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EQUIVARIANT PRINCIPAL BUNDLES AND THEIR CLASSIFYING SPACES

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Abstract. We consider Γ-equivariant principal G-bundles over proper Γ-CW-complexes with prescribed family of local representations. We construct and analyze their classifying spaces for locally compact, second countable topological groups with finite covering dimension Γ and G, such that G is almost connected.

Introduction

Let Γ and G be topological groups. We will introduce the notion of a Γ-*equivariant principle G-bundle* over a Γ-CW-complex, i.e. a principal G-bundle $p\colon E \to B$ together with left Γ-actions on E and B commuting with the right G-action on E such that p is Γ-equivariant. For every $e \in E$ we obtain a *local representation* $\rho_e\colon \Gamma_{p(e)} \to G$ uniquely determined by $\gamma^{-1} \cdot e = e \cdot \rho_e(\gamma)$ for $\gamma \in \Gamma_{p(e)}$, where $\Gamma_{p(e)}$ is the isotropy group of $p(e) \in B$. One can consider such bundles where the family of local representations $\mathcal{R}$ is prescribed, e.g. one may demand that $\Gamma_{p(e)}$ is always compact and allow only certain homomorphisms ρ_e.

Our main technical result is Theorem 8.1, where we prove that a Γ-equivariant principal G-bundle $p\colon E \to B$ is the same as a $\Gamma \times G$-CW-complex E, provided that the family of locally representations satisfies *Condition (H)* introduced in Definition 6.1. This implies the main result of this paper Theorem 11.5 that gives a *universal Γ-equivariant principal G-bundle with respect to a given family of local representations $\mathcal{R}$*, provided that $\mathcal{R}$ satisfies Condition (H).

Condition (H) is needed to ensure homotopy invariance for Γ-equivariant principal G-bundles. It is automatically satisfied, if Γ and G are locally compact second countable groups with finite covering dimensions (e.g. Lie groups), G is almost connected (i.e. G modulo its connected component of the identity is compact), and all base spaces are Γ-CW-complexes with compact isotropy groups, see Theorem 6.3.

Equivariant principal bundles have been studied before by several authors (see [9, 17, 18, 19, 23, 27, 33] and references therein) and our construction generalizes all the previous constructions in the following sense. We isolated the conditions that the groups Γ and G and the family of local representations need to satisfy, in order to show the existence of a universal equivariant bundle associated to the prescribed family of local representation as base; this is the Condition (H) mentioned above. Moreover, in the literature some conditions about the restriction of the bundles to Γ-invariant neighborhoods of points in the base space are demanded; we show that they automatically follow from our setting in Theorem 9.1 if the Condition (H) is satisfied.

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In Section 15 we have included a study of the case $G = \mathcal{PU}(\mathcal{H})$, the projective unitary group, endowed with the norm topology. The main result in Section 15 is Theorem 15.16 which produces a universal Γ-equivariant stable projective unitary bundle for almost free Γ-CW-complexes. This result generalizes [5, Theorem 3.21], where Γ is assumed to be discrete. This universal bundle is relevant for equivariant twisted topological K-theory since with this bundle it can be defined as a parameterized equivariant cohomology theory.

Our results carry directly over to the case, where one allows an intertwining between the Γ and the G-action, i.e. there exists a group homomorphism $\tau\colon \Gamma \to \operatorname{aut}(G)$ and the condition that the Γ and G-action on the total space commute, is replaced by the weaker condition $\gamma\cdot(e\cdot g) = (\gamma\cdot e)\cdot\tau(\gamma)(g)$. A typical example for such a non-trivial intertwining is the case $\Gamma = \mathbb{Z}/2$, $G = U(n)$ and $\tau\colon \mathbb{Z}/2 \to \operatorname{aut}(U(n))$ given by complex conjugation, which leads to real vector bundles in the sense of Atiyah [3]. For the simplicity of the exposition we only treat the case where τ is trivial.

Throughout this paper we will work in the category of compactly generated spaces, see [32] and Section 16, and subgroups are always understood to be closed subgroups. Most of the equivariant CW-complexes under consideration are proper, or, equivalently, have compact isotropy groups.

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1. Principal bundles

We recall some basic facts about principal G-bundles over CW-complexes for a topological group G.

Definition 1.1 (Quasi-regular open set and regular space)**.** An open subset $U \subseteq B$ is called *quasi-regular* if for any $x \in U$ there exists an open neighborhood V_x whose closure in B is contained in U. An space is called *regular* if it separates points from closed subsets.

The main point of this notion is that a quasi-regular open subset equipped with its subspace topology is again compactly generated, see Lemma 16.1 (i), and preimages of quasi-regular open subsets are again quasi-regular open subsets, see Lemma 16.1 (ii).

Definition 1.2 (Principal G-bundle)**.** A *principal G-bundle* $p\colon E \to B$ consists of a space E with right G-action, a space B with trivial G-action and a G-map p such that p is locally trivial, i.e., for any $b \in B$ there exists a quasi-regular open neighborhood U of b in B and a G-homeomorphism $\phi\colon G \times U \to p^{-1}(U)$ satisfying $p \circ \phi = \mathrm{pr}$ for the projection $\mathrm{pr}\colon U \times G \to U$.

In the previous definition we have added the condition that the local trivialization can be done on a quasi-regular open set. On the standard definition of principal bundles the quasi-regularity is not required. Nevertheless, since in this article we will work in the category of equivariant CW-complexes, and any invariant open subset of a equivariant CW-complex is automatically quasi-regular, see Lemma (vi), this extra condition in the definition is innocuous.

Lemma 1.3. *Let B be a CW-complex and let $p\colon E \to B \times [0,1]$ be a principal G-bundle. Let $i_0\colon B = B \times \{0\} \to B \times [0,1]$ be the inclusion.*

Then $i_0^*E \times [0,1] \xrightarrow{i_0^* p \times \mathrm{id}_{[0,1]}} B \times [0,1]$ is a principal G-bundle and there exists an isomorphism of principal G-bundles

$$f\colon i_0^*E \times [0,1] \to E$$

over $B \times [0,1]$ whose restriction to $B \times \{0\}$ is the identity.

Proof. A CW-complex B is paracompact by [24]. Now the proof is analogous to the one of [10, Theorem 9.8 in Chapter 3 on page 51] taking into account that any open subset of B is quasi-regular by Lemma 16.1 (iv) and (v) and that in [10, Theorem 9.8 in Chapter 3 on page 51] the symbol $\times$ stands for the classical product space, where in our setting $\times$ stands for the product within the category of compactly generated spaces. □

Notation 1.4. Given a $\Gamma \times G$-space X, let X_r be the space X but now equipped with the left Γ action given by $\gamma \cdot x = (\gamma, 1) \cdot x$ and the right G-action given by $x \cdot g := (1, g^{-1}) \cdot x$.

Given a space Y with (commuting) left Γ- and right G-action, let Y_l be the same space but now equipped with the left $\Gamma \times G$-action given by $(\gamma, g) \cdot y := \gamma \cdot y \cdot g^{-1}$.

Example 1.5 (Free G-CW-complexes)**.** Let X be a free G-CW-complex. Then $p\colon X_r \to X_r/G$ is a principal G-bundle. Conversely, if $p\colon E \to B$ is a principal G-bundle over a CW-complex B, then E_l carries the structure of a free G-CW-complex coming from the filtration given by the preimages of the skeletons of B.

These claims are proved in [21, 1.24 and 1.25 on page 18], and they will also follow from Theorem 8.1 applied to the special case $\Gamma = \{1\}$.

Example 1.6 (Free proper smooth G-actions on smooth manifolds)**.** Consider a Lie group G with a free proper smooth left G-action on a smooth manifold M. Then M is a proper G-CW-complex by [11] and the projection $M_r \to M_r/G$ is a principal G-bundle.

2. Equivariant principal bundles

We first fix the notions for the objects we want to study.

Definition 2.1 (Γ-equivariant principle G-bundle)**.** A Γ-*equivariant principal G-bundle* $p\colon E \to B$ consists of a principal G-bundle together with left Γ-actions on E and B (commuting with the right G-actions) such that $p\colon E \to B$ is Γ-equivariant.

Note that since the Γ and the G actions commute, then Γ acts on $p\colon E \to B$ through G-bundle maps, cf. [17, §1]. See also [33, Chapter I, Section 8] for this notion and its main properties including universal objects for a compact Lie group Γ and a topological group G, where also a twisting of the left Γ- and right G-actions by a homomorphism $\Gamma \to \operatorname{aut}(G)$ is allowed. Sometimes in the literature some conditions about the restriction of the bundles to Γ-invariant neighborhoods of points in the base space are demanded or can only be proved in the case that G is a compact Lie group, see for instance [33, Proposition 8.10 on page 58]. We will show that they automatically follow from our setting in Theorem 9.1.

We mention some basic properties of Γ-equivariant principal G-bundles.

If $f\colon X \to B$ is a Γ-map of Γ-CW-complexes and $p\colon E \to B$ is a Γ-invariant principal G-bundle, then the map $f^*p\colon f^*E \to X$ obtained from the pullback of p with f

$$\begin{tikzcd} f^*E \arrow[r, "\overline{f}"] \arrow[d, "f^*p"] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

is a Γ-equivariant principal G-bundle again. An *isomorphism of two Γ-invariant principal G-bundles* $p_0\colon E_0 \to X$ and $p_1\colon E_1 \to B$ over the Γ-CW-complex B is a homeomorphism $f\colon E_0 \to E_1$ which is compatible with both the left Γ-action and the right G-action and satisfies $p_1 \circ f = p_0$.

Lemma 2.2. *Let $p_0\colon E_0 \to B$ and $p_1\colon E_1 \to B$ be Γ-equivariant principle G-bundles over the Γ-CW-complex B. Let $f\colon E_0 \to E_1$ be a map which is compatible with both the left Γ-actions and the right G-actions and satisfies $p_1 \circ f = p_0$.*

Then f is an isomorphism of Γ-equivariant principal G-bundles.

Proof. The map f is a homeomorphism because of the local triviality of the principal G-bundles p_0 and p_1. □

3. Families of local representations

In this section we introduce the local representations coming from the left action of the Γ-isotropy group of a point b in the base space and the free right G-action on the fiber over b (after choosing a lift of b to the total space).

Definition 3.1 (Local representations)**.** Let $p\colon E \to B$ be a Γ-equivariant principal G-bundle. Consider $e \in E$. Then we obtain a (continuous) group homomorphism

$$\rho_e\colon \Gamma_{p(e)} \to G \tag{3.2}$$

uniquely determined by $\gamma \cdot e = e \cdot \rho_e(\gamma)$ for $\gamma \in \Gamma_{p(e)}$, where $\Gamma_{p(e)}$ is the isotropy group of $p(e) \in B$.

ρ_e is indeed a homomorphism by the following calculation

$$\begin{aligned} e \cdot \rho_e(\gamma_1 \cdot \gamma_2) &= (\gamma_1 \cdot \gamma_2) \cdot e \\ &= \gamma_1 \cdot (\gamma_2 \cdot e) \\ &= \gamma_1 \cdot \big(e \cdot \rho_e(\gamma_2)\big) \\ &= \big(\gamma_1 \cdot e\big) \cdot \rho_e(\gamma_2) \\ &= \big(e \cdot \rho_e(\gamma_1)\big) \cdot \rho_e(\gamma_2) \\ &= e \cdot \big(\rho_e(\gamma_1) \cdot \rho_e(\gamma_2)\big). \end{aligned}$$

It is continuous since the map $G \to p^{-1}(e),\ g \mapsto e \cdot g$ is a homeomorphism because of the local triviality of the principal G-bundle p and the map $\Gamma_{p(e)} \to p^{-1}(e),\ \gamma \mapsto \gamma \cdot e$ is continuous.

Remark 3.3 (Basic properties of the local representations)**.** If we replace e by eg for some $g \in G$, then $\rho_{eg} = c_{g^{-1}} \circ \rho_e$ for $c_g\colon G \to G$ the conjugation homomorphism sending g' to $gg'g^{-1}$.

If we replace e by γe for some $\gamma \in \Gamma$, then $\Gamma_{p(e)} = \gamma^{-1}\Gamma_{p(\gamma e)}\gamma$ and $\rho_{\gamma e} = \rho_e \circ c_{\gamma^{-1}}$.

If

$$\begin{tikzcd} E_0 \arrow[r, "\overline{f}"] \arrow[d, "p_0"] & E_1 \arrow[d, "p_1"] \\ B_0 \arrow[r, "f"] & B_1 \end{tikzcd}$$

is a morphism of Γ-equivariant principal G-bundles, then

$$\rho_e^{p_0} = \rho_{\overline{f}(e)}^{p_1} \circ i_e$$

holds for all $e \in E_0$, where $i_e\colon \Gamma_{p_0(e)} \to \Gamma_{p_1 \circ \overline{f}(e)}$ is the inclusion.

Definition 3.4 (Family of local representations)**.** A *family $\mathcal{R}$ of local representations for* (Γ, G) is a set of pairs (H, α), where H is a subgroup of Γ and $\alpha\colon H \to G$ is a continuous group homomorphism such that the following conditions are satisfied:

- *Finite intersections*
  Suppose that (H_0, α_0) and (H_1, α_1) belong to $\mathcal{R}$. Define $H := \{h \in H_0 \cap H_1 \mid \alpha_0(h) = \alpha_1(h)\}$ and $\alpha\colon H \to G$ by $\alpha = \alpha_0|_H = \alpha_1|_H$. Then $(H, \alpha) \in \mathcal{R}$;
- *Conjugation in G*
  If (H, α) belongs to $\mathcal{R}$ and $g \in G$, then $(H, c_{g^{-1}} \circ \alpha)$ belongs to $\mathcal{R}$;
- *Conjugation in Γ*
  If (H, α) belongs to $\mathcal{R}$ and $\gamma \in \Gamma$, then $(\gamma H \gamma^{-1}, \alpha \circ c_{\gamma^{-1}})$ belongs to $\mathcal{R}$.

Definition 3.5 ((Pre)family of local representations associated to a Γ-equivariant principal G-bundle)**.** Let $p\colon E \to B$ be a Γ-equivariant principal G-bundle. Define the *prefamily of local representations of p* to be

$$\mathcal{R}'(p) := \{(\Gamma_{p(e)}, \rho_e) \mid e \in E\}.$$

Let $\mathcal{R}(p)$ be the smallest system of local representations containing $\mathcal{R}'(p)$. We call $\mathcal{R}(p)$ the *family of local representations associated to p*.

One easily checks using Remark 3.3 that $\mathcal{R}'(p)$ is closed under conjugation, but not necessary under finite intersections so that $\mathcal{R}'(p)$ itself is not a family of local representations.

We want to deal with families of local representations to ensure that the following lemma is true.

A *family of subgroups of* G is a set of subgroups of G closed under conjugation and taking finite intersections.

Lemma 3.6. *Let $\mathcal{R}$ be a family of local representations for Γ and G. For (H,α) in $\mathcal{R}$ let $K(H,\alpha)$ be the subgroup of $\Gamma \times G$ given by*

$$K(H,\alpha) := \{(\gamma, \alpha(\gamma)) \mid \gamma \in H\}.$$

Put

$$\mathcal{F}(\mathcal{R}) = \{K(H,\alpha) \mid (H,\alpha) \in \mathcal{R}\}.$$

Then $\mathcal{F}(\mathcal{R})$ is a family of subgroups of $\Gamma \times G$.

Proof. We have to check that $\mathcal{F}(\mathcal{R})$ is closed under conjugation and finite intersections.

Consider $K \in \mathcal{F}(\mathcal{R})$ and $(\gamma, g) \in \Gamma \times G$. Choose $(H,\alpha) \in \mathcal{R}$ with $K = K(H,\alpha)$. Then

$$\begin{aligned}
(\gamma,g)^{-1} \cdot K \cdot (\gamma,g) &= (\gamma,g)^{-1} \cdot K(H,\alpha) \cdot (\gamma,g) \\
&= \{(\gamma,g)^{-1} \cdot (h,\alpha(h)) \cdot (\gamma,g) \mid h \in H\} \\
&= \left\{(\gamma^{-1}h\gamma, g^{-1}\alpha(h)g) \mid h \in H\right\} \\
&= \left\{(c_\gamma(h), c_{g^{-1}} \circ \alpha \circ c_\gamma(c_{\gamma^{-1}}(h)) \mid h \in H\right\} \\
&= \left\{h', c_{g^{-1}} \circ \alpha \circ c_\gamma(h') \mid h' \in \gamma^{-1}H\gamma\right\} \\
&= K(\gamma^{-1}H\gamma, c_{g^{-1}} \circ \alpha \circ c_\gamma).
\end{aligned}$$

Since $(\gamma^{-1}H\gamma, c_{g^{-1}} \circ \alpha \circ c_\gamma)$ belongs to $\mathcal{R}$, we conclude that $(\gamma,g)^{-1} \cdot K \cdot (\gamma,g)$ belongs to $\mathcal{F}(\mathcal{R})$.

Consider $K_0, K_1 \in \mathcal{F}(\mathcal{R})$. Choose (H_i,α_i) in $\mathcal{R}$ with $K_i = K(H_i,\alpha_i)$ for $i = 0,1$. Define $H := \{h \in H_0 \cap H_1 \mid \alpha_0(h) = \alpha_1(h)\}$ and $\alpha\colon H \to G$ by $\alpha = \alpha_0|_H = \alpha_1|_H$. Then $K_0 \cap K_1 = K(H,\alpha)$ and $(H,\alpha) \in \mathcal{R}$. This implies $K_0 \cap K_1 \in \mathcal{F}(\mathcal{R})$. □

Remark 3.7 (Families)**.** We later will consider the classifying space $E_{\mathcal{F}}(\Gamma)$ of a family of subgroups of Γ. It can be defined without the condition that $\mathcal{F}$ is closed under finite intersections, being closed under conjugation is enough, but this extra condition is usually required to ensure that for two Γ-spaces X and Y whose isotropy groups belong to $\mathcal{F}$ also the isotropy groups of $X \times Y$ with the diagonal Γ-action belong to $\mathcal{F}$.

It is actually more convenient to require instead of the condition that $\mathcal{F}$ is closed under finite intersections that it is closed under subgroups. However, this rules out one important case, namely, the case of the family of open compact subgroups, which naturally occurs in the context of locally compact second countable totally disconnected groups Γ.

If $\mathcal{R}$ is closed under subgroups, i.e., for $(H,\alpha) \in \mathcal{R}$ and any subgroup $K \subseteq H$ we have $(K, \alpha|_K) \in \mathcal{R}$, then $\mathcal{F}(\mathcal{R})$ is closed under taking subgroups.

Remark 3.8 (Local representations and pullbacks)**.** Let $p\colon E \to B$ be a Γ-equivariant principal G-bundle and let $f\colon A \to B$ be a Γ-map. Let $\mathcal{R}$ be a family of local representations. Suppose that we have $\mathcal{R}(p) \subseteq \mathcal{R}$. Then we get $\mathcal{R}(f^*p) \subseteq \mathcal{R}$ for the pullback f^*p, provided for any $(H,\alpha) \in \mathcal{R}$ and any subgroup $K \subseteq H$ which occurs as isotropy group in A, we have $(K, \alpha|_K) \in \mathcal{R}$. This follows from Remark 3.3.

If we make the assumption that $\mathcal{R}$ is closed under subgroups, then f^*p automatically satisfies $\mathcal{R}(f^*p) \subseteq \mathcal{R}$ if $\mathcal{R}(p) \subseteq \mathcal{R}$ holds.

4. The Condition (S)

Definition 4.1 (Condition (S))**.** Given a topological group Γ and a (closed) subgroup $H \subseteq \Gamma$, we say that the pair (Γ, H) *satisfies Condition (S)* if the projection $\mathrm{pr}\colon \Gamma \to \Gamma/H$ has a local cross section, i.e., there is a quasi-regular open neighborhood U of $1H \in \Gamma/H$ together with a map $\sigma\colon U \to \Gamma$ such that $\mathrm{pr} \circ \sigma = \mathrm{id}_U$.

A topological group Γ *satisfies Condition (S)* if for any subgroup $H \subseteq \Gamma$ the pair (Γ, H) satisfies Condition (S).

The role of the Condition (S) is to ensure the following lemma.

Lemma 4.2. *Let $f\colon E \to \Gamma/H$ be a Γ-map for some subgroup $H \subseteq \Gamma$. Suppose that the pair (Γ, H) satisfies Condition (S).*

Then the Γ-map

$$u\colon \Gamma \times_H f^{-1}(1H) \to E, \quad (\gamma, e) \mapsto \gamma \cdot e$$

is a homeomorphism.

Proof. The map u is clearly a bijective map of sets. Now, the Condition (S) ensures that there is an open neighborhood $U \subseteq \Gamma/H$ of $1H$ and a map $s\colon U \to \Gamma$ whose composite with the projection $\mathrm{pr}\colon \Gamma \to \Gamma/H$ is the identity on U. Moreover, the open subsets $U \subseteq \Gamma/H$, $\mathrm{pr}^{-1}(U) \subseteq \Gamma$ and $f^{-1}(\mathrm{pr}^{-1}(U)) \subseteq E$ equipped with the subspace topologies are compactly generated, see Lemma 16.1 (i) and (ii). Define a map

$$v\colon f^{-1}(\mathrm{pr}^{-1}(U)) \to \mathrm{pr}^{-1}(U) \times_H f^{-1}(1H), \quad e \mapsto (s \circ \pi(e), s \circ \pi(e)^{-1} \cdot e),$$

where $\pi\colon f^{-1}(\mathrm{pr}^{-1}(U)) \to U$ is the map induced by $\mathrm{pr} \circ f$. Let

$$u|_{\mathrm{pr}^{-1}(U) \times_H f^{-1}(1H)}\colon \mathrm{pr}^{-1}(U) \times_H f^{-1}(1H) \to f^{-1}(\mathrm{pr}^{-1}(U))$$

be obtained by restricting u. Then $u|_{\mathrm{pr}^{-1}(U) \times_H f^{-1}(1H)} \circ v = \mathrm{id}_{f^{-1}(\mathrm{pr}^{-1}(U))}$ and $v \circ u|_{\mathrm{pr}^{-1}(U) \times_H f^{-1}(1H)} = \mathrm{id}_{\mathrm{pr}^{-1}(U) \times_H f^{-1}(1H)}$. Hence $u|_{\mathrm{pr}^{-1}(U) \times_H f^{-1}(1H)}$ is a homeomorphism. Since u is Γ-equivariant, $f^{-1}(\mathrm{pr}^{-1}(U)) \subseteq E$ and $\mathrm{pr}^{-1}(U) \times_H f^{-1}(1H) \subseteq \Gamma \times_H f^{-1}(1H)$ are open subsets, $\{\gamma \cdot U \mid \gamma \in \Gamma\}$ is an open covering of Γ/H and u is bijective, then the map u is a homeomorphism. □

Remark 4.3 (Condition (S) and principal bundle structure)**.** If the projection $\mathrm{pr}\colon \Gamma \to \Gamma/H$ is a principal H-bundle, then the local triviality implies that the pair (Γ, H) satisfies Condition (S). The converse is also true, namely, apply Lemma 5.1 (i) in the special case, where the role of Γ, H and G is played by Γ, H, and H, and $Z = \{\bullet\}$, and use the canonical Γ-homeomorphism $\Gamma \times_H H \xrightarrow{\cong} \Gamma, (\gamma, h) \mapsto \gamma \cdot h$.

The Condition (S) is satisfied in many cases.

Lemma 4.4. *(i) Suppose that Γ is completely regular, i.e., for any $x \in \Gamma$ and any neighborhood U of x in Γ, there exists a continuous function $f\colon \Gamma \to [0,1]$ with $f(x) = 0$ and $f(X \setminus U) = 1$. Then for any subgroup $H \subseteq \Gamma$ which is a compact Lie group, the pair (Γ, H) satisfies Condition (S);*

(ii) A topological group Γ satisfies Condition (S) if Γ is discrete, if Γ is a Lie group, or more generally, if Γ is locally compact and second countable and has finite covering dimension.

Proof. (i) This follows from [29], see also [21, Theorem 1.38 on page 27]. Notice that the conditions that Γ is completely regular (and hence regular) and H is compact imply that Γ/H is regular. Hence every open subset of Γ/H is quasi-regular, see Lemma 16.1 (iv).

(ii) This follows from [25]. The metric needed in [25] follows under our assumptions

from Theorem 17.4. Notice that the condition that Γ is locally compact implies that Γ/H is locally compact and hence that any open subset of Γ/H is quasi-regular, see Lemma 16.1 (i) (iv), and (v). $\square$

Example 4.5 (Kac-Moody groups)**.** Kac-Moody groups are not Lie groups but all their compact subgroups are Lie groups, see [14, Theorem 2.4]. Since they are completely regular, Lemma 4.4 (i) applies to them and hence they satisfy Condition (S).

5. Equivariant principal bundles over equivariant cells

In this section we want to analyze Γ-equivariant principal G-bundles over spaces of the type $\Gamma/H \times Z$ for some subgroup $H \subset \Gamma$ and space Z with trivial H-action. Later we will be mainly interested in the case $\Gamma/H \times D^n$.

Let H and G be two topological groups. Equip $\hom(H,G)$ with the subspace topology with respect to the inclusion $\hom(H,G) \subseteq \operatorname{map}(H,G)$ see Subsection 16.5.

Consider a space Z and a map $\sigma\colon Z \to \hom(H,G)$. We have the obvious right H-action on Γ and the left H-action on $Z \times G$ given by $h \cdot (z,g) := \bigl(z, \sigma(z)(h) \cdot g\bigr)$. Let

$$p_\sigma\colon \Gamma \times_H (Z \times G) \to \Gamma/H \times Z$$

be the map induced by the projection $Z \times G \to Z$. It is compatible with the left Γ-action on $\Gamma \times_H (Z \times G)$ given by $\gamma_0 \cdot \bigl(\gamma, (z,g)\bigr) = \bigl(\gamma_0\gamma, (z,g)\bigr)$ and the left Γ-action on $\Gamma/H \times Z$ given by $\gamma_0 \cdot (\gamma \cdot H, z) = (\gamma_0\gamma \cdot H, z)$. It is also compatible with the right G-action on $\Gamma \times_H (Z \times G)$ given by $\bigl(\gamma, (z,g)\bigr) \cdot g_0 = \bigl(\gamma, (z, gg_0)\bigr)$ and the trivial right G-action on $\Gamma \times_H Z$. The left Γ-action and the right G-action on $\Gamma \times_H (Z \times G)$ commute.

Lemma 5.1. *Suppose that the pair (Γ, H) satisfies Condition (S), see Definition 4.1, and let Z be a space. Then*

(i) The map $p_\sigma\colon \Gamma \times_H (Z \times G) \to \Gamma/H \times Z$ is a Γ-equivariant principal G-bundle;

(ii) A Γ-equivariant principal G-bundle $E \to \Gamma/H \times Z$ is isomorphic as Γ-equivariant principal G-bundle to p_σ for an appropriate map $\sigma\colon Z \to \hom(H,G)$, provided that the restriction of p to $\{1H\} \times Z$ is (after forgetting the H-action) a trivial principal G-bundle;

(iii) Given two maps $\sigma_0\colon Z \to \hom(H,G)$ and $\sigma_1\colon Z \to \hom(H,G)$, the Γ-equivariant principal G-bundles p_{σ_0} and p_{σ_1} are isomorphic, if and only if there is a map $\omega\colon Z \to G$ such that

$$\sigma_1(z)(g) = \omega(z) \cdot \sigma_0(z)(h) \cdot \omega(z)^{-1}$$

holds for all $h \in H$ and $z \in Z$;

(iv) Given a map $\sigma\colon Z \to \hom(H,G)$, the homomorphism $\rho_{(\gamma,(z,g))}\colon \Gamma_{(\gamma H,z)} \to G$ associated to p_σ in (3.2) for $\bigl(\gamma, (z,g)\bigr) \in \Gamma \times_H (Z \times G)$ is given by

$$\begin{aligned} \Gamma_{(\gamma H,z)} &= \gamma H \gamma^{-1}; \\ \rho_{(\gamma,(z,g))}(\gamma \cdot h \cdot \gamma^{-1}) &= g^{-1}\sigma(z)(h) \cdot g. \end{aligned}$$

Proof. (i) It remains to show that $p_\sigma\colon \Gamma \times_H (Z \times G) \to \Gamma/H \times Z$ is a principal G-bundle after forgetting the Γ-action. The Condition (S) ensures that there is a quasi-regular open neighborhood $U \subseteq \Gamma/H$ of $1H$ and a map $s\colon U \to \Gamma$ whose composite with the projection $\operatorname{pr}\colon \Gamma \to \Gamma/H$ is the identity. Notice that the open subsets $U \subseteq \Gamma/H$ and $\operatorname{pr}^{-1}(U) \subseteq \Gamma$ equipped with the subspace topology are compactly generated by Lemma 16.1 (i) and (ii).

Define $\overline{s}\colon \operatorname{pr}^{-1}(U) \to H$ by $\overline{s}(\gamma) = s \circ \operatorname{pr}(\gamma)^{-1} \cdot \gamma$. It has the property $\overline{s}(\gamma \cdot h) = \overline{s}(\gamma) \cdot h$ for all $\gamma \in p^{-1}(U)$ and $h \in H$. Define maps

$$\begin{aligned} \alpha\colon (U \times Z) \times G &\to \operatorname{pr}^{-1}(U) \times_H (Z \times G); \\ \beta\colon \operatorname{pr}^{-1}(U) \times_H (Z \times G); &\to (U \times Z) \times G, \end{aligned}$$

by

$$\begin{aligned} \alpha\bigl((\gamma H, z), g\bigr) &:= \bigl(\gamma, (z, \sigma(z)(\overline{s}(\gamma))^{-1} \cdot g)\bigr); \\ \beta\bigl(\gamma, (z, g)\bigr) &:= \bigl((\gamma H, z), \sigma(z)(\overline{s}(\gamma)) \cdot g\bigr). \end{aligned}$$

Then α and β are to one another inverse G-homeomorphisms. They induce isomorphisms of principal G-bundles from p_σ restricted to $U \times Z$ to the trivial principal G-bundle over $U \times Z$. One easily checks using Lemma 16.1 (ii) applied to the standard map $\Gamma/H \times Z = k(\Gamma/H \times_p Z) \to \Gamma/H \times_p Z$, see Subsections 16.2 and 16.3, and Lemma 16.1 (i) that $U \times Z$ is a quasi-regular open subset of $\Gamma/H \times Z$. Since $\gamma H \in \Gamma/H$ is contained in the open subset $\gamma \cdot U$ and p_σ is a Γ-equivariant map, we conclude that p_σ is locally trivial and hence a principal G-bundle.

(ii) In the sequel we identify the subspace $\{1\} \times Z$ of $\Gamma/H \times Z$ with Z. Define the Γ-map

$$u\colon \Gamma \times_H p^{-1}(Z) \to E, \quad (\gamma, e) \to \gamma \cdot e.$$

It is a homeomorphism by Lemma 4.2. It is compatible with the natural right G-actions and commutes with the left Γ-actions. The restriction $p|_{p^{-1}(Z)}\colon p^{-1}(Z) \to Z$ of p to $Z = \{1\} \times Z$ is a principal G-bundle over Z. By assumption $p|_{p^{-1}(Z)}$ is isomorphic to the trivial principal G-bundle over Z. Hence we can choose a G-homeomorphism

$$f\colon Z \times G \xrightarrow{\cong} p^{-1}(Z)$$

such that the composite $p|_{p^{-1}(Z)} \circ f$ is the canonical projection $Z \times G \to Z$. Let $\sigma\colon Z \to \hom(H, G)$ be defined by the map $\rho_{f(z,1)}$, see (3.2), i.e., σ is uniquely determined by $h \cdot f(z, 1) = f(z, 1) \cdot \sigma(z)(h)$. (It is continuous because of the properties of the category of compactly generated spaces listed in Subsection 16.4.) We define a left H-action on $Z \times G$ by $h \cdot (z, g) = \bigl(z, \sigma(z)(h) \cdot g\bigr)$. Then f is compatible with the left H-actions by the following calculation

$$\begin{aligned} f\bigl(h \cdot (z, g)\bigr) &= f\bigl(z, \sigma(z)(h) \cdot g\bigr) \\ &= f(z, 1) \cdot \sigma(z)(h) \cdot g \\ &= h \cdot f(z, 1) \cdot g \\ &= h \cdot f(z, g). \end{aligned}$$

We obtain a homeomorphism compatible with the obvious left Γ-actions and with the obvious right G-actions

$$\Gamma \times_H (Z \times G) \xrightarrow{\operatorname{id}_\Gamma \times_H f} \Gamma \times_H p^{-1}(Z) \xrightarrow{u} E.$$

This is an isomorphism of Γ-equivariant principal G-bundles from $p_\sigma\colon \Gamma \times_H (Z \times G) \to \Gamma/H \times Z$ to $p\colon E \to \Gamma/H \times Z$.

(iii) Let

$$f\colon \Gamma \times_H (Z \times G) \xrightarrow{\cong} \Gamma \times_H (Z \times G)$$

be an isomorphism of Γ-equivariant principal G-bundles from p_{σ_0} to p_{σ_1}. Notice that the left H-actions on $(Z \times G)$ in the source and the target are different, the first one depends on σ_0, the second on σ_1. There is precisely one map $\omega\colon Z \to G$ satisfying $f(1, (z, 1)) = \bigl(1, (z, \omega(z))\bigr)$. (It is continuous because of the properties of the category of compactly generated spaces listed in Subsection 16.4.) Since f is

compatible with the natural left Γ-actions and with the natural right G-actions, we get for $h \in H$ and $z \in Z$

$$\begin{aligned}
\big(1,(z,\sigma_1(z)(h)\cdot\omega(z))\big) &= \big(1,h\cdot(z,\omega(z))\big)\\
&= \big(h,(z,\omega(z))\big)\\
&= h\cdot\big(1,(z,\omega(z))\big)\\
&= h\cdot f\big(1,(z,1)\big)\\
&= f\big(h\cdot(1,(z,1))\big)\\
&= f\big(h,(1,z)\big)\\
&= f\big(1,h\cdot(z,1)\big)\\
&= f\big(1,(z,\sigma_0(z)(h)\big)\\
&= f\big(1,(z,1)\big)\cdot\sigma_0(z)(h)\\
&= (1,z,\omega(z))\cdot\sigma_0(z)(h)\\
&= \big(1,(z,\omega(z)\cdot\sigma_0(z)(h))\big).
\end{aligned}$$

This implies $\sigma_1(z)(h)\cdot\omega(z) = \omega(z)\cdot\sigma_0(z)(h)$ for all $h \in H$ and $z \in Z$.

For the converse, the map

$$\begin{aligned}
\Gamma\times_H(Z\times G) &\to \Gamma\times_H(Z\times G)\\
(\gamma,(z,g)) &\mapsto (\gamma,(z,w(z)g)))
\end{aligned}$$

satisfies the desired properties.

(iv) Consider $(\gamma,(z,g)) \in \Gamma\times_H(Z\times G)$. One easily checks

$$\Gamma_{(\gamma H,z)} = \gamma\cdot\Gamma_{(1H,z)}\cdot\gamma^{-1} = \gamma\cdot H\cdot\gamma^{-1}.$$

We compute for $h \in H$ in $\Gamma\times_H(Z\times G)$

$$\begin{aligned}
(\gamma\cdot h\cdot\gamma^{-1})\cdot\big(\gamma,(z,g)\big) &= \big(\gamma\cdot h,(z,g)\big)\\
&= \big(\gamma,h\cdot(z,g)\big)\\
&= \big(\gamma,(z,\sigma(z)(h)\cdot g)\big)\\
&= \big(\gamma,(z,g\cdot(g^{-1}\cdot\sigma(z)(h)\cdot g))\big)\\
&= \big(\gamma,(z,g)\big)\cdot(g^{-1}\cdot\sigma(z)(h)\cdot g).
\end{aligned}$$

This finishes the proof of Lemma 5.1. □

6. Discussion of homotopy invariance and Condition (H)

Since we want to have a bundle theory for which there exists a universal bundle, we have to ensure homotopy invariance, i.e., if $f_0, f_1\colon B_0 \to B_1$ are Γ-maps and $p\colon E \to B_1$ is a Γ-equivariant principal G-bundle, we want to arrange that the pullbacks f_0^*p and f_1^*p are isomorphic as Γ-equivariant principal G-bundles.

Let Z be a contractible CW-complex and $H \subseteq \Gamma$ be a subgroup. Equip Z with the trivial Γ-action. Then the projection $\mathrm{pr}\colon \Gamma/H\times Z \to \Gamma/H$ is a Γ-homotopy equivalence. Hence in order to guarantee homotopy invariance, we must ensure that every Γ-equivariant principal G-bundle $p\colon E \to \Gamma/H\times Z$ over $\Gamma/H\times Z$ is isomorphic to pr^*E' for some Γ-equivariant principal G-bundle on $p'\colon E' \to \Gamma/H$ over Γ/H.

For $\alpha \in \hom(H,G)$ the *centralizer of α in G* is defined to be the subgroup of G given by

$$C_G(\alpha) := \{g \in G \mid g\alpha(h)g^{-1} = \alpha(h) \text{ for all } h \in H\}.$$

Definition 6.1 (Condition (H))**.** A family $\mathcal{R}$ of local representations in the sense of Definition 3.1 satisfies Condition (H) if the following holds for every $(H,\alpha) \in \mathcal{R}$:

(i) The path component of α in $\hom(H,G)$ is contained in $\{c_g \circ \alpha \mid g \in G\}$;
(ii) The pair $(G, C_G(\alpha))$ satisfies Condition (S) introduced in Definition 4.1;
(iii) The pair (Γ, H) satisfies Condition (S) introduced in Definition 4.1;
(iv) The canonical map

$$\iota_\alpha \colon G/C_G(\alpha) \to \hom(H,G), \quad gC_G(\alpha) \mapsto c_g \circ \alpha$$

is a homeomorphism onto its image.

Lemma 6.2. *Suppose that the family $\mathcal{R}$ of local representations satisfies Condition (H). Let Z be a (non-equivariant) contractible CW-complex and let $p\colon E \to \Gamma/H \times Z$ be a Γ-equivariant principal G-bundle with $\mathcal{R}(p) \subseteq \mathcal{R}$.*

Then p is isomorphic to $\mathrm{pr}^* E'$ for a Γ-equivariant principal G-bundle $p'\colon E' \to \Gamma/H$ for the projection $\mathrm{pr}\colon \Gamma/H \times Z \to \Gamma/H$, or, equivalently, there exists an element (H,α) in $\mathcal{R}$ such that p is isomorphic to the Γ-equivariant principal bundle

$$p_\alpha \colon (\Gamma \times_H G) \times Z \to \Gamma/H \times Z, \quad \big((\gamma, g), z\big) \mapsto (\gamma H, z).$$

Proof. Since Z is contractible, by Lemma 1.3, the restriction bundle $p|_{\{1H\}\times Z} : E|_{\{1H\}\times Z} \to \{1H\} \times Z$ is trivializable. We can then apply Lemma 5.1 (ii) and we obtain that p is isomorphic to p_σ for an appropriate map $\sigma\colon Z \to \hom(H,G)$. Since Z is path connected, the image of σ is contained in a path component of $\alpha \in \hom(H,G)$ if we take $\alpha = \sigma(z)$ for some $z \in Z$. Lemma 5.1 (iv) implies that (H,α) belongs to $\mathcal{R}(p)$ and hence to $\mathcal{R}$. Because of Condition (H) the image of $\sigma\colon Z \to \hom(H,G)$ is contained in the image of $\iota_\alpha\colon G/C_G(\alpha) \to \hom(H,G)$. Since ι_α is a homeomorphism onto its image by Condition (H), we can find a map $\overline{\omega}\colon Z \to G/C_G(\alpha)$ with $\iota_\alpha \circ \overline{\omega} = \sigma$. Because of Condition (H) and Remark 4.3, the projection $G \to G/C_G(\alpha)$ is a principal $C_G(\alpha)$-bundle. Hence its pullback with $\overline{\omega}$ is a principal $C_G(\alpha)$-bundle over the contractible CW-complex Z and hence has a section by Lemma 1.3. Thus we can find a map $\omega\colon Z \to G$ whose composite with the projection $G \to G/C_G(\alpha)$ is $\overline{\omega}$. This implies $\sigma = c_\omega \circ \alpha$. Now apply Lemma 5.1 (iii). □

Theorem 6.3. *Let $\mathcal{R}$ be a family of local representations. Then it satisfies Condition (H) if the following conditions are satisfied:*

(i) The group Γ is locally compact, second countable and has finite covering dimension, e.g., is a Lie groups, or Γ is completely regular and all compact subgroups of Γ are Lie groups, e.g., is a Kac-Moody group;
(ii) The group G is locally compact, second countable and has finite covering dimension, e.g., is a Lie group;
(iii) The group G is almost connected (see Definition 12.3);
(iv) For every element (H,α) the group H is a compact group.

Proof. Condition (i) appearing in Definition 6.1 is proved in Theorem 17.1.

Conditions (ii) and (iii) appearing in Definition 6.1 follow from Lemma 4.4 (ii).

Condition (iv) appearing in Definition 6.1 is proved in Theorem 17.2. □

7. The Slice Theorem for equivariant CW-complexes

In this section we prove the following Slice Theorem for Γ-CW-complexes, generalizing [21, Theorem 1.37].

Theorem 7.1 (Slice Theorem)**.** *Let G be a (compactly generated) topological group and let X be a G-CW-complex. Consider $x \in X$ together with a G_x-invariant neighborhood V_x. Suppose that the pair (G, G_x) satisfies Condition (S), see Definition 4.1.*

Then there exists a G_x-invariant subset S_x of x with the following properties:

(i) *S_x inherited with the subspace topology is compactly generated;*
(ii) *The closure of S_x is contained in V_x;*
(iii) *The inclusion $\{x\} \to S_x$ is a G_x-homotopy equivalence;*
(iv) *The set $U := G \cdot S_x$ is a quasi-regular G-invariant open subset of X and the map*

$$G \times_{G_x} S_x \xrightarrow{\cong} U, \quad (g,s) \mapsto g \cdot s$$

is a G-homeomorphism.

Proof. Let $n_x \geq 0$ be the integer for which $x \in X_{n_x}$ and $x \notin X_{n_x-1}$. We construct inductively for $n = n_x, n_x+1, \ldots$ open G-invariant subsets $U[n] \subseteq X$ such that the following conditions hold if we put $V = G \cdot V_x$:

(1) We have $U[n] \subseteq X_n$, $U[n] \subseteq U[n+1]$, $\overline{U[n]} \subseteq V$ and $x \in U[n]$ for all $n \geq n_x$;
(2) For each $n \geq n_x$ there is a G-map

$$r[n+1] \colon U[n+1] \to U[n]$$

satisfying $r[n+1] \circ i[n+1] = \mathrm{id}_{U[n]}$, where $i[n+1] \colon U[n] \to U[n+1]$ is the inclusion;
(3) For each $n \geq n_x$ there is a G-homotopy

$$h[n+1] \colon U[n+1] \times [0,1] \to U[n+1]$$

satisfying

$$\begin{array}{rcll} h[n+1]_t & = & \mathrm{id}_{U[n+1]} & \text{for } 0 \leq t \leq (n+3)^{-1}; \\ h[n+1]_t & = & i[n+1] \circ r[n+1] & \text{for } (n+2)^{-1} \leq t \leq 1; \\ h[n+1](z,t) & = & z & \text{for } z \in U[n], t \in [0,1]; \\ r[n+1] \circ h[n+1] & = & r[n+1] \circ \mathrm{pr}_{n+1}, & \end{array}$$

where $\mathrm{pr}_{n+1} \colon U[n+1] \times [0,1] \to U[n+1]$ is the projection;
(4) There is a G-map

$$r[n_x] \colon U[n_x] \to Gx$$

such that $r[n_x] \circ i[n_x] = \mathrm{id}_{Gx}$ holds, where $Gx := \{gx \mid g \in G\}$ and $i[n_x] \colon Gx \to U[n_x]$ is the inclusion;
(5) There is a G-homotopy

$$h[n_x] \colon U[n_x] \times [0,1] \to U[n_x]$$

satisfying

$$\begin{array}{rcll} h[n_x]_t & = & \mathrm{id}_{U_{n_x}} & \text{for } t \leq (n_x+2)^{-1}; \\ h[n_x]_t & = & i[n_x] \circ r[n_x] & \text{for } t \geq (n_x+1)^{-1}; \\ h[n_x](gx,t) & = & gx & \text{for } gx \in Gx, t \in [0,1]; \\ r[n_x] \circ h[n_x] & = & r[n_x] \circ \mathrm{pr}_{n_x}, & \end{array}$$

where $\mathrm{pr}_{n_x} \colon U[n_x] \times [0,1] \to U[n_x]$ is the projection.

Recall that we have for each $n \geq 0$ a G-pushout of the form

$$\begin{array}{ccc} \coprod_{i \in I_{n+1}} G/H_i \times S^n & \xrightarrow{\coprod_{i \in I_n} q_i^{n+1}} & X_n \\ \downarrow & & \downarrow \\ \coprod_{i \in I_{n+1}} G/H_i \times D^{n+1} & \xrightarrow{\coprod_{i \in I_n} Q_i^{n+1}} & X_{n+1} \end{array}$$

Next we explain the beginning of the induction. Since x belongs to X_{n_x} but not to X_{n_x-1}, we can find $i \in I_{n_x}$ and $(\gamma H_i, y) \in G/H_i \times \left(D^{n_x} \setminus S^{n_x-1}\right)$ satisfying $Q_i^{n_x}(gH_i, y) = x$. Choose $\delta_i > 0$ such that $G/H_i \times \overline{B_\delta(y)}$ is contained in both

$G/H_i \times D^{n_x} \setminus S^{n_x-1}$ and $(Q_i^{n_x})^{-1}(V_x \cap X_{n_x})$, where $B_\delta(y)$ is the ball of radius δ around y. Define a G-map

$$r[n_x]'\colon G/H_i \times B_\delta(y) \to Gy, \quad (gH_i, z) \mapsto gy,$$

and a G-homotopy

$$h[n_x]'\colon G/H_i \times B_\delta(y) \times [0,1] \to G/H_i \times B_\delta(y)$$

by sending (gH_i, z, t) to

$$\begin{array}{ll} (gH_i, z) & \text{for } t \leq (n_x+2)^{-1}; \\ \left(gH_i, \frac{t-(n_x+2)^{-1}}{(n_x+1)^{-1}-(n_x+2)^{-1}} \cdot y + \left(1 - \frac{t-(n_x+2)^{-1}}{(n_x+1)^{-1}-(n_x+2)^{-1}}\right) \cdot z\right) & \text{for} (n_x+2)^{-1} \leq t \leq (n_x+1)^{-1}; \\ (gH_i, y) & \text{for } t \geq (n_x+1)^{-1}. \end{array}$$

Now define

$$U[n_x] = Q_i^{n_x}\big(G/H_i \times B_\delta(y)\big).$$

Let

$$h[n_x]\colon U[n_x] \times [0,1] \to U[n_x]$$

be the G-homotopy uniquely determined by the property that for every element $(gH_i, z, t) \in G/H_i \times B_\delta(y) \times [0,1]$ we have $Q_i^{n_x} \circ h[n_x]'(gH_i, z, t) = h[n_x](Q_i^{n_x}(gH_i, z), t)$ holds. Define the G-map

$$r[n_x]\colon U[n_x] \to Gx$$

analogously using $r[n_x]'$. One easily checks that $U[n_x]$, $r[n_x]$ and $h[n_x]$ have the desired properties.

Next we explain the induction step from $n \geq n_x$ to $n+1$. For a real number $\epsilon \in (0,1)$ define the subspace

$$S^n[\epsilon] := \{t \cdot z \mid t \in (1-\epsilon, 1], z \in S^n\} \subseteq D^{n+1},$$

a map

$$p[\epsilon]\colon S^n[\epsilon] \to S^n, \quad t \cdot z \mapsto z,$$

and a homotopy

$$l[\epsilon]\colon S^n[\epsilon] \times [0,1] \to S^n[\epsilon], \quad (t \cdot z, s) \mapsto (t \cdot (1-s) + s) \cdot z.$$

If $j[\epsilon]\colon S^n \to S^n[\epsilon]$ is the inclusion, then $l[\epsilon]_0 = \mathrm{id}_{S^n[\epsilon]}$ and $l[\epsilon]_1 = i[\epsilon] \circ p[\epsilon]$. Roughly speaking, the homotopy $l[\epsilon]$ pushes $S^n[\epsilon]$ radially to S^n.

Roughly speaking, we will obtain $U[n+1]$ from $U[n]$ by a thickening into the interior of the various equivariant $(n+1)$-cells such that the thickening is small enough to ensure that the closure of $U[n+1]$ stays within V. In detail: Consider $i \in I_{n+1}$. Then we get inclusions of G-invariants subsets of $\Gamma/H_i \times S^n$.

$$(q_i^{n+1})^{-1}(U[n]) \subseteq (q_i^{n+1})^{-1}(\overline{U[n]}) \subseteq (q_i^{n+1})^{-1}(V)$$

because of the induction hypothesis. Notice that $(Q_i^{n+1})^{-1}(V)$ is a G-invariant open subset of $G/H_i \times D^{n+1}$ containing the closed G-invariant subset $(q_i^{n+1})^{-1}(\overline{U[n]})$. Let $\mathrm{pr}\colon G/H_i \times D^{n+1} \to D^{n+1}$ be the projection. Then $\mathrm{pr}\big((Q_i^{n+1})^{-1}(V)\big)$ is an open subset of D^{n+1} containing the closed subset $\mathrm{pr}\big((q_i^{n+1})^{-1}(\overline{U[n]})\big)$. Since S^n is compact, we can choose $\epsilon_i > 0$ such that

$$p[2 \cdot \epsilon_i]^{-1}\left(\mathrm{pr}\big((q_i^{n+1})^{-1}(\overline{U[n]})\big)\right) \subseteq \mathrm{pr}\left((Q_i^{n+1})^{-1}(V)\right)$$

holds. Define

$$U[n+1] := U[n] \cup \bigcup_{i \in I_{n+1}} Q_i^{n+1}\left(\mathrm{pr}^{-1}\left(p[\epsilon_i]^{-1}\left(\mathrm{pr}\big((q_i^{n+1})^{-1}(\overline{U[n]})\big)\right)\right)\right).$$

One easily checks that $U[n+1]$ is an open G-invariant subset of X_{n+1} such that $\overline{U[n+1]} \subseteq V$ holds. The various G-maps $\mathrm{id}_{G/H_i} \times p[\epsilon_i] \colon G/H_i \times S^n[\epsilon] \to G/H_i \times S^n$ fit together to a G-map

$$r[n+1] \colon U[n+1] \times [0,1] \to U[n+1]$$

such that $r[n+1] \circ i[n+1] = \mathrm{id}_{U[n]}$ holds for the inclusion $i[n+1] \colon U[n] \to U[n+1|$. The various G-homotopies $\mathrm{id}_{G/H_i} \times l[\epsilon_i] \colon G/H_i \times S^n[\epsilon] \to G/H_i \times S^n$ fit together to a G-homotopy

$$h[n+1]' \colon U[n+1] \times [0,1] \to U[n+1]$$

such that $h[n+1]'_0 = \mathrm{id}_{U[n+1]}$, the image of $h[n+1]_1$ lies in $U[n+1] \cap X_n = U[n]$, the restriction of $h[n+1]'_t \colon U[n+1] \to U[n+1]$ to $U[n]$ is the inclusion $U[n] \to U[n+1]$ for all $t \in [0,1]$ and $r[n+1] \circ h[n+1] = r[n+1] \circ \mathrm{pr}_{n+1}$ for the projection $\mathrm{pr}_{n+1} \colon U[n+1] \times [0,1] \to U[n+1]$. Define a map

$$\tau \colon [0,1] \to [0,1], \quad t \mapsto \begin{cases} 0 & 0 \le t \le (n+3)^{-1}; \\ \frac{t-(n+3)^{-1}}{(n+2)^{-1}-(n+3)^{-1}} & (n+3)^{-1} \le t \le (n+2)^{-1}; \\ 1 & (n+2)^{-1} \le t. \end{cases}$$

One easily checks that the G-homotopy

$$h[n+1] := h[n+1]' \circ \left(\mathrm{id}_{U[n+1]} \times \tau\right) \colon U[n+1] \times [0,1] \to U[n+1]$$

has the desired properties. This finishes the induction step.

Now define

$$U := \bigcup_{n \ge n_x} U[n]$$

Since each $U[n] \subseteq X_n$ is an open G-invariant subset of X_n and X has the weak topology with respect to the filtration by its skeletons $\{X_n \mid n \ge -1\}$, the subset U of X is open and G-invariant. Since $\overline{U_n} \subseteq V$ holds for all $n \ge 0$, we conclude $\overline{U} \subseteq V$. Define the G-map

$$\rho[n] := U[n] \to Gx$$

to be the composite $U[n] \xrightarrow{r[n]} U[n-1] \xrightarrow{r[n-1]} \cdots \xrightarrow{r[n_x+1]} U[n_x] \xrightarrow{r[n_x]} Gx$. Since $\rho[n+1]$ restricted to $U[n]$ is $\rho[n]$, we obtain a G-map

$$\rho \colon U \to Gx.$$

Let $\iota \colon Gx \to U$ be the inclusion. Obviously $\rho \circ \iota = \mathrm{id}_{Gx}$. Next we define inductively for $n = n_x, n_x+1, \ldots$ G-homotopies

$$k[n] \colon U[n] \times [0,1] \to U[n]$$

such that $k[n]_t = \mathrm{id}_{U[n]}$ holds for $t \in [0,(n+2)^{-1}]$, we have $k[n]_1 = \rho[n]$, the composite $\rho[n] \circ k[n] \colon U[n] \times [0,1] \to Gx$ factorizes over the projection $U[n] \times [0,1] \to U[n]$ to $\rho[n] \colon U[n] \to Gx$, and the following diagram commutes

$$\begin{array}{ccc} U[n] \times [0,1] & \xrightarrow{k[n]} & U[n] \\ \downarrow{\scriptstyle i[n+1] \times \mathrm{id}_{[0,1]}} & & \downarrow{\scriptstyle i[n+1]} \\ U[n+1] \times [0,1] & \xrightarrow{k[n+1]} & U[n+1] \end{array}$$

In the induction beginning put $k[n_x] = h[n_x]$. In the induction step from $n \ge n_x$ to $n+1$, we define $k[n+1]|_{U[n+1] \times [0,(n+2)^{-1}]}$ to be $h[n+1]|_{U[n+1] \times [0,(n+2)^{-1}]}$, and

define $k[n+1]|_{U[n+1]\times[(n+2)^{-1},1]}$ to be the composite

$$U[n+1]\times[(n+2)^{-1},1] \xrightarrow{r[n+1]\times \mathrm{id}_{[(n+2)^{-1},1]}} U[n]\times[(n+2)^{-1},1] \xrightarrow{k[n]|_{U[n]\times[(n+2)^{-1},1]}} U[n] \xrightarrow{i[n+1]} U[n+1].$$

The homotopies $k[n]$ for $n = n_x, n_x+1, \dots$ fit together to a G-homotopy

$$k\colon U\times[0,1]\to U$$

with $k_0 = \mathrm{id}_U$ and $k_1 = \iota\circ\rho$. To summarize, we have $\rho\circ\iota = \mathrm{id}_{Gx}$ and $k\colon \iota\circ\rho \simeq_G \mathrm{id}_U$, and the composite $\rho\circ k\colon U\times[0,1]\to Gx$ factorizes over the projection $U\times[0,1]\to U[n]$ to ρ.

Let S_x be the preimage of x under the map $\rho\colon U\to Gx$. The G-map $G/G_x \to Gx,\ g\mapsto gx$ is obviously a G-homeomorphism. Since the pair (G,G_x) satisfies Condition (S) by assumption, Lemma 4.2 implies that the map

$$u\colon G\times_{G_x} S_x \xrightarrow{\cong} U, \quad (g,s)\mapsto gs$$

is a G-homeomorphism. The restriction of ι to $\{x\}$ is the inclusion $i\colon\{x\}\to S_x$. The restriction of ρ to S_x is a G_x-homotopy inverse of i since the composite $\rho\circ k\colon U\times[0,1]\to Gx$ factorizes over the projection $U\times[0,1]\to U[n]$ to ρ.

Since V is by definition $G\cdot V_x$ and the closure of U is contained in V, the closure of S_x is contained in V_x.

We conclude from Lemma 16.1 (vi) that $U = \mathrm{pr}^{-1}(\mathrm{pr}(U))$ is quasi-regular and hence U equipped with the subspace topology is compactly generated. Since $S_x\subseteq U$ is closed, also S_x is compactly generated. This finishes the proof of Theorem 7.1. □

8. Equivariant principal bundles versus equivariant CW-complexes

In this section we prove one of our main technical results saying that a Γ-equivariant principal G-bundle $p\colon E\to B$ is the same as a $\Gamma\times G$-CW-complex E with a special structure of its $\Gamma\times G$-isotropy groups provided that $\mathcal{R}(p)\subseteq\mathcal{R}$ holds for a given family of local representations $\mathcal{R}$ satisfying Condition (H).

Theorem 8.1. *Let $\mathcal{R}$ be a family of local representations for (Γ,G) satisfying Condition (H) introduced in Definition 6.1.*

(i) Let $p\colon E\to B$ be a Γ-equivariant principal G-bundle with $\mathcal{R}(p)\subseteq\mathcal{R}$ over a Γ-CW-complex B. Then E_l is a $\Gamma\times G$-CW-complex whose isotropy groups belong to the family $\mathcal{F}(\mathcal{R})$ introduced in Lemma 3.6;

(ii) Let E be a left $\Gamma\times G$-CW-complex whose isotropy groups belong to $\mathcal{F}(\mathcal{R})$. Then $p\colon E_r\to E_r/G$ is a Γ-equivariant principal G-bundle with $\mathcal{R}(p)\subseteq\mathcal{R}$.

Proof. (i) Let B_n be the n-skeleton of the Γ-CW-complex structure on B. Put $E_n := p^{-1}(B_n)$. Let $p_n\colon E_n\to B_n$ be the Γ-equivariant principal G-bundle obtained by restricting p to B_n. Next we will show that the filtration of E by the E_n-s induces the structure of a $\Gamma\times G$-CW-structure on E_l whose isotropy groups belong to $\mathcal{F}(\mathcal{R})$.

Fix $n\geq 0$. Since B is a Γ-CW-complex, there exists a Γ-pushout

$$\begin{array}{ccc} \coprod_{i\in I}\Gamma/H_i\times S^{n-1} & \xrightarrow{\coprod_{i\in I} q_i} & B_{n-1} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle J} \\ \coprod_{i\in I}\Gamma/H_i\times D^n & \xrightarrow[\coprod_{i\in I} Q_i]{} & B_n \end{array}$$

where j and J are the inclusions. Consider the square obtained by the pullback construction applied to $p_n\colon E_n \to B_n$.

$$\begin{array}{ccc} \coprod_{i\in I} q_i^* J^* E_n = j^* Q_i^* E_n & \xrightarrow{\coprod_{i\in I} \overline{q_i}} & J^* E_n \\ \Big\downarrow{\scriptstyle \overline{j}} & & \Big\downarrow{\scriptstyle \overline{J}} \\ \coprod_{i\in I} Q_i^* E_n & \xrightarrow[\coprod_{i\in I} \overline{Q_i}]{} & E_n \end{array}$$

This is a diagram of $\Gamma \times G$-spaces, the various left $\Gamma \times G$-actions come from the left $\Gamma \times G$-action on E_l. It is a $\Gamma \times G$-pushout by Lemma 16.5 (i) and (ii).

We conclude from Remark 3.3 that $\mathcal{R}(Q_i^* E_n) \subseteq \mathcal{R}$ holds. Lemma 6.2 implies that the Γ-equivariant principal G-bundle $Q_i^* E_n$ is isomorphic to the Γ-equivariant principal G-bundle $p_\alpha\colon (\Gamma \times_H G) \times D^n \to \Gamma/H \times D^n$ for some element $(H, \alpha) \in \mathcal{R}$. Hence there exists a $\Gamma \times G$-homeomorphism of left $\Gamma \times G$-pairs

$$(\Gamma \times G)/H_i' \times (D^n, S^{n-1}) \quad \xrightarrow{\cong} \quad \big((Q_i^* E_n)_l, (q_i^* J^* E_n)_l\big) \tag{8.2}$$

for an appropriate subgroup $H_i' \subseteq \Gamma \times G$ belonging to $\mathcal{F}(\mathcal{R})$.

It remains to show that E has the weak topology with respect to the filtration given by the E_n-s. This follows from Lemma 16.6 (i) and (ii). This finishes the proof of assertion (i).

(ii) Let E be a left $\Gamma \times G$-CW-complex whose isotropy groups belong to $\mathcal{F}(\mathcal{R})$. Firstly, we show that E/G is a Γ-CW-complex. Consider $e \in E$. There exists $(H, \alpha) \in \mathcal{R}$ such that $(\Gamma \times G)_e = K(H, \alpha) = \{(h, \alpha(h) \mid h \in H\}$. Hence the image of $(\Gamma \times G)_e$ under the projection $\Gamma \times G \to \Gamma$ is H and hence closed. Lemma 16.7 implies that E/G is a Γ-CW-complex.

Next we show that $p\colon E_r \to E/G$ is a principal G-bundle. Consider $b \in E/G$. Choose $e \in E$ with $p(e) = b$. From the Slice Theorem 7.1 we obtain a $(\Gamma \times G)_e$-invariant (compactly generated) subspace S of E containing e such that the map

$$f\colon (\Gamma \times G) \times_{(\Gamma \times G)_e} S \to (\Gamma \times G) \cdot S, \quad \big((\gamma, g), s\big) \mapsto (\gamma, g) \cdot s$$

is a $(\Gamma \times G)$-homeomorphism and $(\Gamma \times G) \cdot S$ is an open $\Gamma \times G$-invariant (compactly generated) subset of E. Put $V := p((\Gamma \times G) \cdot S)$. Choose $(H, \alpha) \in \mathcal{R}$ such that

$$(\Gamma \times G)_e = K(H, \alpha) = \{(h, \alpha(h) \mid h \in H\}.$$

We equip S with the left H-action given by $h \cdot s := (h, \alpha(h)) \cdot s$. We obtain a Γ-map

$$u\colon \Gamma \times_H S \to V, \quad (\gamma, s) \to p \circ f\big((\gamma, e), s\big).$$

Define the $\Gamma \times G$-map

$$q\colon (\Gamma \times G) \times_{(\Gamma \times G)_e} S \to \Gamma \times_H S, \quad \big((\gamma, g), s\big) \mapsto (\gamma, s),$$

where G acts trivially on the target. Then V is an open Γ-invariant neighborhood of b in E/G which is quasi-regular by Lemma 16.1 (vi). Moreover, the following diagram of right G-spaces commutes, where G acts trivial on $\Gamma \times_H S$ and V, has a G-homeomorphism as upper horizontal map and identifications as vertical maps

$$\begin{array}{ccc} \big((\Gamma \times G) \times_{(\Gamma \times G)_e} S\big)_r & \xrightarrow[\cong]{f} & p^{-1}(V) \\ \Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle p|_{p^{-1}(V)}} \\ \Gamma \times_H S & \xrightarrow[\cong]{u} & V \end{array} \tag{8.3}$$

Since u is a bijective identification, it is a homeomorphisms. Hence it suffices to show that q is a principal G-bundle.

Equip $S \times G$ with the left H-action given by $h \cdot (s,g) := \big(h \cdot s, \alpha(h) \cdot g\big)$. Then

$$p_\alpha \colon \Gamma \times_H (S \times G) \to \Gamma \times_H S, \quad \big(\gamma, (s,g)\big) \mapsto (\gamma, s)$$

is a well-defined map which is compatible with the obvious left Γ-actions on the source and the target and with the obvious right G-action on the source and the trivial right G-action on the target. Define a map

$$\xi \colon \Gamma \times_H (S \times G) \xrightarrow{\cong} \big((\Gamma \times G) \times_{(\Gamma \times G)_e} S\big)_r, \quad \big(\gamma, (s,g)\big) \mapsto \big((\gamma, g^{-1}), s\big).$$

It is well-defined by the following calculation for $\gamma \in \Gamma$, $s \in S$ and $g \in G$

$$\begin{aligned} \big((\gamma h^{-1}, (\alpha(h) \cdot g)^{-1}), h \cdot s\big) &= \big((\gamma \cdot h^{-1}, g^{-1}\alpha(h)^{-1}), h \cdot s\big) \\ &= \big((\gamma, g^{-1}) \cdot (h, \alpha(h))^{-1}, (h, \alpha(h)) \cdot s\big) \\ &= \big((\gamma, g^{-1}), s\big). \end{aligned}$$

The map ξ is a homeomorphisms, an inverse is given by $\big((\gamma, g), s\big) \mapsto \big(\gamma, (s, g^{-1})\big)$. It is compatible with the left Γ-actions and the right G-actions. We obtain a commutative diagram of spaces with left Γ-actions and right G-actions, where the spaces in the lower left and lower right corner carry trivial G-actions

$$\begin{tikzcd} \Gamma \times_H (S \times G) \arrow[r, "\xi", "\cong"'] \arrow[d, "p_\alpha"'] & \big((\Gamma \times G) \times_{(\Gamma \times G)_e} S\big)_r \arrow[d, "u \circ q"] \\ \Gamma \times_H S \arrow[r, "u", "\cong"'] & V \end{tikzcd} \tag{8.4}$$

Hence it remains to show that $p_\alpha \colon \Gamma \times_H (S \times G) \to \Gamma/H \times S$ is a principal G-bundle after forgetting the Γ-action.

The Condition (S) ensures that there is an open quasi-regular neighborhood $U \subseteq \Gamma/H$ of $1H$ and a map $s \colon U \to \Gamma$ whose composite with the projection $\operatorname{pr} \colon \Gamma \to \Gamma/H$ is the identity. Define $\overline{s} \colon \operatorname{pr}^{-1}(U) \to H$ by $\overline{s}(\gamma) = s \circ \operatorname{pr}(\gamma)^{-1} \cdot \gamma$. It has the property $\overline{s}(\gamma \cdot h) = \overline{s}(\gamma) \cdot h$ for all $\gamma \in p^{-1}(U)$. Define maps

$$\begin{aligned} \mu \colon (\operatorname{pr}^{-1}(U) \times_H S) \times G &\to \operatorname{pr}^{-1}(U) \times_H (S \times G); \\ \nu \colon \operatorname{pr}^{-1}(U) \times_H (S \times G) &\to (\operatorname{pr}^{-1}(U) \times_H S) \times G, \end{aligned}$$

by

$$\begin{aligned} \mu\big((\gamma, s), g\big) &:= \big(\gamma, (s, \alpha(\gamma)^{-1} \cdot g)\big); \\ \nu\big(\gamma, (s,g)\big) &:= \big((\gamma, s), \alpha(\gamma) \cdot g\big). \end{aligned}$$

Then μ and ν are to one another inverse G-homeomorphism. They induce a trivialization of p_α restricted to $\operatorname{pr}^{-1}(U) \times_H S$ to the trivial principal G-bundle over $\operatorname{pr}^{-1}(U) \times_H S$.

Since $\gamma H \in \Gamma/H$ is contained in the open subset $\gamma \cdot U$ and p_σ is a Γ-equivariant map, we conclude that p_α is locally trivial and hence a principal G-bundle.

This finishes the proof that $p \colon E_r \to E/G$ is a principal G-bundle. Since p is obviously Γ-equivariant, p is a Γ-equivariant principal G-bundle. It remains to prove $\mathcal{R}'(p) \subseteq \mathcal{R}$ since then $\mathcal{R}(p) \subseteq \mathcal{R}$ holds.

Consider $e \in E$. Its isotropy subgroup $(\Gamma \times G)_e$ belongs to $\mathcal{F}(\mathcal{R})$ by assumption. Hence there exists $(H, \alpha) \in \mathcal{R}$ with $(\Gamma \times G)_e = K(H, \alpha) := \{(h, \alpha(h) \mid h \in H\}$. This implies that $\Gamma_{p(e)} = H$. We conclude that $\rho_e \colon H \to G$ is given by α since for $h \in H$ we have $e = (h, \alpha(h))^{-1} \cdot e = (h^{-1}, 1) \cdot (1, \alpha(h^{-1})) \cdot e$ in the $\Gamma \times G$-space E and hence we get $h \cdot e = e \cdot \alpha(h)$ in E_r. This finishes the proof of Theorem 8.1.

□

9. Local structure

Next we deal with the structure of a Γ-equivariant principal G-bundles on small open Γ_b-invariant neighborhoods of points b in the base space.

Theorem 9.1 (Local structure)**.** *Let $\mathcal{R}$ be a family of local representations for (Γ, G) satisfying Condition (H) introduced in Definition 6.1. Let $p\colon E \to B$ be a Γ-equivariant principal G-bundle with $\mathcal{R}(p) \subseteq \mathcal{R}$. Consider any point $b \in B$ and any Γ_b-invariant open neighborhood W of $b \in B$.*

Then for every b in B the exists a commutative diagram

$$\begin{array}{ccc} \Gamma \times_{\Gamma_b} (T \times G) & \xrightarrow[\cong]{f} & p^{-1}(V) \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p|_{p^{-1}(V)}} \\ \Gamma \times_{\Gamma_b} T & \xrightarrow[\cong]{u} & V \end{array}$$

with the following properties:

(i) The subset $T \subseteq B$ contains b, satisfies $T \subseteq W$, is Γ_b-invariant and Γ_b-contractible;

(ii) The subset $V \subseteq B$ is an open Γ-invariant neighborhood of b;

(iii) The group Γ_b acts from the right on $T \times G$ by

$$\gamma \cdot (t, g) := \big(\gamma \cdot t, \rho_e(\gamma) \cdot g\big),$$

where (Γ_e, ρ_e) is the local representation of p associated to a fixed element $e \in E$ with $p(e) = b$, see (3.2);

(iv) The upper vertical map is a homeomorphism compatible with the left Γ-actions and the right G-actions, which at the source is given by

$$\gamma' \cdot \big((\gamma, t), g\big) \cdot g' = \big((\gamma'\gamma, t), gg'\big);$$

(v) The lower horizontal arrow is a homeomorphism compatible with the left Γ-actions, and q sends $\big((\gamma, t), g\big)$ to (γ, t).

Proof. Because of Theorem 8.1 (i) we can interpret E_l as a $\Gamma \times G$-CW-complex. Then the claim follows from the Slice Theorem 7.1 and the proof of Theorem 8.1 (ii). More precisely, the desired diagram is the diagram of left $\Gamma \times G$-spaces coming from combining the diagrams (8.3) and (8.4), if we replace S by its image T under $p\colon E \to B$. This is possible since the projection $\Gamma \times G \to \Gamma$ induces an isomorphism $\psi_e\colon (\Gamma \times G)_e \xrightarrow{\cong} \Gamma_b$ and $p|_S\colon S \to T$ is a ψ_e-homeomorphism. $\square$

10. Homotopy invariance

Next we show that the pullback of a Γ-equivariant principal G-bundle with Γ-homotopic maps yields isomorphic Γ-equivariant principal G-bundles.

Theorem 10.1 (Homotopy invariance)**.** *Let $\mathcal{R}$ be a family of local representations for (Γ, G) satisfying Condition (H) introduced in Definition 6.1. Let B be a Γ-CW-complex and let $p\colon E \to B \times [0,1]$ be a Γ-equivariant principal G-bundle with $\mathcal{R}(p) \subseteq \mathcal{R}$. Let $i_0\colon B = B \times \{0\} \to B \times [0,1]$ be the inclusion.*

Then $i_0^*E \times [0,1] \xrightarrow{i_0^*p \times \mathrm{id}_{[0,1]}} B \times [0,1]$ is a Γ-equivariant principal G-bundle and there exists an isomorphism of Γ-equivariant principal G-bundles

$$f\colon i_0^*E \times [0,1] \to E$$

over $B \times [0,1]$ whose restriction to $B \times \{0\}$ is the identity.

Proof. Let $p_n\colon E_n \to B_n$ be the restriction of i_0^*E to the n-skeleton B_n of B. We will construct inductively over n an isomorphism of Γ-equivariant principal G-bundles

$$f_n\colon E_n \times [0,1] \xrightarrow{\cong} E|_{B_n \times [0,1]}$$

such that the restriction of f_n to $B_n \times \{0\}$ is the identity and the restriction of f_n to $B_{n-1} \times [0,1]$ is f_{n-1}. Then we can define the desired isomorphism f by requiring that $f|_{B_n \times [0,1]} = f_n$, since i_0^*E has the weak topology with respect to the filtration by the E_n-s by Lemma 16.6 (i) and (ii).

The induction beginning $n = -1$ is trivial, the induction step from $(n-1)$ to n done as follows. Choose a Γ-pushout

$$\begin{array}{ccc} \coprod_{i\in I} \Gamma/H_i \times S^{n-1} & \xrightarrow{\coprod_{i\in I} q_i} & B_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I} \Gamma/H_i \times D^n & \xrightarrow{\coprod_{i\in I} Q_i} & B_n \end{array}$$

From Lemma 16.5 (i) and (ii) we obtain a $\Gamma \times G$-pushout

$$\begin{array}{ccc} \coprod_{i\in I} q_i^* E_{n-1} & \xrightarrow{\coprod_{i\in I} \overline{q_i}} & E_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I} Q_i^* E_n & \xrightarrow{\coprod_{i\in I} \overline{Q_i}} & E_n \end{array}$$

and thus a $\Gamma \times G$-pushout

$$\begin{array}{ccc} \coprod_{i\in I} q_i^* E_{n-1} \times [0,1] & \xrightarrow{\coprod_{i\in I} \overline{q_i} \times \mathrm{id}_{[0,1]}} & E_{n-1} \times [0,1] \\ \downarrow & & \downarrow \\ \coprod_{i\in I} Q_i^* E_n \times [0,1] & \xrightarrow{\coprod_{i\in I} \overline{Q_i} \times \mathrm{id}_{[0,1]}} & E_n \times [0,1] \end{array}$$

Hence it suffices to extend for each $i \in I$ the map of Γ-equivariant principal G-bundles over $\Gamma/H_i \times S^{n-1} \times [0,1]$

$$x_i\colon q_i^* E_{n-1} \times [0,1] \xrightarrow{\overline{q_i} \times [0,1]} E_{n-1} \times [0,1] \xrightarrow{f_{n-1}} E|_{B_{n-1} \times [0,1]}$$

covering $q_i \times \mathrm{id}_{[0,1]}\colon \Gamma/H_i \times S^{n-1} \times [0,1] \to B_{n-1} \times [0,1]$ to a map of Γ-equivariant principal G-bundles over $\Gamma/H_i \times D^n \times [0,1]$

$$y_i\colon Q_i^* E_n \times [0,1] \to E|_{B_n \times [0,1]}$$

covering $Q_i \times \mathrm{id}_{[0,1]}\colon \Gamma/H_i \times D^n \times [0,1] \to B_n \times [0,1]$ such that the restriction of y_i to $\Gamma/H_i \times D^n \times \{0\}$ is fiberwise the identity. We obtain from x_i a map of Γ-equivariant principal G-bundles over $\Gamma/H_i \times S^{n-1} \times [0,1]$

$$x_i'\colon q_i^* E_{n-1} \times [0,1] \to (Q_i \times \mathrm{id}_{[0,1]})^* E|_{\Gamma/H_i \times S^{n-1} \times [0,1]}$$

covering the identity $\mathrm{id}\colon \Gamma/H_i \times S^{n-1} \times [0,1] \to \Gamma/H_i \times S^{n-1} \times [0,1]$ such that the restriction of x_i' to $\Gamma/H_i \times S^{n-1} \times \{0\}$ is the identity. It remains to extend x_i' to map of Γ-equivariant principal G-bundles over $\Gamma/H_i \times D^n \times [0,1]$

$$y_i'\colon Q_i^* E_n \times [0,1] \to (Q_i \times \mathrm{id}_{[0,1]})^* E$$

covering the identity $\mathrm{id}\colon \Gamma/H_i \times D^n \times [0,1] \to \Gamma/H_i \times D^n \times [0,1]$ such that the restriction of y_i' to $\Gamma/H_i \times D^n \times \{0\}$ is the identity. We obtain from (8.2), now

applied to $D^n \times [0,1]$ instead of D^n, isomorphisms of Γ-equivariant principal G-bundles over $\Gamma/H_i \times D^n \times [0,1]$

$$\begin{aligned} a\colon (\Gamma \times G)/H_i \times D^n \times [0,1] &\xrightarrow{\cong} Q_i^* E_n \times [0,1]; \\ b\colon (\Gamma \times G)/H_i \times D^n \times [0,1] &\xrightarrow{\cong} (Q_i \times \mathrm{id}_{[0,1]})^* E, \end{aligned}$$

for an appropriate subgroup $H_i \subseteq \Gamma \times G$ if we convert the left G-action into a right G-action in the usual way. By conjugation with the restrictions of a and b^{-1} to $\Gamma/H_i \times S^{n-1} \times [0,1]$, we obtain from x_i' an isomorphism of Γ-equivariant principal G-bundles over $\Gamma/H_i \times S^{n-1} \times [0,1]$

$$x_i''\colon (\Gamma \times G)/H_i \times S^{n-1} \times [0,1] \xrightarrow{\cong} (\Gamma \times G)/H_i \times S^{n-1} \times [0,1].$$

The remaining problem is to extend x_i'' to an isomorphism of Γ-equivariant principal G-bundles over $\Gamma/H_i \times D^n \times [0,1]$

$$y_i''\colon (\Gamma \times G)/H_i \times D^n \times [0,1] \xrightarrow{\cong} (\Gamma \times G)/H_i \times D^n \times [0,1]$$

whose restriction to $D^n \times \{0\}$ is the restriction of $b^{-1} \circ a$ to $D^n \times \{0\}$. Notice that x_i'' is the same as a map $S^{n-1} \times [0,1] \to \mathrm{map}_{\Gamma \times G}\big((\Gamma \times G)/H_i, (\Gamma \times G)/H_i\big)$ and y_i'' is the same as map $D^n \times [0,1] \to \mathrm{map}_{\Gamma \times G}\big((\Gamma \times G)/H_i, (\Gamma \times G)/H_i\big)$. Hence the remaining problem is to extend a given map

$$S^{n-1} \times [0,1] \cup D^n \times \{0\} \to \mathrm{map}_{\Gamma \times G}\big((\Gamma \times G)/H_i, (\Gamma \times G)/H_i\big)$$

to a map

$$D^n \times [0,1] \to \mathrm{map}_{\Gamma \times G}\big((\Gamma \times G)/H_i, (\Gamma \times G)/H_i\big).$$

This is possible since there is a retraction $D^n \times [0,1] \to S^{n-1} \times [0,1] \cup D^n \times \{0\}$. $\square$

11. Universal equivariant principal bundles

Fix a family of local representations $\mathcal{R}$ satisfying Condition (H) introduced in Definition 6.1. In this section we construct the universal Γ-equivariant principal G-bundle and in particular the classifying space for Γ-equivariant principal G-bundles with respect to $\mathcal{R}$.

Definition 11.1 (Compatibility)**.** We call $\mathcal{R}$ *compatible with the* Γ-*CW*-*complex* X if for any $x \in X$ and $(H, \alpha) \in \mathcal{R}$ with $\Gamma_x \subseteq H$ the pair $(\Gamma_x, \alpha|_{\Gamma_x})$ belongs to $\mathcal{R}$ again.

Remark 11.2. Notice that this condition is automatically satisfied for every Γ-CW if $\mathcal{R}$ is closed under taking subgroups, i.e., for (H, α) in $\mathcal{R}$ and $K \subseteq H$ we have $(K, \alpha|_K) \in \mathcal{R}$.

Consider Γ-CW-complexes X and B and a Γ-equivariant principal G-bundle $p\colon E \to B$ over the Γ-CW-complex B with $\mathcal{R}(p) \subseteq \mathcal{R}$. Let $\mathrm{Bundle}_{\Gamma,G,\mathcal{R}}(X)$ be the set of isomorphism classes of Γ-equivariant principal G-bundles $q\colon E \to X$ with $\mathcal{R}(q) \subseteq \mathcal{R}$. Suppose that $\mathcal{R}$ is compatible with X. Then for any Γ-map $f\colon X \to B$ the pull back f^*p is a Γ-equivariant principal G-bundle with $\mathcal{R}(f^*p) \subseteq \mathcal{R}$. Because of Theorem 10.1 the pullback construction yields a well-defined map

$$c\colon [X, B]^\Gamma \to \mathrm{Bundle}_{\Gamma,G,\mathcal{R}}(X), \quad [f] \mapsto [f^*p]. \tag{11.3}$$

Notation 11.4 (Classifying space of a family of subgroups)**.** Given group G and a family $\mathcal{F}$ of subgroups of G, denote by $E_{\mathcal{F}}(G)$ the *classifying space of the family* $\mathcal{F}$.

Recall that $E_{\mathcal{F}}(G)$ is a G-CW-complex whose isotropy groups belong to $\mathcal{F}$ and for which the H-fixed point set $E_{\mathcal{F}}(G)^H$ is non-empty and weakly contractible for every $H \in \mathcal{F}$. A model always exists. For any G-CW-complex X whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $X \to E_{\mathcal{F}}(G)$. In particular two models for $E_{\mathcal{F}}(G)$ are G-homotopy equivalent. For more information about classifying spaces of a family we refer for instance to [22].

Theorem 11.5 (Classifying space for Γ-equivariant principal G-bundles with family of local representatives contained in $\mathcal{R}$ satisfying Condition (H))**.** *Let $\mathcal{R}$ be a family of local representations for (Γ, G) satisfying Condition (H) introduced in Definition 6.1. Define*

$$\begin{aligned} E(\Gamma, G, \mathcal{R}) &:= E_{\mathcal{F}(\mathcal{R})}(\Gamma \times G)_r; \\ B(\Gamma, G, \mathcal{R}) &:= E_{\mathcal{F}(\mathcal{R})}(\Gamma \times G)_r/G, \end{aligned}$$

where $\mathcal{F}(\mathcal{R})$ is the family of subgroups of $\Gamma \times G$ introduced in Lemma 3.6. Let $p\colon E(\Gamma, G, \mathcal{R}) \to B(\Gamma, G, \mathcal{R})$ be the projection. Let X be a Γ-CW-complex such that $\mathcal{R}$ is compatible with X in the sense of Definition 11.1.

Then $p\colon E(\Gamma, G, \mathcal{R}) \to B(\Gamma, G, \mathcal{R})$ is a Γ-equivariant principal G-bundle and the map c defined in (11.3)

$$c\colon [X, B]^{\Gamma} \to \mathrm{Bundle}_{\Gamma, G, \mathcal{R}}(X), \quad [f] \mapsto [f^*p].$$

is bijective.

Proof. We conclude from Theorem 8.1 (ii) that $p\colon E(\Gamma, G, \mathcal{R}) \to B(\Gamma, G, \mathcal{R})$ is a Γ-equivariant principal G-bundle.

We construct an inverse map

$$d\colon \mathrm{Bundle}_{\Gamma, G, \mathcal{R}}(X) \to [X, B]^{\Gamma}$$

as follows.

Let $q\colon E \to B$ be a Γ-equivariant principal G-bundle whose family of local representations is contained in $\mathcal{R}$. We conclude from Theorem 8.1 (i) that E_l is a left $\Gamma \times G$-CW-complex whose isotropy groups belong to $\mathcal{F}(\mathcal{R})$. Hence there is up to $\Gamma \times G$-homotopy precisely one $\Gamma \times G$-map $\overline{f}\colon E_l \to E_{\mathcal{F}(\mathcal{R})}(\Gamma \times G)$ which is the same a map $\overline{f}\colon E \to E(\Gamma, G, \mathcal{R})$ compatible with the left Γ-actions and right G-actions. Taking the G-quotient yields a map $f\colon B \to B(\Gamma, G, \mathcal{R})$ which is unique up to Γ-homotopy and for which the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\overline{f}} & E(\Gamma, G, \mathcal{R}) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[f]{} & B(\Gamma, G, \mathcal{R}) \end{array}$$

Hence q is up to isomorphism the pullback of the universal bundle p with the Γ-map $f\colon B \to B(\Gamma, G, \mathcal{R})$ and the construction of this Γ-map f is unique up to Γ-homotopy. Define $d([q])$ by $[f]$. This is well-defined since obviously $[f]$ depends only on the isomorphism class $[q]$ of q. Since f^*p is isomorphic to q, we get $c \circ d = \mathrm{id}_{\mathrm{Bundle}_{\Gamma, G, \mathcal{R}}(X)}$. One easily checks $d \circ c = \mathrm{id}_{[X, B]^{\Gamma}}$.

This finishes the proof of Theorem 11.5. □

We will call $p\colon E(\Gamma, G, \mathcal{R}) \to B(\Gamma, G, \mathcal{R})$ appearing in Theorem 11.5 the *universal Γ-equivariant principal G-bundle with respect to the family of local representations $\mathcal{R}$.*

If $p'\colon E(\gamma, G, \mathcal{R})' \to B(\gamma, G, \mathcal{R})'$ is another such universal bundle, then there is a commutative diagram of such bundles covering a Γ-homotopy equivalence f

$$\begin{array}{ccc} E(\Gamma, G, \mathcal{R}) & \xrightarrow{\overline{f}} & E(\Gamma, G, \mathcal{R})' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B(\Gamma, G, \mathcal{R}) & \xrightarrow{f} & B(\Gamma, G, \mathcal{R})' \end{array}$$

i.e., $\overline{f}$ is compatible with the left Γ and the right G-actions and is a homeomorphism. Such a diagram is unique up to $\Gamma \times G$-homotopy, in particular the Γ-map f is unique up to Γ-homotopy.

12. Reduction of the structure group to a maximal compact subgroup

If K is a subgroup of G and $p_0\colon E_0 \to B$ is a Γ-equivariant principal K-bundle, then

$$\overline{p_0}\colon E_0 \times_K G \to B, \quad (e, g) \mapsto p_0(e)$$

inherits the structure of a Γ-equivariant principal G-bundle in the obvious way. One has just to verify local triviality of the underlying principal G-bundle $\overline{p_0}$.

Definition 12.1 (Reduction)**.** Let $p\colon E \to B$ be a Γ-equivariant principal G-bundle. Given a subgroup $K \subseteq G$, a *reduction* to a Γ-equivariant principal K-bundle is a Γ-equivariant principal K-bundle $p_0\colon E_0 \to B$ together with an isomorphism of Γ-equivariant principal G-bundles from $\overline{p_0}$ to p.

Lemma 12.2. *Let $p\colon E \to B$ be a Γ-equivariant principal G-bundle and $K \subseteq G$ be a subgroup. Let $f\colon E \to G/K$ be a $\Gamma \times G$-map, where we equip the target with the $\Gamma \times G$-action given by $(\gamma, g) \cdot g'K = gg'K$. Suppose that the pair (G, K) satisfies Condition (S), see Definition 4.1. Put $E_f = f^{-1}(1K)$. Let $p_f\colon E_f \to B$ be the restriction of p to E_f. Then:*

- *(i) The map $p_f\colon E_f \to B$ is a Γ-equivariant principal K-bundle which is a K-reduction of p;*
- *(ii) Every K-reduction of $p\colon E \to B$ is up to isomorphism of Γ-equivariant principal K-bundles of the form $p_f\colon E_f \to B$ for appropriate f;*
- *(iii) If $f_0, f_1\colon E \to G/K$ are $\Gamma \times G$-maps which are $\Gamma \times G$-homotopic, then $p_{f_0}\colon E_{f_0} \to B$ and $p_{f_1}\colon E_{f_1} \to B$ are isomorphic as Γ-equivariant principal K-bundles.*

Proof. (i) From Lemma 4.4 we obtain a $\Gamma \times G$-homeomorphism

$$u\colon E_0 \times_K G \xrightarrow{\cong} E, \quad (g, e) \mapsto e \cdot g.$$

It remains to show that $p_0\colon E_0 \to B$ is a principal K-bundle, i.e., to show local triviality. Since p is locally trivial, it suffices to treat the case, where $E = G \times B$ and $p\colon G \times B \to B$ is the projection. Choose a quasi-regular open subset $U \subseteq G/K$ with $1K \in U$ and a map $s\colon U \to G$ satisfying $\mathrm{pr} \circ s = \mathrm{id}_U$, where $\mathrm{pr}\colon G \to G/K$ is the projection. Let V be the open subset of B given by $p(f^{-1}(U))$. Since V is Γ-invariant, it is quasi-regular by Lemma 16.1 (vi). Let $\alpha\colon V \to G$ be the map sending v to $s \circ f(v, 1)$. Then we obtain an automorphism of the trivial principal G-bundle $G \times B \to G \times B$

$$\overline{\alpha}\colon G \times V \xrightarrow{\cong} G \times V, \quad (g, v) \mapsto (g \cdot \alpha(v), v)$$

If $\pi\colon G \times V \to G/K$ sends (g, v) to gK, then $\pi \circ \overline{\alpha} = f$. Hence $\overline{\alpha}$ induces a commutative diagram of K-spaces

$$\begin{tikzcd} p_f^{-1}(V) \arrow[rr, "\cong"] \arrow[dr, "p_f|_{p_f^{-1}(V)}"'] & & V \times K \arrow[dl] \\ & V & \end{tikzcd}$$

such that the horizontal arrow is a K-homeomorphism and the right vertical arrow is the projection.

(ii) Let $p_0\colon E_0 \to B$ be a K-reduction of p. Choose an isomorphism of Γ-equivariant principal G-bundles $u\colon G \times_K E_0 \xrightarrow{\cong} E$. If we take $f\colon E \to G/K$ to be the composite of u^{-1} with the obvious projection $G \times_K E_0 \to G/K$, then p_0 is p_f.

(iii) Fix a $\Gamma \times G$-homotopy $h\colon E \times [0,1] \to G/K$ with $h_k = f_k$ for $k = 0, 1$. Let $q\colon E \times [0,1] \to B \times [0,1]$ be the Γ-equivariant principal G-bundle given by $p \times \mathrm{id}_{[0,1]}$. Then we obtain a Γ-equivariant principal K-bundle $q_h\colon (E \times [0,1])_h \to B \times [0,1]$ whose restriction to $B \times \{k\}$ is p_{f_k} for $k = 0, 1$. Now the claim follows from Theorem 10.1. □

Definition 12.3 (Almost connected group)**.** Given a group G, let G^0 be its *component of the identity* and define the *component group* $\overline{G}$ by $\overline{G} = G/G^0$. We call G *almost connected* if its component group $\overline{G}$ is compact.

The next result is due to Abels [1, Corollary 4.14].

Theorem 12.4 (Almost connected groups)**.** *Let G be a locally compact Hausdorff topological group. Suppose that G is almost connected.*

Then G contains a maximal compact subgroup K which is unique up to conjugation, and the H-fixed point set $(G/K)^H$ is contractible for every compact subgroup $H \subseteq K$.

Theorem 12.5 (Existence of a K-reduction)**.** *Let G and Γ be locally compact second countable topological groups. Suppose that G is almost connected. Let $K \subseteq G$ be a maximal compact subgroup. Let $\mathcal{R}$ be a family of local representation for (Γ, G) such that for each element (H, α) in $\mathcal{R}$ the subgroup $H \subseteq \Gamma$ is compact. Let $\mathcal{R}_K$ be the family of local representations of (Γ, K) which is given by $\mathcal{R}_K = \{(H, \alpha) \mid \alpha(H) \subseteq K, (H, \alpha) \in \mathcal{R}\}$.*

Then every Γ-equivariant principal G-bundle $p\colon E \to B$ with $\mathcal{R}(p) \subseteq \mathcal{R}$ has a (preferred) K-reduction $p_0\colon E_0 \to B$ which is unique up to isomorphism of Γ-equivariant principal K-bundles and satisfies $\mathcal{R}(p_0) \subseteq \mathcal{R}_K$.

Proof. Condition (H) is always satisfied under the conditions of Lemma 12.5 because of Theorem 6.3. It suffices to prove the claim for the universal Γ-equivariant principal G-bundle because of Theorem 11.5, since K-reductions are compatible with pullbacks.

Let $p\colon E(\Gamma, G, \mathcal{R}) \to B(\Gamma, G, \mathcal{R})$ be the universal Γ-equivariant principal G-bundle with respect to the family $\mathcal{R}$, see Theorem 11.5. Recall that $E(\Gamma, G, \mathcal{R})$ is a Γ-G-CW-complex whose isotropy groups belong to the family of subgroups $\mathcal{F}(\mathcal{R})$ associated to $\mathcal{R}$. For any element in $K(H, \alpha) \in \mathcal{F}$ the image of $K(H, \alpha)$ under the projection $\Gamma \times G \to G$ is compact and hence a closed subgroup of G. Lemma 16.7 implies that $E(\Gamma, G, \mathcal{R})/\Gamma$ is a G-CW-complex. Because of Theorem 12.4 the G-CW-complex G/K is the classifying space for proper G-actions. Hence there is up to G-homotopy precisely one map $E(\Gamma, G, \mathcal{R})/\Gamma \to G/K$. We conclude that there is up to $\Gamma \times G$-homotopy precisely one $\Gamma \times G$-map $f\colon E(\Gamma, G, \mathcal{R}) \to G/K$. From

Lemma 12.2 we obtain a K-reduction p_f of p which is unique up to isomorphism of Γ-equivariant principal K-bundles. One easily checks $\mathcal{R}(p_f) \subseteq \mathcal{R}_K$. $\square$

Example 12.6 (Equivariant vector bundles and Riemannian metrics)**.** In the case $G = GL_n(\mathbb{R})$, we have as maximal compact subgroup $O(n)$ and Theorem 12.5 implies the well-known the statement that any equivariant vector bundle ξ over a proper Γ-CW-complex can be equipped with a Γ-invariant Riemannian metric and that two equivariant vector bundles over a proper Γ-CW-complex with Γ-invariant Riemannian metrics admit an isomorphism respecting the Γ-invariant Riemannian metrics if and only the equivariant vector bundles are isomorphic (after forgetting the invariant Riemannian metrics).

13. On the homotopy type of the classifying space

In this section we want to establish for $H \subseteq \Gamma$ a weak homotopy equivalence

$$\bigsqcup_{[\alpha] \in \hom_{\mathcal{R}}(H,G)/G} BC_G(\alpha) \quad \simeq \quad B(\Gamma, G, \mathcal{R})^H,$$

where $[\alpha]$ runs over the G-conjugacy classes of homomorphisms $\alpha\colon H \to G$ with $(H, \alpha) \in \mathcal{R}$. This will follow from Theorem 13.1.

Let $\hom(H, G)$ be the space of homomorphisms of topological groups $H \to G$, endowed with the subspace topology from $\hom(H, G) \subseteq \operatorname{map}(H; G)$, see Subsection 16.5. Denote by $\hom(H, G)/G$ the quotient space under the conjugation action of G, i.e., the left G-action sending $(g, \alpha) \in G \times \hom(H, G)$ to $c_g \circ \alpha$, where $c_g\colon G \to G$ sends g' to $gg'g^{-1}$, see Subsection 16.5. Recall that the centralizer of $\alpha \in \hom(H, G)$ is

$$C_G(\alpha) := \{g \in G \mid g\alpha(h)g^{-1} = \alpha(h) \text{ for all } h \in H\}.$$

For $\mathcal{R}$ a family of local representations for (Γ, G), and (H, α) in $\mathcal{R}$ define

$$\hom_{\mathcal{R}}(H, G) := \{\alpha \in \hom(H, G) | (H, \alpha) \in \mathcal{R}\}$$

and note that $\hom_{\mathcal{R}}(H, G)$ is closed under the conjugation of G.

Theorem 13.1 (Fixed point sets of $B(\Gamma, G, \mathcal{R})$)**.** *Let $\mathcal{R}$ be a family of local representations for (Γ, G) satisfying Condition (H) introduced in Definition 6.1. Consider an element (H, α) in $\mathcal{R}$.*

(i) We obtain a bijection

$$\hom_{\mathcal{R}}(H, G)/G \xrightarrow{\cong} \pi_0\big(B(\Gamma, G, \mathcal{R})^H\big);$$

(ii) Given (H, α) in $\mathcal{R}$, let $B(\Gamma, G, \mathcal{R})^H_\alpha$ be the path component of $B(\Gamma, G, \mathcal{R})^H$ that corresponds under the bijection of assertion (i) to the class of α in $\hom_{\mathcal{R}}(H, G)/G$.

Then there exists a weak homotopy equivalence

$$BC_G(\alpha) \xrightarrow{\simeq} B(\Gamma, G, \mathcal{R})^H_\alpha.$$

Proof. (i) We obtain from Theorem 11.5 bijections

$$\pi_0\big(B(\Gamma, G, \mathcal{R})^H\big) \xrightarrow{\cong} [\Gamma/H, B(\Gamma, G, \mathcal{R})]^\Gamma \xrightarrow{\cong} \operatorname{Bundle}_{\Gamma, G, \mathcal{R}}(\Gamma/H).$$

Given $\alpha \in \hom_{\mathcal{R}}(H, G)$, we obtain a Γ-equivariant principal G-bundle

$$\xi_\alpha\colon \Gamma \times_\alpha G \to \Gamma/H, \quad (\gamma, g) \mapsto \gamma H,$$

where $\Gamma \times_\alpha G$ is the quotient of $\Gamma \times G$ under the left H-action given by $h \cdot (\gamma, g) = (\gamma \cdot h^{-1}, \alpha(h) \cdot g)$. We conclude from Lemma 5.1 (ii) and (iii) that any Γ-equivariant principal G-bundle $q\colon E \to \Gamma/H$ with $\mathcal{R}(q) \subset \mathcal{R}$ is isomorphic to ξ_α for some $\alpha \in \hom_{\mathcal{R}}(H, G)$, and for two elements $\alpha, \beta \in \hom_{\mathcal{R}}(H, G)$ the Γ-equivariant

principal G-bundles ξ_α and ξ_β are isomorphic if and only if the classes of α and β in $\hom_{\mathcal{R}}(H,G)/G$ agree. Hence we obtain a bijection

$$\hom_{\mathcal{R}}(H,G)/G \xrightarrow{\cong} \mathrm{Bundle}_{\Gamma,G,\mathcal{R}}(\Gamma/H), \quad [\alpha] \mapsto [\xi_\alpha].$$

(ii) Let $p\colon E(\Gamma,G,\mathcal{R}) \to B(\Gamma,G,\mathcal{R})$ be the canonical projection which is a principal G-bundle. We want to show that p induces a principal $C_G(\alpha)$-bundle

$$p_{(H,\alpha)}\colon E(\Gamma,G,\mathcal{R})^{K(H,\alpha)} \to B(\Gamma,G,\mathcal{R})^H_\alpha.$$

Abbreviate $E = E(\Gamma,G,\mathcal{R})$. Consider $(H,\alpha) \in \mathcal{R}$. Let $E^{\langle H\rangle}$ be the subspace of E consisting of those elements $e \in E$ such that for each $h \in H$ there exists $g \in G$ with $h \cdot e = e \cdot g$. For each $e \in E$ define $\rho_e \in \hom(H,G)$ by requiring $h \cdot e = e \cdot \rho_e(h)$. Thus we can define a map of sets

$$\rho\colon E^{\langle H\rangle} \to \hom(H,G).$$

Next we show that ρ is continuous. We apply Theorem 9.1 at the point $b = p(e)$ and obtain a commutative diagram

$$\begin{tikzcd}
\Gamma \times_{\Gamma_b} (T \times G) \arrow[r, "f", "\cong"'] \arrow[d, "q"'] & p^{-1}(V) \arrow[d, "p|_{p^{-1}(V)}"] \\
\Gamma \times_{\Gamma_b} T \arrow[r, "u", "\cong"'] & V
\end{tikzcd}$$

It suffices to show that the composite of ρ with the map $f^{\langle H\rangle}\colon \big(\Gamma \times_{\Gamma_b} (T \times G)\big)^{\langle H\rangle} \to E^{\langle H\rangle}$ is continuous.

Put $\Gamma^{\langle H\rangle} = \{\gamma \in \Gamma \mid \gamma^{-1} \cdot H \cdot \gamma \subseteq \Gamma_b\}$. Consider the commutative diagram

$$\begin{tikzcd}
\Gamma^{\langle H\rangle} \arrow[r, "\nu'"] \arrow[d, hook] & \hom(H,\Gamma_b) \arrow[d, hook] \\
\Gamma \arrow[r, "\nu"'] & \hom(H,\Gamma)
\end{tikzcd}$$

where ν' and ν respectively are given by conjugating the inclusion homomorphism $i\colon H \to \Gamma_b$ with $\gamma \in \Gamma^{\langle H\rangle}$ and $\gamma \in \Gamma$ respectively, the left vertical arrow is the inclusion of subgroups and the right vertical arrow is the injection induced by the inclusion $\Gamma_b \to \Gamma$. Since $\mathrm{map}(H,\Gamma_b)$ is the preimage of the constant map under $\mathrm{map}(H,\Gamma) \to \mathrm{map}(H,\Gamma/\Gamma_b)$, we conclude from Subsection 16.2 that the right vertical arrow is the inclusion of a closed subspace. The conjugation map $\Gamma \times \hom(H,\Gamma) \to \hom(H,\Gamma)$ is continuous as explained in Subsection 16.2, We conclude that ν and hence ν' are continuous. Define a map of sets

$$\mu\colon \Gamma^{\langle H\rangle} \times G \to \hom(H,G)$$

by sending (γ,g) to the composite $H \xrightarrow{c_\gamma \circ i} \Gamma_b \xrightarrow{\rho_e} G \xrightarrow{c_g} G$, where $i\colon H \to \Gamma_b$ is the inclusion. Since ν' is continuous, μ is continuous. The map μ factorizes through the obvious projection $\Gamma^{\langle H\rangle} \times G \to \big(\Gamma \times_{\Gamma_b} G\big)^{\langle H\rangle}$, which is an identification, to a continuous map $\overline{\mu}\colon \big(\Gamma \times_{\Gamma_b} G\big)^{\langle H\rangle} \to \hom(H,G)$ making the following diagram commutative

$$\begin{tikzcd}
\Gamma^{\langle H\rangle} \times G \arrow[d] \arrow[dr, "\mu"] & \\
\big(\Gamma \times_{\Gamma_b} G\big)^{\langle H\rangle} \arrow[r, "\overline{\mu}"] & \hom(H,G)
\end{tikzcd}$$

If $\operatorname{pr}\colon \Gamma \times_{\Gamma_b} (T \times G) \to \Gamma \times_{\Gamma_b} G$ is the projection, the following diagram is commutative

$$\begin{array}{ccc} \big(\Gamma \times_{\Gamma_b} (T \times G)\big)^{\langle H \rangle} & \xrightarrow{f^{\langle H \rangle}} & E^{\langle H \rangle} \\ \Big\downarrow \operatorname{pr}^{\langle H \rangle} & & \Big\downarrow \rho \\ \big(\Gamma \times_{\Gamma_b} G\big)^{\langle H \rangle} & \xrightarrow{\overline{\mu}} & \hom(H, G) \end{array}$$

Hence $\overline{\mu} \circ \operatorname{pr}^{\langle H \rangle} = \rho \circ f^{\langle H \rangle}$. This implies that $\rho \circ f^{\langle H \rangle}$ and hence $\rho\colon E^{\langle H \rangle} \to \hom(H, G)$ is continuous. One easily checks that ρ is a G-map.

Let $E^{\langle H, \alpha \rangle}$ be the preimage under ρ of the orbit $G \cdot \alpha \subseteq \hom(H, G)$ with respect to the action of G on $\hom(H, G)$ given by composing with conjugation automorphisms. Hence ρ induces a G-map

$$\rho'_\alpha\colon E^{\langle H, \alpha \rangle} \to G \cdot \alpha.$$

Since $\mathcal{R}$ satisfies Condition (H), see Definition 6.1, the G-map $\iota_\alpha\colon G/C_G(\alpha) \to G \cdot \alpha, g \mapsto c_g \circ \alpha$ is a homeomorphism. Define the G-map

$$\rho_\alpha = \iota_\alpha^{-1} \circ \rho'_\alpha\colon E^{\langle H, \alpha \rangle} \to G/C_G(\alpha).$$

One easily checks that the preimage of $1 \cdot C_G(\alpha) \in G/C_G(\alpha)$ under ρ_α is $E^{K(H,\alpha)}$. Since $(G, C_G(\alpha))$ satisfies Condition (S), it implies that the canonical G-map

$$G \times_{C_G(\alpha)} E^{K(H,\alpha)} \to E^{\langle H, \alpha \rangle}$$

is a G-homeomorphism. One easily checks that $E^{\langle H, \alpha \rangle}$ is the preimage of $B(\Gamma, G, \mathcal{R})^H_\alpha$ under $p\colon E(\Gamma, G, \mathcal{R}) \to B(\Gamma, G, \mathcal{R})$. Hence p induces a $C_G(\alpha)$-map

$$p_{(H,\alpha)}\colon E(\Gamma, G, \mathcal{R})^{K(H,\alpha)} \to B(\Gamma, G, \mathcal{R})^H_\alpha$$

for which the following diagram of G-spaces commutes

$$\begin{array}{ccc} G \times_{C_G(\alpha)} E^{K(H,\alpha)} & \xrightarrow{\cong} & E^{\langle H, \alpha \rangle} \\ & {\scriptstyle G \times_{C_G(\alpha)} p_{(H,\alpha)}} \searrow \quad \swarrow {\scriptstyle p|_{p^{-1}(B(\Gamma,G,\mathcal{R})^H_\alpha)}} & \\ & B(\Gamma, G, \mathcal{R})^H_\alpha & \end{array}$$

Since the right vertical arrow is a principal G-bundle, Lemma 12.2 implies that $p_{(H,\alpha)}\colon E(\Gamma, G, \mathcal{R})^{K(H,\alpha)} \to B(\Gamma, G, \mathcal{R})^H_\alpha$ is a principal $C_G(\alpha)$-bundle.

Let X be a CW-complex and $f\colon X \to B(\Gamma, G, \mathcal{R})^H_\alpha$ be a weak homotopy equivalence. Let $q\colon f^*E(\Gamma, G, \mathcal{R})^{K(H,\alpha)} \to X$ be the pullback of the principal $C_G(\alpha)$-bundle $p_{(H,\alpha)}$. Every principal G-bundle is a fibration. We conclude from the long exact sequence of homotopy groups associated to $p_{(H,\alpha)}$ and q and the Five-Lemma that the induced map $f^*E(\Gamma, G, \mathcal{R})^{K(H,\alpha)} \to E(\Gamma, G, \mathcal{R})^{K(H,\alpha)}$ is a weak homotopy equivalence. Since $E(\Gamma, G, \mathcal{R})^{K(H,\alpha)}$ is weakly contractible, the same is true for $f^*E(\Gamma, G, \mathcal{R})^{K(H,\alpha)}$. Since q is a principal G-bundle over the CW-complex X with weakly contractible total space, it is a model for the universal principal $C_G(\alpha)$-bundle and X is a model for $BC_G(\alpha)$. □

14. Examples

14.1. Some special cases for G, Γ, and $\mathcal{R}$.

Example 14.1 (Trivial Γ)**.** Suppose Γ is trivial. Then there is only one family $\mathcal{R}$ of local representations consisting of $\{1\} \to G$, and the universal Γ-equivariant principal G-bundle with respect to the family of local representations $\mathcal{R}$ is the same as the universal principal G-bundle $EG \to BG$.

Example 14.2 (Trivial G)**.** Suppose G is trivial. Then a family of local representations is the same as a family $\mathcal{F}$ of subgroups of Γ, and the universal Γ-equivariant principal G-bundle with respect to the family of local representations $\mathcal{R} = \mathcal{F}$ is the identity $E_{\mathcal{F}}(\Gamma) \to E_{\mathcal{F}}(\Gamma)$.

Example 14.3 (Trivial local representations)**.** Let $\mathcal{F}$ be a family of subgroups of Γ. Let $\mathcal{TR}(\mathcal{F})$ be the system of local representations given by pairs (H, ρ_H), where H belongs to $\mathcal{F}$ and ρ_H is the trivial representation. Then a Γ-equivariant principal G-bundle $q\colon E \to B$ satisfies $\mathcal{R}(p) \subseteq \mathcal{TR}(\mathcal{F})$ if and only if all isotropy groups of B belong to $\mathcal{F}$, the induced map $q/\Gamma\colon E/\Gamma \to B/\Gamma$ is a principal G-bundle, and q is the pullback of q/Γ with the projection $\mathrm{pr}\colon B \to B/\Gamma$. This follows from Theorem 9.1.

A model for the universal Γ-equivariant principal G-bundle associated to $\mathcal{TR}(\mathcal{F})$ is

$$\mathrm{id} \times p\colon E_{\mathcal{F}}(\Gamma) \times EG \to E_{\mathcal{F}}(\Gamma) \times BG,$$

where $p\colon EG \to BG$ is the universal principal G-bundle, the left Γ- and right G-actions are given on the total space by $\gamma \cdot (x, e) \cdot g = (\gamma \cdot x, e \cdot g)$ and on the base space by $\gamma \cdot (x, b) \cdot g = (\gamma \cdot x, b)$.

Example 14.4 ($\mathcal{R}(\mathcal{F})$)**.** Consider a family $\mathcal{F}$ of subgroups of Γ. Define the associated family of local representations

$$\mathcal{R}(\mathcal{F}) := \{(H, \alpha) \mid H \in \mathcal{F} \text{ and } \alpha\colon H \to G \text{ any group homomorphism}\},$$

in other words, H runs through elements in $\mathcal{F}$ and α through all possible group homomorphisms.

Then a Γ-equivariant principle G-bundle $p\colon E \to B$ over the Γ-CW-complex B satisfies $\mathcal{R}(p) \subseteq \mathcal{R}(\mathcal{F})$ if and only if all isotropy groups of B belong to $\mathcal{F}$. If we suppose that $\mathcal{R}(\mathcal{F})$ satisfies Condition (H) introduced in Definition 6.1, then in this situation the universal Γ-equivariant principal G-bundle with respect to $\mathcal{R}(\mathcal{F})$ classifies Γ-equivariant principal G-bundles over Γ-CW-complexes whose isotropy groups belong to $\mathcal{F}$.

If we choose $\mathcal{F}$ to be the family of compact subgroups and $\mathcal{R}(\mathcal{F})$ satisfies Condition (H), these are precisely the Γ-equivariant principal G-bundles over proper Γ-CW-complexes.

Example 14.5 (n-dimensional complex vector bundles with Hermitian metric over proper Γ-CW-complexes)**.** Let Γ be a topological group. Assume that all compact subgroups of Γ are Lie groups and Γ is completely regular, e.g., Γ is a Kac-Moody group, or that Γ is locally compact, second countable and has finite covering dimension, e.g., is a Lie groups. Let $G = \mathcal{U}(n)$ be the Lie group of unitary automorphisms of $\mathbb{C}^n$. Consider the family of representations

$$\mathcal{R} := \{(H, \alpha) \mid H \text{ is compact and } \alpha\colon H \to G \text{ any group homomorphism}\},$$

which is the family $\mathcal{R}(\mathcal{COM})$ of Example 14.4 associated to the family $\mathcal{COM}$ of compact subgroups.

In this case $\hom(H, \mathcal{U}(n))$ is isomorphic to the space of unitary representations of H on $\mathbb{C}^n$, and $\hom(H, \mathcal{U}(n))/\mathcal{U}(n)$ is isomorphic to the set of isomorphism classes of n-dimensional unitary H-representations. Denote by $V_1, V_2, \dots, V_k, \dots$ the irreducible unitary representations of H and $d_i := \dim_{\mathbb{C}} V_i$. The set of isomorphism classes unitary n-dimensional H-representations can be parametrized with the set of partitions of n using the dimensions d_i, i.e.,

$$\hom(H, \mathcal{U}(n))/\mathcal{U}(n) \cong \{(n_1, n_2, \dots, n_k, \dots) \mid n_1 d_1 + \dots + n_k d_k + \dots = n, n_i \geq 0\}.$$

If the homomorphism $\alpha\colon H \to \mathcal{U}(n)$ induces the representation $V = \bigoplus_i V_i^{\oplus n_i}$, then the isotropy group

$$C_{\mathcal{U}(n)}(\alpha) \cong \prod_i \mathcal{U}(n_i)$$

and therefore we have that

$$B(\Gamma, \mathcal{U}(n), \mathcal{R})^H \cong \bigsqcup_{\{(n_1,n_2,\ldots,n_k,\ldots)|n_1d_1+\cdots+n_kd_k+\cdots=n, n_i\geq 0\}} \prod_i B\mathcal{U}(n_i).$$

14.2. **Compact abelian Lie group G.** Let Γ be a topological group. Assume that all compact subgroups of Γ are Lie groups and Γ is completely regular, e.g., Γ is a Kac-Moody group, or that Γ is locally compact, second countable and has finite covering dimension, e.g., is a Lie groups. Let G be a compact abelian Lie group.

Consider the family of representations

$$\mathcal{R} := \{(H, \alpha) \mid H \text{ is compact and } \alpha\colon H \to G \text{ any group homomorphism}\},$$

which is the family $\mathcal{R}(\mathcal{COM})$ of Example 14.4 associated to the family $\mathcal{COM}$ of compact subgroups. We have

$$\hom_{\mathcal{R}}(H, G) = \hom(H, G)$$

and therefore we obtain from Theorem 13.1 a weak homotopy equivalence

$$\bigsqcup_{\hom(H,G)} BG \simeq B(\Gamma, G, \mathcal{R})^H.$$

Whenever $G = S^1$ we have that $\hom(H, S^1) \cong H^2(BH, \mathbb{Z})$ and $BS^1 \simeq K(\mathbb{Z}, 2)$. Therefore we obtain a weak homotopy equivalence

$$\bigsqcup_{H^2(BH,\mathbb{Z})} K(\mathbb{Z}, 2) \simeq B(\Gamma, S^1, \mathcal{R})^H.$$

Now make the stronger assumption that Γ is a Lie group. Then for every compact subgroup $H \subseteq \Gamma$ the homogeneous space Γ/H is a smooth manifold and hence a CW-complex. This implies that $E\Gamma \times_\Gamma B(\Gamma, G, \mathcal{R})$ is a CW-complex. Because of Theorem 6.3 and Theorem 11.5 we have the universal Γ-equivariant G-bundle with respect to $\mathcal{R}$

$$p\colon E(\Gamma, G, \mathcal{R}) \to B(\Gamma, G, \mathcal{R})$$

Applying the homotopy quotient with respect to the group Γ we obtain a principal G-bundle over a CW-complex

$$G \to E\Gamma \times_\Gamma E(\Gamma, G, \mathcal{R}) \to E\Gamma \times_\Gamma B(\Gamma, G, \mathcal{R})$$

which can be classified by a map

$$E\Gamma \times_\Gamma B(\Gamma, G, \mathcal{R}) \to BG.$$

This map induces an adjoint map

$$\psi\colon B(\Gamma, G, \mathcal{R}) \to \operatorname{map}(E\Gamma, BG)$$

which is Γ-equivariant. In general the map ψ is not a Γ-equivariant homotopy equivalence, but under the specific choices of Γ, G and $\mathcal{R}$ above indeed they are.

Theorem 14.6. *Let Γ be a Lie group and let G be a compact abelian Lie group. Put $\mathcal{R} = \mathcal{R}(\mathcal{COM})$ for $\mathcal{COM}$ the family of compact subgroups of Γ. Let X be a proper Γ-CW complex.*

Then the map

$$\operatorname{map}(\mathrm{id}_X, \psi)\colon \operatorname{map}(X, B(\Gamma, G, \mathcal{R})) \to \operatorname{map}(X, \operatorname{map}(E\Gamma, BG)), \quad f \mapsto \psi \circ f$$

is a weak Γ-homotopy equivalence.

In particular we obtain bijections

$$\mathrm{Bundle}_{\Gamma,G,\mathcal{R}}(X) \cong [X, B(\Gamma,G,\mathcal{R}))]^{\Gamma} \xrightarrow{\cong} [X, \mathrm{map}(E\Gamma, BG)]^{\Gamma} = [E\Gamma \times_{\Gamma} X, BG],$$

and, when $G = S^1$

$$\mathrm{Bundle}_{\Gamma,S^1,\mathcal{R}}(X) \cong H^2(E\Gamma \times_{\Gamma} X, \mathbb{Z}).$$

Proof. For H a compact subgroup of Γ, we claim that the induced map

$$\psi^H \colon B(\Gamma,G,\mathcal{R})^H \to \mathrm{map}(E\Gamma, BG)^H \simeq \mathrm{map}(BH, BG)$$

is a weak homotopy equivalence; its proof is based on the proofs and results of [19, Theorem 2 and Proposition4]. From Theorem 13.1 we know that

$$B(\Gamma,G,\mathcal{R})^H \cong \coprod_{\alpha \in \hom(H,G)} B(\Gamma,G,\mathcal{R})^H_{\alpha},$$

and since G is abelian we know that

$$E(\Gamma,G,\mathcal{R})^{K(H,\alpha)} \to B(\Gamma,G,\mathcal{R})^H_{\alpha}$$

is a G-principal bundle. Therefore we have a commutative square of principal G-bundles

$$\begin{tikzcd} E(\Gamma,G,\mathcal{R})^{K(H,\alpha)} \arrow[r] \arrow[d] & EG \arrow[d] \\ B(\Gamma,G,\mathcal{R})^H_{\alpha} \arrow[r, "\simeq"] & BG \end{tikzcd}$$

where the bottom arrow is a homotopy equivalence. This square induces another square of associated principal G-bundles

$$\begin{tikzcd} E\Gamma \times_{K(H,\alpha)} E(\Gamma,G,\mathcal{R})^{K(H,\alpha)} \arrow[r] \arrow[d] & E\Gamma \times_{H,\alpha} EG \arrow[r] \arrow[d] & EG \arrow[d] \\ E\Gamma/H \times B(\Gamma,G,\mathcal{R})^H_{\alpha} \arrow[r, "\simeq"] & E\Gamma/H \times BG \arrow[r] & BG \end{tikzcd} \tag{14.7}$$

where on the upper left corner the group $K(H,\alpha)$ acts on $E\Gamma$ via the canonical isomorphism $H \cong K(H,\alpha)$, on the upper middle term the group H acts on EG via the homomorphism $\alpha \colon H \to G$ and the horizontal maps on the right hand side are classifying maps. In [19, Proof of Theorem 2, page 173] it is shown that the adjoint map of the lower horizontal maps of diagram (14.7)

$$BG \to \mathrm{map}(E\Gamma, EG)^H \simeq \mathrm{map}(BH, BG)$$

is a weak equivalence on basepoint components; therefore the map

$$\psi^H|_{B(\Gamma,G,\mathcal{R})^H_{\alpha}} \colon B(\Gamma,G,\mathcal{R})^H_{\alpha} \to \mathrm{map}(E\Gamma, BG)^H \simeq \mathrm{map}(BH, BG)$$

is also a weak equivalence of basepoint components. In [19, Proposition 4] it is shown that the map

$$B \colon \hom(H,G) \to [BH, BG]$$

given by the classifying space functor is an isomorphism, and since the bundle $E\Gamma \times_{H,\alpha} EG$ is constructed through the action defined by the homomorphism α, then we conclude that the map

$$\psi^H \colon B(\Gamma,G,\mathcal{R})^H \to \mathrm{map}(E\Gamma, BG)^H \simeq \mathrm{map}(BH, BG)$$

is indeed a weak homotopy equivalence.

Hence for every proper Γ-CW-complex Y the induced map

$$\psi_* \colon [Y, B(\Gamma,G,\mathcal{R})]^{\Gamma} \to [Y, \mathrm{map}(E\Gamma, BG)]^{\Gamma}, \quad [f] \mapsto [\psi \circ f] \tag{14.8}$$

is bijective, see [21, Proposition 2.3 on page 35].

In order to show that $\operatorname{map}(\mathrm{id}_X, \psi)$ is a weak Γ-homotopy equivalence, it suffices to show for any Γ-CW-complex Z that the induced map

$$\operatorname{map}(\mathrm{id}_X, \psi)_* \colon [Z, \operatorname{map}(X, B(\Gamma, G, \mathcal{R}))]^\Gamma \to [Z, \operatorname{map}(X, \operatorname{map}(E\Gamma, BG))]^\Gamma, \qquad [f] \mapsto [\operatorname{map}(\mathrm{id}_X, \psi) \circ f]$$

is bijective, see [21, Proposition 2.3 on page 35]. Because of the adjunctions appearing in Subsections 16.4 and the identification $\pi_0(\operatorname{map}(A, B)^\Gamma) = [A, B]^\Gamma$ for Γ-spaces A and B, this is equivalent to showing that adjoint map

$$\psi_* \colon [X \times Z, B(\Gamma, G, \mathcal{R})]^\Gamma \to [X \times Z, \operatorname{map}(E\Gamma, BG)]^\Gamma, \quad [f] \mapsto [\psi \circ f]$$

is bijective, where Γ acts diagonally on $X \times Z$. Since Γ is a Lie group and X is a proper Γ-CW-complex, $X \times Y$ is Γ-homotopy equivalent to a proper Γ-CW-complex Y. Hence $\operatorname{map}(\mathrm{id}_X, \psi)$ is a weak Γ-homotopy equivalence because of the bijectivity of (14.8).

The other claims follow using Theorem 13.1. □

15. The case $G = \mathcal{PU}(\mathcal{H})$ the projective unitary group.

Twisted versions of K-theory may be defined via a specific type of projective unitary bundles. The key point is that the space $\operatorname{Fred}(\mathcal{H})$ of Fredholm operators on a separable Hilbert space $\mathcal{H}$ endowed with the norm topology, which itself has the homotopy type of $\mathbb{Z} \times BU$ [4, 12], carries a conjugation action by the group $\mathcal{PU}(\mathcal{H})$ of projective unitary operators. Thus, to a pair (X, P) of a CW-complex X together with a principal $\mathcal{PU}(\mathcal{H})$-bundle

$$\mathcal{PU}(\mathcal{H}) \to P \to X$$

over X, one can associate the *twisted K-theory groups* $K^{-i}(X, P)$ (see [2]) defined as the homotopy groups

$$K^{-i}(X, P) := \pi_i\big(\Gamma(P \times_{\mathcal{PU}(\mathcal{H})} \operatorname{Fred}(\mathcal{H}))\big)$$

of the space of sections of the associated $\operatorname{Fred}(\mathcal{H})$-bundle

$$\operatorname{Fred}(\mathcal{H}) \to P \times_{\mathcal{PU}(\mathcal{H})} \operatorname{Fred}(\mathcal{H}) \to X.$$

The equivariant version of the previous construction requires equivariant projective unitary bundles of a certain kind, and in order to construct their universal and classifying space we need to show that the group $\mathcal{PU}(\mathcal{H})$ satisfies items (i), (ii) and (iv) of Condition (H) introduced in Definition 6.1. In what follows, we will show that $\mathcal{PU}(\mathcal{H})$ satisfies items (i), (ii) and (iv) of Condition (H), whenever we consider homomorphisms $\alpha \colon H \to \mathcal{PU}(\mathcal{H})$ from finite groups H.

15.1. **Existence of local cross sections for H finite.** Let $\mathcal{U}(\mathcal{H})$ and $\mathcal{PU}(\mathcal{H})$ be respectively the unitary and projective unitary groups of a separable Hilbert space $\mathcal{H}$. The group $\mathcal{U}(\mathcal{H})$ is defined as

$$\mathcal{U}(\mathcal{H}) := \{U \in \mathcal{B}(\mathcal{H}) \mid UU^* = U^*U = 1\}$$

where $\mathcal{B}(\mathcal{H})$ denotes the space of bounded operators on $\mathcal{H}$, its center $Z(\mathcal{U}(\mathcal{H}))$ is S^1, and $\mathcal{PU}(\mathcal{H})$ is the quotient $\mathcal{U}(\mathcal{H})/S^1$. We endow $\mathcal{U}(\mathcal{H})$ with the norm topology, i.e., a sub-base for the topology is given by the sets

$$B_\epsilon(T) := \{S \in \mathcal{U}(\mathcal{H}) \mid \|S - T\| < \epsilon\}$$

where

$$\|T\| := \sup\{\|Tx\| \mid x \in \mathcal{H} \text{ such that } \|x\| \leq 1\}.$$

Endow $\mathcal{PU}(\mathcal{H})$ with the quotient topology and note this topology can be recovered with the metric defined by the distance between the S^1-orbits, i.e., for $T, U \in \mathcal{PU}(\mathcal{H})$ define

$$(T,U)_{\min} := \min\{\|\widetilde{T} - \widetilde{U}\| \mid \widetilde{T}, \widetilde{U} \in \mathcal{U}(\mathcal{H}) \text{ lifts of } T, U \text{ respectively}\}.$$

With these topologies the groups $\mathcal{U}(\mathcal{H})$ and $\mathcal{PU}(\mathcal{H})$ become a topological groups. The short exact sequence of topological groups

$$1 \to S^1 \to \mathcal{U}(\mathcal{H}) \xrightarrow{p} \mathcal{PU}(\mathcal{H}) \to 1$$

is a S^1-principal bundle (cf. [31]).

The group $\mathcal{U}(\mathcal{H})$ endowed with the norm topology is moreover a Banach Lie group*; namely, $\mathcal{U}(\mathcal{H})$ is a Banach manifold whose structural maps are maps of Banach manifolds, see for instance [28, Example V.1.6, page 391]. It can be modeled locally by the vector space $\mathcal{A}$ of skew-adjoint operators

$$\mathcal{A} := \{L \in \mathcal{B}(\mathcal{H}) \mid L + L^* = 0\};$$

via the exponential map

$$\exp\colon \mathcal{A} \to \mathcal{U}(\mathcal{H}), \quad L \mapsto \exp(L);$$

in this way we could think of the skew-adjoint operators as the tangent space of $\mathcal{U}(\mathcal{H})$ at the identity, $T_1\mathcal{U}(\mathcal{H}) = \mathcal{A}$.

This in particular implies that a base of open sets around $1 \in \mathcal{U}(\mathcal{H})$ may be obtained by the image of the exponential map of a base of open sets around $0 \in \mathcal{A}$. Denoting by $i\mathbb{R}$ the operators of the form $r\sqrt{-1} \cdot \mathrm{id}_{\mathcal{H}}$ for $r \in \mathbb{R}$, we obtain the diagram

$$\begin{tikzcd}
0 \arrow[r] & i\mathbb{R} \arrow[r] \arrow[d, "\exp"] & \mathcal{A} \arrow[r, "\pi"] \arrow[d, "\exp"] & \mathcal{A}/i\mathbb{R} \arrow[r] & 0 \\
1 \arrow[r] & S^1 \arrow[r] & \mathcal{U}(\mathcal{H}) \arrow[r, "p"] & \mathcal{PU}(\mathcal{H}) \arrow[r] & 1
\end{tikzcd} \tag{15.1}$$

which in particular permits us to model locally $\mathcal{PU}(\mathcal{H})$ through any section

$$\begin{tikzcd}
0 \arrow[r] & i\mathbb{R} \arrow[r] & \mathcal{A} \arrow[r, "\pi"'] & \mathcal{A}/i\mathbb{R} \arrow[l, bend right, "\sigma"'] \arrow[r] & 0
\end{tikzcd}$$

by the composition

$$p \circ \exp \circ \sigma\colon \mathcal{A}/i\mathbb{R} \to \mathcal{PU}(\mathcal{H})$$

making the following diagram commutative

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\pi"'] \arrow[d, "\exp"] & \mathcal{A}/i\mathbb{R} \arrow[l, bend right, "\sigma"'] \arrow[d, "p\circ\exp\circ\sigma"] \\
\mathcal{U}(\mathcal{H}) \arrow[r, "p"] & \mathcal{PU}(\mathcal{H}).
\end{tikzcd}$$

Now let H be a finite group. Recall that $\hom(H, \mathcal{PU}(\mathcal{H}))$ obtains the subspace topology from its embedding into $\mathrm{map}(H, \mathcal{PU}(\mathcal{H}))$, see Subsection 16.5; since $\mathcal{PU}(\mathcal{H})$ is metric, this topology can also be defined with the supremums metric of (17.3).

*A reference for the foundations of Banach manifolds may be found on [16], and a reference on the properties the properties of Banach Lie groups and their Banach Lie algebras may be found on [7, 28] and the references therein.

Consider the conjugation action

$$\begin{aligned} \mathcal{PU}(\mathcal{H}) \times \hom(H, \mathcal{PU}(\mathcal{H})) &\to \hom(H, \mathcal{PU}(\mathcal{H})) \\ (g, \alpha) &\mapsto g\alpha g^{-1} \end{aligned}$$

and for $\alpha \in \hom(H, \mathcal{PU}(\mathcal{H}))$ denote by

$$\mathcal{PU}(\mathcal{H}) \cdot \alpha := \{g\alpha g^{-1} \mid g \in \mathcal{PU}(\mathcal{H})\}$$

the orbit of α under the conjugation action. We claim

Theorem 15.2. *For any finite group H and any $\alpha \in \hom(H, \mathcal{PU}(\mathcal{H}))$, the projection*

$$\mathrm{pr}\colon \mathcal{PU}(\mathcal{H}) \to \mathcal{PU}(H)/C_{\mathcal{PU}(\mathcal{H}))}(\alpha)$$

is a principal $C_{\mathcal{PU}(\mathcal{H})}(\alpha)$-bundle and the canonical map

$$\iota_\alpha\colon \mathcal{PU}(H)/C_{\mathcal{PU}(\mathcal{H})}(\alpha) \xrightarrow{\cong} \mathcal{PU}(H) \cdot \alpha$$

is a homeomorphism.

Its proof needs some preparation. For $\alpha\colon H \to \mathcal{PU}(\mathcal{H})$ denote by

$$\widetilde{H} := \alpha^* \mathcal{U}(\mathcal{H})$$

the central S^1-extension of H defined by the pullback of α making the following diagram commutative

$$\begin{tikzcd} 1 \arrow[r] & S^1 \arrow[r] \arrow[d, "="] & \widetilde{H} \arrow[r, "\gamma"] \arrow[d, "\widetilde{\alpha}"] & H \arrow[r] \arrow[d, "\alpha"] & 1 \\ 1 \arrow[r] & S^1 \arrow[r] & \mathcal{U}(\mathcal{H}) \arrow[r] & \mathcal{PU}(\mathcal{H}) \arrow[r] & 1, \end{tikzcd} \tag{15.3}$$

Denote the space of homomorphisms from $\widetilde{H}$ to $\mathcal{U}(\mathcal{H})$ on which the kernel of the map $\widetilde{H} \xrightarrow{\gamma} H$ acts by multiplication

$$\hom_{S^1}(\widetilde{H}, \mathcal{U}(\mathcal{H})) := \{f \in \hom(\widetilde{H}, \mathcal{U}(\mathcal{H})) \mid f(x) = x \text{ for all } x \in \ker(\gamma)\},$$

and endow it with subspace topology $\hom_{S^1}(\widetilde{H}, \mathcal{U}(\mathcal{H})) \subseteq \hom(\widetilde{H}, \mathcal{U}(\mathcal{H}))$. (This is automatically compactly generated by the next result.)

Lemma 15.4. *The space $\hom_{S^1}(\widetilde{H}, \mathcal{U}(\mathcal{H}))$ is closed and open in $\hom(\widetilde{H}, \mathcal{U}(\mathcal{H}))$.*

Proof. Consider the restriction map

$$\begin{aligned} A : \hom(\widetilde{H}, \mathcal{U}(\mathcal{H})) &\to \hom(S^1, \mathcal{U}(\mathcal{H})) \\ f &\mapsto f|_{\ker(\gamma)} \end{aligned}$$

and note that $\hom_{S^1}(\widetilde{H}, \mathcal{U}(\mathcal{H})) = A^{-1}(\rho)$ where $\rho(\lambda)(x) = \lambda \cdot x$ for any $x \in \mathcal{H}$.

Take any other $\beta \in \hom(S^1, \mathcal{U}(\mathcal{H}))$ and since it defines a representation of S^1 different from ρ, there must exists an element $x \in \mathcal{H}$ of norm 1 and an integer $k \neq 1$ such that $\beta(\lambda)x = \lambda^k x$. We have then

$$\|\rho, \beta\|_{\sup} \geq \sup_{\lambda \in S^1} \|\rho(\lambda)x - \beta(\lambda)x\| = \sup_{\lambda \in S^1} |1 - \lambda^{k-1}| > 1$$

and therefore we can conclude that ρ is an isolated point in $\hom(S^1, \mathcal{U}(\mathcal{H}))$. Hence $A^{-1}(\rho)$ is closed and open. $\square$

Define also the space of homomorphisms from H to $\mathcal{PU}(\mathcal{H})$ which induce isomorphic central S^1-extensions of H

$$\hom(H,\mathcal{PU}(\mathcal{H}))_{\widetilde{H}} := \{\beta\colon H \to \mathcal{PU}(\mathcal{H}) \mid \beta^*\mathcal{U}(\mathcal{H}) \cong \widetilde{H} \text{ as central } S^1\text{-extensions of } H\}$$

and endow it with the subspace topology of $\hom(H,\mathcal{PU}(\mathcal{H}))$. (This is automatically compactly generated by the next result.)

Lemma 15.5. *The space* $\hom(H,\mathcal{PU}(\mathcal{H}))_{\widetilde{H}}$ *is closed and open in* $\hom(H,\mathcal{PU}(\mathcal{H}))$.

Proof. The isomorphic classes of central extensions of a finite group are in one to one correspondence with the group $H^2(H,S^1)$. This cohomology group can be calculated by the continuous cohomology [30, Example 1.4] as the quotient of the topological groups

$$\frac{\ker\big(\mathrm{map}(H^2,S^1) \xrightarrow{\delta} \mathrm{map}(H^3,S^1)\big)}{\mathrm{im}\big(\mathrm{map}(H,S^1) \xrightarrow{\delta} \mathrm{map}(H^2,S^1)\big)}$$

where δ denotes the standard group cohomology differential. For a homomorphism $\alpha \in \hom(H,\mathcal{PU}(\mathcal{H}))$ define the 2-cocycle $\widetilde{c}_\alpha \in \mathrm{map}(H^2,S^1)$

$$\widetilde{c}_\alpha(g,h) := \widetilde{\alpha(g)}\widetilde{\alpha(h)}\widetilde{\alpha(gh)}^{-1}$$

where $\widetilde{\alpha(g)} \in \mathcal{U}(\mathcal{H})$ denotes fixed lifts of $\alpha(g)$ for all $g \in H$. If

$$\overline{c}_\alpha(g,h) := \overline{\alpha(g)}\ \overline{\alpha(h)}\ \overline{\alpha(gh)}^{-1}$$

denotes the 2-cocycle defined by a different choice of lifts, then, since $\overline{\alpha(g)}\widetilde{\alpha(g)}^{-1}$ is in the center of $\mathcal{U}(\mathcal{H})$ for all $g \in H$, the map $e \in \mathrm{map}(H,S^1)$

$$e(g) := \overline{\alpha(g)}\widetilde{\alpha(g)}^{-1}$$

satisfies the equation $\delta(e)\cdot\widetilde{c}_\alpha = \overline{c}_\alpha$. Therefore we have the equality cohomology classes $[\widetilde{c}_\alpha] = [\overline{c}_\alpha]$. Thus we can define a map

$$\begin{aligned}\rho : \hom(H,\mathcal{PU}(\mathcal{H})) &\to H^2(H,S^1)\\ \alpha &\mapsto [\widetilde{c}_\alpha]\end{aligned}$$

Next we prove that it is continuous. Since H is finite and $\mathcal{U}(\mathcal{H}) \to \mathcal{PU}(\mathcal{H})$ is a principal S^1-bundle, we can define a small neighborhood V of α in $\hom(H,\mathcal{PU}(\mathcal{H}))$ such that for every h there exists a continuous map $V \to \mathcal{U}$, $\beta \mapsto \widetilde{\beta(h)}$ whose value for $\beta = \alpha$ is the given lift $\widetilde{\alpha(h)}$ and whose composite with the projection $\mathcal{U}(H) \to \mathcal{PU}(H)$ is the evaluation map $\beta \mapsto \beta(h)$. Now we can define a continuous map

$$\begin{aligned}&V \to \mathrm{map}(H^2,S^1)_0 := \ker(\mathrm{map}(H^2,S^1) \xrightarrow{\delta} \mathrm{map}(H^3,S^1))\\ &\beta \mapsto \widetilde{c}_\beta := \widetilde{\beta(g)}\widetilde{\beta(h)}\widetilde{\beta(gh)}^{-1}.\end{aligned}$$

Since the quotient map

$$\mathrm{map}(H^2,S^1)_0 \to \mathrm{map}(H^2,S^1)_0/Im(\delta) = H^2(H,S^1)$$

is continuous, and $H^2(H,S^1)$ is finite, we conclude that the map ρ is continuous.

Any 2-cocycle $\widetilde{c}_\alpha$ defines a S^1-central extension $H \times_{\widetilde{c}_\alpha} S^1$ of H, and it is classical result in group theory that $\widetilde{H}$ and $H \times_{\widetilde{c}_\alpha} S^1$ are isomorphic as central extensions of H.

Therefore $\hom(H,\mathcal{PU}(\mathcal{H}))_{\widetilde{H}} = \rho^{-1}([\widetilde{c}_\alpha])$ and hence it is open and closed in $\hom(H,\mathcal{PU}(\mathcal{H}))$.

□

Before we proceed with the study of the canonical map $\phi\colon \hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H})) \to \hom(H,\mathcal{PU}(\mathcal{H}))_{\widetilde{H}}$, we will make use of Banach Lie group structure of $\mathcal{U}(\mathcal{H})$ to prove the following lemma.

Lemma 15.6. *For $n > 1$ let $\mathcal{U}(\mathcal{H})_n := \{U \in \mathcal{U}(\mathcal{H})\colon U^n = 1\}$ and $\mathcal{PU}(\mathcal{H})_n := \{U \in \mathcal{PU}(\mathcal{H}) \mid U^n = 1\}$. Then the projection map*

$$p|_{\mathcal{U}(\mathcal{H})_n}\colon \mathcal{U}(\mathcal{H})_n \to \mathcal{PU}(\mathcal{H})_n$$

is a local homeomorphism.

Proof. The subspace $\mathcal{U}(\mathcal{H})_n$ is a Banach submanifold of $\mathcal{U}(\mathcal{H})$ since it is the inverse image $f^{-1}(\{1\}) = \mathcal{U}(\mathcal{H})_n$ for the function

$$f\colon \mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H}), \quad U \mapsto U^n,$$

that is analytical. Take $U \in \mathcal{U}(\mathcal{H})_n$ and consider the isomorphism of tangent spaces

$$R_{U^{-1}}\colon T_U\mathcal{U}(\mathcal{H}) \xrightarrow{\cong} T_1\mathcal{U}(\mathcal{H}), \quad L \mapsto LU^{-1}.$$

Since the tangent space of $U \in \mathcal{U}(\mathcal{H})_n$ at U is

$$T_U\mathcal{U}(\mathcal{H})_n = \{L \in \mathcal{B}(\mathcal{H}) \mid LU^* + UL^* = 0 \text{ and } \sum_{k=0}^{n-1} U^k L U^{n-k-1} = 0\},$$

its image under $R_{U^{-1}}$ becomes

$$R_{U^{-1}}T_U\mathcal{U}(\mathcal{H})_n = \{A + A^* = 0 \text{ and } \sum_{k=0}^{n-1} U^k A U^{n-k} = 0\}.$$

The vector space $i\mathbb{R}$ of diagram (15.1) is not included in $R_{U^{-1}}T_U\mathcal{U}(\mathcal{H})_n$ since for $A = r\sqrt{-1}$ with $r \neq 0$ the sum $\sum_{k=0}^{n-1} U^k A U^{n-k} = nr\sqrt{-1} \neq 0$. Therefore the vector space $R_{U^{-1}}T_U\mathcal{U}(\mathcal{H})_n$ maps isomorphically to $\pi(R_{U^{-1}}T_U\mathcal{U}(\mathcal{H})_n)$ under the map π of diagram (15.1). By the inverse function theorem for Banach manifolds [16, Theorem 1 §I.5, page 13] the map p restricted to $\mathcal{U}(\mathcal{H})_n \cdot U^{-1}$ has a local inverse around 1, and hence the map $p|_{\mathcal{U}(\mathcal{H})_n}\colon \mathcal{U}(\mathcal{H})_n \to \mathcal{PU}(\mathcal{H})_n$ has a local inverse around a neighborhood of U in $\mathcal{U}(\mathcal{H})_n$. □

Proposition 15.7. *The canonical map*

$$\phi\colon \hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H})) \to \hom(H,\mathcal{PU}(\mathcal{H}))_{\widetilde{H}}$$
$$f\colon \widetilde{H} \to \mathcal{U}(\mathcal{H}) \mapsto f/S^1\colon \widetilde{H}/S^1 \to \mathcal{U}(\mathcal{H})/S^1$$

is a principal $\hom(H,S^1)$*-bundle, and since* $\hom(H,S^1)$ *is finite, it is also a local homeomorphism.*

Proof. The map $\phi\colon \hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H})) \to \hom(H,\mathcal{PU}(\mathcal{H}))_{\widetilde{H}}$ is continuous because of Lemma 16.4. One easily checks that it is surjective.

Define the $\hom(H,S^1)$-action on $\hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H}))$ by the map

$$\hom(H,S^1) \times \hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H})) \to \hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H})), \quad (r,f) \mapsto r \cdot f,$$

where $(r \cdot f)(\widetilde{h}) := r(\gamma(\widetilde{h})) \cdot f(\widetilde{h})$ for all $\widetilde{h} \in \widetilde{H}$. One easily checks that the action is free and that ϕ and $\phi(f) = \phi(r \cdot f)$ holds for all $r \in \hom(H,S^1)$ and $f \in \hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H}))$. Hence ϕ induces a surjective map

$$\overline{\phi}\colon \hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H}))/\hom(H,S^1) \to \hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H}))$$

Next we show that $\overline{\phi}$ is injective and hence ϕ is locally bijective. Consider $f, g \in \hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H}))$ with $\phi(f) = \phi(g)$. For all $h \in H$, take any lift $\widetilde{h} \in \widetilde{H}$ such that $\gamma(\widetilde{h}) = h$ and define the element $r(h) \in S^1$ by the equation $f(\widetilde{h}) = r(h)g(\widetilde{h})$. Note

that $r(h)$ does not depend on the choice of lift and therefore we obtain a map $r\colon H \to S^1$. For $h, k \in H$ with respective lifts $\widetilde{h}, \widetilde{k} \in \widetilde{H}$ one has

$$r(hk)g(\widetilde{h}\widetilde{k}) = f(\widetilde{h}\widetilde{k}) = r(h)g(\widetilde{h})r(k)g(\widetilde{k}) = r(h)r(k)g(\widetilde{h}\widetilde{k}),$$

and therefore we see that $r \in \hom(H, S^1)$ and $r \cdot f = g$.

We are left with showing that ϕ is a local homeomorphism.

For $h \in H$ denote by $|h|$ its order and by $\langle h \rangle$ the cyclic group it generates. Since all central S^1-extensions of cyclic groups are trivializable, the restriction of $\widetilde{H}$ to $\langle h \rangle$ must be isomorphic to $S^1 \times \langle h \rangle$; therefore we can choose a lift $\widetilde{h} \in \widetilde{H}$ for each $h \in H$ such that $\widetilde{h}$ generates a cyclic group of order $|h|$.

Define the restriction maps

$$\Psi\colon \hom_{S^1}(\widetilde{H}, \mathcal{U}(\mathcal{H})) \to \prod_{h \in H} \hom(\langle \widetilde{h} \rangle, \mathcal{U}(\mathcal{H})), \qquad \widetilde{\alpha} \mapsto \prod_{h \in H} \widetilde{\alpha}|_{\langle \widetilde{h} \rangle};$$

$$\psi\colon \hom(H, \mathcal{PU}(\mathcal{H}))_{\widetilde{H}} \to \prod_{h \in H} \hom(\langle h \rangle, \mathcal{PU}(\mathcal{H})), \qquad \alpha \mapsto \prod_{h \in H} \alpha|_{\langle h \rangle},$$

and note that both maps Ψ and ψ are injective. We claim moreover that the map Ψ induces a homeomorphism onto its image. The proof is as follows: for $\epsilon > 0$ and $\widetilde{\alpha} \in \hom_{S^1}(\widetilde{H}, \mathcal{U}(\mathcal{H}))$, consider the open ball of radius ϵ defined by the supremum metric of (17.3)

$$B_\epsilon(\widetilde{\alpha}) := \{f \in \hom_{S^1}(\widetilde{H}, \mathcal{U}(\mathcal{H})) \mid \|\widetilde{\alpha}(g) - f(g)\| < \epsilon \text{ for all } g \in \widetilde{H}\},$$

and note that this open ball can also be defined as

$$B_\epsilon(\widetilde{\alpha}) := \{f \in \hom_{S^1}(\widetilde{H}, \mathcal{U}(\mathcal{H})) \mid \|\widetilde{\alpha}(\widetilde{h}) - f(\widetilde{h})\| < \epsilon \text{ for all } h \in H\},$$

since for all $\lambda \in S^1$ we have that

$$\|\widetilde{\alpha}(\widetilde{h}) - f(\widetilde{h})\| = \|\widetilde{\alpha}(\lambda \cdot \widetilde{h}) - f(\lambda \cdot \widetilde{h})\|.$$

For the restricted homomorphisms $\widetilde{\alpha}|_{\langle \widetilde{h} \rangle}$ we can also consider the open balls of radius ϵ

$$B_\epsilon(\widetilde{\alpha}|_{\langle \widetilde{h} \rangle}) := \{f \in \hom(\langle \widetilde{h} \rangle, \mathcal{U}(\mathcal{H})) \mid \|\widetilde{\alpha}(g) - f(g)\| < \epsilon \text{ for all } g \in \langle \widetilde{h} \rangle\},$$

and therefore we get the following equality of sets

$$\Psi(B_\epsilon(\widetilde{\alpha})) = \left(\prod_{h \in H} B_\epsilon(\widetilde{\alpha}|_{\langle \widetilde{h} \rangle}) \right) \cap \Psi\big(\hom_{S^1}(\widetilde{H}, \mathcal{U}(\mathcal{H}))\big).$$

This implies that the map Ψ induces an open map onto its image, and since it is injective and continuous, it induces a homeomorphism onto its image.

Denoting by $\phi_h\colon \hom(\langle \widetilde{h} \rangle, \mathcal{U}(\mathcal{H})) \to \hom(\langle h \rangle, \mathcal{PU}(\mathcal{H}))$ the canonical map defined by $\phi_h(\gamma)(h) = \gamma(\widetilde{h})$, we obtain the following commutative diagram

$$\begin{tikzcd} \hom_{S^1}(\widetilde{H}, \mathcal{U}(\mathcal{H})) \arrow[r, "\Psi"] \arrow[d, "\phi"'] & \prod_{h \in H} \hom(\langle \widetilde{h} \rangle, \mathcal{U}(\mathcal{H})) \arrow[d, "\prod_h \phi_h"] \\ \hom(H, \mathcal{PU}(\mathcal{H}))_{\widetilde{H}} \arrow[r, "\psi"'] & \prod_{h \in H} \hom(\langle h \rangle, \mathcal{PU}(\mathcal{H})). \end{tikzcd} \tag{15.8}$$

Since there are canonical homeomorphisms

$$\hom(\langle \widetilde{h} \rangle, \mathcal{U}(\mathcal{H})) \xrightarrow{\cong} \mathcal{U}(\mathcal{H}))_{|h|} \quad f \mapsto f(\widetilde{h}),$$

by Lemma 15.6 we know that there must exist ϵ such that the maps

$$\phi_h\colon B_\epsilon(\widetilde{\alpha}|_{\langle \widetilde{h} \rangle}) \to \phi_h(B_\epsilon(\widetilde{\alpha}|_{\langle \widetilde{h} \rangle}))$$

are homeomorphisms for all $h \in H$; denote by

$$\tau_h : \phi_h(B_\epsilon(\widetilde{\alpha}|_{\langle \widetilde{h} \rangle})) \to B_\epsilon(\widetilde{\alpha}|_{\langle \widetilde{h} \rangle})$$

these inverse maps. Restricting to the open subset $B_\epsilon(\widetilde{\alpha})$ of the top left corner of the diagram (15.8), we obtain the following diagram

$$\begin{tikzcd}
B_\epsilon(\widetilde{\alpha}) \arrow[d, "\phi"'] & \Psi(B_\epsilon(\widetilde{\alpha})) \arrow[l, "(\Psi|_{B_\epsilon(\widetilde{\alpha})})^{-1}"', "\cong"] \\
\phi(B_\epsilon(\widetilde{\alpha})) \arrow[r, "\psi"'] & \psi(\phi(B_\epsilon(\widetilde{\alpha}))) \arrow[u, "\cong", "(\prod_h \tau_h)|_{\psi(\phi(B_\epsilon(\widetilde{\alpha})))}"']
\end{tikzcd}$$

where the right hand side vertical arrow is the homeomorphism that the maps τ_h induce once restricted to the open set

$$\psi(\phi(B_\epsilon(\widetilde{\alpha}))) = \left(\prod_h \phi_h(B_\epsilon(\widetilde{\alpha}|_{\langle \widetilde{h} \rangle})) \right) \cap \big(\prod_h \phi_h\big)\Psi(B_\epsilon(\widetilde{\alpha})),$$

and the upper horizontal arrow is the inverse of Ψ restricted to $B_\epsilon(\widetilde{\alpha})$.

Since the map ψ is injective, we can define the map

$$(\Psi|_{B_\epsilon(\widetilde{\alpha})})^{-1} \circ \big(\prod_h \tau_h\big)|_{\psi(\phi(B_\epsilon(\widetilde{\alpha})))} \circ \psi \colon \phi(B_\epsilon(\widetilde{\alpha})) \to B_\epsilon(\widetilde{\alpha})$$

which clearly is the inverse map of ϕ once it is restricted to $B_\epsilon(\widetilde{\alpha})$. This proves that ϕ is a local homeomorphism. □

The conjugation action of $\mathcal{U}(\mathcal{H})$ on $\hom_{S^1}(\widetilde{H}, \mathcal{U}(\mathcal{H}))$ factors through the projection $\mathcal{U}(\mathcal{H}) \to \mathcal{PU}(\mathcal{H})$ to an action

$$\mathcal{PU}(\mathcal{H})) \times \hom_{S^1}(\widetilde{H}, \mathcal{U}(\mathcal{H})) \to \hom_{S^1}(\widetilde{H}, \mathcal{U}(\mathcal{H})),$$

since the conjugation action restricted to the center of $\mathcal{U}(\mathcal{H})$ is trivial.

Lemma 15.9. *Let $\alpha \colon H \to \mathcal{PU}(\mathcal{H})$ and $\widetilde{\alpha} \colon \widetilde{H} \to \mathcal{U}(\mathcal{H})$ be as in diagram* (15.3). *Then the existence of a local cross section for the conjugation map $\mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H}) \cdot \widetilde{\alpha}$ implies the existence of a local cross section of the conjugation map $\mathcal{PU}(\mathcal{H}) \to \mathcal{PU}(\mathcal{H}) \cdot \alpha$.*

Proof. This follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{U}(\mathcal{H}) \arrow[r] \arrow[d] & \mathcal{PU}(\mathcal{H}) \arrow[r, "="] \arrow[d] & \mathcal{PU}(\mathcal{H}) \arrow[d] \\
\mathcal{U}(\mathcal{H}) \cdot \widetilde{\alpha} \arrow[r, "="] & \mathcal{PU}(\mathcal{H}) \cdot \widetilde{\alpha} \arrow[r, "\phi"] & \mathcal{PU}(\mathcal{H}) \cdot \alpha,
\end{tikzcd}$$

where the vertical arrows are defined by conjugation on $\widetilde{\alpha}$ and α respectively, and the conclusion of Proposition 15.7 that the map ϕ is a local homeomorphism. □

Lemma 15.10. *For $\widetilde{\alpha} \in \hom_{S^1}(\widetilde{H}, \mathcal{U}(\mathcal{H}))$ the map induced by the conjugation action*

$$\mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H}) \cdot \widetilde{\alpha}, \quad g \mapsto g\widetilde{\alpha}g^{-1}$$

has a local cross section.

Proof. For $\beta, \gamma \in \hom_{S^1}(\widetilde{H}, \mathcal{U}(\mathcal{H}))$ let

$$T_{\beta,\gamma} := \int_{\widetilde{H}} \beta(\widetilde{h}) \circ \gamma(\widetilde{h}^{-1})\, d\widetilde{h} \quad \in \mathcal{U}(\mathcal{H})$$

be the intertwining operator between β and γ appearing in [8, VII.1.Proposition 21] where $d\widetilde{h}$ is the normalized Haar measure on $\widetilde{H}$. We have

$$\begin{aligned} \beta \circ T_{\beta,\gamma} &= T_{\beta,\gamma} \circ \gamma; \\ T^*_{\beta,\gamma} &= T_{\gamma,\beta}, \end{aligned}$$

since for all $a \in \widetilde{H}$ one has

$$\begin{aligned} T_{\beta,\gamma} \circ \gamma(a) &= \int_{\widetilde{H}} \beta(\widetilde{h}) \circ \gamma(\widetilde{h}^{-1}a)\, d\widetilde{h} \\ &= \int_{\widetilde{H}} \beta(a\widetilde{h}) \circ \gamma(\widetilde{h}^{-1})\, d\widetilde{h} \\ &= \beta(a) \circ T_{\beta.\gamma}, \end{aligned}$$

and

$$\begin{aligned} T^*_{\beta,\gamma} &= \int_{\widetilde{H}} \left(\beta(\widetilde{h}) \circ \gamma(\widetilde{h}^{-1})\right)^* d\widetilde{h} \\ &= \int_{\widetilde{H}} \gamma(\widetilde{h}^{-1})^* \cdot \beta(\widetilde{h})^*\, d\widetilde{h} \\ &= \int_{\widetilde{H}} \gamma(\widetilde{h}^{-1})^{-1} \cdot \beta(\widetilde{h})^{-1}\, d\widetilde{h} \\ &= \int_{\widetilde{H}} \gamma(\widetilde{h}) \cdot \beta(\widetilde{h}^{-1})\, d\widetilde{h} \\ &= T_{\gamma,\beta}. \end{aligned}$$

Notice that since β and γ belong to $\hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H}))$, then the intertwiner $T_{\beta,\gamma}$ can also be defined by the finite sum

$$T_{\beta,\gamma} := \frac{1}{|H|} \cdot \sum_{h \in H} \beta(\widetilde{h}) \circ \gamma(\widetilde{h}^{-1}) \tag{15.11}$$

where $\widetilde{h}$ is any fixed choice of lift of h in $\widetilde{H}$. We can choose an open neighborhood of $\widetilde{\alpha}$

$$V := \{\beta \in \hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H})) \mid \|\alpha(\widetilde{h}) - \beta(\widetilde{h})\| < \frac{1}{|H|} \text{ for all } h \in H\}.$$

The triangle inequality implies that for all β in V we have $\|1 - T_{\beta,\widetilde{\alpha}}\| < 1$ and therefore the operator $T_{\beta,\widetilde{\alpha}}$ is invertible. Hence we get a map

$$\tau\colon V \to GL(\mathcal{H}), \quad \beta \mapsto T_{\beta,\widetilde{\alpha}},$$

whose continuity follows from the finite sum definition of $T_{\beta,\widetilde{\alpha}}$ of (15.11). It satisfies for every $\beta \in V$

$$\begin{aligned} \tau(\beta) \circ \widetilde{\alpha} \circ \tau(\beta)^{-1} &= \beta \\ \tau(\beta)^* \circ \beta &= \widetilde{\alpha} \circ \tau(\beta)^*. \end{aligned}$$

Composing the map τ with the retraction

$$\rho\colon GL(\mathcal{H}) \to \mathcal{U}(\mathcal{H}), \quad T \mapsto T\sqrt{(T^*T)}^{-1}$$

defined in [15, Chapter 4], we obtain a continuous map

$$\sigma := \rho \circ \tau\colon V \to \mathcal{U}(\mathcal{H})$$

from a open neighborhood V of $\widetilde{\alpha}$ in $\hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H}))$ to $\mathcal{U}(\mathcal{H})$ with the desired property

$$\sigma(\beta) \circ \widetilde{\alpha} \circ \sigma(\beta))^{-1} = \beta.$$

Since the action of $\mathcal{U}(\mathcal{H})$ is transitive on $\mathcal{U}(\mathcal{H})\cdot\widetilde{\alpha}$, the translations of the previously defined local cross section define local cross sections around any point in the orbit space. $\square$

Corollary 15.12. *The space* $\hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H}))$ *is homeomorphic to the disjoint union of its orbits under the conjugation action of* $\mathcal{U}(\mathcal{H})$*. Hence each orbit* $\mathcal{U}(\mathcal{H})\cdot\widetilde{\alpha}$ *is open and closed in* $\hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H}))$*.*

Proof. From the proof of Lemma 15.10 we know that for $\alpha,\beta\in\hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H}))$ lying in different orbits there must exist $h\in H$ such that for all lifts $\widetilde{h}$

$$\|\alpha(\widetilde{h})-\beta(\widetilde{h})\|\geq\frac{1}{|H|}$$

and therefore the distance between α and β is greater or equal than $\frac{1}{|H|}$. This implies that different orbits are separated by at least a distance of $\frac{1}{|H|}$ and therefore $\hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H}))$ is homeomorphic to the disjoint union of its orbits under the conjugation action of $\mathcal{U}(\mathcal{H})$. $\square$

Proof of Theorem 15.2. Lemma 15.9 and Lemma 15.10 imply the existence of local cross sections for the map

$$\mathcal{PU}(\mathcal{H})\to\mathcal{PU}(\mathcal{H})\cdot\alpha,\quad g\mapsto g\alpha g^{-1}.$$

This map is the composite of the projection $\mathrm{pr}\colon\mathcal{PU}(\mathcal{H})\to\mathcal{PU}(H)/C_{\mathcal{PU}(\mathcal{H}))}(\alpha)$ with the bijective continuous map $\iota_\alpha\colon\mathcal{PU}(H)/C_{\mathcal{PU}(\mathcal{H}))}(\alpha)\to\mathcal{PU}(H)\cdot\alpha$. This implies that ι_α has local cross sections and hence is a homeomorphism. We conclude that $\mathrm{pr}\colon\mathcal{PU}(\mathcal{H})\to\mathcal{PU}(H)/C_{\mathcal{PU}(\mathcal{H}))}(\alpha)$ has local cross sections and hence is a principal $C_{\mathcal{PU}(\mathcal{H}))}(\alpha)$-bundle by Remark 4.3. This finishes the proof of Theorem 15.2. $\square$

Proposition 15.13. *Let* H *be a finite group and* $\alpha\in\hom(H,\mathcal{PU}(\mathcal{H}))$*. Then items (i), (ii) and (iv) of Condition (H) are satisfied:*

(i) *The path component of* α *in* $\hom(H,\mathcal{PU}(\mathcal{H}))$ *is contained in the orbit* $\mathcal{PU}(\mathcal{H})\cdot\alpha$*;*
(ii) *The canonical map* $\mathcal{PU}(\mathcal{H})\to\mathcal{PU}(\mathcal{H})/C_{\mathcal{PU}(\mathcal{H})}(\alpha)$ *is a* $C_{\mathcal{PU}(\mathcal{H})}(\alpha)$*-principal bundle;*
(iv) *The canonical map*

$$\iota_\alpha\colon\mathcal{PU}(\mathcal{H})/C_{\mathcal{PU}(\mathcal{H})}(\alpha)\to\hom(H,\mathcal{PU}(\mathcal{H})),\quad gC_{\mathcal{PU}(\mathcal{H})}(\alpha)\mapsto g\alpha g^{-1}$$

is a homeomorphism onto its image.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathcal{U}(\mathcal{H})\cdot\widetilde{\alpha} & \hookrightarrow & \hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H})) & \hookrightarrow & \hom(\widetilde{H},\mathcal{U}(\mathcal{H}))\\
\downarrow\phi & & \downarrow\phi & & \downarrow\\
\mathcal{PU}(\mathcal{H})\cdot\alpha & \hookrightarrow & \hom(H,\mathcal{PU}(\mathcal{H}))_{\widetilde{H}} & \hookrightarrow & \hom(H,\mathcal{PU}(\mathcal{H}).
\end{array}$$

By Lemma 15.5 we know that $\hom(H,\mathcal{PU}(\mathcal{H}))_{\widetilde{H}}$ is open and closed in $\hom(H,\mathcal{PU}(\mathcal{H})$, by Proposition 15.7 we know that the map ϕ is a principal $\hom(H,S^1)$-bundle and by Corollary 15.12 we know that $\mathcal{U}(\mathcal{H})\cdot\widetilde{\alpha}$ is both open and closed in $\hom_{S^1}(\widetilde{H},\mathcal{U}(\mathcal{H}))$. This implies that $\mathcal{PU}(\mathcal{H})\cdot\alpha$ is open and closed in $\hom(H,\mathcal{PU}(\mathcal{H})$; this proves (i).

The other two conditions (ii) and (iii) have already been proved in Theorem 15.2. $\square$

15.2. **Almost free Γ-equivariant stable projective unitary bundles.** Let Γ be a Hausdorff topological group and consider the family of all finite subgroups $\mathcal{FIN}(\Gamma)$ of Γ. Let $\mathcal{R} = \mathcal{R}(\mathcal{FIN}(\Gamma))$ be the associated family of local representations

$$\mathcal{R} = \{(H, \alpha) \mid H \in \mathcal{FIN}(\Gamma) \text{ and } \alpha\colon H \to \mathcal{PU}(\mathcal{H}) \text{ any group homomorphism}\}.$$

A Γ-CW-complex X is called *almost free* if all its isotropy groups are finite.

Theorem 15.14 (Universal Γ-equivariant projective unitary bundle for almost free Γ-CW-complexes)**.** *Let Γ be a Hausdorff topological group. Then the bundle*

$$\mathcal{PU}(\mathcal{H}) \to E(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{R}) \xrightarrow{p} B(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{R})$$

is a universal Γ-equivariant projective unitary bundle for almost free Γ-CW-complexes

Proof. The result follows from Theorem 11.5 since items (i), (ii) and (iv) of Condition (H) were proved in Proposition 15.13 and item (iii) follows from the fact that Γ is Hausdorff and the subgroups $H \in \mathcal{FIN}(\Gamma)$ are finite. □

Certainly one can associate a $\mathrm{Fred}(\mathcal{H})$-bundle to any Γ-equivariant projective unitary bundle and one can take the homotopy groups of its Γ-equivariant sections. This procedure applied to some Γ-equivariant projective unitary bundles may fail to produce the expected twisted equivariant K-theory groups, let us see why: for H finite subgroup of Γ and $\alpha\colon H \to \mathcal{PU}(\mathcal{H})$ any group homomorphism, we can construct the Γ-equivariant principal $\mathcal{PU}(\mathcal{H})$-bundle

$$\Gamma \times_\alpha \mathcal{PU}(\mathcal{H}) \to \Gamma/H$$

where $\Gamma \times_\alpha \mathcal{PU}(\mathcal{H})$ is the quotient of $\Gamma \times \mathcal{PU}(\mathcal{H})$ under the left H-action given by $h \cdot (\gamma, g) = (\gamma h^{-1}, \alpha(h)g)$. The associated bundle

$$(\Gamma \times_\alpha \mathcal{PU}(\mathcal{H})) \times_{\mathcal{PU}(\mathcal{H})} \mathrm{Fred}(\mathcal{H}) \cong \Gamma \times_\alpha \mathrm{Fred}(\mathcal{H})$$

is equivalent to the quotient of $\Gamma \times \mathrm{Fred}(\mathcal{H})$ under the left H-action given by $h \cdot (\gamma, F) = (\gamma h^{-1}, \alpha(h) F \alpha(h)^{-1})$. Therefore the space of Γ-equivariant sections of the bundle

$$\Gamma \times_\alpha \mathrm{Fred}(\mathcal{H}) \to \Gamma/H$$

is homeomorphic to the space of α-invariant operators

$$\mathrm{Fred}(\mathcal{H})^\alpha := \{F \in \mathrm{Fred}(\mathcal{H}) \mid \alpha(h) F \alpha(h)^{-1} = F \text{ for all } h \in H\}.$$

Now, the index map

$$\mathrm{index}\colon \mathrm{Fred}(\mathcal{H})^\alpha \to R(\widetilde{H}) \qquad F \mapsto \ker(F) - \mathrm{coker}(F)$$

maps an α-invariant operator to an element of the Grothendieck ring of representations of $\widetilde{H} := \alpha^* \mathcal{U}(\mathcal{H})$, group which was defined in diagram (15.3).

In order for the homotopy groups of $\mathrm{Fred}(\mathcal{H})^\alpha$ to represent the local coefficients for twisted equivariant K-theory, we know that we need to impose two conditions on the homomorphism $\alpha\colon H \to \mathcal{PU}(\mathcal{H})$:

- The image of the index map is a subgroup of $R(\widetilde{H})$; this in order for the connected components $\pi_0(\mathrm{Fred}(\mathcal{H})^\alpha)$ to be a group, and
- All the representations of $\widetilde{H}$, on which $S^1 = \ker(\widetilde{H} \to H)$ acts by multiplication, must appear on the image of the index map; this in order to obtain a theory compatible with restriction of representations.

If a homomorphism α satisfies these two conditions, then the index map

$$\mathrm{index}\colon \pi_0(\mathrm{Fred}(\mathcal{H})^\alpha) \xrightarrow{\cong} R_\alpha(H),$$

induces an isomorphism between the path components of $\mathrm{Fred}(\mathcal{H})^\alpha$ and the Grothendieck group

$$R_\alpha(H) := \{V \in R(\widetilde{H}) \mid S^1 = \ker(\widetilde{H} \to H) \text{ acts by multiplication on } V\}$$

of so called α-twisted H-representations

Homomorphisms that satisfy these two conditions will be called *stable*, cf. [2, Section 6].

Definition 15.15 (Stable homomorphism)**.** A homomorphism $\alpha\colon H \to \mathcal{PU}(\mathcal{H})$ is *stable* if all irreducible representations of $\widetilde{H}$ on which $S^1 = \ker(\widetilde{H} \to H)$ acts by multiplication appear infinitely number of times.

Define the set $\mathcal{S}$ of stable local representations for almost free Γ-actions

$$\mathcal{S} = \{(H, \alpha) \mid H \in \mathcal{FIN}(\Gamma) \text{ and } \alpha\colon H \to \mathcal{PU}(\mathcal{H}) \text{ is a stable homomorphism}\}.$$

Note that $\mathcal{S}$ is indeed a family of local representations since conjugation of stable homomorphisms is also stable and the restriction of a stable homomorphism to a subgroup is stable; this last statement follows from the Frobenius Reciprocity Theorem since all irreducible $\alpha|_K$-twisted representations of K subgroup of H may be obtained from the α-twisted representations of H.

Now let B be an almost free Γ-CW-complex. Then a Γ-equivariant principle $\mathcal{PU}(\mathcal{H})$-bundle $p\colon E \to B$ whose family of local representations is contained in $\mathcal{S}$ is the same as a Γ-equivariant stable projective unitary bundle over B in the sense of [5, Definition 2.2]. Note that the existence of the local data appearing in [5, Definition 2.2] is automatically satisfied by Theorem 9.1.

We obtain from Theorem 11.5 a universal Γ-equivariant stable projective unitary bundle $p\colon E(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S}) \to B(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S})$ such that for any almost free Γ-CW-complex X the pullback construction yields a bijection from $\big[X, B(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S})\big]^\Gamma$ to the set of isomorphism classes of Γ-equivariant stable projective unitary bundles over X. This generalizes [5, Theorem 3.4], where Γ is assumed to be discrete. Summarizing:

Theorem 15.16 (Universal Γ-equivariant stable projective bundle)**.** *Let Γ be a Hausdorff topological group. Then the bundle*

$$\mathcal{PU}(\mathcal{H}) \to E(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S}) \xrightarrow{p} B(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S})$$

is a universal Γ-equivariant stable projective unitary bundle for almost free Γ-CW-complexes.

For X a proper Γ-CW-complex, the set of the isomorphism classes of projective unitary Γ-equivariant stable bundles over X has been denoted by $\mathrm{Bun}^\Gamma_{st}(X, \mathcal{PU}(\mathcal{H}))$ in [5, Definition 2.2]; we have then the canonical isomorphisms of sets

$$\mathrm{Bun}^\Gamma_{st}(X, \mathcal{PU}(\mathcal{H})) = \mathrm{Bundle}_{\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S}}(X) \xrightarrow{\cong} \big[X, B(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S})\big]^\Gamma.$$

Therefore the Γ-equivariant stable projective unitary bundle

$$\mathcal{PU}(\mathcal{H}) \to E(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S}) \xrightarrow{p} B(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S})$$

is the universal twist in equivariant K-theory for almost free Γ-CW-complexes. In particular, for a fixed Γ-equivariant map $f\colon X \to B(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S})$ with X almost free Γ-CW-complex, the twisted equivariant K-theory groups of the pair (X, f) can be defined as

$$K^{-i}_\Gamma(X, f) := \pi_i\big(\mathrm{Sections}(f^* E(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S}) \times_{\mathcal{PU}(\mathcal{H})} \mathrm{Fred}(\mathcal{H}))^\Gamma\big),$$

namely, as the homotopy groups of the space of Γ-equivariant sections of the associated $\mathrm{Fred}(\mathcal{H})$-bundle; see [5, Appendix A] for further properties of the twisted equivariant K-theory groups defined in this way.

Now, in the case that Γ is furthermore a Lie group we can show that there is an isomorphism between the set $\mathrm{Bundle}^{\Gamma}_{st}(X, \mathcal{PU}(\mathcal{H}))$ and $H^3(E\Gamma \times_\Gamma X, \mathbb{Z})$.

Theorem 15.17. *Let Γ be a Lie group and X be an almost free Γ-CW complex.*

Then the map

$$\mathrm{map}(\mathrm{id}_X, \psi)\colon \mathrm{map}(X, B(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S})) \to \mathrm{map}(X, \mathrm{map}(E\Gamma, B\mathcal{PU}(\mathcal{H}))), \quad f \mapsto \psi \circ f$$

is a weak Γ-homotopy equivalence.

In particular we obtain bijections

$$\mathrm{Bundle}_{\Gamma, \mathcal{PU}(\mathcal{H}, \mathcal{S}}(X) \cong [X, B(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S})]^\Gamma \xrightarrow{\cong} [X, \mathrm{map}(E\Gamma, B\mathcal{PU}(\mathcal{H}))]^\Gamma$$
$$[E\Gamma \times_\Gamma X, B\mathcal{PU}(\mathcal{H})] \cong H^3(E\Gamma \times_\Gamma X, \mathbb{Z}).$$

Proof. For H a finite group of Γ we just need to show that the induced map

$$\psi^H\colon B(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S})^H \to \mathrm{map}(E\Gamma, B\mathcal{PU}(\mathcal{H}))^H \simeq \mathrm{map}(BH, B\mathcal{PU}(\mathcal{H}))$$

is a weak homotopy equivalence; the rest of the proof is formal and it is equivalent to the one that appears in the proof of Theorem 14.6.

Let $\hom_{\mathcal{S}}(H, \mathcal{PU}(\mathcal{H}))$ be the set of stable homomorphisms. In [5, Proposition 1.5] it is shown that the map

$$\hom_{\mathcal{S}}(H, \mathcal{PU}(\mathcal{H}))/\mathcal{PU}(\mathcal{H}) \xrightarrow{\cong} \mathrm{Ext}(H, S^1), \quad \alpha \mapsto \widetilde{H} = \alpha^* \mathcal{U}(\mathcal{H})$$

is an isomorphism of sets, where $\mathrm{Ext}(H, S^1)$ denotes the set of isomorphism classes of central S^1-extensions of H. Now, we have the isomorphism of sets

$$\mathrm{Ext}(H, S^1) \cong H^3(BH, \mathbb{Z}) \cong [BH, B\mathcal{PU}(\mathcal{H})]$$

since $\mathcal{PU}(\mathcal{H})$ is $K(\mathbb{Z}, 2)$-space by Kuiper's Theorem [15]. Since by Theorem 13.1 we know that

$$\hom_{\mathcal{S}}(H, \mathcal{PU}(\mathcal{H}))/\mathcal{PU}(\mathcal{H}) \xrightarrow{\cong} \pi_0(B(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S})^H),$$

we obtain the desired isomorphism at the level of connected components

$$\pi_0(\psi^H)\colon \pi_0(B(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S})^H) \xrightarrow{\cong} \pi_0(\mathrm{map}(E\Gamma, B\mathcal{PU}(\mathcal{H}))^H).$$

Take now $\alpha \in \hom_{\mathcal{S}}(H, \mathcal{PU}(\mathcal{H}))$. By Theorem 13.1 we have a weak homotopy equivalence

$$BC_{\mathcal{PU}(\mathcal{H})}(\alpha) \xrightarrow{\simeq} B(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S})^H_\alpha;$$

by [5, Theorem 1.8] we know that the homotopy groups of $C_{\mathcal{PU}(\mathcal{H})}(\alpha)$ are

$$\pi_j(C_{\mathcal{PU}(\mathcal{H})}(\alpha)) = \begin{cases} \hom(H, S^1) & \text{for} \quad j = 0; \\ \mathbb{Z} & \text{for} \quad j = 2; \\ 0 & \text{otherwise.} \end{cases}$$

By a simple calculation the homotopy groups for $j > 0$ of $\mathrm{map}(E\Gamma, B\mathcal{PU}(\mathcal{H}))^H$ are

$$\pi_j(\mathrm{map}(E\Gamma, B\mathcal{PU}(\mathcal{H}))^H) = \begin{cases} H^2(BH, \mathbb{Z}) = \hom(H, S^1) & \text{for} \quad j = 1; \\ H^0(BH, \mathbb{Z}) = \mathbb{Z} & \text{for} \quad j = 3; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we see that the homotopy groups of $B(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S})^H$ and $\mathrm{map}(E\Gamma, B\mathcal{PU}(\mathcal{H}))^H$ are isomorphic and concentrated in degrees $j = 0$ and $j = 2$.

Now, in order to show that indeed ψ^H induces the desired isomorphism on homotopy groups we need to resort to the proof of Theorem 13.1. For $E = E(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S})$ we know that

$$E|_{B(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S})^H_\alpha} \cong \mathcal{PU}(\mathcal{H}) \times_{C_{\mathcal{PU}(\mathcal{H})}(\alpha)} E^{K(H, \alpha)}$$

with $E^{K(H,\alpha)} \to B(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S})^H_\alpha$ a principal $C_{\mathcal{PU}(\mathcal{H})}(\alpha)$-bundle. Therefore the map that defines ψ^H

$$E\Gamma/H \times B(\Gamma, \mathcal{PU}(\mathcal{H}), \mathcal{S})^H_\alpha \to B\mathcal{PU}(\mathcal{H})$$

is weakly homotopically equivalent to the map obtained by applying the classifying space functor

$$BH \times BC_{\mathcal{PU}(\mathcal{H})}(\alpha) \to B\mathcal{PU}(\mathcal{H})$$

to the group homomorphism

$$\phi\colon H \times C_{\mathcal{PU}(\mathcal{H})}(\alpha) \to \mathcal{PU}(\mathcal{H}). \quad (h, g) \mapsto \alpha(h)g.$$

Since all the $\widetilde{H}$ representations defined by α appear infinitely number of times, then it follows that the inclusion of groups $C_{\mathcal{PU}(\mathcal{H})}(\alpha) \subset \mathcal{PU}(\mathcal{H})$ induces an isomorphism on the second homotopy groups π_2; this implies that ψ^H induces an isomorphism on the third homotopy groups π_3. It remains to show that ψ^H induces an isomorphism at the level of fundamental groups.

The map ϕ permits to define a homeomorphism

$$C_{\mathcal{PU}(\mathcal{H})}(\alpha) \to \hom_\alpha(\mathbb{Z} \times H, \mathcal{PU}(\mathcal{H})), \quad g \mapsto [(n, h) \mapsto g^n \alpha(h)]\,.$$

where

$$\hom_\alpha(\mathbb{Z} \times H, \mathcal{PU}(\mathcal{H})) := \{\beta \in \hom_\alpha(\mathbb{Z} \times H, \mathcal{PU}(\mathcal{H})) \mid \beta(0, h) = \alpha(h)\}.$$

Moreover we can define a group homomorphism (see [5, Lemma 1.6])

$$C_{\mathcal{PU}(\mathcal{H})}(\alpha) \to \hom(H, S^1), \quad g \mapsto (h \mapsto \widetilde{\alpha}(\widetilde{h}) \circ \overline{g} \circ \widetilde{\alpha}(\widetilde{h})^{-1} \circ \overline{g}^{-1}),$$

where $\widetilde{h}$ is any lift of h in $\widetilde{H}$ and $\overline{g}$ is any lift of g on $\mathcal{U}(\mathcal{H})$, such that it becomes an isomorphism of groups on path connected components

$$\pi_0(C_{\mathcal{PU}(\mathcal{H})}(\alpha)) \xrightarrow{\cong} \hom(H, S^1).$$

Therefore we obtain the following commutative diagram

$$\begin{array}{ccccc}
\pi_0(C_{\mathcal{PU}(\mathcal{H})}(\alpha)) & \xrightarrow{\cong} & \pi_0(\hom_\alpha(\mathbb{Z} \times H, \mathcal{PU}(\mathcal{H}))) & \longrightarrow & \pi_0(\mathrm{map}_{B\alpha}(B\mathbb{Z} \times BH, B\mathcal{PU}(\mathcal{H}))) \\
\downarrow \cong & & & & \downarrow \cong \\
\hom(H, S^1) & \xrightarrow{\cong} & [BH, BS^1] & \xrightarrow{\cong} & [BH, \Omega B\mathcal{PU}(\mathcal{H})]
\end{array}$$

where

$$\mathrm{map}_{B\alpha}(B\mathbb{Z} \times BH, B\mathcal{PU}(\mathcal{H})) := \{f\colon B\mathbb{Z} \times BH \to B\mathcal{PU}(\mathcal{H})\colon f(*, y) = B\alpha(y)\},$$

the right vertical arrow is given by the adjoint map since $B\mathbb{Z} \simeq S^1$ and Ω denotes the based loop space functor, and the bottom right map is defined via the homotopy equivalence $BS^1 \xrightarrow{\simeq} \Omega B\mathcal{PU}(\mathcal{H})$ which induces the homotopy equivalence $BS^1 \to \mathcal{PU}(\mathcal{H})$. Hence we conclude that at the level of fundamental groups the map ϕ induces an isomorphism

$$\pi_1(BC_{\mathcal{PU}(\mathcal{H})}(\alpha)) \xrightarrow{\cong} \pi_1(\mathrm{map}(BH, B\mathcal{PU}(\mathcal{H})))$$

and therefore ψ^H induces the desired isomorphism at the level of fundamental groups. $\square$

16. Appendix A:Compactly generated spaces

We briefly recall some basics about compactly generated spaces. More information and proofs can be found in [32]. A topological space X is *compactly generated* if it is a Hausdorff space and a set $A \subseteq X$ is closed if and only if for any compact subset $C \subset X$ the intersection $C \cap A$ is a closed subspace of C.

Every locally compact space, and every space satisfying the first axiom of countability, e.g., a metrizable space, is compactly generated. If $p\colon X \to Y$ is an identification of topological spaces and X is compactly generated and Y is Hausdorff, then Y is compactly generated. A closed subset of a compactly generated space is again compactly generated. For open subsets one has to be careful as it is explained in Subsection 16.1.

16.1. **Open subsets.** Recall that a topological space X is called *regular* if for any point $x \in X$ and closed set $A \subseteq X$ there exists open subsets U and V with $x \in U$, $A \subseteq V$ and $U \cap V = \emptyset$. A space is called *locally compact* if every $x \in X$ possesses a compact neighborhood. Equivalently, for every $x \in X$ and open neighborhood U there exist an open neighborhood V of x such that the closure of V in X is compact and contained in U, see [26, Lemma 8.2 in Section 3-8 on page 185]. Recall the Definition 1.1 saying that an open subset $U \subseteq X$ is called *quasi-regular* if for any $x \in U$ there exists an open neighborhood V_x whose closure in X is contained in U.

Lemma 16.1. *(i) Let B be a (compactly generated) space. A quasi-regular open subset $U \subseteq B$ equipped with the subspace topology is compactly generated;*

(ii) Let $f\colon X \to Y$ be a (continuous) map between (not necessarily compactly generated) spaces. If $V \subseteq Y$ is a quasi-regular open subset, then $f^{-1}(V) \subseteq X$ is a quasi-regular open subset.

(iii) The intersection of finitely many quasi-regular open subsets is again a quasi-regular open subset;

(iv) A space is regular if and only if every open subset is quasi-regular;

(v) Any locally compact space, any metrizable space and every CW-complex are regular;

(vi) Every Γ-invariant open subset of a Γ-CW-complex is quasi-regular and, equipped with the subspace topology, compactly generated.

Proof. (i) See [32, page 135].

(ii) Consider a point $x \in f^{-1}(V)$. Choose an open set W of Y such that $f(x) \in W$ and the closure of W in B is contained in V. Then $f^{-1}(W)$ is an open subset of X which contains x and whose closure in X is contained in $f^{-1}(V)$.

(iii) Let $U_1, U_2, \ldots, U_r$ be quasi-regular open subsets. Consider $x \in U := \bigcap_{i=1}^r U_i$. Choose for every $i = 1, 2 \ldots, r$ an open subset V_i with $x \in V_i$ such that the closure $\overline{V_i}$ of V_i in B is contained in U_i. Put $V := \bigcap_{i=1}^r V_i$. Then $x \in V$ and $\overline{V} \subseteq \cap_{i=1}^r \overline{V_i} \subseteq U$. Hence U is a quasi-regular open subset.

(iv) See [26, Lemma 2.1 in Section 4-2 on page 196].

(v) This is obvious for locally compact spaces. Metrizable spaces are treated in [26, Theorem 2.3 in Section 4-2 on page 198]. Every CW-complex is paracompact, see [24], and hence in particular regular, see [26, Theorem 4.1 in Section 6-4 on page 255].

(vi) The projection $X \to X/\Gamma$ is open and X/Γ is a CW-complex by Lemma 16.7. Now apply assertions (i), (iv) and (v) $\square$

16.2. **The retraction functor k.** There is a construction which assigns to a topological Hausdorff space X a new topological space $k(X)$ such that X and $k(X)$ have the same underlying sets, $k(X)$ is compactly generated, X and $k(X)$ have the same compact subsets, the identity $k(X) \to X$ is continuous and is a homeomorphism if and only if X is compactly generated. Namely, define the new topology on $k(X)$ by declaring a subset $A \subseteq X$ to be closed if and only if for every compact subset of X the intersection $A \cap C$ is a closed subset of C.

16.3. **Mapping spaces, product spaces and subspaces.** Given two compactly generated spaces X and Y, denote by $\operatorname{map}(X,Y)_{k.o.}$ the set of maps $X \to Y$ with the compact-open-topology, i.e., a subbasis for the compact-open-topology is given by the sets $W(C,U) = \{f\colon X \to Y \mid f(C) \subseteq U\}$, where C runs through the compact subsets of X and U runs though the open subsets of Y. Notice that $\operatorname{map}(X,Y)_{k.o.}$ is not compactly generated in general. We denote by $\operatorname{map}(X,Y)$ the topological space given by $k(\operatorname{map}(X,Y)_{k.o.})$.

If X and Y are compactly generated spaces, then $X \times Y$ stands for $k(X \times_p X)$, where $X \times_p Y$ is the topological space with respect to the "classical" product topology.

If $A \subseteq X$ is a subset of a compactly generated space, the subspace topology means that we take $k(A_{st})$ for A_{st} the topology space given by the "classical" subspace topology on A.

Roughly speaking, all the usual constructions of topologies are made compactly generated by passing from Y to $k(Y)$ in order to stay within the category of compactly generated spaces.

16.4. **Basic features of the category of compactly generated spaces.** The category of compactly generated spaces has the following convenient features:

- A map $f\colon X \to Y$ of compactly generated spaces is continuous if and only if its restriction $f|_C\colon C \to Y$ to any compact subset $C \subseteq X$ is continuous;
- The product of two identifications is again an identification;
- If X is locally compact and Y compactly generated, then $X \times Y$ and $X \times_p Y$ are the same topological spaces;
- The product of a Γ_1-CW-complex and of a Γ_2-CW-complex is a $\Gamma_1 \times \Gamma_2$-CW-complex;
- If X, Y, and Z are compactly generated spaces, then the obvious maps
$$\begin{aligned} \operatorname{map}(X, \operatorname{map}(Y,Z)) &\xrightarrow{\cong} \operatorname{map}(X \times Y, Z); \\ \operatorname{map}(X, Y \times Z) &\xrightarrow{\cong} \operatorname{map}(X,Y) \times \operatorname{map}(X,Z). \end{aligned}$$
are homeomorphisms and the map given by composition
$$\operatorname{map}(X,Y) \times \operatorname{map}(Y;Z) \to \operatorname{map}(X,Z)$$
is continuous;
- Given a pushout in the category of compactly generated spaces, its product with a compactly generated space is again a pushout in the category of compactly generated spaces.

16.5. **Space of homomorphisms.** Let H and G be (compactly generated) topological groups. Let $\hom(H,G)$ be the set of group homomorphisms from H to G. It is obviously a subset of $\operatorname{map}(H,G)$. The proof of the next result is a typical formal proof using the convenient basic properties of the category of compactly generated spaces.

Lemma 16.2. *The subset* $\hom(H,G)$ *of* $\operatorname{map}(H,G)$ *is closed.*

Proof. Consider the map

$$u\colon H \times H \times \operatorname{map}(H,G) \to G, \quad (h_1,h_2,\alpha) \mapsto \alpha(h_1)\cdot\alpha(h_2)\cdot\alpha(h_1h_2)^{-1}.$$

It is continuous since it can be written as a composition of maps each of which is obviously continuous by the basic features presented in Subsection 16.4.

$$H \times H \times \operatorname{map}(H,G) \xrightarrow{u_1} H \times \operatorname{map}(H,G) \times H \times \operatorname{map}(H,G) \times H \times \operatorname{map}(H,G) \xrightarrow{u_2} G \times G \times G \xrightarrow{u_3} G$$

where

$$\begin{aligned} u_1(h_1,h_1,\alpha) &= (h_1,\alpha,h_2,\alpha,h_1h_2,\alpha); \\ u_2(h_1,\alpha_1,h_2,\alpha_2,h_3,\alpha_3) &= (\alpha_1(h_1),\alpha_2(h_2),\alpha_3(h_3)); \\ \alpha_3(g_1,g_2,g_3) &= g_1g_2g_3^{-1}. \end{aligned}$$

Its adjoint is the continuous map

$$v\colon \operatorname{map}(H,G) \to \operatorname{map}(H\times H,G), \quad \alpha \mapsto v(\alpha)$$

given by $v(\alpha)(h_1,h_2) := \alpha(h_1)\cdot\alpha(h_2)\cdot\alpha(h_1h_2)^{-1}$. The evaluation map

$$\operatorname{ev}\colon \operatorname{map}(H,G) \to G, \quad \alpha \mapsto \alpha(1)$$

is continuous by Subsection 16.4. Since $\hom(H,G)$ is the intersection of $v^{-1}(\{1\})$ and $\operatorname{ev}^{-1}(\{1\})$ and $\{1\} \subseteq G$ is closed, $\hom(H,G)$ is a closed subset of $\operatorname{map}(H,G)$. □

We will equip $\hom(H;G)$ with the subspace topology coming from $\operatorname{map}(H,G)$. Notice that it is automatically compactly generated so that $\hom(H;G)$ agrees with $k(\hom(H;G))$.

We leave the formal proof to the reader that the conjugation action

$$G \times \hom(H,G) \to \hom(H,G), \quad (g,\alpha) \mapsto c_g \circ \alpha$$

is continuous, where $c_g\colon G \to G$ sends g' to $gg'g^{-1}$.

We also include the following results

Lemma 16.3. *Let $p\colon X \to Y$ be an identification and Z be a space. Then the induced map*

$$p^*\colon \operatorname{map}(Y,Z) \to \operatorname{map}(X,Z), \quad f \mapsto f\circ p$$

is a closed embedding, i.e., has closed image and the induced map $q\colon \operatorname{map}(Y,Z) \to \operatorname{im}(p^*)$ is a homeomorphism.

Proof. Obviously p^* is injective and continuous.

Next we show that the image of p^* is closed. Consider $f\colon X \to Z$ which is not in the image of p^*. Then there exists x_0 and x_1 in X with $f(x_0) \neq f(x_1)$ and $p(x_0) = p(x_1)$. Since the map

$$\operatorname{ev}\colon \operatorname{map}(X,Z) \mapsto Z \times Z, \quad f \mapsto \big(f(x_0), f(x_1)\big)$$

is continuous, the preimage of the open set $\{(z_0,z_1) \in Z\times Z \mid z_0 \neq z_1\}$ under ev is an open set which contains f and does not meet $\operatorname{im}(p^*)$. Hence $\operatorname{im}(p^*)$ is closed in $\operatorname{map}(X,Z)$.

It remains to show that the inverse of q

$$q^{-1}\colon \operatorname{im}(p^*) \to \operatorname{map}(Y,Z)$$

is continuous. Consider an open subset $V \subset \operatorname{map}(Y,Z)_{k.o.}$. By the definition of the compact open topology we can find an index set I, for every element $i \in I$ a

finite index set J_i, and for every $i \in I$ and $j_i \in J_i$ an open subset $U_{i,j_i} \subseteq Y$ and a compact subset $C_{i,j_i} \subseteq Z$ such that

$$V = \bigcup_{i \in I} \bigcap_{j_i \in J_i} W(U_{i,j_i}, C_{i,j_i})$$

where $W(U_{i,j_i}, C_{i,j_i}) = \{f \in \operatorname{map}(Y,Z) \mid f(U_{i,j_i}) \subseteq C_{i,j_i}\}$. We have

$$\begin{aligned} q(V) &= p^* \left(\bigcup_{i \in I} \bigcap_{j_i \in J_i} W(U_{i,j_i}, C_{i,j_i}) \right) \\ &= \bigcup_{i \in I} \bigcap_{j_i \in J_i} p^*\big(W(U_{i,j_i}, C_{i,j_i})\big) \\ &= \bigcup_{i \in I} \bigcap_{j_i \in J_i} \big(W\big(p^{-1}(U_{i,j_i}), C_{i,j_i}\big) \cap \operatorname{im}(p^*)\big) \\ &= \left(\bigcup_{i \in I} \bigcap_{j_i \in J_i} W\big(p^{-1}(U_{i,j_i}), C_{i,j_i}\big) \right) \cap \operatorname{im}(p^*) \end{aligned}$$

where $W\big(p^{-1}(U_{i,j_i}), C_{i,j_i}\big) = \{f \in \operatorname{map}(X,Z) \mid f(p^{-1}(U_{i,j_i})) \subseteq C_{i,j_i}\}$. Since $\bigcup_{i \in I} \bigcap_{j_i \in J_i} W\big(p^{-1}(U_{i,j_i}), C_{i,j_i}\big)$ is an open subset of $\operatorname{map}(Y,Z)_{k.o.}$, the subset $q(V)$ of $\operatorname{im}(p^*)_{k.o.}$ is open where the topology on $\operatorname{im}(p^*)_{k.o.}$ is the subspace topology with respect to the inclusion $\operatorname{im}(p^*) \to \operatorname{map}(X,Z)_{k.o.}$. Hence the map $q_{k.o.}^{-1}\colon \operatorname{im}(p^*)_{k.o.} \to \operatorname{map}(Y,Z)_{k.o.}$ is continuous. This implies that the map

$$k(q_{k.o.}^{-1})\colon k\big(\operatorname{im}(p^*)_{k.o.}\big) \to k\big(\operatorname{map}(Y,Z)_{k.o.}\big) = \operatorname{map}(Y,Z)$$

is continuous. The identity induces a continuous map $i_0\colon \operatorname{map}(X,Z) = k(\operatorname{map}(X,Z)_{k.o.}) \to \operatorname{map}(X,Z)_{k.o.}$ and hence by restriction a continuous map $i_1\colon \operatorname{im}(p^*) \to \operatorname{im}(p^*)_{k,o.}$. If we apply k to it and use the fact that $\operatorname{im}(p^*)$ is compactly generated, we obtain a continuous map

$$i_1\colon \operatorname{im}(p^*) \to k(\operatorname{im}(p^*)_{k.o.}).$$

Since $q^{-1}\colon \operatorname{im}(p^*) \to \operatorname{map}(Y,Z)$ is the composite of the two continuous maps $k(q_{k.o.}^{-1})$ and i_1 above, it is continuous itself. This finishes the proof of Lemma 16.3. □

Lemma 16.4. *For every group G and G-spaces Y and Z the map*

$$Q_{Y,Z}\colon \operatorname{map}_G(Y,Z) \to \operatorname{map}_G(Y/G, Z/G), \quad f \mapsto f/G$$

is continuous.

Proof. Let $p_Y\colon Y \to Y/G$ and $p_Z\colon Z \to Z/G$ be the projections. The map

$$(p_Z)_*\colon \operatorname{map}(Y,Z) \to \operatorname{map}(Y, Z/G), \quad f \mapsto p_Z \circ f$$

is continuous and hence induces a continuous map

$$p_Z\colon \operatorname{map}_G(Y,Z) \to \operatorname{map}_G(Y, Z/G), \quad f \mapsto p_Z \circ f.$$

Therefore the claim follows if we can prove that the map

$$Q_{Y,Z/G}\colon \operatorname{map}_G(Y, Z/G) \to \operatorname{map}(Y/G, Z/G), \quad f \mapsto f/G$$

is continuous. Its inverse is the map

$$p_Y^*\colon \operatorname{map}(Y/G, Z/G) \to \operatorname{map}_G(Y, Z/G) \quad f \mapsto f \circ p_Y$$

which is a homeomorphism by Lemma 16.3. This finishes the proof of Lemma 16.4. □

16.6. **Further features of the category of compactly generated spaces.**

Lemma 16.5. *Consider a commutative square of* Γ*-spaces*

$$\begin{tikzcd} B_0 \arrow[r, "i_1"] \arrow[d, "i_2"] & B_1 \arrow[d, "j_1"] \\ B_2 \arrow[r, "j_2"] & B \end{tikzcd}$$

Then

(i) The square is a Γ*-pushout if and only if it is pushout after forgetting the group action.*

(ii) If the square above is a Γ*-pushout and* $f\colon E \to B$ *is a* Γ*-map, then the square obtained by the pullback construction*

$$\begin{tikzcd} E_0 \arrow[r, "\overline{i_1}"] \arrow[d, "\overline{i_2}"] & E_1 \arrow[d, "\overline{j_1}"] \\ E_2 \arrow[r, "\overline{j_2}"] & E \end{tikzcd}$$

is a Γ*-pushout;*

(iii) If the given square is a Γ*-pushout, the square*

$$\begin{tikzcd} B_0/\Gamma \arrow[r, "i_1/\Gamma"] \arrow[d, "i_2/\Gamma"] & B_1/\Gamma \arrow[d, "j_1/\Gamma"] \\ B_2/\Gamma \arrow[r, "j_2/\Gamma"] & B/\Gamma \end{tikzcd}$$

is a pushout of spaces.

Proof. (i) Suppose that the square is a (non-equivariant) pushout. We want to show that it is a Γ-pushout. Consider Γ-maps $f_k\colon B_k \to Y$ for $k = 0, 1, 2$ satisfying $f_k \circ i_k = f_0$ for $k = 1, 2$. Then there is precisely one map $f\colon B \to X$ with $f \circ j_k = f_k$ for $k = 1, 2$. It remains to show that f is Γ-equivariant. This follows from the fact that the product of Γ with the square above is again a (non-equivariant) pushout. This follows formally from the adjunctions of mapping spaces and the universal property of pushouts.

Now suppose that the square is a Γ-pushout. Then it is a pushout after forgetting the group action, since there is a bijection

$$a\colon \operatorname{map}(B, X) \xrightarrow{\cong} \operatorname{map}_\Gamma(B, \operatorname{map}(\Gamma, X)), \quad f \mapsto a(f),$$

where $a(f)$ sends $b \in B$ to the map $\Gamma \to X$, $\gamma \mapsto f(\gamma \cdot b)$, X is a (non-equivariant) space and $\operatorname{map}(\Gamma, X)$ becomes a Γ-space $\gamma \cdot f(h) := f(h\gamma)$.

(ii) Because of assertion (i), we can assume without loss of generality that Γ is trivial. The map $j_1 \amalg j_2\colon B_1 \amalg B_2 \to B$ is an identification since the given square is a pushout. Then $\mathrm{id}_E \times j_1 \amalg \mathrm{id}_E \times j_2\colon E \times B_1 \amalg E \times B_2 \to E \times B$ is an identification, see [32, Theorem 4.4]. Consider E as a closed subspace of $E \times B$ by identifying e with $(e, f(e))$. Then the restriction of $\mathrm{id}_E \times j_1 \amalg \mathrm{id}_E \times j_2$ to the preimage of E is again an identification which can be identified with $\overline{j_1} \amalg \overline{j_2}\colon E_1 \amalg E_2 \to E$. Obviously the square obtained by the pullback construction is a pushout of sets. Hence it is a pushout of spaces.

(iii) This follows from the universal properties of a pushout and the projection maps $B \to B/\Gamma$. $\square$

Lemma 16.6. *Let $B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq B$ be a filtration of the Γ-space B by closed Γ-invariant subspaces.*

(i) We have $B = \operatorname{colim}_{n\to\infty} B_n$ in the category of Γ-spaces if and only if we have after forgetting the Γ-actions $B = \operatorname{colim}_{n\to\infty} B_n$ in the category of spaces;

(ii) Suppose that we have $B = \operatorname{colim}_{n\to\infty} B_n$ in the category of Γ-spaces. Let $f\colon E \to B$ be a Γ-map. Then we obtain a filtration

$$E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E$$

of E by closed Γ-invariant subspaces $E_n = f^{-1}(B_n)$ with the property that $E = \operatorname{colim}_{n\to\infty} E_n$ holds in the category of Γ-spaces;

(iii) Suppose that $B = \operatorname{colim}_{n\to\infty} B_n$ holds in the category of Γ-spaces. Then

$$B_0/\Gamma \subseteq B_1/\Gamma \subseteq B_2/\Gamma \subseteq \cdots \subseteq B/\Gamma$$

is a filtration of the space $B\Gamma$ by closed subspaces with the property that $B/\Gamma = \operatorname{colim}_{n\to\infty} B_n/\Gamma$ holds in the category of spaces.

Proof. (i) The proof is similar to the one of Lemma 16.5 (i), using the fact that after forgetting the group actions we obtain a filtration by closed subspaces $\Gamma\times B_0 \subseteq \Gamma \times B_1 \subseteq \Gamma \times B_2 \subseteq \cdots \subseteq \Gamma \times B$ such that $\Gamma \times B = \operatorname{colim}_{n\to\infty} \Gamma \times B_n$ holds in the category of Γ-spaces, see [32, Theorem 10.3].

(ii) Let $A \subseteq E$ be a subset. Suppose that $A \cap E_n$ is closed. We have to show that $A \subseteq E$ is closed, or, equivalently, that for every compact subset $C \subset E$ the space $C \cap A$ is a closed subspace of C. Since $f(C)$ is compact, there exists a natural number n with $f(C) \subseteq B_n$. This implies $C \subseteq E_n$, see [32, Lemma 9.3]. Now the claim follows from $A \cap C = A \cap (C \cap E_n) = (A \cap E_n) \cap C$ since $A \cap E_n$ is closed in E_n and hence $(A \cap E_n) \cap C$ is closed in C.

(iii) This follows from the universal properties of $\operatorname{colim}_{n\to\infty}$ and of the quotient maps $B_k \to B_k/\Gamma$ and $B \to B/\Gamma_k$. □

Lemma 16.7. *If X is a G-CW-complex. Let $N \subseteq G$ a normal subgroup and Q be the topological group G/N and $\operatorname{pr}\colon G \to Q$ be the projection. Suppose for any $x \in X$ that $\operatorname{pr}(G_x)$ is closed in Q. (This assumption is automatically satisfied if N is compact or Q is trivial.)*

Then X/N is a Q-CW-complex.

Proof. Let $\operatorname{pr}\colon G \to Q$ be the projection. Consider a subgroup $H \subseteq G$ such that $\operatorname{pr}(H)$ is closed in Q. Then we obtain a Q-homeomorphism $N\backslash\big(G/H\big) \xrightarrow{\cong} Q/\operatorname{pr}(H)$. Now the claim follows from Lemma 16.5 (i) and (iii) and Lemma 16.6 (i) and (iii). □

17. Appendix B: Some properties of locally compact groups

Let H and G be topological groups. For α in $\hom(H, G)$, let $C_G(\alpha)$ be the centralizer of α, i.e., $C_G(\alpha) = \{g \in G \mid c_g \circ \alpha = \alpha\}$, and denote by $G \cdot \alpha$ the orbit of α under the G-action. Then the map

$$\iota_\alpha\colon G/C_G(\alpha) \to G \cdot \alpha, \quad gC_G(\alpha) \mapsto c_g \circ \alpha$$

is bijective, continuous and G-equivariant. The next two theorems are the main results of this section.

Theorem 17.1. *Let H be a compact group, let G a locally compact second countable group and $\alpha \in \hom(H, G)$.*

Then the connected component $\mathcal{C}_\alpha$ of α in $\hom(H, G)$ is contained in the orbit of α under G, i.e. $\mathcal{C}_\alpha \subset G \cdot \alpha$.

Theorem 17.2. *Let G be a second countable, locally compact group which is almost connected. Let H be a compact group.*

Then $\iota_\alpha\colon G/C_G(\alpha) \to G\cdot\alpha$ is a G-homeomorphism.

Their respective proofs need some preparation. For a compact space X and a complete metric space (Y,d) we can equip the set of continuous maps from X to Y with the supremums metric

$$d_{\sup}(\alpha,\beta) := \sup\{d(\alpha(x),\beta(x)) \mid x\in X\}, \tag{17.3}$$

and obtain a complete metric space, see [26, Theorem 1.4 in Section 7-1 on page 267]. Moreover, the topology induced by the supremums metric agrees with the compact-open topology by [26, Theorem 4.6 in Section 7-4 on page 283 and Theorem 5.1 in Section 7.5 on page 286]. In particular $\mathrm{map}(X,Y) = \mathrm{map}(X,Y)_{c.o.}$.

The following result is due to Birkhoff [6] and Kakutani [13].

Theorem 17.4. *A Hausdorff topological group is metrizable if and only if it is first countable. In this case the metric can be taken to be left invariant.*

Theorem 17.4 implies that for a locally compact first countable group G we can choose a left invariant metric d_G inducing the given topology such that G with this metric is complete. Note that completeness follows from local compactness. Namely, we then can find an $\epsilon > 0$ such that the closed ball $\overline{B}_\epsilon(1)$ around 1 is compact and hence the closed ball $\overline{B}_\epsilon(g)$ around any $g\in G$ is compact which implies that every Cauchy sequence contains a subsequence contained in $B_\epsilon(g)$ for some $g\in G$ and hence contains a convergent subsequence, by [26, Theorem 7.4 in Section 3-7 on page 181], and finally we can apply [26, Lemma 1.1 in Section 7-1 on page 264]

Recall from Subsection 16.5 that we have equipped $\hom(H,G)$ with the subspace topology of $\mathrm{map}(H,G)$, that $\hom(H,G)\subseteq \mathrm{map}(H,G)$ is closed and that the conjugation action

$$G\times \hom(H,G)\to \hom(H,G),\quad (g,\alpha)\mapsto c_g\circ\alpha$$

is continuous. Hence this topology on $\hom(H,G)$ agrees with the topology coming from the compact-open topology on $\mathrm{map}(H,G)$ restricted to $\hom(H;G)$ as well with the topology associated to the supremums metric restricted to $\hom(H,G)$, and $\hom(H,G)$ is a complete metric space with the supremums metric.

The next Theorem is taken from [20, Theorem I] and obviously implies Theorem 17.1.

Theorem 17.5. *Let H be a compact group and G a locally compact second countable group. Let $\mathcal{C}\subset \hom(H,G)$ be a connected component of the space $\hom(H,G)$. Then if θ and ϕ are in $\mathcal{C}$, then there exists $g\in G$ with $\theta = c_g\circ\phi$.*

The following result is taken from Lee-Wu [20, Theorem II]. Notice that the condition of being almost connected is necessary, see [20, Example, page 412].

Theorem 17.6. *Let L be a locally compact group which is almost connected. Let F be a compact subgroup of L and denote by $i\colon F\to L$ the inclusion. If $\{x_\lambda \mid \lambda\in\Lambda\}$ is a net in L such that the homomorphisms $c_{x_\lambda}\circ i$ converge to an element $\theta\in\hom(F,L)$, then there exists an element $y\in L$ such that $\theta = c_y\circ i$.*

Theorem 17.7. *Let G be a locally compact group which is almost connected. Let H be a compact group and $\alpha\in\hom(H,G)$. Then the orbit $G\cdot\alpha$ of α under the conjugation action of G is closed in $\hom(H,G)$.*

Proof. Suppose we have a net $\{g_\lambda\in G \mid \lambda\in\Lambda\}$ of elements of G such that the homomorphisms $c_{g_\lambda}\circ\alpha$ converge to an element $\phi\in\hom(H,G)$. We want to show that ϕ belongs to $G\cdot\alpha$.

Let us take $L = H \times G$ and $K(H,\alpha) := \{(h,\alpha(h) \mid h \in H\}$ the associated compact subgroup of $H \times G$ defined by α. We have then that L is locally compact and L is almost connected. Denote by $i\colon K(H,\alpha) \to H \times G$ the inclusion and take the net $\{(1,g_\lambda) \in H \times G \mid \lambda \in \Lambda\}$ of elements in $H \times G$ induced by the elements $g_\lambda \in G$. Note that since the homomorphisms $c_{g_\lambda} \circ \alpha$ converge to $\phi \in \hom(H,G)$, the homomorphisms

$$c_{(1,g_\lambda)} \circ i \in \hom(K(H,\alpha), H \times G)$$

converge to the homomorphism $\theta_\phi \in \hom(K(H,\alpha), H \times G)$ defined by

$$\theta_\phi(h,\alpha(h)) := (h,\phi(h)).$$

Applying Theorem 17.6 we know that there exists a pair $(y,g) \in H \times G$ such that $\theta_\phi = c_{(y,g)} \circ i$; this implies that for all $h \in H$

$$(h,\phi(h)) = (yhy^{-1}, g\alpha(h)g^{-1})$$

and therefore we can conclude that $\phi = c_g \circ \alpha$. Hence ϕ belongs to the orbit $G \cdot \alpha$. □

Finally we give the proof Theorem 17.2.

Proof of Theorem 17.2. Since G is second countable, it is a Lindelöf space, i.e., every open covering contains a countable subcovering, see [26, Theorem 1.3 in Section 4-1 on page 191].

By Theorem 17.7 we know that the orbit $G \cdot \alpha$ is closed in $\hom(H,G)$. We have already explained that $\hom(H,G)$ is a complete metric space. We conclude that the orbit $G \cdot \alpha$ is a complete metric space and hence by the Baire Category Theorem, see [26, Theorem 7.2 in Section 7.7 on page 294], also a Baire space, i.e., every countable union of closed sets each of which has empty interior has itself empty interior.

Next we show that the map ι_α is an open map. Since we know that it is a bijective continuous G-map, this will imply that ι_α is indeed a G-homeomorphism. Since the image and source of ι_α are transitive G-spaces, it suffices to show for every open set U of G containing $1_G \in G$ that there exists an open neighborhood of α which is contained in $\iota_\alpha(U)$.

Since G is locally compact and the map $\mu\colon G \times G \to G,\ (g_1,g_2) \mapsto g_1^{-1}g_2$ is continuous, we can find a non-empty open set $V \subset U$ such that its closure $K = \overline{V}$ is compact and satisfies

$$K^{-1}K := \{k_1^{-1} \cdot k_2 \mid k_1, k_2 \in K\} \subset U;$$

any nonempty open set $V \subset U$ such that $\overline{V} \times \overline{V} \subset \mu^{-1}(U) \cap U \times U$ satisfy this condition.

Since G is a Lindelöf space, it can be covered by countably many G-translates of the open subset V and hence also by countably many G-translates of the subset K. This implies that $G/C_G(\alpha)$ can be covered by countably many G-translates of $KC_G(\alpha)/C_G(\alpha)$. Since each

$$\iota_\alpha(gKC_G(\alpha)/C_G(\alpha)) = gK \cdot \alpha$$

is a closed set in $G \cdot \alpha$ and $G \cdot \alpha$ is a Baire space, one of the G-translates of $KC_G(\alpha)/C_G(\alpha)$ must have non empty interior. This implies that $K \cdot \alpha$ has non empty interior, and therefore the set $K^{-1}K \cdot \alpha$ contains an open neighborhood of α. Since $K^{-1}K\alpha$ is contained in $i_\alpha(U)$, the claim follows. □

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