho^{(2)}(\Phi \circ \Psi) = \rho^{(2)}(\Phi) + \rho^{(2)}(\Psi)$ is *not* true in general for two det-finite group

automorphisms Φ and Ψ of the same $\mathcal{FIN}$ -group G . For example for a pseudo-Anosov diffeomorphism f of a closed surface the mapping tori T_f and $T_{f^{-1}}$ are homeomorphic and hence have the same non-zero L^2 -torsion, but the L^2 -torsion of $T_{f \circ f^{-1}} = T_{\text{id}}$ vanishes.

Let S be a compact connected orientable 2-dimensional manifold, possibly with boundary. Let $f: S \rightarrow S$ be an orientation preserving homeomorphism. The mapping torus T_f is a compact connected orientable 3-manifold whose boundary is empty or a disjoint union of 2-dimensional tori. By the Nielsen–Thurston decomposition for mapping classes and Thurston’s hyperbolisation theorem there is a maximal family of embedded incompressible tori, which are pairwise not isotopic and not boundary parallel, such that it decomposes T_f into pieces, which are Seifert or hyperbolic. Let $M_1, M_2, \dots, M_r$ be the hyperbolic pieces. They all have finite volume $\text{vol}(M_i)$.

Choose a base point $x \in S$ and a path $w: I \rightarrow S$ from x to $f(x)$. Let $t_w: \pi_1(S, f(x)) \xrightarrow{\cong} \pi_1(S, x)$ be the isomorphism sending the class $[v]$ of a loop v in S at $f(x)$ to the class $[w * v * w^-]$ of the loop $w * v * w^-$ at x given by concatenation of paths, where w^- is the inverse of w given by $w^-(t) = w(1 - t)$. Let $\Phi: \pi_1(S, x) \xrightarrow{\cong} \pi_1(S, x)$ be the automorphism given by the composite $\pi_1(S, x) \xrightarrow{\pi_1(f, x)} \pi_1(S, f(x)) \xrightarrow{t_w} \pi_1(S, x)$. Then $\pi_1(S, x)$ is det-finite and the real number $\rho^{(2)}(\Phi)$ is defined. Theorem 5.3 (i) and (vii) imply that $\rho^{(2)}(\Phi)$ is independent of the choices of $x \in S$ and w .

Theorem 5.5 (Surface automorphisms). *In the situation described above, the following statement true:*

- (i) If S is S^2 , D^2 , or T^2 , then $\rho^{(2)}(\Phi) = 0$;
- (ii) If S is not S^2 , D^2 , or T^2 , then

$$\rho^{(2)}(\Phi) = \frac{-1}{6\pi} \cdot \sum_{i=1}^r \text{vol}(M_i).$$

Proof. This follows from [33, Theorem 7.28 on page 307]. □

Let $\Phi: G \rightarrow G$ be an automorphism of the $\mathcal{FIN}$ -finite group G . Let X be a finite G -CW-complex such that for any finite subgroup $H \subseteq G$ the H -fixed point set X^H is contractible. Let $a: X \rightarrow X$ be a Φ -homeomorphism such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{a} & X \\ & \text{pr} \searrow & \swarrow \text{pr} \\ & X/G & \end{array}$$

where pr is the canonical projection. Suppose that the isotropy group G_x for each $x \in X$ is $\mathcal{FIN}$ -finite.

For $\bar{c} \in \pi_0(X_n \setminus X_{n-1})/G$ choose an element $c \in \pi_0(X_n \setminus X_{n-1})$ representing $\bar{c}$, an element $x \in c$, and an element $g \in G$ satisfying $a(x) = gx$. Note that the element g exists because of the assumption $\text{pr} \circ a = \text{pr}$. The isotropy group of $a(x) \in X$ agrees with the isotropy group $G_{gx} = gG_xg^{-1}$ of $gx \in X$ and is given by

$$\begin{aligned} \{g' \in G \mid g'a(x) = a(x)\} &= \{g' \in G \mid a(\Phi^{-1}(g')x) = a(x)\} \\ &= \{g' \in G \mid \Phi^{-1}(g')x = x\} = \{g' \in G \mid \Phi^{-1}(g') \in G_x\} = \Phi(G_x). \end{aligned}$$

Hence we get $gG_xg^{-1} = \Phi(G_x)$. Therefore we can define an automorphism of the $\mathcal{FIN}$ -finite group G_x

$$\Phi_{c,x,g}: G_x \rightarrow G_x, \quad g' \mapsto g^{-1}\Phi(g')g.$$

Now suppose that $\Phi_{c,x,g}$ is det-finite. Then the real number $\rho^{(2)}(\Phi_{c,x,g}; \mathcal{N}(G_x))$ is defined. The condition that $\Phi_{c,x,g}$ is det-finite and the real number $\rho^{(2)}(\Phi_{c,x,g}; \mathcal{N}(G_x))$ depend only on $\bar{c}$ and are independent of the choices of c , x , and g by Theorem 5.3 (i) and (vii).

Consider $\bar{c} \in \pi_0(X_n \setminus X_{n-1})/G$. Then we say that $\Phi_{\bar{c}}$ is det-finite and in this case can define

$$(5.6) \quad \rho^{(2)}(\Phi_{\bar{c}}) \in \mathbb{R}$$

by requiring that $\Phi_{c,x,g}$ is det-finite and putting $\rho^{(2)}(\Phi_{\bar{c}}) = \rho^{(2)}(\Phi_{c,x,g}; \mathcal{N}(G_x))$ for one (and hence every) choice of c , x , and g .

Theorem 5.7. *Let $\Phi: G \rightarrow G$ be an automorphism of the $\mathcal{FIN}$ -finite group G . Let X be a finite G -CW-complex such that for any finite subgroup $H \subseteq G$ the H -fixed point set X^H is contractible and there is a cellular Φ -homeomorphism $a: X \rightarrow X$ satisfying $\text{pr} \circ a = \text{pr}$ for the projection $\text{pr}: X \rightarrow X/G$. Suppose that the isotropy group G_x for each $x \in X$ is $\mathcal{FIN}$ -finite and that the automorphisms $\Phi_{\bar{c}}$ is det-finite for every $\bar{c} \in \pi_0(X_n \setminus X_{n-1})/G$ and $n \in \mathbb{Z}^{\geq 0}$.*

Then the group G is $\mathcal{FIN}$ -finite, $\Phi_{\bar{c}}$ is det-finite, and we get

$$\rho^{(2)}(\Phi) = \sum_{n \geq 0} (-1)^n \cdot \sum_{\bar{c} \in \pi_0(X_n \setminus X_{n-1})/G} \rho^{(2)}(\Phi_{\bar{c}}),$$

where $\rho^{(2)}(\Phi_{\bar{c}})$ has been defined in (5.6).

Proof. Choose a cellular Φ -homotopy equivalence $f: \underline{EG} \rightarrow \underline{EG}$. If $\underline{EG}$ is a $\mathcal{FIN}$ -finite G -space, then we get from the definitions that Φ is det-finite if and only if $(f; \Phi)$ is det-finite, and in this case

$$(5.8) \quad \rho^{(2)}(\Phi) = \rho^{(2)}(f; \Phi).$$

Let $I_m = \pi_0(X_m \setminus X_{m-1})/G$ be the set of equivariant m -cells of X for $m = 0, 1, 2, \dots, \dim(X)$. We show by induction for $n = -1, 0, 1, 2, \dots, \dim(X)$ that $\underline{EG} \times X_n$ is a $\mathcal{FIN}$ -finite G -space, $(f \times a|_{X_n}; \Phi)$ is det-finite, and we have

$$(5.9) \quad \rho^{(2)}(f \times a|_{X_n}; \Phi) = \sum_{m=0}^n (-1)^m \cdot \sum_{\bar{c} \in I_m} \rho^{(2)}(\Phi_{\bar{c}}).$$

The induction beginning $n = -1$ is trivial, since X_{-1} is empty. The induction step from $(n-1)$ to $0 \leq n \leq \dim(X)$ is done as follows.

Choose a cellular G -pushout

$$(5.10) \quad \begin{array}{ccc} \coprod_{\bar{c} \in I_n} G/H_{\bar{c}} \times S^{n-1} & \xrightarrow{q = \coprod_{i \in \bar{c}} q_{\bar{c}}} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{\bar{c} \in I_n} G/H_{\bar{c}} \times D & \xrightarrow{Q = \coprod_{i \in \bar{c}} Q_{\bar{c}}} & X_n. \end{array}$$

Then we get a G -pushout by taking the cross product with $\underline{E}G$ and the diagonal G -actions

$$(5.11) \quad \begin{array}{ccc} \coprod_{\bar{c} \in I_n} \underline{E}G \times G/H_{\bar{c}} \times S^{n-1} & \longrightarrow & \underline{E}G \times X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{\bar{c} \in I_n} \underline{E}G \times G/H_{\bar{c}} \times D & \longrightarrow & \underline{E}G \times X_n. \end{array}$$

Let $x_{\bar{c}} \in X_n$ be the image of $(eH_{\bar{c}}, 0)$ for $0 \in D^n$ the origin under the characteristic map $Q_{\bar{c}}$ appearing the G -pushout (5.10). Choose $g_{\bar{c}} \in G$ such that $a(x_{\bar{c}}) = g_{\bar{c}}x_{\bar{c}}$ holds. Then we get a Φ -map $u_{\bar{c}}: G/H_{\bar{c}} \rightarrow G/H_{\bar{c}}$ sending $gH_{\bar{c}}$ to $\Phi(g)g_{\bar{c}}H_{\bar{c}}$ such that the following diagram commutes

(5.12)

$$\begin{array}{ccccc} \coprod_{\bar{c} \in I_n} G/H_{\bar{c}} \times S^{n-1} & \xrightarrow{q} & X_{n-1} & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \coprod_{\bar{c} \in I_n} G/H_{\bar{c}} \times D^n & \xrightarrow{Q} & X_n & \xrightarrow{a|_{X_{n-1}}} & X_{n-1} \\ & \searrow & \downarrow & \searrow & \\ & & \coprod_{\bar{c} \in I_n} G/H_{\bar{c}} \times S^{n-1} & \xrightarrow{q} & X_{n-1} \\ & \searrow & \downarrow & \searrow & \\ & & \coprod_{\bar{c} \in I_n} G/H_{\bar{c}} \times D^n & \xrightarrow{Q} & X_n \end{array}$$

$\coprod_{\bar{c} \in I_n} u_{\bar{c}} \times \text{id}_{S^{n-1}}$ (top right arrow), a (middle right arrow), $\coprod_{\bar{c} \in I_n} u_{\bar{c}} \times \text{id}_{D^n}$ (bottom left arrow), Q (bottom right arrow), q (top right arrow), $a|_{X_{n-1}}$ (middle right arrow).

We obtain a G -homeomorphisms

$$s: G \times_{H_{\bar{c}}} (\text{res}_G^{H_{\bar{c}}} \underline{E}G) \xrightarrow{\cong} \underline{E}G \times G/H_{\bar{c}}$$

sending (g, x) to $(gx, gH_{\bar{c}})$. The following diagram of G -spaces commutes

$$(5.13) \quad \begin{array}{ccc} G \times_{H_{\bar{c}}} \underline{E}G & \xrightarrow{s} & \underline{E}G \times G/H_{\bar{c}} \\ \downarrow v & & \downarrow f \times u_{\bar{c}} \\ G \times_{H_{\bar{c}}} \underline{E}G & \xrightarrow{s} & \underline{E}G \times G/H_{\bar{c}} \end{array}$$

where the vertical maps are Φ -maps and v sends (g, x) to $(\Phi(g)g_{\bar{c}}, g_{\bar{c}}^{-1}f(x))$. The map $w: \underline{E}G \rightarrow \underline{E}G$ sending x to $g_{\bar{c}}^{-1}f(x)$ satisfies

$$w(hx) = g_{\bar{c}}^{-1}f(hx) = g_{\bar{c}}^{-1}\Phi(h)f(x) = g_{\bar{c}}^{-1}\Phi(h)g_{\bar{c}}g_{\bar{c}}^{-1}f(x) = g_{\bar{c}}^{-1}\Phi(h)g_{\bar{c}}w(x)$$

and hence is $\Phi_{\bar{c}}$ -equivariant, where $\Phi_{\bar{c}}: H_{\bar{c}} \rightarrow H_{\bar{c}}$ sends h to $g_{\bar{c}}^{-1}\Phi(h)g_{\bar{c}}$. Note that $\text{res}_G^{H_{\bar{c}}} \underline{E}G$ is a model for $\underline{E}H_{\bar{c}}$ and $\underline{E}H_{\bar{c}}$ is by assumption a $\mathcal{FIN}$ -finite G -space. Since $\Phi_{\bar{c}}$ is det-finite by assumption, $(w; \Phi_{\bar{c}})$ is det-finite. Now Theorem 4.8 (v) implies that $\underline{E}G \times G/H_{\bar{c}}$ is $\mathcal{FIN}$ - G -space, $(f \times u_{\bar{c}}; \Phi)$ is det-finite, and we get

$$(5.14) \quad \rho^{(2)}(\Phi_{\bar{c}}) = \rho^{(2)}(f \times u_{\bar{c}}; \Phi).$$

By the induction hypothesis the G -space $\underline{E}G \times X_{n-1}$ is $\mathcal{FIN}$ -finite, $f \times a|_{X_{n-1}}; \Phi$ is det-finite, and we have

$$(5.15) \quad \rho^{(2)}(f \times a|_{X_{n-1}}; \Phi) = \sum_{m=0}^{n-1} (-1)^m \cdot \sum_{\bar{c} \in I_m} \rho^{(2)}(\Phi_{\bar{c}}).$$

We conclude from Theorem 4.8 (viii) and (ix) and the G -pushout (5.11) that the G -space $\underline{E}G \times X_n$ is $\mathcal{FIN}$ -finite, $(f \times a|_{X_n}; \Phi)$ is det-finite, and we get

$$(5.16) \quad \rho^{(2)}(f \times a|_{X_n}; \Phi) = \rho^{(2)}(f \times a|_{X_{n-1}}; \Phi) + (-1)^n \cdot \sum_{\bar{e} \in I_m} \rho^{(2)}(\Phi_{\bar{e}}).$$

Now the induction step from $(n-1)$ to n follows from (5.15) and (5.16).

If we apply (5.9) in the case $n = \dim(X)$, we conclude that $(f \times a; \Phi)$ is det-finite and we have

$$(5.17) \quad \rho^{(2)}(f \times a; \Phi) = \sum_{m \geq 0} (-1)^m \cdot \sum_{\bar{e} \in I_m} \rho^{(2)}(\Phi_{\bar{e}}).$$

Since the projection $\underline{E}G \times X \rightarrow \underline{E}G$ is a G -homotopy equivalence and $(f \times a; \Phi)$ is det-finite, Theorem 4.8 (ii) implies that $(f; \Phi)$ is det-finite and we have $\rho^{(2)}(f \times a; \Phi) = \rho^{(2)}(f; \Phi) = \rho^{(2)}(\Phi)$. Hence Theorem 5.7 follows from (5.17) $\square$

Although one may only be interested in $\rho^{(2)}(\Phi)$, it is useful that we have the L^2 -torsion $\rho^{(2)}(f; \Phi)$ of Definition 4.6 at hand, since in the proof of Theorem 5.7 one needs to consider this more general notion $\rho^{(2)}(f; \Phi)$, see (5.9).

Example 5.18 (Group extensions). Suppose that we can write G as an extension $1 \rightarrow K \rightarrow G \xrightarrow{p} Q \rightarrow 1$ and there is a finite model for $\underline{E}Q$. Then we can consider the G -CW-complex $X = p^* \underline{E}Q$ obtained from the finite Q -CW-complex $\underline{E}Q$ by restriction with p . Obviously $X^H = \underline{E}Q^{p(H)}$ is contractible for any finite subgroup $H \subseteq G$. There is a bijective correspondence between the open equivariant cells of X and the open equivariant cells of $\underline{E}Q$ given by $c \mapsto p(c)$. We have $\dim(c) = \dim(p(c))$ and $G_c = p^{-1}(Q_{p(c)})$. Hence G_c contains K as a subgroup of finite index $[G_c : K] = |Q_{p(c)}|$. Suppose that G_c is $\mathcal{FIN}$ -finite for every open cell c . (Note that it is not true that a group is a $\mathcal{FIN}$ -finite group if it contains a $\mathcal{FIN}$ -finite subgroup of finite index, see [30].) Consider a group automorphism $\Phi: G \xrightarrow{\cong} G$ with $\Phi(K) = K$ and $p \circ \Phi = p$ such that $\Phi|_{G_c}$ is det-finite. We conclude from Theorem 5.3 (v) that $|Q_{p(c)}| \cdot \rho^{(2)}(G_c) = \rho^{(2)}(K)$. Recall the orbifold Euler characteristic of $\underline{E}Q$

$$\chi_{\text{orb}}(\underline{E}Q) = \sum_c (-1)^{\dim(c)} \cdot \frac{1}{|Q_{p(c)}|}.$$

which agrees with the L^2 -Euler characteristic $\chi^{(2)}(\underline{E}Q; \mathcal{N}(Q))$.

Theorem 5.7 implies that G is $\mathcal{FIN}$ -finite, Φ is det-finite, and we have

$$(5.19) \quad \rho^{(2)}(\Phi) = \chi_{\text{orb}}(\underline{E}Q) \cdot \rho^{(2)}(\Phi|_K) = \chi^{(2)}(\underline{E}Q; \mathcal{N}(Q)) \cdot \rho^{(2)}(\Phi|_K).$$

In particular we get $\rho^{(2)}(\Phi) = 0$ if $\chi^{(2)}(\underline{E}Q; \mathcal{N}(Q))$ vanishes.

Now assume that K is an infinite amenable $\mathcal{FIN}$ -finite group and that each group G_c is $\mathcal{FIN}$ -finite. Consider a group automorphism $\Phi: G \xrightarrow{\cong} G$ with $\Phi(K) = K$ and $p \circ \Phi = p$. Then K and G_c for every equivariant cell c are infinite amenable $\mathcal{FIN}$ -finite groups and satisfy the Determinant Conjecture. Hence $\Phi|_K$ and $\Phi|_{G_c}$ for every equivariant cell c are det-finite. Moreover, $\underline{E}K$ is det- L^2 -acyclic and we get $\rho^{(2)}(\Phi|_K) = 0$ by Theorem 5.3 (viii) and [33, Theorem 6.54 (8) on page 266 and Theorem 6.75 on page 274]. Hence Φ is det-finite and satisfies $\rho^{(2)}(\Phi) = 0$ by (5.19).

Note that Theorem 5.3 (iv) is a special case of this example.

Example 5.20 (Graph of groups). Let Y be a connected non-empty graph in the sense of [44, Definition 1 in Section 2.1 on page 13]. Let (G, Y) be a graph of groups in the sense of [44, Definition 8 in Section 4.4. on page 37]. (In the sequel we use the notation of [44]). An automorphism $\varphi: (G, T) \xrightarrow{\cong} (G, T)$ consists

of a collection of automorphisms $\varphi_P: G_P \xrightarrow{\cong} G_P$ for every $P \in \text{vert}(T)$ and an automorphism $\varphi_y: G_y \xrightarrow{\cong} G_y$ for every $y \in \text{edge}(T)$ which are compatible with the monomorphisms $G_y \rightarrow G_{t(y)}$ and satisfy $\varphi_{\tilde{y}} = \varphi_y$.

Let P_0 be an element in $\text{vert}(Y)$ and let T be a maximal tree of Y . Denote by $\overline{\text{edge}(Y)}$ the quotient of $\text{edge}(Y)$ under the involution $y \mapsto \tilde{y}$. Note that Y considered as a CW -complex has the set $\text{vert}(T)$ as set of 0-cells and $\overline{\text{edge}(Y)}$ as set of 1-cells. Since by definition $G_y = G_{\tilde{y}}$ holds, we can define for $\bar{y} \in \overline{\text{edge}(Y)}$ the group $G_{\bar{y}}$ to be G_y for a representative $y \in \text{edge}(Y)$ of $\bar{y}$ and analogously define the automorphism $\varphi_{\bar{y}}: G_{\bar{y}} \xrightarrow{\cong} G_{\bar{y}}$.

Let $\pi_1(G, Y, P_0)$ and $\pi_1(G, Y, T)$ be the fundamental groups in the sense of [44, page 42]. Note that $\pi_1(G, Y, P_0)$ and $\pi_1(G, Y, T)$ are isomorphic, see [44, Proposition 20 in Section 5.1. on page 44]. Then there exists

- A graph $\tilde{X} = \tilde{X}(G, Y, T)$;
- An action of $\pi = \pi_1(G, Y, T)$ on $\tilde{X}$;
- A morphism $p: \tilde{X} \rightarrow X$ inducing an isomorphism $\pi \backslash \tilde{X} \rightarrow Y$;

such that the following is true

- $\tilde{X}$ is a tree;
- $\tilde{X}^H$ is contractible for every finite subgroup $H \subseteq \pi$;
- $\tilde{X}$ is a 1-dimensional π - CW -complex for which there exists π -pushout

$$\begin{array}{ccc} \coprod_{P \in \text{vert}(T)} \pi / \pi_P \times S^0 & \longrightarrow & \coprod_{\bar{y} \in \overline{\text{edge}(Y)}} \pi / \pi_{\bar{y}} \\ \downarrow & & \downarrow \\ \coprod_{P \in \text{vert}(T)} \pi / \pi_P \times D^1 & \longrightarrow & \tilde{X} \end{array}$$

such that $\pi_P \cong G_P$ for $P \in \text{vert}(T)$ and $\pi_{\bar{y}} = G_{\bar{y}}$ for $\bar{y} \in \overline{\text{edge}(Y)}$ holds.

- φ induces an automorphism $\Phi: \pi \xrightarrow{\cong} \pi$ and a cellular Φ -homeomorphism $f: \tilde{X} \rightarrow \tilde{X}$.

All these claims follows from [44, Section 5.3 and Theorem 15 in Section 6.1 on page 58].

Now suppose that Y is finite, each of the groups G_P for $P \in \text{vert}(T)$ and $G_{\bar{y}}$ for $\bar{y} \in \overline{\text{edge}(Y)}$ are $\mathcal{FIN}$ -finite, and each of the automorphisms φ_P for $P \in \text{vert}(T)$ and $\varphi_{\bar{y}}$ for $\bar{y} \in \overline{\text{edge}(Y)}$ is det-finite

Then we conclude from Theorem 5.7 that π is $\mathcal{FIN}$ -finite, Φ is det-finite, and we get

$$\rho^{(2)}(\Phi) = \sum_{P \in \text{vert}(T)} \rho^{(2)}(\varphi_P) - \sum_{\bar{y} \in \overline{\text{edge}(Y)}} \rho^{(2)}(\varphi_{\bar{y}}).$$

Example 5.21 (Amalgamated Products). Let G_0 be a common subgroup of the group G_1 and the group G_2 . Denote by $G = G_1 *_{G_0} G_2$ the amalgamated product. Let $\Phi_i: G_i \xrightarrow{\cong} G_i$ be an automorphism for $i = 0, 1, 2$ such that $\Phi_j|_{G_0} = \Phi_0$ holds for $j = 1, 2$. Denote by $\Phi: G \xrightarrow{\cong} G$ be the automorphism of G induced by the automorphisms Φ_i for $i = 0, 1, 2$. Suppose that G_i is $\mathcal{FIN}$ -finite and Φ_i is det-finite for $i = 0, 1, 2$.

Then G is $\mathcal{FIN}$ -finite, Φ is det-finite, and we get

$$\rho^{(2)}(\Phi) = \rho^{(2)}(\Phi_1) + \rho^{(2)}(\Phi_2) - \rho^{(2)}(\Phi_0).$$

This follows from Example 5.20 applied to the graph of groups associated to $G_1 *_{G_0} G_2$, see [44, page 43].

6. A SHORT DISCUSSION OF THE FINITENESS ASSUMPTIONS

We have introduced the notion of $\mathcal{FLN}$ -finite group in Definition 2.9, of $\mathcal{FLN}$ -finite G -space in Definition 4.1, of a det-finite G -space in Definition 4.2, of a det-finite Φ -homotopy equivalence $(f; \Phi)$ of a $\mathcal{FLN}$ - G -space in Definition 4.5, and of a det-finite automorphism of a $\mathcal{FLN}$ -finite group in Definition 5.1. We have introduced the notions of the L^2 -torsion $\rho^{(2)}(f; \Phi)$ and $\rho^{(2)}(\Phi)$ in Definitions 4.6 and 5.2. The conditions det-finite were needed to make sense of $\rho^{(2)}(f; \Phi)$ and $\rho^{(2)}(\Phi)$.

Definition 6.1 (Det-finite group). We call a group G *det-finite* if it satisfies the following conditions:

- (i) There exists a finite G -CW-model for the classifying space of proper actions $\underline{E}G$;
- (ii) For one (and hence all) finite G -CW-models X for $\underline{E}G$, the G -CW-complex X is of determinant class.

Lemma 6.2. *Let G be a group and X be a G -space.*

- (i) *The group G is $\mathcal{FLN}$ -finite if and only if the G -space $\underline{E}G$ is $\mathcal{FLN}$ -finite;*
- (ii) *The group G is det-finite if and only if the G -space $\underline{E}G$ is det-finite;*
- (iii) *Suppose that the group G is det-finite. Then every automorphism $\Phi: G \rightarrow G$ is det-finite;*
- (iv) *Suppose that G is a $\mathcal{FLN}$ -finite-group and satisfies the Determinant Conjecture. Consider any automorphism $\Phi: G \rightarrow G$. Then Φ is det-finite, the group $G \rtimes_{\Phi} \mathbb{Z}$ is det- L^2 -acyclic, and we get*

$$\rho^{(2)}(\Phi) = \rho^{(2)}(G \rtimes_{\Phi} \mathbb{Z});$$

- (v) *If the group G is sofic, then it satisfies the Determinant Conjecture. In particular any automorphism Φ of a sofic $\mathcal{FLN}$ -finite group G is det-finite, $G \rtimes_{\Phi} \mathbb{Z}$ is det- L^2 -acyclic, and we get*

$$\rho^{(2)}(\Phi) = \rho^{(2)}(G \rtimes_{\Phi} \mathbb{Z}).$$

Proof. (i) and (ii) follow directly from the definitions.

(iii) This follows from Lemma 4.4 (ii)

(iv) Since G satisfies the Determinant Conjecture, $G \rtimes_{\Phi} \mathbb{Z}$ satisfies the Determinant Conjecture, see [33, Proposition 13.39 on page 469]. (Note that proof of [33, Proposition 13.39 on page 469] is not correct in the generality as stated, but still works in the special case that $H \subseteq G$ is a normal subgroup with amenable quotient G/H .) Now the claim follows from Theorem 4.8 (i) since $T_{\hat{f}; \Phi}$ is a model for $\underline{E}(G \rtimes_{\Phi} \mathbb{Z})$.

(v) See [14, Theorem 5]. $\square$

Remark 6.3. Note that in the setting of Section 3 we had always to assume that the group G for which we want to make a computation in terms of subgroups had to satisfy the condition (DFJ). The advantage of the setup of Section 5 is that we only have to make assumptions about the restrictions of an automorphism to certain subgroups and then the necessary assumptions are automatically satisfied for the automorphism Φ itself. For instance, it is not known whether a graph of groups has a sofic fundamental group if all edge and vertex groups are sofic and whether any hyperbolic group is sofic.

This advantage is essentially due to the fact that in the definition of $\rho^{(2)}(f)$ we use the finite proper $G \rtimes_{\Phi} \mathbb{Z}$ -CW-complex $T_{\hat{f}; \Phi}$. Note that $T_{\hat{f}; \Phi}$ is a model for $\underline{E}(G \rtimes_{\Phi} \mathbb{Z})$. If we additionally assume that G is sofic, we could define $\rho^{(2)}(\Phi)$ just by $\rho^{(2)}(G \rtimes_{\Phi} \mathbb{Z})$, see Lemma 6.2 (v).

7. COMPUTATIONS

In this section we use Theorem 3.6 (vii) and the theory developed in the later sections to compute the L^2 -torsion for a large range of groups and automorphisms. We expect that there are many more applications of the formula so have selected applications which require both additional techniques and which may appeal to a range of audiences.

An essential point in some of our arguments in the subsequent sections involves verifying that the groups involved satisfy the Determinant Conjecture. In some of the cases this amounts to proving that the group is sofic and then appealing to [14, Theorem 5]. However, for automorphisms of groups $\Phi: G \rightarrow G$ we try to avoid the use of Determinant Conjecture for the group G and only assume it for the restrictions to isotropy groups of cells. In this we do not know whether G itself or $G \rtimes_{\Phi} \mathbb{Z}$ satisfies the Determinant Conjecture.

7.1. Lattices. Let G be a second countable locally compact group with Haar measure μ . We say a discrete subgroup $\Gamma \leq G$ is a *lattice* if $\mu(\Gamma \backslash G)$ is finite. We say a lattice Γ is *uniform* if $\Gamma \backslash G$ is compact.

Lemma 7.1. *Let G and H be second countable locally compact groups admitting lattices and suppose that there exists a lattice in G which is L^2 -acyclic. If Γ is a lattice in $G \times H$, then Γ is L^2 -acyclic.*

Proof. Let Λ be an L^2 -acyclic lattice in G and let Λ' be any lattice in H . The group $L := \Lambda \times \Lambda'$ is a lattice in $G \times H$ and hence is measure equivalent to Γ . From the Künneth formula for L^2 -Betti numbers, see [33, Theorem 6.54 (5) on page 266], we conclude that L is L^2 -acyclic. Now, by Gaboriau's invariance of L^2 -Betti numbers under measure equivalence [17], we see that Γ is L^2 -acyclic. $\square$

For a locally finite CAT(0) polyhedral complex X we denote by $\text{Isom}(X)$ the set of isometries of X such that if g fixes a cell $\sigma \in X$ setwise, then g fixes σ pointwise. We say $\text{Isom}(X)$ acts *minimally* if there is no non-empty $\text{Isom}(X)$ -invariant proper subspace $Y \subset X$. Following [33, Definition 5.6], for a symmetric space $M = G/K$ with $G = \text{Isom}_0(M)$ semi-simple, we define the *fundamental rank* of M to be

$$\text{fr}(M) = \text{rk}_{\mathbb{C}}(G) - \text{rk}_{\mathbb{C}}(K).$$

Proposition 7.2. *Let $n \geq 0$ and let $M = M_1 \times \cdots \times M_k \times \mathbb{E}^n$ be a symmetric space with each M_i irreducible of non-compact type. Let X a locally finite CAT(0) polyhedral complex with $\text{Isom}(X)$ acting minimally and cocompactly. Let Γ be satisfy (DFJ) or be virtually torsion-free. If Γ is a uniform lattice in $\text{Isom}(M) \times \text{Isom}^+(X)$ and either*

- (i) $\text{fr}(M_i) \geq 2$ for some i ,
- (ii) or $n \geq 1$,

then $\rho^{(2)}(\Gamma) = 0$.

The hypothesis that Γ satisfies (DFJ) could be dropped if we either knew the Determinant Conjecture for CAT(0) groups, or if we knew the Farrell–Jones Conjecture for the Weyl groups of finite subgroups of CAT(0) groups. Note there are non-virtually torsion-free CAT(0) groups, see e.g., [26].

Proof. If Γ is virtually torsion-free, then it satisfies (DFJ). Indeed, Γ satisfies the Farrell–Jones Conjecture by [8, 49, 27]. By [25, Theorem A], any stabiliser Γ_{σ} in Γ of a cell σ in X fits into short exact sequence $1 \rightarrow F_{\sigma} \rightarrow \Gamma_{\sigma} \rightarrow \Lambda_{\sigma} \rightarrow 1$, where F_{σ} is finite and Λ_{σ} is a uniform lattice in $\text{Isom}(M)$. We have $\rho^{(2)}(\Gamma_{\sigma}) = \frac{1}{|F_{\sigma}|} \rho^{(2)}(\Lambda_{\sigma})$ by Theorem 2.7(vi). By [41], either condition in the theorem ensures that $\rho^{(2)}(\Lambda_{\sigma}) = 0$. The result follows from Theorem 3.6 (vii). $\square$

Example 7.3 (*S*-arithmetic subgroups of $\mathrm{GL}_n(\mathbb{C})$). Let k be a number field, let S be a finite set of places containing the archimedean ones, and let $\mathbf{G}$ be a simply connected simple k -group. Let $\Gamma < \mathbf{G}(k)$ be an *S*-arithmetic subgroup. By the general theory of *S*-arithmetic lattices Γ acts on a product of symmetric spaces $M = M_1 \times \dots \times M_k$ with each M_i non-compact and irreducible and a product of Euclidean buildings $X = X_1 \times \dots \times X_\ell$ such that the diagonal action on $X \times M$ is cocompact. If $\mathrm{fr}(M_i) \geq 2$ for some i , then $\rho^{(2)}(\Gamma) = 0$. Note that the lattices here often have a strict fundamental domain on the building and so one could instead apply [40, Theorem 6.1].

We also provide an example where the theorem applies to a non-residually finite $\mathrm{CAT}(0)$ lattice.

Example 7.4 (Leary–Minasyan groups). Let L' and L'' denote a pair of finite index subgroup of $L = \mathbb{Z}^n$, and let $A \in \mathrm{GL}_n(\mathbb{Q})$ such that multiplication by A defines an isomorphism $L' \rightarrow L''$. We form the HNN extension

$$\mathrm{LM}(A, L') = \langle x_1, \dots, x_n, t \mid [x_i, x_j], txt^{-1} = A(x) \ \forall x \in L' \rangle,$$

where $\langle x_1, \dots, x_n \rangle = L$. Let $\mathcal{T}$ denote the Bass–Serre tree of the HNN extension and note that every vertex has valence $|L : L'| + |L : L''|$. By [29, Theorem 7.5], if A is conjugate in $\mathrm{GL}_n(\mathbb{R})$ to an orthogonal matrix, then $\mathrm{LM}(A, L')$ is a lattice in $\mathrm{Isom}(\mathbb{E}^n) \times \mathrm{Aut}(\mathcal{T})$. By [29, Theorem 1.1(1)], the group $\mathrm{LM}(A, L')$ is residually finite if and only if A is conjugate in $\mathrm{GL}_n(\mathbb{Q})$ to a matrix in $\mathrm{GL}_n(\mathbb{Z})$. Now, by Theorem 3.6 (vii) we have $\rho^{(2)}(\mathrm{LM}(A, L')) = 0$.

We mention that a zoo of non-residually finite examples of $\mathrm{CAT}(0)$ lattices can be constructed using the tools from [25].

7.2. Higher dimensional graph manifolds. Higher dimensional graph manifolds were introduced in [16] and studied from the point of view of the Borel Conjecture and quasi-isometric rigidity. We now recall the construction:

Let $n \geq 3$ and let Γ be a finite graph. For each vertex v of Γ , let M_v be a compact manifold of dimension n_v whose interior is a complete hyperbolic manifold of finite volume, and hence has toral cusps. The boundary of M_v is a collection of tori, where $2 \leq n_v \leq n$. Write $N_v = T^{n-n_v} \times N_v$, where T^ℓ denotes the ℓ -torus. We call each N_v a *piece*. Its boundary is a collection of $(n-1)$ -dimensional tori. An *extended graph manifold* M is any manifold obtained from pieces as above by gluing their torus boundaries by affine diffeomorphisms such that each edge of Γ corresponds to some gluing. For a piece N_v of M , if $n - n_v = 0$, then we call N_v a *hyperbolic piece*. We denote the set of hyperbolic pieces in M by $\mathcal{H}$.

Recall that the L^2 -torsion of a closed hyperbolic manifold of odd dimension $(2d+1)$ has been computed by Hess–Schick [24] and of a compact manifold of odd dimension $(2d+1)$ whose interior is a complete hyperbolic manifold of finite volume by Lück–Schick [38, Theorem 0.5], namely, it is proportional by a dimension constant $C_{2n+1} \neq 0$ to its volume.

We may decompose a graph manifold as a graph of spaces: each vertex space M_v is a piece N_i and each edge space M_e is the collar of the torus boundary that is obtained by gluing two geometric pieces via an affine diffeomorphism. Note that on the level of fundamental groups this decomposes $\pi_1 M$ as a graph of groups with vertex groups $\pi_1 M_v = \pi_1 N_i$ and edge groups $\mathbb{Z}^{n-1}$.

Theorem 7.5. *Let M be a $(2n+1)$ -dimensional extended graph manifold with hyperbolic pieces $\mathcal{H}$. Then, $\pi_1(M)$ is sofic, satisfies the Determinant Conjecture, and*

$$\rho^{(2)}(\widetilde{M}) = \sum_{M_v \in \mathcal{H}} \rho^{(2)}(\widetilde{M}_v).$$

Proof. We first establish that $\pi_1(M)$ is sofic satisfies the Determinant Conjecture. We have that $\pi_1 M$ admits a decomposition as a graph of groups such that: Every edge group G_e is $\mathbb{Z}^{2n}$, and every vertex group G_v splits as a direct product $A_v \times Q_v$ where A_v is a (possibly trivial) abelian group and Q_v is the fundamental group of a compact manifold of dimension $2n + 1 - \dim A_v$ whose interior is a complete hyperbolic manifold of finite volume. In the case that A_v is trivial, we have $G_v \in \mathcal{H}$. It follows that every edge group is amenable and every vertex group is residually finite. Thus, $\pi_1 M$ is sofic by [12, Theorem 1.2] and so satisfies the Determinant Conjecture by [14, Theorem 5]. Note that this implies the stabiliser of any vertex or edge satisfies Determinant Conjecture.

Note that $\widetilde{M} \simeq E\pi_1(M)$ is a cocompact $\pi_1(M)$ -space. In the graph of groups decomposition for $\pi_1(M)$ we have that every vertex group either contains an infinite finitely generated abelian normal subgroup and is L^2 -acyclic by [33, Theorem 7.4(1),(2)], or is the fundamental group of a compact manifold of odd dimension whose interior is a complete hyperbolic manifold of finite volume and is L^2 -acyclic by [38, Corollary 6.5].

Now, we have that $\rho^{(2)}(\widetilde{M}) = \rho^{(2)}(\pi_1 M)$. For an edge e , the group G_e satisfies $\rho^{(2)}(G_e) = 0$ by [48]. Each vertex group G_v is either isomorphic to $\pi_1(M_v)$ for some $M_v \in \mathcal{H}$, or has an infinite abelian normal subgroup. In the later case, the L^2 -torsion of G_v vanishes by [48]. The result follows from Theorem 3.6 (vii). $\square$

Corollary 7.6. *Let M be an n -dimensional graph manifold. If the graph decomposition of M contains a hyperbolic piece, then M does not admit any non-trivial action of the circle S^1 .*

Proof. If M is even dimensional, then one easily sees that M has a non-vanishing L^2 -Betti number by the L^2 -Mayer–Vietoris sequence and [10]. The non-vanishing L^2 -Betti of an aspherical number obstructs non-trivial circle actions by [33, Corollary 1.43]. If M is odd-dimensional, then by Theorem 7.5, M has non-vanishing L^2 -torsion. Non-zero L^2 -torsion of (the universal cover of) an aspherical space obstructs non-trivial circle actions by [33, Theorem 3.105]. $\square$

7.3. Relatively hyperbolic groups.

Definition 7.7. We say a group G is *one-ended relative to a collection of subgroups* $\mathcal{P}$ if there does not exist a splitting of G over finite subgroups such that each group in $\mathcal{P}$ is conjugate into some vertex group. Note that a one-ended group is one-ended relative to every collection of subgroups. We write $\text{Aut}(G; \mathcal{P})$ to denote the group of automorphisms of G which preserve the conjugacy classes of every subgroup $P \in \mathcal{P}$.

Lemma 7.8. *Let G be hyperbolic relative to a finite collection $\mathcal{P}$ of $\mathcal{FIN}$ -finite groups. Then, G is $\mathcal{FIN}$ -finite.*

Proof. Let $\mathcal{F}$ denote the family of subgroups of G generated by $\mathcal{FIN}$ and $\mathcal{P}$. The main theorem of [39] states that there exists a finite model for the space $E_{\mathcal{F}}G$. Applying Theorem 3.6 to each element of $\mathcal{P}$ we obtain a finite model for $E_{\mathcal{FIN}}G$. That is G is $\mathcal{FIN}$ -finite. $\square$

Suppose that G is one-ended and hyperbolic relative to $\mathcal{P}$. By the work of Guirardel and Levitt [23, Corollary 9.20] (see also [22, Section 3.3]), there is a canonical *JSJ tree* $\mathcal{T}_G$ for G . We denote the quotient classes of vertices in $G \backslash \mathcal{T}_G$ by $\text{JSJ}(G)$. More precisely, $\mathcal{T}_G$ is a simplicial G -tree such that the G -equivariant homeomorphism class of $\mathcal{T}_G$ is preserved by the elements of $\text{Aut}(G; \mathcal{P})$.

Note that if each $P \in \mathcal{P}$ satisfies the Farrell–Jones Conjecture, then G above satisfies the Farrell–Jones Conjecture as well [7].

Proposition 7.9. *Let G be an L^2 -acyclic group which is hyperbolic and one-ended relative to a finite collection $\mathcal{P}$ of $\mathcal{FIN}$ -finite groups. If G satisfies (DFJ), then*

$$\rho^{(2)}(G) = \sum_{v \in \text{JSJ}(G)} \rho^{(2)}(G_v),$$

where $\text{JSJ}(G)$ is the set of vertex groups in some JSJ decomposition for G .

Proof. This follows immediately from Theorem 3.6 (vii) applied to the JSJ tree, noting the groups involved are $\mathcal{FIN}$ -finite by hypothesis or by Lemma 7.8. Note that the passage from an action on the tree to a graph of groups is explained in Example 5.20. $\square$

7.4. Automorphisms of one-ended relatively hyperbolic groups. Recall that a *toral relatively hyperbolic group* is a torsion-free group hyperbolic relative to a finite collection of finitely generated abelian groups. These are a natural generalisation of hyperbolic manifolds with toral cusps. In the next section we show that L^2 -torsion of an automorphism of a class containing all toral relatively hyperbolic groups reduces to the sum of the L^2 -torsion of the restrictions of the automorphisms to certain surface subgroups in the JSJ decomposition. Said differently, the L^2 -torsion of an automorphism of a toral relatively hyperbolic group is carried by its restriction to surface subgroups.

Let G be {hyperbolic and one-ended} relative to $\mathcal{P}$ and let $\mathcal{T}_G$ denote the canonical JSJ tree of G . The group $\text{Aut}(G; \mathcal{P})$ has a finite index subgroup $\mathcal{K}(\mathcal{T}_G)$ whose action on $\mathcal{T}_G$ descends to the identity on the quotient graph $G \backslash \mathcal{T}_G$. Note that if the groups in $\mathcal{P}$ are not relatively hyperbolic groups, e.g., if they are virtually polycyclic groups that are not virtually cyclic, then $\text{Aut}(G; \mathcal{P})$ has finite index in $\text{Aut}(G)$.

Theorem 7.10. *Let G be a group hyperbolic and one-ended relative to a finite collection $\mathcal{P}$ of virtually polycyclic groups, let $\Phi \in \mathcal{K}(\mathcal{T}_G)$, and let $\Gamma = G \rtimes_{\Phi} \mathbb{Z}$. Then*

$$\rho^{(2)}(\Phi) = \sum_{v \in \text{Flex}(G)} \rho^{(2)}(G_v \rtimes_{\Phi|_{G_v}} \mathbb{Z}),$$

where $\text{Flex}(G)$ is the set of flexible vertices in a JSJ decomposition of G .

Proof. We are in the setting of Example 5.20 with the tree given by $\mathcal{T}_G$. We need to argue that each of the vertex and edge stabilisers are $\mathcal{FIN}$ -finite and each of the automorphisms Φ_v, Φ_e for $v \in V(T)$ and $e \in E(T)$ are det-finite.

The stabiliser G_v in G of a vertex v in $\mathcal{T}_G$ satisfies one of the following:

- (i) v is *flexible*: the group G_v fits into a short exact sequence $1 \rightarrow F_v \rightarrow G_v \rightarrow Q_v \rightarrow 1$, where F_v is finite and Q_v is isomorphic to the fundamental group of a compact hyperbolic orbifold S and the image of the natural homomorphism $\mathcal{K}(\mathcal{T}_G) \rightarrow \text{Out}(G_v)$ is contained in the mapping class group $\text{Mod}(S)$ of S ;
- (ii) there exists $P_i \in \mathcal{P}$ with $G_v = P = P_i^g$ for some $g \in G$. In particular, the group G_v is virtually polycyclic;
- (iii) the image of the natural homomorphism $\mathcal{K}(\mathcal{T}_G) \rightarrow \text{Out}(G_v)$ is finite.

In the first case G_v is commensurable with the fundamental group of a compact hyperbolic Riemann surface. One way to see this is to note that Q_v is a Fuchsian group which, by [20], are ‘good’ in the sense of Serre [45, Page 16]. So G_v is residually finite by [45, Page 16] and hence contains a torsion-free subgroup of finite index—this subgroup is the desired surface subgroup. We claim that G_v is $\mathcal{FIN}$ -finite (a model for $E_{\mathcal{FIN}}G_v$ is given by the hyperbolic plane with action factoring through Q_v). Since G_v is residually finite we see that it is sofic and hence

satisfies the Determinant Conjecture [14]. Thus, in this case we have $\rho^{(2)}(\Phi_v) = \rho^{(2)}(G_v \rtimes_{\Phi_v} \mathbb{Z})$.

In the second case (or in the case of an edge group) we have

$$\rho^{(2)}(\Phi_v) = \rho^{(2)}(G_v \rtimes_{\Phi_v} \mathbb{Z}) = 0$$

by [48], because $G_v \rtimes \mathbb{Z}$ is {virtually polycyclic}-by- $\mathbb{Z}$ which is again virtually polycyclic. Note that virtually polycyclic groups are $\mathcal{FIN}$ -finite. Hence, Φ_v or Φ_e in this case is det-finite.

In the third case we have that G_v is a group hyperbolic relative to finitely many virtually polycyclic groups and so is $\mathcal{FIN}$ -finite by Lemma 7.8. Moreover, we have that Φ_v is periodic. In particular, Φ_v is det-finite and $\rho^{(2)}(\Gamma_v) = 0$ by Theorem 5.3 (iii).

The formula claimed in the proposition now follows from Theorem 3.6 (vii). $\square$

Remark 7.11. Let G and Φ be as in the Proposition 7.10. If $\Gamma = G \rtimes_{\Phi} \mathbb{Z}$ satisfies (DFJ), then $\rho^{(2)}(\Gamma) = \rho^{(2)}(\Phi)$. In particular, this holds whenever G is toral relatively hyperbolic. Indeed, in this case Γ is torsion-free and satisfies the Farrell–Jones Conjecture [2]. However, in the case where Γ has non-trivial torsion we do not know that the Farrell–Jones Conjecture holds for the Weyl groups of finite subgroups of Γ .

7.5. Polynomially growing automorphisms. Let G be a group generated by a finite set S . Let $|\cdot|$ denote the word metric on G with respect to S . We say that $\Phi \in \text{Aut}(G)$ has *polynomial growth* (or *grows polynomially*) of *degree* at most d if for each $g \in G$ there is a constant C such that $|\Phi^n(g)| < Cn^d + C$ for all $n \in \mathbb{N}$.

For a conjugacy class $[g]$ in G let $||[c]||$ denote the length of a shortest representative. We say $\Phi \in \text{Out}(G)$ is *polynomially growing* if for each conjugacy class $[g]$ of G there is a constant C such that $||\Phi^n([g])|| < Cn^d + C$ for all $n \in \mathbb{N}$.

We remark that for any two finite generating sets S_1 and S_2 the corresponding word metrics on G are bi-Lipschitz equivalent. It follows that the two definitions of growth are independent of the choice of a finite generating set.

Throughout the next few subsections we will prove vanishing of L^2 -torsion for various polynomially growing automorphisms. We collect these results in Theorem 7.19 and remark that many of the arguments here are inspired by arguments in [3] proving vanishing of torsion homology growth using the cheap rebuilding property [1].

7.5.1. Free products. For any splitting of a group G as a free product $*_{i=1}^k G_i$ with each G_i non-trivial and not necessarily freely irreducible, we call the collection of conjugacy classes of the G_i a *free factor system*.

Let $G_1, \dots, G_k$ be non-trivial finitely generated groups and let F_N denote a free group of rank N . Let $G = *_{i=1}^k G_i * F_N$ and denote by $\mathcal{F}$ the set of conjugacy classes of the subgroups G_i in G .

We say that the pair $(G, \mathcal{F})$ is a *sporadic free product* if one of the following holds:

- (i) $k = 0$ and $G = \mathbb{Z}$;
- (ii) $k = 1$ and $G = G_1$ or $G = G_1 * \mathbb{Z}$; or
- (iii) $k = 2$ and $G = G_1 * G_2$.

The key point for us is that automorphisms of sporadic free products have a canonical G -tree that they preserve. We recall this result here noting that it built on work of Guirardel and Horbez [21]

Proposition 7.12. [3, Proposition 2.1] *Let $(G, \mathcal{F})$ be a free product and let $\Phi \in \text{Out}(G, \mathcal{F})$ be polynomially growing. There is a free factor system $\mathcal{F}'$ of $(G, \mathcal{F})$*

and $k \in \mathbb{N}$ such that Φ^k preserves $\mathcal{F}'$ and $(G, \mathcal{F}')$ is sporadic. In particular, Φ^k preserves a Bass–Serre tree associated to $\mathcal{F}'$.

We are now ready to prove a vanishing result for the L^2 -torsion of polynomially growing automorphisms of free products.

Proposition 7.13. *Let $G = G_1 * \dots * G_k * F_N$ be a free product of groups satisfying condition (DFJ), let Φ be a polynomially-growing automorphism of G which preserves the conjugacy classes of the factors G_i , and let $\Phi_i: G_i \rightarrow G_i$ be the appropriate restriction of Φ up to conjugacy. Suppose that for every $i \in \{1, \dots, k\}$, we have $\rho^{(2)}(\Phi_i) = 0$, then $\rho^{(2)}(\Phi) = 0$.*

Proof. This proof follows a similar structure to [3, Theorem 3.1]. We proceed by induction on the Grushko rank $k + N$. If $k = 1$ and $N = 0$, then $\rho^{(2)}(\Gamma) = 0$, since $\Gamma \cong \mathbb{Z}$ by hypothesis. If $k = 0$ and $N = 1$, then Γ has finite index subgroup isomorphic to $\mathbb{Z}^2$, in particular, $\rho^{(2)}(\Gamma) = 0$. Suppose now $k + N \geq 2$. Let Φ be the image of Φ in $\text{Out}(G)$ and let $\mathcal{F}$ be the sporadic free factor system given by Proposition 7.12. Let $\mathcal{T}$ be the Bass–Serre tree of G associated to the free factor system $\mathcal{F}$ and note that there is a positive power of Φ which preserves $\mathcal{T}$. The vertex stabilisers of G acting on $\mathcal{T}$ are proper free factors of G and hence have smaller Grushko rank than G and the edge stabilisers are trivial. It follows that a finite index subgroup Γ' of Γ acts on $\mathcal{T}$ with vertex stabilisers of the form $\Gamma'_v = G_v \rtimes_{\Phi^n|_{G_v}} \mathbb{Z}$ and with infinite cyclic edge stabilisers. Now, Φ^n is det-finite and $\rho^{(2)}(\Phi^n) = 0$ by the inductive hypothesis and $\rho^{(2)}(\mathbb{Z}) = 0$. Thus, the result follows from Example 5.20 and Theorem 5.3 (ii). $\square$

We remark that in the special case of free-by-cyclic groups this recovers a result of Clay [13].

Corollary 7.14. *If $\Phi \in \text{Aut}(F_N)$ is polynomially growing, then $\rho^{(2)}(F_N \rtimes_{\Phi} \mathbb{Z}) = 0$.*

7.5.2. Relatively hyperbolic groups. We first deal with the one-ended case.

Theorem 7.15. *Let G be a group hyperbolic and one-ended relative to a finite collection $\mathcal{P}$ of virtually polycyclic groups and let $\Phi \in \text{Aut}(G)$ be polynomially growing. Then $\rho^{(2)}(\Phi) = 0$.*

Proof. Let $\Gamma = G \rtimes_{\Phi} \mathbb{Z}$. Let H be a characteristic finite index torsion-free subgroup of G , let $n > 0$ be such that $\Phi^n \in \mathcal{K}(\mathcal{T}_G)$, and let $\Lambda = G \rtimes_{\Phi^n} \mathbb{Z}$. Note that Λ has finite index in Γ . We denote Φ^n by Ψ .

By Proposition 7.10, the only possible non-zero contribution to $\rho^{(2)}(\Psi)$ is from the terms $\rho^{(2)}(G_v \rtimes_{\Psi|_{G_v}} \mathbb{Z})$ where v runs over flexible vertices. In this case, $\Lambda_v = G_v \rtimes_{\Psi|_{G_v}} \mathbb{Z}$, where G_v is the fundamental group of a compact hyperbolic surface S_v and $\Psi|_{G_v}$ is an element of $\text{Mod}(S_v)$ with polynomial growth. Hence, $\Psi^n|_{G_v}$ is periodic or a Dehn twist. Thus, by [38, Theorem 0.7] we have $\rho^{(2)}(\Lambda_v) = 0$. The result now follows from Theorem 5.3 (ii) $\square$

Proposition 7.16. *Let G be a group hyperbolic relative to a finite collection $\mathcal{P}$ of virtually polycyclic groups and let $\Phi \in \text{Aut}(G)$ be polynomially growing. Then $\rho^{(2)}(\Phi) = 0$.*

Proof. Let $\Gamma = G \rtimes_{\Phi} \mathbb{Z}$. If G has finitely many ends we are done by Proposition 7.15. If not, then since G is finitely presented, G admits a finite index characteristic subgroup H which splits as a free product of a (possibly trivial) free group F_N and finitely many groups G_i , each hyperbolic relative to a finite collection of virtually polycyclic groups and each with finitely many ends. We may pass to a large power ℓ of Φ such that Φ^ℓ preserves the conjugacy classes of the G_i . Now, by Proposition 7.15 we have $\rho^{(2)}(G_i \rtimes_{\Phi^\ell} \mathbb{Z}) = 0$ for each i . We have now verified the

hypothesis of Proposition 7.13 applied to the group $H \rtimes_{\Phi^\ell} \mathbb{Z}$. The result follows from Theorem 2.7 (iv). $\square$

7.5.3. Right-angled Artin and Coxeter groups. Let L be a flag complex. The *right-angled Artin group* (RAAG) A_L is defined to be the group with presentation

$$\langle L^{(0)} \mid [v, w] \text{ if } \{v, w\} \in L^{(1)} \rangle.$$

The *right-angled Coxeter group* (RACG) is the group

$$W_L = A_L / \langle\langle v^2 \mid v \in L^{(0)} \rangle\rangle.$$

We define a number of automorphisms of RAAGs and RACGs:

- (i) *graph automorphisms*, that is automorphism induced from L ;
- (ii) *inversions*, which send $v \mapsto v^{-1}$ and $u \mapsto u$ for $u \neq v$ and $u, v \in L^{(0)}$;
- (iii) *partial conjugations* $k_{w,C}$ for $w \in L^{(0)}$ and a connected component C of $L \setminus \text{st}(w)$, which are defined by $k_{w,C}(u) = w^{-1}uw$ if $u \in C^{(0)}$ and $k_{w,C}(u) = u$ if $u \in L^{(0)} \setminus C$;
- (iv) *folds* $t_{v,w}$ for any $v, w \in L^{(0)}$ with $\text{lk}(v) \subseteq \text{lk}(w)$, which are defined by $t_{v,w}(v) = vw$ and $t_{v,w}(u) = u$ for all $u \in L^{(0)} \setminus \{v\}$.

We say an automorphism of A_L (resp. W_L) is *untwisted* if it is contained in the subgroup $U(A_L) \leq \text{Aut}(A_L)$ (resp. $U(W_L) \leq \text{Aut}(W_L)$) which is generated by the graph automorphisms, inversions, partial conjugations, and folds.

By [15, Proposition A(3)], the untwisted automorphisms of A_L are exactly the automorphism which preserve the standard coarse median structure on A_L . By [43], the subgroup of untwisted automorphisms of $\text{Aut}(W_L)$ has finite index.

Proposition 7.17. *Let L be a flag complex on $[m]$ and let $\Gamma = A_L \rtimes_{\Phi} \mathbb{Z}$. If Φ is an untwisted and polynomially growing automorphism of A_L , then $\rho^{(2)}(\Gamma) = 0$.*

We note that the following argument is structurally very similar to the proof of [3, Theorem 5.1]

Proof. We proceed by induction on m . The base case, when $m = 1$, implies Γ is virtually isomorphic to $\mathbb{Z}^2$. In this case we have $\rho^{(2)}(\Gamma) = 0$ as required. We now suppose $m > 1$. Note that if K is a full subcomplex of L then any untwisted automorphism of A_L preserving $A_K \leq A_L$ restricts to an untwisted automorphism of A_K . There are three cases to consider:

The first case is when A_L is freely reducible. In this case $A_L = A_{K_1} * \dots * A_{K_k} * F_n$ and each K_i and $[n]$ is a full subcomplex of L . Note that each K_i and if $n \neq 0$ the complex $[n]$ all contain at least one vertex and strictly less than m vertices. Now, pass to a sufficiently high power ℓ of Φ which preserves the conjugacy classes of the A_{K_i} . Then, by the inductive hypothesis $\rho^{(2)}(A_{K_i} \rtimes_{\Phi^\ell} \mathbb{Z}) = 0$. The case then follows from Proposition 7.13 and Theorem 2.7 (iv).

The second case is when A_L is both freely and directly irreducible. In this case, by [15, Proposition D] we have that $A_L = A_{K_1} *_{A_{K_3}} A_{K_2}$ with $K_3 = K_1 \cap K_2$ and each K_1, K_2, K_3 a non-empty proper full subcomplex of L . Passing to a large enough power ℓ of Φ , the Bass-Serre tree $\mathcal{T}$ of the splitting is Φ^ℓ invariant and Φ^ℓ preserves the stabilisers, that is $\Phi^\ell(A_{K_i}) = A_{K_i}$. Thus, Γ admits a finite index subgroup Λ acts on $\mathcal{T}$ with stabilisers of the form $A_{K_i} \rtimes_{\Phi^\ell} \mathbb{Z}$. By induction these stabilisers have vanishing L^2 -torsion. Thus, the case follows from Theorem 3.6 (vii) and Theorem 2.7 (iv).

The final case is when A_L is directly reducible. In this case $A_L \cong \prod_{i=1}^n A_{K_i} \times \mathbb{Z}^k$ where each K_i is a full subcomplex of L and A_{K_i} is directly irreducible and non-cyclic. If $k \geq 0$, then A_L is L^2 -acyclic and so $\rho^{(2)}(\Gamma) = 0$ by Theorem 5.3 (viii). Thus, we may suppose $k = 0$. Now, we have that A_L acts on a product of trees

$X = \prod_{i=1}^n \mathcal{T}_i$, where $\mathcal{T}_i$ is a tree either provided by Proposition 7.12 or by [15, Proposition D] depending on if A_{K_i} is freely reducible or not. We now pass to a sufficiently large power ℓ of Φ such that $\Phi^\ell|_{A_{K_i}}$ preserves each tree $\mathcal{T}_i$. In particular, if σ is a cell of X we have $\Phi^\ell(\text{Stab}_{A_L}(\sigma)) = \text{Stab}_{A_L}(\sigma)$. Moreover, each stabiliser of a cell in X under the action of A_L is a product of the stabilisers of some of the groups A_{K_i} acting on $\mathcal{T}_i$ and the remaining A_{K_j} . In particular, each stabiliser of $A_L \rtimes_{\Phi^\ell} \mathbb{Z}$ acting on X is of the form $A_{J_\sigma} \rtimes_{\Phi^\ell|_{A_{J_\sigma}}} \mathbb{Z}$ where J_σ is a full proper non-empty subcomplex of L . Hence, the L^2 -torsion of the stabilisers vanishes by induction and the proposition follows from Theorem 3.6 (vii) and Theorem 2.7 (iv). $\square$

The following proposition is proved verbatim taking into account the remarks after Theorem E and at the start of Section 5 [15] and after noting the subgroup of untwisted automorphisms of $\text{Aut}(W_L)$ has finite index [43].

Proposition 7.18. *Let L be a flag complex on $[m]$ and let $\Gamma = W_L \rtimes_{\Phi} \mathbb{Z}$. If Φ is a polynomially growing automorphism of A_L , then $\rho^{(2)}(\Gamma) = 0$.*

7.5.4. Anthology. We collect the above results on polynomially growing automorphisms, namely Propositions 7.16, 7.17 and 7.18, into one theorem. This answers [3, Question 1.2].

Theorem 7.19. *Let Γ be a group isomorphic to one of*

- $G \rtimes_{\Phi} \mathbb{Z}$ with G residually finite and hyperbolic;
- $G \rtimes_{\Phi} \mathbb{Z}$ with G residually finite and hyperbolic relative to a finite collection of virtually polycyclic groups;
- $A_L \rtimes_{\Phi} \mathbb{Z}$ where A_L is a right-angled Artin group and $\Phi \in \text{Aut}(A_L)$ is untwisted; or
- $W_L \rtimes_{\Phi} \mathbb{Z}$ where W_L is a right-angled Coxeter group.

If Φ is polynomially growing, then $\rho^{(2)}(\Gamma) = 0$.

7.6. Handlebody groups. Let V_g denote a genus g handlebody and let $\text{Mod}(V_g)$ denote its mapping class group, *the genus g handlebody group*. The reader is referred to [4] for more information. Our final result answers [4, Problem 28].

Theorem 7.20. *Let $g \geq 2$ and let V_g denote the genus g handlebody. Then,*

$$\rho^{(2)}(\text{Mod}(V_g)) = 0.$$

Proof. Let X denote the disc complex for $\text{Mod}(V_g)$. We denote by G the intersection of the handlebody group with the pure mapping class group in $\text{Mod}(S_g)$. This is a finite index torsion-free subgroup of $\text{Mod}(V_g)$ which acts on the disc complex X cocompactly and cellularly such that the set-wise stabiliser of any cell is equal to its point-wise stabiliser. In particular, X is a finite G -CW-complex. We claim G is admissible, indeed, EG is L^2 -acyclic by [4, Theorem 6.1]. Since G is a subgroup of the mapping class group it is residually finite [19]. Hence, sofic and so satisfies the Determinant Conjecture [14, Theorem 5].

By [4, §1.A], for every cell $\sigma \in X$, the stabiliser G_σ fits into an exact sequence

$$1 \rightarrow \mathbb{Z}^{n_\sigma} \rightarrow G_\sigma \rightarrow H_\sigma \rightarrow 1$$

where H_σ has finite cohomological dimension and is a torsion-free finite index subgroup of a group of type VF. In particular, $\rho^{(2)}(G_\sigma) = 0$ by [48]. Applying this to Theorem 3.6 (vii) we obtain the vanishing of the L^2 -torsion of $\text{Mod}(V_g)$. The vanishing of the torsion homology growth is [4, Theorem 6.1]. $\square$

REFERENCES

- [1] M. Abert, N. Bergeron, M. Fraczyk, and D. Gaboriau. On homology torsion growth. *J. Eur. Math. Soc. (JEMS)*, 27(6):2293–2357, 2025. Cited on Pages 2 and 35.
- [2] N. Andrew, Y. Guerch, and S. Hughes. Automorphisms of relatively hyperbolic groups and the Farrell–Jones Conjecture. Preprint, arXiv:2311.14036 [math.KT], 2023. Cited on Page 35.
- [3] N. Andrew, Y. Guerch, S. Hughes, and M. Kudlinska. Homology growth of polynomially growing mapping tori. Preprint, arXiv:2305.10410 [math.GR], to appear in Groups Geometry Dynamics, published online first, <https://ems.press/journals/ggd/articles/14298718>, 2023. Cited on Pages 3, 35, 36, 37, and 38.
- [4] N. Andrew, S. Hensel, S. Hughes, and R. D. Wade. Problems on handlebody groups. Preprint, arXiv:2502.21177 [math.GR], appear in Royal Society Open Science, 2025. Cited on Pages 3 and 38.
- [5] N. Andrew, S. Hughes, and M. Kudlinska. Torsion homology growth of polynomially growing free-by-cyclic groups. *Rocky Mountain J. Math.*, 54(4):933–941, 2024. Cited on Page 2.
- [6] G. Avramidi, B. Okun, and K. Schreve. Mod p and torsion homology growth in nonpositive curvature. *Invent. Math.*, 226(3):711–723, 2021. Cited on Page 2.
- [7] A. Bartels. Coarse flow spaces for relatively hyperbolic groups. *Compos. Math.*, 153(4):745–779, 2017. Cited on Page 33.
- [8] A. Bartels and W. Lück. The Borel conjecture for hyperbolic and CAT(0)-groups. *Ann. of Math. (2)*, 175:631–689, 2012. Cited on Page 31.
- [9] N. Bergeron and A. Venkatesh. The asymptotic growth of torsion homology for arithmetic groups. *J. Inst. Math. Jussieu*, 12(2):391–447, 2013. Cited on Page 2.
- [10] A. Borel. The L^2 -cohomology of negatively curved Riemannian symmetric spaces. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 10:95–105, 1985. Cited on Page 33.
- [11] C. Bregman and M. Clay. Minimal volume entropy of free-by-cyclic groups and 2-dimensional right-angled Artin groups. *Math. Ann.*, 381(3-4):1253–1281, 2021. Cited on Page 2.
- [12] L. Ciobanu, D. F. Holt, and S. Rees. Sofic groups: graph products and graphs of groups. *Pacific J. Math.*, 271(1):53–64, 2014. Cited on Page 33.
- [13] M. Clay. ℓ^2 -torsion of free-by-cyclic groups. *Q. J. Math.*, 68(2):617–634, 2017. Cited on Pages 1, 2, and 36.
- [14] G. Elek and E. Szabó. Hyperlinearity, essentially free actions and L^2 -invariants. The sofic property. *Math. Ann.*, 332(2):421–441, 2005. Cited on Pages 30, 31, 33, 35, and 38.
- [15] E. Fioravanti. Coarse-median preserving automorphisms. *Geom. Topol.*, 28(1):161–266, 2024. Cited on Pages 37 and 38.
- [16] R. Frigerio, J.-F. Lafont, and A. Sisto. Rigidity of high dimensional graph manifolds. *Astérisque*, 372:xxi+177, 2015. Cited on Page 32.
- [17] D. Gaboriau. Invariants l^2 de relations d’équivalence et de groupes. *Publ. Math. Inst. Hautes Études Sci.*, 95:93–150, 2002. Cited on Page 31.
- [18] M. Gromov. Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.*, 56:5–99 (1983), 1982. Cited on Pages 1 and 4.
- [19] E. K. Grossman. On the residual finiteness of certain mapping class groups. *J. London Math. Soc. (2)*, 9:160–164, 1974/75. Cited on Page 38.
- [20] F. Grunewald, A. Jaikin-Zapirain, and P. A. Zalesskii. Cohomological goodness and the profinite completion of Bianchi groups. *Duke Math. J.*, 144(1):53–72, 2008. Cited on Page 34.
- [21] V. Guirardel and C. Horbez. Boundaries of relative factor graphs and subgroup classification for automorphisms of free products. *Geom. Topol.*, 26(1):71–126, 2022. Cited on Page 35.
- [22] V. Guirardel and G. Levitt. Splittings and automorphisms of relatively hyperbolic groups. *Groups Geom. Dyn.*, 9(2):599–663, 2015. Cited on Page 33.
- [23] V. Guirardel and G. Levitt. JSJ decompositions of groups. *Astérisque*, 395:vii+165, 2017. Cited on Pages 3 and 33.
- [24] E. Hess and T. Schick. L^2 -torsion of hyperbolic manifolds. *Manuscripta Math.*, 97(3):329–334, 1998. Cited on Page 32.
- [25] S. Hughes. Graphs and complexes of lattices. Preprint, arXiv:2104.13728 [math.GR], 2021. Cited on Pages 31 and 32.
- [26] S. Hughes. Lattices in a product of trees, hierarchically hyperbolic groups and virtual torsion-freeness. *Bull. Lond. Math. Soc.*, 54(4):1413–1419, 2022. Cited on Page 31.
- [27] D. Kasprowski and H. Rüping. Long and thin covers for flow spaces. *Groups Geom. Dyn.*, 11(4):1201–1229, 2017. Cited on Page 31.
- [28] D. Kirstein, C. Kremer, and W. Lück. Some problems and conjectures about L^2 -invariants. In *Geometry and topology of aspherical manifolds*, volume 816 of *Contemp. Math.*, pages 3–43. Amer. Math. Soc., Providence, RI, [2025] ©2025. Cited on Page 2.

- [29] I. J. Leary and A. Minasyan. Commensurating HNN extensions: nonpositive curvature and biautomaticity. *Geom. Topol.*, 25(4):1819–1860, 2021. Cited on Page 32.
- [30] I. J. Leary and B. E. A. Nucinkis. Some groups of type VF . *Invent. Math.*, 151(1):135–165, 2003. Cited on Page 28.
- [31] C. Löh, K. Li, M. Moraschini, R. Sauer, and M. Uschold. The algebraic cheap rebuilding property. Preprint, arXiv:2409.05774 [math.GR], 2024. Cited on Page 2.
- [32] W. Lück. *Transformation groups and algebraic K-theory*, volume 1408 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989. Cited on Pages 8, 13, 17, 20, and 21.
- [33] W. Lück. *L^2 -Invariants: Theory and Applications to Geometry and K-Theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002. Cited on Pages 1, 2, 4, 5, 6, 8, 11, 12, 14, 15, 22, 25, 28, 30, 31, and 33.
- [34] W. Lück. Approximating L^2 -invariants and homology growth. *Geom. Funct. Anal.*, 23(2):622–663, 2013. Cited on Page 2.
- [35] W. Lück. Approximating L^2 -invariants by their classical counterparts. *EMS Surv. Math. Sci.*, 3(2):269–344, 2016. Cited on Page 2.
- [36] W. Lück. Twisting l^2 -invariants with finite-dimensional representations. *J. Topol. Anal.*, 10(4):723–816, 2018. Cited on Pages 4 and 6.
- [37] W. Lück. *Isomorphism Conjectures in K- and L-theory*, volume 78 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer, 2025. With contributions by Christopher Winkes. Cited on Pages 5, 6, and 8.
- [38] W. Lück and T. Schick. L^2 -torsion of hyperbolic manifolds of finite volume. *GAFA*, 9:518–567, 1999. Cited on Pages 1, 4, 32, 33, and 36.
- [39] E. Martínez-Pedroza and P. Przytycki. Dismantlable classifying space for the family of parabolic subgroups of a relatively hyperbolic group. *J. Inst. Math. Jussieu*, 18(2):329–345, 2019. Cited on Page 33.
- [40] B. Okun and K. Schreve. Torsion invariants of complexes of groups. *Duke Math. J.*, 173(2):391–418, 2024. Cited on Page 32.
- [41] M. Olbrich. L^2 -invariants of locally symmetric spaces. *Doc. Math.*, 7:219–237, 2002. Cited on Pages 1 and 31.
- [42] E. Pieroni. Minimal entropy of 3-manifolds. Preprint, arXiv:1902.09190 [math.DG], 2019. Cited on Page 1.
- [43] A. Sale and T. Susse. Outer automorphism groups of right-angled Coxeter groups are either large or virtually abelian. *Trans. Amer. Math. Soc.*, 372(11):7785–7803, 2019. Cited on Pages 37 and 38.
- [44] J.-P. Serre. *Trees*. Springer-Verlag, Berlin, 1980. Translated from the French by J. Stillwell. Cited on Pages 12, 13, 28, and 29.
- [45] J.-P. Serre. *Galois cohomology*. Springer-Verlag, Berlin, 1997. Translated from the French by Patrick Ion and revised by the author. Cited on Page 34.
- [46] T. Soma. The Gromov invariant of links. *Invent. Math.*, 64(3):445–454, 1981. Cited on Pages 1 and 4.
- [47] W. P. Thurston. Geometry and topology of 3-manifolds. Lecture notes, Princeton. <http://www.msri.org/publications/books/gt3m>, 1978. Cited on Pages 1 and 4.
- [48] C. Wegner. L^2 -invariants of finite aspherical CW-complexes. *Manuscripta Math.*, 128(4):469–481, 2009. Cited on Pages 1, 33, 35, and 38.
- [49] C. Wegner. The K -theoretic Farrell-Jones conjecture for $\text{CAT}(0)$ -groups. *Proc. Amer. Math. Soc.*, 140(3):779–793, 2012. Cited on Page 31.

MATHEMATICIANS INSTITUT DER UNIVERSITÄT BONN, ENDENICHER ALLEE 60, 53115 BONN, GERMANY

Email address: sam.hughes.maths@gmail.com; hughes@math.uni-bonn.de

URL: <https://samhughesmaths.github.io>

MATHEMATICIANS INSTITUT DER UNIVERSITÄT BONN, ENDENICHER ALLEE 60, 53115 BONN, GERMANY

Email address: wolfgang.lueck@him.uni-bonn.de

URL: <http://www.him.uni-bonn.de/lueck>