Charakterisierung projektiver Mengen durch Topologieverfeinerungen

(Characterization of projective sets by finer topologies)

Diplomarbeit

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Contents

Zusammenfassung

Die vorliegende Diplomarbeit beschäftigt sich mit zwei unveröffentlichten Artikeln von Professor Howard S. Becker von der University of South Carolina in Columbia. Ausgangspunkt ist folgende klassische Charakterisierung von Borel Mengen in polnischen Räumen durch Topologieverfeinerungen:

> Eine Teilmenge eines polnischen Raumes ist genau dann eine Borel-Menge, wenn eine polnische Topologie auf dieser Teilmenge existiert, die die Teilraumtopologie verfeinert.

Professor Becker diskutiert in seinen Aufzeichnungen "Finer topologies on pointsets in Polish spaces" vom März 1991 und "Playing around with finer topologies" vom Januar 1992 mögliche Verallgemeinerungen dieser Charakterisierung für komplexere Teilmengen polnischer Räume, insbesondere für projektive Mengen in polnischen Räumen. In dieser Diplomarbeit werden seine Resultate mit ausführlichen Beweisen und der Bereitstellung aller Grundlagen präsentiert.

Die Diplomarbeit gliedert sich in zwei Teile. Im ersten Teil werden alle für diese Arbeit notwendigen Definitionen und Resultate aus der deskriptiven Mengenlehre eingeführt. Der zweite Teil befaßt sich dann mit dem eigentlichen Thema dieser Arbeit, der Charakterisierung projektiver Mengen durch Topologieverfeinerungen.

Die klassische deskriptive Mengenlehre beschäftigt sich mit "definierbaren Teilmengen" der reellen Zahlen und deren Eigenschaften. Die reellen Zahlen sind ein topologischer Raum, dessen Topologie von einer vollständigen Metrik induziert wird. Desweiteren liefert die abzählbar dichte Teilmenge der rationalen Zahlen eine abzählbare Basis für diese Topologie. Solche topologischen Räume nennt man *polnische Räume.* Man kann zeigen, dass die Definierbarkeitshierarchien auf den reellen Zahlen topologischen Hierarchien entsprechen. Deswegen beschäftigt sich die deskriptive Mengenlehre heutzutage oft allgemeiner mit definierbaren Teilmengen von polnischen Räumen.

Wir beginnen deshalb in Teil 1 dieser Arbeit mit einem kurzen Kapitel über polnische Räume. Es werden die grundlegenden Definitionen wiederholt und es wird gezeigt, dass Summen und Produkte (in der Kategorie der topologischen Räume) von polnischen Räumen wieder polnische Räume sind. Weiter erwähnen wir, dass genau die *G_δ-Mengen* (d.h. abzählbare Schnitte offener Mengen) versehen mit der Teilraumtopologie wieder polnische Räume sind.

Als wichtigstes Beispiel eines polnischen Raumes (neben $\mathbb{R}$) führen wir den Baire-Raum ω^ω ein. Als topologischer Raum ist dies das topologische Produkt der Mengen ω versehen mit der diskreten Topologie. Mit Hilfe von Bäumen können wir eine Basis der Topologie des Baire-Raumes angeben. Bäume spielen in dieser Arbeit eine herausragende Rolle und werden zusammen mit einigen damit verwandten Begriffen in Kapitel 2 eingeführt. Ein *Baum* auf ω besteht aus endlichen Folgen natürlicher Zahlen, so dass jedes Anfangsstück solch einer Folge auch ein Element des Baumes ist. Besonders wichtig für den Baire-Raum sind unendliche Äste durch einen solchen Baum auf ω. Ein *unendlicher Ast* durch einen Baum auf ω ist eine abzählbare Folge von natürlichen Zahlen, also ein Element von ω^ω, so dass alle endlichen Teilfolgen im Baum sind. Ein einfaches aber wichtiges Resultat in diesem Zusammenhang ist die Charakterisierung einer abgschlossenen Teilmenge des Baire-Raumes als Menge der unendlichen Äste durch einen Baum auf ω. In einem Unterkapitel von Kapitel 2 wird die Wichtigkeit des Baire-Raumes deutlich, da wir für jeden polnischen Raum eine stetige Surjektion des Baire-Raumes in den polnischen Raum finden.

Von entscheidender Bedeutung für die deskriptive Mengenlehre und insbesonders für unsere Arbeit hier ist eine weitere Darstellung von Teilmengen des Baire-Raumes durch Bäume. Wir definieren Bäume auf dem Produkt von ω mit einer Ordinalzahl λ und nennen die Mengen, welche sich durch eine Projektion der Menge der unendlichen Äste auf ω^ω darstellen lassen λ-*Suslin-Mengen*. Dies wird die entscheidende Definition in Kapitel zwei sein und wir diskutieren die λ-Suslin-Mengen entsprechend. Eng verknüpft damit ist das Konzept einer *Skala*. Dafür betrachten wir eine Folge von *Normen* (dies sind Abbildungen von Teilmengen des Baire-Raumes in die Ordinalzahlen) mit gewissen Eigenschaften. Sind alle Normen einer Skala Abbildungen, deren Bilder beschränkt sind durch eine Ordinalzahl λ, so sprechen wir von λ-*Skalen* und wir zeigen, dass Teilmengen des Baire-Raumes genau dann eine λ-Skala besitzen, wenn die Mengen λ-Suslin sind. Wir schließen Kapitel 2 mit der Definition von *Borel*-, und in Verallgemeinerung λ-*Borel-Mengen*. Auch hier wird der Zusammenhang mit λ-Suslin-Mengen diskutiert werden.

In Kapitel 3 führen wir die *Borel-Hierarchie* und die *projektive Hierarchie* ein. Die deskriptive Mengenlehre klassifiziert Teilmengen polnischer Räume in Hierarchien in Bezug auf die Komplexität der Menge. Zum Beispiel besteht die unterste Ebene der Borel-Hierarchie aus den offenen und abgeschlossenen Teilmengen. Die nächste Ebene enthält nun abzählbare Vereinigungen abgeschlossener Mengen (F_σ-*Mengen*) und abzählbare Schnitte offener Mengen (G_δ-*Mengen*). Um zur nächsten Ebene zu kommen betrachtet man wiederum abzählbare Vereinigungen von G_δ-Mengen bzw. abzählbare Schnitte von F_σ-Mengen und so weiter. Die Vereinigung aller Ebenen dieser Hierarchie liefert die Klasse aller Borel-Mengen. Borel-Mengen sind abgeschlossen unter Komplementbildung und abzählbaren Vereinigungen und Schnitten. Allerdings nicht unter Projektionen. Wir nutzen diese Tatsache zur Definition der projektiven Hierarchie. Wir nennen Projektionen von Borel-Mengen *analytische* oder $\boldsymbol{\Sigma}^1_1$-*Mengen* und zusammen mit ihren Komplementen (den $\boldsymbol{\Pi}^1_1$-*Mengen*) bilden sie die erste Stufe der projektiven Hierarchie. Projektionen von Komplementen von analytischen Mengen bilden dann (zusammen wieder mit deren Komplementen) die nächste Stufe der projektiven Hierarchie (die $\mathbf{\Sigma}^1_2$- bzw. $\mathbf{\Pi}^1_2$-Mengen). Dies lässt sich so abzählbar oft fortsetzen, d.h. wir erhalten die Klassen $\mathbf{\Sigma}^1_n$ und $\mathbf{\Pi}^1_n$ für $n \in \omega$. Die Mengen dieser Hierarchie nennt man *projektive Mengen* und für diese Mengen geben wir in Teil zwei dieser Diplomarbeit eine topologische Charakterisierung.

Im Kapitel 4 kommen wir dann zu einem moderneren Gebiet der deskriptiven Mengenlehre, nämlich zu *Spielen* und der *Determiniertheit von Spielen*. Als Prototyp für die Spiele, die wir betrachten, dient folgendes Spiel auf den natürlichen Zahlen. Es wird zunächst eine Teilmenge des Baire-Raumes als *Gewinnmenge* festgelegt. Zwei Spieler I und II wählen nun abwechselnd abzählbar oft natürliche Zahlen. Das Ergebnis dieses Spiels ist dann also eine abzählbare Folge natürlicher Zahlen und somit ein Element des Baire-Raumes. Wir sagen, dass Spieler I das Spiel gewinnt, falls die Folge in der Gewinnmenge liegt. Anderenfalls hat Spieler II gewonnen. Mit Hilfe von Bäumen definieren wir *Strategien* für die einzelnen Spieler, die dem Spieler in jedem Zug mitteilen, mit welcher natürlichen Zahl er auf eine bis dahin gespielte Folge antworten soll. Eine solche Strategie heißt *Gewinnstrategie*, falls der entsprechende Spieler jeden Spielverlauf gewinnt, indem er der Strategie folgt. Es ist klar, dass die Existenz einer Gewinnstrategie immer von der Gewinnmenge abhängt und es ist auch klar, dass es Gewinnmengen gibt, für die man sehr einfach Gewinnstrategien für einen der Spieler angeben kann. Eine Gewinnmenge nennt man *determiniert*, falls für einen der Spieler eine Gewinnstrategie existiert. Es ist ein schwieriges und interessantes Problem, welche Klassen von Teilmengen determiniert sind; wir beschäftigen uns hier allerdings nicht damit, sondern führen neue Axiome ein, die die Determiniertheit von Mengen postulieren. Das Axiom der projektiven Determiniertheit **PD** garantiert die Determiniertheit aller projektiven Mengen des Baire-Raumes. Das stärkere Axiom der Determiniertheit **AD** besagt, daß alle Teilmengen des Baire-Raumes determiniert sind. Später werden wir dann sogar das Axiom $\mathbf{AD}_{\mathbb{R}}$ voraussetzen. Hierzu werden Spiele auf Elementen des Baire-Raumes betrachtet. Die Gewinnmenge ist dann eine Teilmenge von $(\omega^\omega)^\omega$ und es werden abwechselnd Elemente von ω^ω gespielt. Ansonsten werden die obigen Definitionen in offensichtlicher Weise auf diese Spiele übertragen und $\mathbf{AD}_{\mathbb{R}}$ ist dann das Axiom, welches besagt, dass für alle Gewinnmengen solcher Spiele eine Gewinnstrategie für einen der Spieler existiert.

Wir schließen in Kapitel 4 mit einer Charakterisierung der polnischen Räume durch starke Choquet-Spiele. Dies sind Spiele für zwei Personen in obigem Sinn, nur werden diesmal nichtleere offene Mengen eines polnischen Raumes gespielt, so dass eine absteigende Folge von ineinander enthaltenen offenen Mengen entsteht und Spieler II gewinnt dieses *Choquet-Spiel*, wenn der Schnitt aller offenen gespielten Mengen nichtleer ist. Im *starken Choquet-Spiel* wird zusätzlich von Spieler I jeweils ein Punkt in seiner offenen Menge gespielt und Spieler zwei muss dann eine offene Umgebung um diesen Punkt spielen, welche in der offenen Menge von I enthalten ist. Auch hier gewinnt II, wenn der Schnitt aller offenen Mengen nicht leer ist. Ein topologischer Raum heißt *starker Choquet-Raum*, falls Spieler II eine Gewinnstrategie im starken Choquet-Spiel hat. Beispiele für solche starken Choquet-Räume sind unter anderem die polnischen Räume. Insbesondere sind polnische Räume reguläre starke Choquet-Räume mit abzählbarer Basis und es gilt die Hausdorff Trennungseigenschaft. Diese Eigenschaften von polnischen Räumen benutzen wir für unsere Charakterisierung der projektiven Mengen.

Die ersten vier Kapitel benutzen als Voraussetzung nur die Theorie **ZF+DC** und an einigen wenigen Stellen zusätzlich das volle Auswahlaxiom **AC**. Diese Theorien sind nicht geeignet für die vollständige topologische Charakterisierung der projektiven Mengen. Aus diesem Grunde haben wir in Kapitel 4 die Axiome der Determiniertheit eingeführt. In Kapitel 5 zeigen wir einige Resultate unter Annahme dieser Axiome. Entscheidend für die Beweise der Theoreme über die Charakterisierung der projektiven Mengen ist, dass die projektiven Mengen λ-Suslin sind. Dies gilt unter **PD** und wird in Kapitel 5 bewiesen. Die Ordinalzahl λ hängt eng mit den Längen von bestimmten Normen zusammen. Jeder Norm läßt sich nämlich eine fundierte Relation zuordnen, deren Länge durch das Bild einer zugehörigen Norm (der Rangfunktion) definiert ist. Wir definieren für $n \in \omega$ die *projektiven Ordinalzahlen* $\boldsymbol{\delta}^1_n$ als das Supremum aller Längen von solch fundierten Relation, die zusätzlich noch in $\boldsymbol{\Sigma}^1_n$ und $\boldsymbol{\Pi}^1_n$ liegen. Die projektiven Ordinalzahlen untersuchen wir im Rahmen diese Kapitels unter der Annahme **AD** genauer. Damit ist dann der erste Teil dieser Diplomarbeit abgeschlossen.

Der zweite Teil behandelt nun die eigentliche Charakterisierung der projektiven Mengen durch feinere Topologien. In Kapitel 6 beweisen wir zuerst das oben angegebene Resultat über die Borel-Mengen. Darauf folgt die Charkterisierung der analytischen Mengen, die folgendermaßen lautet:

> Eine Teilmenge eines polnischen Raumes ist genau dann analytisch, wenn es eine starke Choquet-Topologie mit abzählbarer Basis auf der Teilmenge gibt, welche die Teilraumtopologie verfeinert.

Das letzte Kapitel, Kapitel 7, gibt eine Charakterisierung dieser Art dann für jede $\boldsymbol{\Sigma}^1_n$-Menge.

> Eine Teilmenge eines polnischen Raumes ist genau dann in $\boldsymbol{\Sigma}^1_n$, wenn es eine starke Choquet-Topologie mit Basis der Länge kleiner als $\boldsymbol{\delta}^1_n$ auf dieser Teilmenge gibt, welche die Teilraumtopologie verfeinert.

Für diese Charakterisierung arbeiten wir unter der Theorie **ZF+DC+AD**$_{\mathbb{R}}$. Damit haben wir, wenn auch unter der sehr starken Annahme von **AD**$_{\mathbb{R}}$, eine vollständige Charakterisierung der projektiven Mengen durch Topologieverfeinerungen erreicht.

Introduction

A characterization of Borel sets by finer topologies is the starting point for this work. The following is a fundamental fact about Borel sets in Polish spaces:

> For every Borel set in a Polish space exists a finer Polish topology for the space, such that the Borel set is open and closed with respect to this finer topology.

This fact implies very easily a remarkable result for one of the classical, if not *the* classical, problem in early set theory, the Continuum Hypothesis (**CH**) by Cantor. Cantors conjecture was that every subset of the reals (that he called the continuum) is either at most countable or has the cardinality of the continuum (cf. [Cant78]).

Of course, nowadays we know that this problem can not be decided in Zermelo Fraenkel set theory. But Cantor tried very hard to find a proof for his conjecture and one of the most promising attempts for him was the proof of the perfect set property for closed subsets of the reals (see [Cant84]). This fact is known today under the name Cantor-Bendixson Theorem and asserts that every uncountable closed subset of the reals contains a perfect subset, that is, a nonempty closed subset with no isolated points. Perfect subsets have the cardinality of the continuum. So by the Cantor-Bendixson theorem the Continuum Hypothesis is true for closed subsets of the reals. Cantor was convinced that he can expand the result for all sets. Of course he could not succeed, but about 30 years later Felix Hausdorff, who was Professor here at the University of Bonn from 1910 until 1932, could prove the Continuum Hypothesis for Borel sets in [Haus16]:

> "Jede Borelsche Menge ist entweder endlich oder abzählbar oder von der Mächtigkeit des Kontinuums"

Hausdorff's proof can be described as "going down the Borel hierarchy". Roughly his idea is the following. An uncountable Borel set is in some $\mathbf{\Sigma}^0_\alpha$ for an ordinal α. Since this is a countable union of sets from lower stages of the Borel hierarchy one of these sets from the union is uncountable. This set is again a countable union of sets from lower stages of the Borel hierarchy and so on. So finally he arrives at closed sets there the result is known by Cantor's result.

With the above fact about Borel sets in Polish spaces (and an immediate generalization of the Cantor-Bendixson Theorem to Polish spaces) the proof that uncountable Borel sets are of cardinality of the continuum is trivial. Because then uncountable Borel sets are closed sets in a Polish space and have therefore by the Cantor-Bendixson Theorem the cardinality of the continuum.

Another nice application of the fact about Borel sets is that we can characterize analytic sets as continuous images of the Baire space. We will prove this in Proposition 6.1.6 in this thesis.

So this result about Borel sets is really an interesting one. By a well-known Theorem from Lusin that the image of a Borel set under an injective continuous mapping is again Borel we can prove the converse of this result by applying it to the identity mapping from the Polish space with the finer topology to the Polish space with its original Polish topology. So we get indeed a characterization of Borel sets by finer topologies. We can state this characterization as follows:

> A subset of a Polish space is a Borel set iff there exists a Polish topology on this subset that is finer than the restriction of the topology of the Polish space to the subset.

One could ask if we get such characterizations for other classes of sets than the Borel sets. Or, seen from another point of view, one can ask what class of subsets do we get by dropping some properties of the finer topology. Professor Howard S. Becker from the University of South Carolina in Columbia discussed this question in two unpublished notes. The goal of this thesis is to present the results from Professor Becker. In "Finer topologies of pointsets in Polish spaces" from March 1991 he found a characterization for $\mathbf{\Sigma}^1_1$ sets in the theory $\mathbf{ZF} + \mathbf{DC}$ and more general for all sets from the projective hierarchy in his notes "Playing around with finer topologies" from January 1992 under the axioms $\mathbf{ZF} + \mathbf{DC} + \mathbf{AD}_{\mathbb{R}}$.

This thesis is divided now in two parts. In the first part we introduce all notions and results necessary for the proofs of the main theorems. It starts with a short chapter about Polish spaces. In the second chapter we discuss the basic concepts of trees and λ-Suslin sets that are fundamental for the characterization of the projective sets. In this connection we examine the relation of the λ-Suslin sets with λ-scales and λ-Borel sets. Chapter 3 recalls the concepts of the Borel and the projective hierarchy and its main properties. Since the characterization for pointsets of higher classes of the projective hierarchy requires the axiom of determinacy of the reals we introduce games and the concept of determinacy in chapter four. This chapter also includes a characterization of Polish spaces as strong Choquet spaces.

For this we need the notion of a strong Choquet game, that is, a two person game in which the players take turns in playing nonempty open sets of the topological space, such that each set is contained in the sets played before. In addition player I has to play a point in his open set and player II is obliged to play an open set such that it contains also this point played by player I. Player II wins this game if the intersection of all open sets is nonempty.

A topological space is called strong Choquet space if player II has a winning strategy in the strong Choquet space. We prove that Polish spaces are second countable, regular, strong Choquet spaces with the Hausdorff property and use this properties in Part 2 for the characterization of the projective sets by finer topologies. But before we come to this part we close Part 1 with a chapter about the scale property and about projective ordinals under the axioms **PD** and **AD**.

In Part 2 we give proofs for all results about the characterization of the projective sets. We start in Chapter 6 with the proof of the above characterization of the Borel sets. The theory **ZF** + **DC** is sufficient to prove then a corresponding result for the analytic sets:

> A subset of a Polish space is analytic iff there exists a second countable, strong Choquet topology on this subset that is finer than the restriction of the topology of the Polish space to the subset.

A construction of such a finer topology for all $\mathbf{\Sigma}^1_n$ sets is immediate if we work under the additional axiom **PD**. This is proved in the beginning of Chapter 7. Crucial for this is that $\mathbf{\Sigma}^1_n$ sets are κ-Suslin for a cardinal κ less than the projective ordinal $\boldsymbol{\delta}^1_n$ as an ordinal. We thus construct finer strong Choquet topolgies on such sets with a basis of lenth less than the associated projective ordinals. The prove of the converse is a lot more difficult. We have to introduce some new notions about reliable ordinals and honest subsets of reliable ordinals before we finish in Chapter 7 with the following theorem:

> A subset of a Polish space is $\mathbf{\Sigma}^1_n$ iff there exists a strong Choquet topology with a basis of length less than $\boldsymbol{\delta}^1_n$ on this subset that is finer than the restriction of the topology of the Polish space to the subset.

The proof of this theorem requires the very strong axiom $\mathbf{AD}_{\mathbb{R}}$. But assuming this we have in fact found a topological characterization of all projective sets.

Our notation is close to the notation in [Kech95] and [Mosc80]. The basic theory for this paper is the Zermelo-Fraenkel set theory together with the axiom of dependent choice **DC**.

Part I

Facts from descriptive set theory

In this first part we will introduce all of the basic concepts that will be necessary for the characterization of the projective sets and the proofs for it. The topological spaces we consider are the Polish spaces. So in the first chapter we define the **Polish spaces** and will take a look at sums and products as well as certain subsets of Polish spaces.

By far the most important Polish space for our approach is the **Baire space**, i.e., the space ω^ω seen as the topological product of the discrete topological spaces ω. In the forthcoming we will call elements of ω **integers** and elements of the Baire space **reals**. To examine the Baire space, the concept of a tree is of help. Trees are a fundamental tool for descriptive set theory and in particular in our work here. In Chapter 2 we thus introduce the notion of trees and many concepts related to it. A **tree** on ω consists of finite sequences of integers such that each initial segment of such a finite sequence is again in the tree. By an **infinite branch** through such a tree we understand an uncountable sequence of integers, an element of the Baire space, such that all finite inital segments of this sequence are also in the tree. Closed subsets of the Baire space are characterised by the set of all infinite branches of a tree on ω. This easy but important result is the starting point for the consideration of representations of subsets from the Baire space by trees. This leads in particular to the proof that for each Polish space exists a continuous mapping from the Baire space onto the considered Polish space. This explains the special role the Baire space plays in the category of Polish spaces.

Another tree representation is the main definition in Chapter 2. We consider trees on the product of ω and an ordinal λ. Subsets of the Baire space that can be characterized as the projection of the infinite branches of such a tree to the Baire space are called **λ-Suslin sets**. The existence of such a representation will turn out to be crucial for our topological characterisation of projective sets. So in the rest of Chapter 2 we discuss these sets. In particular we examine the connection between λ-scales and λ-Suslin sets. A **λ-scale** on a subset of the Baire space is a sequence of **λ-norms**, i.e., a sequence of mappings from the subset to λ, with additional properties. We will prove that each subset that admits a λ-scale is λ-Suslin. We finish Chapter 2 by introducing **Borel** and **λ-Borel sets** and discussing the relation between these sets and the λ-Suslin sets.

Chapter 3 gives a short overview about the **Borel** and the **projective hierarchy**. We will define these hierarchies and state the main properties. In the second part of this chapter we introduce the effective analogs of these hierarchies together with their main properties.

In Chapter 4 we turn to the concept of **games** and **determinacy**. We consider two person games for example on the integers. For a subset of the Baire space, called the payoff set, such a game works as follows. The two players I and II take turns in playing integers. After ω moves, the outcome of such a game is an uncountable sequence of integers, therefore an element of the Baire space. We say, player I has won the game if the outcome of this run of the game is in the payoff set. Otherwise II has won. A **strategy** for one of the players tells the player which move to make in every round of the game depending on the finite sequence played so far. Such a strategy is called a **winning strategy** if the player wins all runs of the game by following his strategy. We call a subset of the Baire space **determined**, if in the associated game with this subset as the payoff set one of the players has a winning strategy.

It is an interesting problem which pointsets of the Baire space are determined. We are here not interested in this problem but rather postulate the determinacy of certain pointsets. We introduce the axiom **PD** (which asserts that all projective pointsets are determined) and the axiom **AD** (which asserts that all pointsets of the Baire space are determined). Furthermore we will need the axiom $\mathbf{AD}_{\mathbb{R}}$ that asserts that in a game on the reals (on the Baire space) every pointset is determined. We will work under the assumption of these axiom to prove the characterization of the projective sets.

As described in the introduction we will also consider the strong Choquet game and prove the characterization of Polish spaces as strong Choquet spaces in the second part of Chapter 4.

In Chapter 5 we will show that the projective sets admit certain scales if we work under determinacy axioms as described in Chapter 4. Therefore we conclude that the projective sets are λ-Suslin sets. The ordinal λ will be closely related to the **projective ordinals**, which are defined as the supremum of all the lengths of norms on the Baire space which are in $\mathbf{\Delta}^1_n$. Chapter 5 ends with an analysis of these projective ordinals under **AD**.

The basic theory for this chapter is the Zermelo-Fraenkel set theory together with the **Principle of dependent choices (DC)**:

> (**DC**) For every binary relation $R \subseteq X \times X$ on a nonempty set X the following holds:
>
> $$\forall x \in X\, \exists y \in X\, (x, y) \in R \Rightarrow \exists f : \omega \longrightarrow X\, \forall n((f(n), f(n+1)) \in R$$

Often we need just the weaker **Axiom of Countable Choice ($\mathbf{AC}_\omega$)**:

> ($\mathbf{AC}_\omega$) Every countable set consisting of nonempty sets has a choice function.

The axiom **DC** implies $\mathbf{AC}_\omega$, for a proof see for example [Rohd01, Lemma 1.7].

If one of our results needs additional assumptions it will be specified.

Chapter 1

Polish spaces

We want to start off with the definition and some basic facts about Polish spaces. We assume familiarity with the basic concepts of topological and metric spaces but repeat first a few properties of it and introduce notation.

Definition 1.1. Let $(X, \mathcal{T})$ be a topological space.

1. $(X, \mathcal{T})$ is **separable** if there exists a countable dense subset of X, that is, a subset that has a nonempty intersection with every nonempty open set.

2. A **basis** $\mathcal{B}$ for $\mathcal{T}$ is a collection $\mathcal{B} \subseteq \mathcal{T}$ such that every nonempty set in $\mathcal{T}$ can be written as a union of sets from $\mathcal{B}$. The **length of a basis** $\mathcal{B}$ for $\mathcal{T}$ is the cardinality of $\mathcal{B}$.

3. $(X, \mathcal{T})$ is **second countable** if $(X, \mathcal{T})$ has a countable basis.

4. $(X, \mathcal{T})$ is called a **T1 space** if for every two distinct points $x, y \in X$ there exists an open set U of X such that $x \in U$ and $y \notin U$.

5. $(X, \mathcal{T})$ is called a **Hausdorff space** if for every two distinct points $x, y \in X$ there exist open neighborhoods U of x and V of y such that $U \cap V = \emptyset$.

6. $(X, \mathcal{T})$ is called **regular** if for every point $x \in X$ and every open neighborhood U of x there is an open neighborhood V of x such that the closure of V is contained in U. We denote the closure of a subsets V of X by $\mathrm{cl}_{\mathcal{T}}(V)$.

Polish spaces are topological spaces $(X, \mathcal{T})$ where the topology is induced by a metric d on X. That means the open balls $B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$ for all $x \in X$ and all radius $\varepsilon \geq 0$ serve as a basis for the topology. A topological space $(X, \mathcal{T})$ is called **metrizable** if there exists a metric d on X such that $\mathcal{T}$ is the topology induced by the metric d. The space $(X, \mathcal{T})$ is called **completely metrizable** if the topology $\mathcal{T}$ is induced by a complete metric d. In general this metric d is not unique. We say a (complete) **metric d is compatible** for a (completely) metrizable topological space $(X, \mathcal{T})$ if this d induces the topology.

Lemma 1.2 (AC_ω). *Every second countable topological space X is separable. Every metrizable, separable topological space X is second countable. In particular, for metrizable spaces separable is equivalent to second countable.*

Proof. Let X be a topological space with a countable basis $\{B_i \mid i \in \omega\}$. By AC_ω we can choose a point in each basic set. The set of all these points is countable and dense in X.

Let X be a separable space where the topology comes from a metric d. Let D be a countable dense subset of X. We claim that a basis for this topology is given by the open balls with center the points of D and rational radius (and by AC_ω this basis is countable). To see this, let U be an open set in X. Let $x \in U$. Since U is open there exists an open ball around x which is completely in U. Let $B(x, \varepsilon)$ be such a ball. Since D is dense in X there is a point $y \in D$ and a rational δ with $d(x, y) < \delta < \frac{\varepsilon}{2}$. Then $x \in B(y, \delta)$ and $B(y, \delta) \subseteq B(x, \varepsilon)$, since for $z \in B(y, \delta)$ we have $d(x, z) \leq d(x, y) + d(y, z) < 2\delta < \varepsilon$. So we can find for each point in U a neigborhood that has the form $B(y, \delta)$ with $y \in D$ and δ rational and lies completely in U. So U is the union of all these balls, which proves what we claimed. $\square$

Lemma 1.3. *Every metrizable space is a regular Hausdorff space. So in particular a T1 space.*

Proof. Let $(X, \mathcal{T})$ be a metrizable space and d be a compatible metric for $(X, \mathcal{T})$. First we want to prove the Hausdorff property. For this let x, y be two distinct points in X with $d(x, y) = \varepsilon > 0$. Then $B(x, \frac{\varepsilon}{4})$ and $B(y, \frac{\varepsilon}{4})$ are open sets that separate these two points, i.e., the intersection of these two open sets is empty.

To prove the regularity let U be an open neighborhood of a point x. Then there is an open ball $B(x, \varepsilon)$ contained in U and $B(x, \frac{\varepsilon}{2})$ is an open neighborhood of x with $\mathrm{cl}_{\mathcal{T}}(B(x, \frac{\varepsilon}{2})) \subseteq B(x, \varepsilon) \subseteq U$. $\square$

Definition 1.4. A topological space $(X, \mathcal{T})$ is called a **Polish space** if $(X, \mathcal{T})$ is a separable, completely metrizable space.

Example 1.5. (i) $\mathbb{R}$ with the usual metric is a Polish spaces.
(ii) Any set X with the discrete topology is a completely metrizable space. A compatible metric is given for example by the discrete metric δ, defined by

$$\delta(x, y) = 1 \text{ if } x \neq y \text{ and } \delta(x, y) = 0 \text{ if } x = y$$

The set X together with the discrete topology is a Polish space iff X is countable.

In the category of topological spaces exists products and sums (coproducts). It turns out that the product in the category of topological spaces of two Polish spaces is again Polish and also the sum of two Polish spaces is again Polish. We want to prove this next. It is necessary for the proof that the compatible metric d of a Polish space X is bounded by 1, i.e., $d(x, y) \leq 1$ for all $x, y \in X$. We already noted that the compatible metric is not unique and we show first, that there is indeed always a metric bounded by 1 that is compatible for the Polish space.

Two metrics d and d' on a set X are called **equivalent** if they induce the same topology. Since in a metric space the closed sets are exactly those sets in which the limit point of a convergent sequence in the set is again in the set, it suffices to show that two metrics d and d' on X induce the same notion of convergence in X, i.e., for every $x \in X$ and every sequence $(x_i)_{i\in\omega}$ in X the conditions $\lim_{i\to\omega} d(x, x_i) = 0$ and $\lim_{i\to\omega} d'(x, x_i) = 0$ are equivalent, to prove that d and d' are equivalent. We use this fact to show that in a metrizable space we can choose the metric that induces the topology to be bounded by 1.

Lemma 1.6. *In every metric space (X, d) the metric $d' = \frac{d}{1+d}$ is equivalent to d.*

Proof. Let (X, d) be a metric space. First we have to check that d' really is a metric. It is obvious that $d'(x, y) = 0$ iff $x = y$ and that $d'(x, y) = d'(y, x)$. To prove the triangle inequality consider the following equivalence in which I omitted the easy calculations. Let x, y, z be in X.

$$\begin{aligned} & d'(x,z) \leq d'(x,y) + d'(y,z) \\ \Leftrightarrow\ & d(xy) + d(y,z) - d(x,z) + 2d(x,y)d(y,z) + d(x,y)d(x,z)d(y,z) \geq 0 \end{aligned}$$

But the second line is true since $d(x, y) + d(y, z) - d(x, z) \geq 0$ by the triangle inequality for d. So d' is a metric and it is now trivial that d and d' induce the same notion of convergence. □

Proposition 1.7. *i) The product of a countable sequence of Polish spaces is Polish.*
ii) The sum of a sequence of Polish spaces is Polish.

Proof. (i) Let $(X_n)_{n\in\omega}$ be a sequence of metrizable spaces. For all $n \in \omega$ let d_n be a compatible metric for X_n with d_n bounded by 1. A metric on $\prod_{n=0}^{\omega} X_n$ is given by

$$d(x,y) = \sum_{n=0}^{\omega} \frac{1}{2^{n+1}} d_n(x_n, y_n)$$

where $x = (x_n), y = (y_n)$. This is obviously a metric.
(1) The topology induced from d on $\prod_{n=0}^{\omega} X_n$ is the same as the product topology.

Proof: The product topology is the smallest topology on $\prod_{n=0}^{\omega} X_n$ such that all projections $p_i : \prod_{n=0}^{\omega} X_n \to X_i$ are continuous. So if all projections p_i are continuous with respect to the topology induced by the metric d we know that this topology is finer than the product topology. But $p_i : (\prod_{n=0}^{\omega} X_n, d) \to (X_i, d_i)$ is in fact continuous for all i: Let $x = (x_n) \in \prod_{n=0}^{\omega} X_n$, let $\varepsilon > 0$. Then $d(x, y) < \frac{\varepsilon}{2^{i+1}}$ implies $d_i(p_i(x), p_i(y)) = d_i(x_i, y_i) < \varepsilon$. Thus the p_i's are continuous.
Let conversely $B(x, \varepsilon)$ be an open ball around $x = (x_n) \in \prod_{n=0}^{\omega} X_n$ with respect to the metric d. Let i be a natural number such that $\sum_{n=i}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2^i} < \varepsilon$.

Consider for $n < i$ the balls $B_n = B(x_n, \frac{\varepsilon}{2})$ with respect to the metric d_n. Then $\bigcap_{n=0}^{i} p_n^{-1}(B_n)$ is by definition of the product topology open and contains x. Let $y = (y_n) \in \bigcap_{n=0}^{i} p_n^{-1}(B_n)$. Then

$$\begin{aligned} d(x,y) &= \sum_{n=0}^{i-1} \frac{1}{2^{n+1}} d_n(x_n, y_n) + \sum_{n=i}^{\omega} \frac{1}{2^{n+1}} d_n(x_n, y_n) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

So $y \in B(x, \varepsilon)$. Therefore $\bigcap_{n=0}^{i} p_n^{-1}(B_n) \subseteq B(x, \varepsilon)$ and (1) is proved. q.e.d.(1)

A basis for the product topology is given by products $\prod_n U_n$ where $U_n = X_n$ except for finitely many i for which U_i is a basic set of X_i. So if all X_n's are separable the product space $\prod_{n=0}^{\omega} X_n$ is separable.

The last we have to check is that if all d_n are complete metrics then d is a complete metric. For this let (x^i) be a Cauchy sequence in X. Then $(p_n(x^i))_i = (x_n^i)_i$ is a Cauchy sequence in X_n for all n. Since all the X_n's are complete spaces the sequence $(x_n^i)_i$ converges against a $x_n \in X_n$ for all n. Thus $x = (x_n) \in \prod_{n=0}^{\omega} X_n$ and it is easy to see that the sequence (x^i) converges to the point x.

(ii) Let $(X_n)_{n \in \omega}$ be a sequence of metrizable spaces. For any n let d_n be a compatible metric on X_n bounded by 1. We may assume that the sets X_n are pairwise disjoint. Now define a metric on $X = \bigoplus_{n=0}^{\infty} X_n$ by

$$d(x,y) = \begin{cases} d_i(x,y) & \text{if } x, y \in X_i \text{ for some } i \in \omega \\ 1 & \text{otherwise} \end{cases}$$

The only thing to check that this is indeed a metric is the triangle inequality. Let $x, y, z \in X$. If $x, z \in X_i$ for some i then if y is also in X_i we have $d(x,z) = d_i(x,z) \leq d_i(x,y) + d_i(y,z) = d(x,y) + d(y,z)$ by the triangle inequality for d_i, otherwise $d(x,z) = d_i(x,z) < 1 < 2 = d(x,y) + d(y,z)$. If $x \in X_i, z \in X_j$ for $i \neq j$ we have $d(x,z) = 1$. But if $y \in X_i$ we have $d(x,z) = 1 \leq d(x,y) + 1$, if $y \in X_j$ we have $d(x,z) = 1 \leq 1 + d(y,z)$, and otherwise $d(x,z) = 1 < 2 = d(x,y) + d(y,z)$.

To show that the topology induced by d is the same as the sum topology, note that an open ball in X_i around an $x \in X_i$ with radius $\varepsilon < 1$ with respect to d_i is equal to an open ball in X around x with radius ε with respect to d. With this in mind everything that remains to show is obvious.

If all the X_n are separable spaces the sum is separable since the union of all the bases of the X_n is a basis for X.

If all d_n are complete then d is complete since a Cauchy sequence in X with respect to d will finally be in one X_i and we have the convergence there. □

Example 1.8. (i) $\mathbb{R}^n, n \in \omega$ and $\mathbb{R}^\omega$ with the usual metric are Polish spaces.
(ii) Let X be any set viewed as a topological space with the discrete topology. We already mentioned that this is a completely metrizable space and it is a Polish space iff X is countable. By the above Theorem 1.7(i) the product space X^ω of countable many copies of the discrete topological space X is again a

completely metrizable space. In the next chapter, having the notion of a tree, we will define a complete compatible metric for such spaces. If X is countable, X^ω is Polish. For example is ω^ω a Polish space and this space is called the Baire space. It is of great importance for our work here and we will come back to this space at various points.

Definition 1.9. The space ω^ω viewed as the product space of countable many copies of the discrete topological space ω is called **Baire space** and is denoted by $\mathcal{N}$.

Remark 1.10. It is common use in descriptive set theory to call the elements of the Baire space reals. This is justified by the fact that the Baire space is homeomorphic to the set of irrationals with the relative topology (for a definition of relative topology see below). Since the set of the rationals is countable, meager and from Lebesgue measure zero, the difference between the reals and the irrational plays no important role for many results in descriptive set theory.

We are now interested in subspaces of Polish spaces that are again Polish. We define the topology on a subspace Y of a topological space $(X, \mathcal{T})$ by the **relative topology** $\mathcal{T}|Y = \{U \cap Y \mid U \in \mathcal{T}\}$. It is easy to see that closed subsets of Polish spaces are again Polish with respect to the relative topology by taking the restriction of the complete metric to the closed subset. It is also possible to prove that open subsets of Polish spaces are again Polish but more difficult to find the correct metric. We do not want to prove this here but state instead a more general Theorem that tells us that the subsets of a Polish space with the relativized topology that are also Polish are exactly the G_δ sets.

Definition 1.11. Let $(X, \mathcal{T})$ be a topological space.
$G \subseteq X$ is called an G_δ set if G is an intersection of countable many open subsets of X. $F \subseteq X$ is called an F_σ set if F is a union of countable many closed sets of X.

Example 1.12. The open sets of a topological space are G_δ sets, the closed sets of a topological space are F_σ sets.

In Polish spaces the closed sets are G_δ sets.

To prove that a closed set in a Polish space is a G_δ set we have to introduce the **distance of a point from a subset** in a metric space (X, d). We define for a point $x \in X$ and a subset $A \subseteq X$ the distance of x from A by

$$d(x, A) = \inf\{d(x, y) \mid y \in A\}$$

Lemma 1.13. *Let X be a metrizable space. Then every closed subset of X is G_δ.*

Proof. Let d be a compatible metric for X. Let A be a closed set in X. We show that for $\varepsilon > 0$ the ε-ball around A, $B(A, \varepsilon) = \{x \in X \mid d(x, A) < \varepsilon\}$, is open. To see this let $y \in B(A, \varepsilon)$. Then $d(y, A) < \varepsilon$, say $d(y, A) = \overline{\varepsilon} < \varepsilon$. The ball $B(y, \varepsilon - \overline{\varepsilon})$ is contained in $B(A, \varepsilon)$, since for $z \in B(y, \epsilon - \overline{\epsilon})$ we have $d(z, A) \leq d(z, y) + d(y, A) < (\varepsilon - \overline{\varepsilon}) + \overline{\varepsilon} = \varepsilon$.
But now we can write $A = \bigcap_n B(A, \frac{1}{n+1})$ and thus A is a G_δ set. $\square$

We state now the Theorem about the subsets which are Polish with respect to the relative topology we mentioned above. For a proof see [Kech95, Ch.1 §3, Theorem 3.11].

Theorem 1.14. *A subspace of a Polish space with its relativized topology is Polish iff it is* G_δ.

So in particular the open subsets of a Polish space and by Lemma 1.13 the closed subsets of a Polish space are again Polish.

Chapter 2

Trees

A basic tool in descriptive set theory and for a better understanding of the Baire space is the notion of a tree. We begin with some notations.

Let X be a set. X^n is the set of all **finite sequences** $s = (s_0, \dots, s_{n-1})$ in X **of length** n. For $n = 0$ let $X^0 = \{\emptyset\}$, where $\emptyset$ denotes the empty sequence. For $s = (s_0, \dots, s_{m-1}) \in X^m$ and $t = (t_0, \dots, t_{n-1}) \in X^n$ we define the **concatenation** of s and t to be the finite sequence $s^\frown t = (s_0, \dots s_{m-1}, t_0, \dots, t_{n-1}) \in X^{n+m}$. In abuse of notation we write for $t = (x)$, a sequence of length 1, $s^\frown x$ instead of $s^\frown(x)$. A finite sequence s is an **initial segment** of the sequence t, $s \subseteq t$, if $m = \text{length}(s) \leq \text{length}(t) = n$ and $s = t|m = (t_0, \dots, t_{m-1})$. Two such finite sequences are called **compatible** if one is an initial segment of the other. Otherwise we will call them incompatible and denote this by $s \perp t$. If $x = (x_n)_{n\in\omega} \in X^\omega$ is an infinite sequence, we say a finite sequence s is an initial segment of x if there is an $m \in \omega$ such that $s = x|m = (x_0, \dots, x_{m-1})$. We denote this also by $s \subseteq x$. Finally $X^{<\omega} = \bigcup_{n\in\omega} X^n$ is the set of all finite sequences.

Definition 2.1. A **tree** T on X is a set of finite sequences in X closed under initial segments, i.e., $T \subseteq X^{\leq\omega}$ and if $t \in T$ and $s \subseteq t$ then $s \in T$.

An **infinite branch** of T is an infinite sequence $x \in X^\omega$ such that for all $n \in \omega$ the sequence $x|n = (x_0, \dots x_{n-1}) \in T$. The set of all infinite branches of T is denoted by $[T]$, so $[T] = \{x \in X^\omega \mid \forall n\ x|n \in T\}$.

2.1 The topology of the Baire space

We will now define a metric that induces the topology of the Baire space and also leads to a definition of a countable basis. Instead of just working with the Baire space we consider the more general context of metrizable spaces of the form X^ω seen as the product of countable many copies of the discrete topological space X.

Lemma 2.1.1. *Let X be a set. X^ω viewed as the product space of countable many copies of the discrete topological space X is metrizable with the complete* *metric*

$$d(x,y) = \begin{cases} 2^{-(\min\{n\in\omega \mid x|n\neq y|n\}+1)} & \textit{if } x \neq y \\ 0 & \textit{otherwise} \end{cases}$$

A basis for the topology of X^ω is then given by the sets

$$N_s = \{x \in X^\omega \mid s \subseteq x\} \ , \ s \in X^{<\omega}$$

Proof. It is easy to see that d is a metric.

A basic for the product topology of X^ω is given by sets of the form $\prod_{i\in\omega} U_i$ where $U_i = X$ except for finitely many i for which $U_i = \{x_i\}$ for an $x_i \in X$. The topology on X^ω induced by the metric d has by definition a basis consisting of sets $N_s, s \in X^{<\omega}$. Note that for $s \subseteq t$ we have $N_s \cap N_t = N_t$, and if $s \perp t$ we have $N_s \cap N_t = \emptyset$. It suffices to show, that these two topologies are the same. For this it is enough that each set of the basis of the one topology is open with respect to the other topology.

Let $U = \prod_{i\in\omega} U_i$ with $U_{i_0} = \{x_0\}, \ldots, U_{i_{n-1}} = \{x_{n-1}\}, i_0 < \ldots i_{n-1}$ and all other $U_i = X$. Then $U = \bigcup\{N_s \mid \text{length}(s) = i_{n-1} \text{ and } s_{i_0} = x_0, \ldots, s_{i_n} = x_n\}$.

Conversely, is $s = (s_0, \ldots, s_{n-1})$, then $N_s = \prod_{i\in\omega} U_i$ with $U_i = \{s_i\}$ for $i \leq n-1$, $U_i = X$ otherwise.

To see, that d is complete consider first the following equivalence:

(1) Let $(x^n)_{n\in\omega}$ be a sequence in X^ω. Then $x^n \to x$ iff $\forall i\ (x^n(i) \to x(i))$.

Proof: "$\Rightarrow$" Let $i \in \omega$. Let $\varepsilon < \frac{1}{2^{i+1}}$. Since $x^n \to x$ there exists a $N \in \omega$ such that $d(x^n, x) < \varepsilon$ for all $n > N$. But

$$d(x^n, x) = \frac{1}{2^{(\min\{k\in\omega \mid x^n|k\neq x|k\}+1)}} < \varepsilon < \frac{1}{2^{i+1}}$$

implies $x^n(i) = x(i)$ for $n > N$. So $x^n(i) \to x(i)$.

"$\Leftarrow$" Let $\varepsilon > 0$. Let $i \in \omega$ such that $\frac{1}{2^{i+1}} < \varepsilon$. For $j \leq i$ exists an $N_j \in \omega$ such that $x^n(j) = x(j)$ for $n > N_j$ by the assumption. Let $N = \max\{N_j \mid j \leq i\}$. So for any $n > N$ we have $\min\{j \in \omega \mid x^n(j) \neq x(j)\} > i$. Therefore

$$d(x^n, x) = \frac{1}{2^{(\min\{k\in\omega \mid x^n|k\neq x|k\}+1)}} \leq \frac{1}{2^{i+1}} < \varepsilon$$

for every $n > N$. q.e.d (1)

Let now $(x^n)_{n\in\omega}$ be a Cauchy sequence in X^ω. Let $i \in \omega$ and fix $\varepsilon > 0$ with $\varepsilon < \frac{1}{2^{i+1}}$. Then there exists an $N \in \omega$ such that $d(x^n, x^m) < \varepsilon$ for $n, m > N$. By the choice of ε we have $x^n(i) = x^m(i)$ for all $n, m > N$. So in particular $\delta(x^n(i), x^m(i)) = 0$ for $n, m > N$ and therefore $(x^n)_{n\in\omega}$ is a Cauchy seqence. This sequence becomes eventually constant and converges against this constant point. Since i was arbitrary, we are done by (1). □

By Proposition 1.7 the products $(X^\omega)^n$, $n \in \omega$, and $(X^\omega)^\omega$ are again metrizable spaces. But the next lemma tells that these are not really new spaces since they are all homeomorphic to X^ω.

Lemma 2.1.2. *(i) For every $n \in \omega$ the product space $(X^\omega)^n$ is homeomorphic to X^ω.*
(ii) $(X^\omega)^\omega$ is homeomorphic to X^ω.

Proof. (i) Let $n \in \omega$. Let

$$\begin{array}{rcll} f : X^\omega & \longrightarrow & (X^\omega)^n & \\ x & \longmapsto & (x_0, \dots, n_{n-1}) & \quad \text{with } x_i(j) = x(nj+i) \text{ for } i < n \end{array}$$

This f is clearly a bijection. It is continuous, since for $N_{s_0} \times \ldots \times N_{s_{n-1}}$ a basic open set in $(X^\omega)^n$ we have $f^{-1}(N_{s_0}, \ldots \times N_{s_{n-1}}) = \bigcup\{N_s \mid s(nj+i) = s_i(j) \text{ if } j \leq \text{length}(s_i)\}$. f is open, since $f(N_s) = \bigcup\{N_{s_0} \times \ldots, N_{s_{n-1}} \mid s_i(j) = s(nj+i) \text{ if defined}\}$.

(ii) Fix a bijection $\langle\, , \rangle : \omega^2 \longrightarrow \omega$. Let

$$\begin{array}{rcll} f : X^\omega & \longrightarrow & (X^\omega)^\omega & \\ x & \longmapsto & (x_i)_i & \quad \text{with } x_i(j) = x(\langle i, j \rangle) \end{array}$$

This is clearly a bijection. Let $\prod_i U_i$ be a basic open set in $(X^\omega)^\omega$, say $U_{i_0} = N_{s_0}, \ldots, U_{i_m} = N_{s_{m-1}}$ and all other $U_i = X^\omega$. Then $f^{-1}(\prod_i U_i) = \bigcup\{N_s \mid s(\langle i_k, j\rangle) = s_{i_k}(j) \text{ if } j \leq \text{length}(s_{i_k}) \text{ and } k \leq m-1\}$. Thus f is continuous. On the other hand let $s = (s_0, \ldots, s_{m-1})$ and let i_k, j_k such that $\langle i_k, j_k \rangle = k$ for $k \leq m-1$. Then $f(N_s) = \prod_i U_i$ with all $U_i = X^\omega$ except for U_{i_k} with $U_{i_k} = \bigcup\{N_{s_{i_k}} \mid s_{i_k} = s_k \text{ if defined }\}$ for $k \leq m-1$. Thus f is open. □

An example for the importance of the trees in describing the metrizable spaces of the form X^ω is the following propositions that infinite branches of a tree on X are exactly the closed sets.

Proposition 2.1.3. *A set $C \subseteq X^\omega$ is closed iff there is a tree on X such that $C = [T]$.*

Proof. Let C be a closed set in X^ω. Consider the tree $T_C = \{x|m \mid x \in C \wedge m \in \omega\}$. Clearly this is a tree and $C \subseteq [T_C]$. If $y \notin C$, there exists an open neighborhood of y not in C. So by Lemma 2.1.1 there exists an $m \in \omega$ such that $N_{y|m} \cap C = \emptyset$. Therefore $y \notin [T_C]$. Hence $C = [T_C]$.

Now let T be a tree on X and $x \notin [T]$. Then there exists an $m \in \omega$ such that $x|m \notin T$. Therefore $N_{x|m} \cap [T] = \emptyset$ and $X^\omega \setminus [T]$ is open. □

There is also a connection between "nice" maps between trees on two sets and continuous functions on the product spaces of these sets.

Definition 2.1.4. Let S be a tree on a set A, T be a tree on a set B. A map $\varphi : S \longrightarrow T$ is called **monotone** if $s \subseteq t$ in S implies $\varphi(s) \subseteq \varphi(t)$.
For such φ let $D(\varphi) = \{x \in [S] \mid \lim_{n\in\omega} \text{length}(\varphi(x|n)) = \infty\}$. For $x \in D(\varphi)$ let $f_\varphi(x) = \bigcup_{n\in\omega} \varphi(x|n)$. φ is called **proper**, if $D(\varphi) = [S]$.

Proposition 2.1.5. *Let $\varphi : S \longrightarrow T$ be a monotone map on trees S, T on sets A, B. The the set $D(\varphi)$ is G_δ and $f_\varphi : D(\varphi) \longrightarrow [T]$ is continuous.*

Proof. (1) $D(\varphi)$ is G_δ:

We have $x \in D(\varphi) \Leftrightarrow \forall n \exists m\, (\text{length}(\varphi(x|m)) \geq n)$. So $D(\varphi) = \bigcap_{n\in\omega} U_n$ with $U_n = \{x \in [S] \mid \exists m \text{ length}(\varphi(x|m)) \geq n\}$. But these sets are open, since, if $y \in U_n$, there is an $m \in \omega$ with $\text{length}(\varphi(y|m)) \geq n$. Therefore $N_{y|m} \subseteq U_n$.
(2) f is continuous:

Let $V_t = N_t \cap [T]$ be a set from the basis of the topology of $[T]$. Then

$$\begin{aligned}
f_\varphi^{-1}(V_t) &= \{x \in D(\varphi) \mid f_\varphi(x) \in N_t \cap [T]\} \\
&= \{x \in D(\varphi) \mid f_\varphi(x) \supseteq t\} \\
&= \{x \in D(\varphi \mid \bigcup_{n\in\omega} \varphi(x|n) \supseteq t\} \\
&= \{x \in D(\varphi) \mid \exists s \in S, s \subseteq x, \varphi(s) \supseteq t\} \\
&= \bigcup \{N_s \cap D(\varphi) \mid s \in S, \varphi(s) \supseteq t\}
\end{aligned}$$

□

Definition 2.1.6. Let (X, τ) be a topological space. A closed set $F \subseteq X$ is a **retract** of X if there is a continuous surjection $f : X \longrightarrow F$ such that $f(x) = x$ for $x \in F$.

Proposition 2.1.7. *Let A be a countable set. Let $F \subseteq H$ be two closed subsets of A^ω. Then F is a retract of H.*

Proof. Since F, H are closed in A^ω there are trees S, T on A such that $F = [S], H = [T]$. Without loss of generality we can assume that these trees are **pruned**, that is, every sequence s in each tree has a proper extension $t \supsetneq s$. (Cutting off all finite branches without proper extension in S, T leads to the same $[S], [T]$.) We will define a monotone proper $\varphi : T \longrightarrow S$ with $\varphi(s) = s$ for $s \in S$. Then the continuous map f_φ is a witness for F being a retract of H. We define $\varphi(t)$ by induction on length(t). Let $\varphi(\emptyset) = \emptyset$. Now let $t \in T$ and $\varphi(t)$ be given. Let $a \in A$ such that $t^\frown a \in T$. If $t^\frown a \in S$, let $\varphi(t^\frown a) = t^\frown a$. If $t^\frown a \notin S$, let $\varphi(t^\frown a)$ be some $\varphi(t)^\frown b \in S$, and this exists since S is pruned. [Under the assumption of the Axiom of Choice this result holds for any set A, not only for countable ones.] □

2.2 Polish spaces as surjective images of the Baire space

The Baire space $\mathcal{N}$ plays a special role in the category of Polish spaces, since for every Polish space there exists always a continuous surjection of the Baire space in the Polish space. For a proof we first define the concept of a Lusin scheme.

Definition 2.2.1. A **Lusin scheme** on a set X is a family $(A_s)_{s\in\omega^{<\omega}}$ of subsets of X such that

(i) $A_{s^\frown i} \cap A_{s^\frown j} = \emptyset$ for $s \in \omega^{<\omega}, i \neq j \in \omega$

(ii) $A_{s^\frown i} \subseteq A_s$ for $s \in \omega^{<\omega}, i \in \omega$.

By (ii) in the defintion of a Lusin scheme the subsets A_s get smaller than the length of the sequence gets longer. In applications of the Lusin scheme we often construct subsets that get arbitrarily small. For this we use the notion of the diameter of a subset. In a metric space (X, d) we define the **diameter of a subset** A of X by

$$\operatorname{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}$$

Proposition 2.2.2. *Let $(A_s)_{s\in\omega^{<\omega}}$ be a Lusin scheme on a metric space (X, d) with $\lim_{n\to\omega} \operatorname{diam}(A_{x|n}) = 0$ for all $x \in \mathcal{N}$. Let $D = \{x \in \mathcal{N} \mid \bigcap_{n\in\omega} A_{x|n} \neq \emptyset\}$ and define $f : D \longrightarrow X$ by $\{f(x)\} = \bigcap_{n\in\omega} A_{x|n}$. Then f is injective and continuous. If (X, d) is complete and each A_s is closed, then D is closed.*

Proof. Note first that f is welldefined: Let $x \in D$. Since $\bigcap_{n\in\omega} A_{x|n} \neq \emptyset$, there is a $z \in \bigcap_{n\in\omega} A_{x|n}$. Let $z' \neq z$. Since X is a metric space, $d(z, z') > 0$, say $d(z, z') = \varepsilon$. But $\lim_{n\in\omega} \operatorname{diam}(A_{x|n}) = 0$, so there is an $m \in \omega$ such that $z \in A_{x|m}$ and $\operatorname{diam}(A_{x|m}) < \varepsilon$. Therefore $z' \notin A_{x|m} \supseteq \bigcap_{n\in\omega} A_{x|n}$.

(1) f is injective:

Let $x \neq y \in D$, Then there is an initial segment s (possibly the empty sequence) of x and y and $i \neq j \in \omega$, such that $s^\frown i \subseteq x, s^\frown i \nsubseteq y, s^\frown j \subseteq y, s^\frown j \nsubseteq x$. Then $A_{s^\frown i} \cap A_{s^\frown j} = \emptyset$, thus $\bigcap_{n\in\omega} A_{x|n} \cap \bigcap_{n\in\omega} A_{y|n} = \emptyset$. So $f(x) \neq f(y)$.

(2) f is continuous:

Let $d_{\mathcal{N}}$ be the metric from Lemma 2.1.1. Let $x \in D$. We have to show that for all $\varepsilon > 0$ exists an $\delta > 0$ such that $d_{\mathcal{N}}(x, y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$. Let $\varepsilon > 0$ be given. We have to find a proper δ. Since $\lim_{n\to\omega} \operatorname{diam}(A_{x|n}) = 0$, there is an $N \in \omega$ such that $\operatorname{diam}(A_{x|m}) < \varepsilon$ for all $m \geq N$. Take now $\delta = \frac{1}{2^{N+2}}$. Now let $y \in D$ such that $d_{\mathcal{N}}(x, y) < \delta$. Then $x|N = y|N$. Therefore $f(x), f(y) \in A_{x|N}$. Thus $d(f(x), f(y)) \leq \operatorname{diam}(A_{x|N}) < \varepsilon$.

(3) Now let d be a compatible complete metric on X and let each A_s be closed. Let $(x_n)_{n\in\omega}$ be a sequence in D with $x_n \to x$. We want to show first that $(f(x_n))_{n\in\omega}$ is a Cauchy sequence. Let for this $\varepsilon > 0$. Then there is a $N \in \omega$ with $\operatorname{diam}(A_{x|N}) < \varepsilon$. Since $x_n \to x$, there is an $M \in \omega$ such that $x_m|N = x|N$ for all $m > M$. So $f(x_m), f(x_n) \in A_{x|N}$ for $n, m > M$, hence $d(f(x_m), f(x_n)) < \varepsilon$ for $n, m > M$. So $(f(x_n))_{n\in\omega}$ converges against an $z \in X$. We have already seen that the sequence $(f(x_n))_{n\in\omega}$ is eventually in every $A_{x|N}$ for $N \in \omega$. Since these sets are closed, $z \in A_{x|N}$ for all $N \in \omega$. Thus $z \in \bigcap_{N\in\omega} A_{x|N}$, so we have $x \in D$. Thus D is closed. □

Theorem 2.2.3. *Let $(X, \mathcal{T})$ be a Polish space. Then there is a closed set $F \subseteq \mathcal{N}$ and a continuous bijection $f : F \longrightarrow X$. If X is nonempty, f can be extended to a continuous surjection $g : \mathcal{N} \longrightarrow X$.*

Proof. If we have such an f, the second assumption follows from Proposition 2.1.7.

Fix a compatible complete metric $d \leq 1$ on X. We will construct a Lusin scheme $(F_s)_{s\in\omega^{<\omega}}$ on X such that

(i) $F_\emptyset = X$

(ii) F_s is an F_σ set, i.e., a countable union of closed sets

(iii) $F_s = \bigcup_i F_{s^\frown i} = \bigcup_i \mathrm{cl}_{\mathcal{T}}(F_{s^\frown i}$

(iv) $\mathrm{diam}(F_s) \leq 2^{-\mathrm{length}(s)}$.

If we have defined such a scheme, consider the associated continuous map $f : D \longrightarrow X$ as in the above Proposition 2.2.2.

(1) $f(D) = X$

Proof: Let $z \in X$. We use induction to find a unique $x \in \mathcal{N}$ such that $f(x) = z$. Since X is the disjoint union of the $F_{(i)}$'s, there is exactly one $j \in \omega$ with $z \in F_{(j)}$. Let $x(0) = x_0 = j$.
If $s = (x_0, \ldots, x_{n-1})$ is the only sequence of length n such that $z \in F_s$, and F_s is the disjoint union of the $F_{s^\frown i}$, then there is exactly one $k \in \omega$ such that $z \in F_{s^\frown k}, z \notin F_{s^\frown i}$ for $i \neq k$. Let $x(n) = k$. This construction obviously leads to an $x \in \mathcal{N}$ such that $f(x) = z$. q.e.d. (1)

(2) D is closed

Proof: Let $(x_n)_{n\in\omega}$ be a sequence in D, $x_n \to x$. We show that $(f(x_n))_{n\in\omega}$ is a Cauchy sequence and thus converges in X, say $\lim_{n\in\omega} f(x_n) = y$. To see this, let $\varepsilon > 0$. Let $N \in \omega$ such that $\mathrm{diam}(F_{x|N}) < \varepsilon$. Since $x_n \to x$ there is an $M \in \omega$ such that $x_m|N = x|N$ for all $m > M$. Therefore $f(x_m), f(x_n) \in F_{x|N}$ for $m, n > M$ and $d(f(x_m), f(x_n)) < \varepsilon$ for $n, m > M$. In particular, the sequence $(f(x_n))_{n\in\omega}$ is eventually in $F_{x|N}$, thus $y \in \mathrm{cl}_{\mathcal{T}}(F_{x|N})$. N was chosen arbitrarily, thus $y \in \bigcap_{N\in\omega} \mathrm{cl}_{\mathcal{T}}(F_{x|N})$. But since $F_{x|N} = \bigcup_{i\in\omega} F_{x|N^\frown i} = \bigcup_i \mathrm{cl}_{\mathcal{T}}(F_{x|N^\frown i})$ and there is an $j \in \omega$ such that $x|N+1 = x|N^\frown j$, we also have $y \in \bigcup_{N\in\omega} F_{x|N}$. So $x \in D$ and $f(x) = y$. q.e.d. (2)

To construct now the Lusin scheme (F_s) it is enough to show that for every F_σ set $F \subseteq X$ and every $\varepsilon > 0$ we can write $F = \bigcup_{i\in\omega} F_i$, where the F_i are pairwise disjoint F_σ sets of diameter $< \varepsilon$, such that $\mathrm{cl}_{\mathcal{T}}(F_i) \subseteq F$. For notational simplicity we denote the complement of a subset D in X by $\sim D$. Note first, that if C, D are closed sets, then $C \setminus D$ is F_σ since

$$\begin{aligned} C \setminus D &= C \cap \sim D \\ &= C \cap \sim \bigcap_{n\in\omega} B(D, \frac{1}{n}) \\ &= C \cap \bigcup_{n\in\omega} \sim B(D, \frac{1}{n}) \\ &= \bigcup_{n\in\omega} C \cap \sim B(D, \frac{1}{n}) \end{aligned}$$

with $B(D, \frac{1}{n})$ the open balls around D (cf. the proof of Proposition 1.13). Now let $F = \bigcup_{i\in\omega} C_i, C_i$ closed, be an F_σ set. We can assume that $C_i \subseteq C_{i+1}$ for every $i \in \omega$, since we can write $F = \bigcup_{i\in\omega} C_i^*$ with $C_i^* = \bigcup_{n=0}^{i} C_n$ the closed sets. Then F can be written as a disjoint union of F_σ sets, $F = \bigcup_{i\in\omega} C_i \setminus C_{i-1}, C_{-1} = \emptyset$.

Now let $\{U_i \mid i \in \omega\}$ be a basis for the topology of X. It is clear that we can assume that all U_i have diameter $< \varepsilon$. Then $X = \bigcup_{i\in\omega} U_i$ and also $X = \bigcup_{i\in\omega} \mathrm{cl}_{\mathcal{T}}(U_i)$. Let $U_0^* = \mathrm{cl}_{\mathcal{T}}(U_0), U_{i+1}^* = \mathrm{cl}_{\mathcal{T}}(U_{i+1}) \setminus \bigcup_{j=0}^{i} \mathrm{cl}_{\mathcal{T}}(U_j)$. These are all pairwise disjoint F_σ sets of diameter $< \varepsilon$ and $\bigcup_{i\in\omega} U_i^* = X$. So we can write F as a union of pairwise disjoint F_σ sets of diameter $< \varepsilon$, $F = \bigcup_{i,j\in\omega}(C_i \setminus C_{i-1}) \cap U_j^*$, and $\mathrm{cl}_{\mathcal{T}}((C_i \setminus C_{i-1}) \cap U_j^*) \subseteq \mathrm{cl}_{\mathcal{T}}(C_i \setminus C_{i-1}) \subseteq C_i \subseteq F$. □

2.3 λ-Suslin sets and λ-scales

We are often interested in trees on products of two (or more) sets A and B. Let T be a tree on $A \times B$. The elements of $[T]$ are then elements of $(A \times B)^\omega$. But by using the canonical bijection

$$\begin{array}{rcl} (A \times B)^\omega & \longrightarrow & A^\omega \times B^\omega \\ ((a_0, b_0), (a_1, b_1), \ldots) & \longmapsto & ((a_0, a_1, \ldots), (b_0, b_1, \ldots)) \end{array}$$

we can view elements of $[T]$ as elements of $A^\omega \times B^\omega$. We sometimes also write finite sequence of T as $((a_0, a_1, \ldots, a_{n-1}), (b_0, b_1, \ldots, b_{n-1}))$ instead of $((a_0, b_0), (a_1, b_1), \ldots, (a_{n-1}, b_{n-1}))$. It makes now sense to apply the projection on A^ω to the set of the infinite sequences. We define

$$p[T] = \{x \in A^\omega \mid \exists y \in B^\omega\, (x, y) \in [T]\}$$

For example the projection of a closed set $C \subseteq \mathcal{N} \times \mathcal{N}$, that is given by the infinite sequences $[T]$ of a tree T on $\omega \times \omega$, to its first component is given by

$$\mathrm{proj}_{\mathcal{N}}[C] = \{x \in \mathcal{N} \mid \exists y \in \mathcal{N}\, (x, y) \in C\} = p[T] = \{x \in \mathcal{N} \mid \exists y \in \mathcal{N}\, (x, y) \in [T]\}$$

We call projections of closed sets of $\mathcal{N} \times \mathcal{N}$ **analytic sets** of the Baire space and they are exactly the sets that have the form $p[T]$ for some tree T on $\omega \times \omega$ following Proposition 2.1.3. We will come back to the analytic sets in the next section.

It will turn out that having sets as a projection of (the infinite branches of) a tree is fundamental for proving our main theorem and also in many other areas of descriptive set theory. In particular trees on wellfounded sets will be of special interest. The important definition in this context is thus the following.

Definition 2.3.1. Let λ be an infinite ordinal. $A \subseteq \mathcal{N}^k$ is called a **λ-Suslin set** if there is a tree T on $\omega^k \times \lambda$ such that $A = p[T]$.

In this notation the analytic sets are exactly the ω-Suslin sets. So far these are the only examples we have for λ-Suslin sets. We will show below that all sets that admit λ-scales are λ-Suslin sets. Before we introduce the scales we will show that λ-Suslin sets are closed under projections in the following sense.

Proposition 2.3.2. *Let $A \subseteq \mathcal{N}^{k+1}$ for $k \geq 1$ be a λ-Suslin set. Then $p[A] = \{(x_1, \dots, x_k) \mid \exists x_{k+1}\ (x_1, \dots, x_k) \in A\}$ is also λ-Suslin.*

Proof. Let $A \subseteq \mathcal{N}^{k+1}$ be λ-Suslin witnessed by a tree T on $\omega^{k+1} \times \lambda$, i.e., $A = p[T]$. Fix a bijection

$$f : \omega \times \lambda \longrightarrow \lambda$$

This leads to a bijection

$$f^* : (\omega \times \lambda)^{<\omega} \longrightarrow \lambda^{<\omega}$$

We define a tree T' on $\omega^k \times \lambda$ by

$$(s_1, \dots, s_k, \eta) \in T' :\Leftrightarrow (s_1, \dots, s_k, f^{*-1}(\eta)) \in T$$

Claim $p[T'] = p[A]$
Proof:

$$\begin{aligned}
(x_1, \dots, x_k) \in p[T'] &\Leftrightarrow \exists u \in \lambda^\omega\ (x_1, \dots, x_k, u) \in [T'] \\
&\Leftrightarrow \exists u \in \lambda^\omega\ \forall n (x_1|n, , \dots, x_k|n, u|n) \in T' \\
&\Leftrightarrow \exists u \in \lambda^\omega\ \forall n (x_1|n, \dots, x_y|n, f^{*-1}(u|n)) \in T \\
&\Leftrightarrow \exists u \in \lambda^\omega \exists x_{k+1} \in \mathcal{N}\ \forall n (x_1|n, \dots, x_{k+1}|n, u|n) \in T \\
&\Leftrightarrow \exists u \in \lambda^\omega \exists x_{k+1} \in \mathcal{N}\ (x_1, \dots, x_{k+1}, u) \in T \\
&\Leftrightarrow \exists x_{k+1} \in \mathcal{N} (x_1, \dots, x_{k+1}) \in p[T] = A \\
&\Leftrightarrow (x_1, \dots, x_k) \in p[A]
\end{aligned}$$

□

Proposition 2.3.2 will be important later.

Given a λ-Suslin set $A \subseteq \mathcal{N}^k$ note that using a bijection between the ordinal λ and its cardinality $\kappa = \overline{\overline{\lambda}}$ we get a tree T' on $\omega^k \times \kappa$ such that $A = p[T']$ and thus A is κ-Suslin. So, often one considers just κ-Suslin sets where κ is a cardinal. It seems more natural for the upcoming definition to introduce here the more general notion.

Before we start defining λ-scales and prove that there is a close relation between sets that admit λ-scales and sets that are λ-Suslin we have to introduce the notion of norms and prewellorderings.

We first recall the concept of wellfounded relations. Let $\preceq$ be a binary relation on a set X. The strict part $\prec$ of the relation $\preceq$ is defined by

$$x \prec y \ \Leftrightarrow\ x \preceq y \wedge \neg (y \preceq x).$$

We call the relation $\preceq$ a **wellfounded relation** if each nonempty subset A of X has a $\prec$-minimal element, that is, there exists an element $x \in A$ such that $\neg y \prec x$ for all other $y \in A$. Under **DC** this is equivalent to the fact that no infinite descending chain with respect to $\prec$ exists, i.e., there exists no infinite sequence

$$x_0 \succ x_1 \succ x_2 \dots$$

One can apply the concepts of induction and recursion to wellfounded relations (see for example [BuKo96, Ch.5.5]). In particular one can define the length of a wellfounded relation by defining a canonical rank function on X. A **rank function** on X with respect to the wellfounded relation $\preceq$ is a function $\rho : X \longrightarrow \mathrm{Ord}$ such that if $x \prec y$ for $x, y \in X$ then $f(x) < f(y)$. A canonical rank function $\rho_{\preceq}$ for X with respect to a wellfounded relation $\preceq$ is defined by recursion in the following way:

$$\begin{aligned} \rho_{\preceq} : X &\longrightarrow \mathrm{Ord} \\ x &\longmapsto \sup\{\rho(y) + 1 \mid y \prec x\} \end{aligned}$$

One can prove that such a canonical rank function exists (see for example [Jech97, Part I, Ch.2, Theorem 5]). The range of this canonical rank function $\rho_{\preceq}$ is an ordinal and this ordinal is called the **length of the wellfounded relation** $\preceq$ and is denoted by $|\preceq|$.

A prewellordering is now just a wellfounded relation with additional properties. The concept of a norm is closely related to prewellorderings, since it will be pretty obvious how to get a prewellordering out of a norm.

Definition 2.3.3. Let X be a set. A **norm on X** is a map $\varphi : X \longrightarrow \mathrm{Ord}$. A norm is called **regular** if $\varphi[X]$ is an ordinal, that is, φ maps X onto some ordinal λ.

A **prewellordering** on a set X is a wellfounded relation $\leq$ on X which is reflexive, transitive and connected, which means for every $x, y \in X$ we have $x \leq y$ or $y \leq x$.

It is very easy to see that for each norm φ on a set X the relation $\leq_{\varphi}$ defined by

$$x \leq_{\varphi} y \;\Leftrightarrow\; \varphi(x) \leq \varphi(y)$$

is a prewellordering. Conversely, one can define the canonical rank function on each prewellordering and gets a norm. So the concepts of a norm and of a prewellordering coincide. The following proposition states this fact.

Proposition 2.3.4. *Let X be a set. If $\varphi : X \longrightarrow \mathrm{Ord}$ is a norm, then $\leq_{\varphi}$ defined by $x \leq_{\varphi} y :\Leftrightarrow \varphi(x) \leq \varphi(y)$ is a prewellordering on X. If $\preceq$ is a prewellordering of X, then there exists a unique regular norm φ on X with $\preceq = \leq_{\varphi}$.*

Proof. If φ is a norm on X one proves easily that the relation $\leq_{\varphi}$ is a prewellordering on X.

If a prewellordering $\preceq$ of X is given one defines by recursion on the wellfounded relation $\prec$ the canonical rank function ρ by $\rho(x) = \sup(\{\rho_{\preceq}(y)+1 \mid y \prec x\}$. The rank function is a surjection on some ordinal and it is easy to see that we get back our prewellordering $\preceq$ as $\leq_{\rho}$. So it remains to show that this norm ρ is unique. Assume there is a distinct surjection τ from X onto some ordinal such that $\preceq = \leq_{\tau}$. Let x be minimal with respect to $\preceq$ such that $\rho(x) \neq \tau(x)$ and without loss of generality let $\alpha = \rho(x) < \tau(x)$. Since τ is surjective there exists an $y \in X$ such that $\tau(y) = \alpha < \tau(x)$. Therefore we have $y \prec x$. But then we have $\rho(y) = \tau(y) = \alpha$ and thus $x \leq_\rho y$, so $x \preceq y$. This contradicts $y \prec x$. □

We call two norms φ, ψ on a set X **equivalent** if $\leq_\varphi = \leq_\psi$. Clearly every norm is equivalent to a unique regular norm (consider the associated prewellordering and the canonical rank function of this prewellordering). The **length of a prewellordering** $\leq$ is the range of the associated regular norm, denoted by $|\leq|$.

Of course there exist a lot of trivial norms for a set. The concept becomes interesting if we put definability conditions on a norm. We will come back to this in Chapter 5.

A (semi-)scale is now a sequence of norms in the following in sense:

Definition 2.3.5. (a) A **semi-scale** on a subset A of a Polish space X is a sequence of norms $(\varphi_n)_{n\in\omega}$ on A, such that for every sequence $(x_i)_{i\in\omega}$ in A for which the following holds
1. $\lim_{i\to\omega} x_i = x$
2. for all n there is a $\lambda_n \in \mathrm{Ord}$ such that $\varphi_n(x_i) = \lambda_n$ for all i large enough

we have $x \in A$.
It is a **scale** if in addition $\varphi_n(x) \leq \lambda_n$ for all n.
(b) A (semi-)scale $(\varphi_n)_{n\in\omega}$ is a **λ-(semi-)scale** if for all $n \in \omega$ the length of φ_n is less or equal λ.

Similar to the norms the concept of scales becomes more interesting then we put definablity conditions on it. This will play a crucial role in proving our main theorem and we will also come back to it in Chapter 5. But subsets of the Baire space that admit λ-semi-scales are of interest in there own sense since they are λ-Suslin sets. The next theorem assures that the converse is also true, i.e., λ-Suslin sets admit λ-semi-scales. We introduce one more notion for the proof of it.

Definition 2.3.6. Let T be a tree on a set A. For a finite sequence $s \in A^{<\omega}$ we define

$$T_s = \{t \in T \mid t \text{ is compatible with } s\} = \{t \in T \mid t \subseteq s \vee s \subseteq t\}$$

Theorem 2.3.7. *A subset A of the Baire space $\mathcal{N}$ is λ-Suslin iff A admits a λ-semi-scale.*

Proof. Let first $A \subseteq \mathcal{N}$ be λ-Suslin. Fix a tree T on $\omega \times \kappa$ such that $A = p[T]$. For $x \in A$ we want to pick now one branch $(x, f) \in T$ without using any choice. For this we need the notion of a leftmost branch of a tree. We define the **leftmost branch** (x, f_x) of $[T]$ by recursion as follows:
First let $\prec$ be a wellordering on $\omega \times \lambda$ defined by

$$(k, \alpha) \prec (\ell, \beta) \Leftrightarrow \alpha < \beta \vee (\alpha = \beta \wedge k < \ell)$$

If $((x(0), \ldots, x(n-1)), (f_x(0), \ldots, f_x(n-1))$ is already defined (possibly the empty sequence), let $(x(n), f_x(n))$ be the $\prec$-least element (k, α) of $\omega \times \lambda$ such that $[T_{x|n^\frown k, f_x|n^\frown \alpha}] \neq \emptyset$.

Now let for $x \in A$ the leftmost branch of T be given by (x, f_x). Let $\varphi_n(x) = f_x(n)$ for $n \in \omega$. So φ_n is a λ-norm on A. To prove it is a semi-scale let $(x_i)_{i \in \omega}$ be a sequence in A such that $x_i \to x$ and $\varphi_n(x_i) = \lambda_n$ for i large enough and for all n. We have therefore

$$(x_i, f_{x_i}) = (x_i, (\varphi_n(x_i))_{n \in \omega}) \in [T]$$

and

$$(x_i, (\varphi_n(x_i))_{n \in \omega}) \to (x, (\lambda_n)_{n \in \omega})$$

Since $[T]$ is closed $(x, (\lambda_n)_{n \in \omega}) \in [T]$, thus $x \in p[T] = A$. This proves that the norms φ_n form indeed a semi-scale.

Let now conversely $(\varphi_n)_{n \in \omega}$ be a λ-semi-scale on $A \subseteq \mathcal{N}$. The tree T on $\omega \times \lambda$ associated to this semi-scale is given by:

$$\begin{gathered} ((k_0, \ldots, k_n), (\xi_0, \ldots, \xi_n)) \in T :\Leftrightarrow \\ \exists x \in A \text{ such that } x(i) = k_i \text{ and } \varphi_i(x) = \xi_i \text{ for all } i \leq n \end{gathered}$$

(1) $A = p[T]$
Proof: "$\subseteq$" Let x be in A. Then obviously $(x, (\varphi_i(x))_i) \in [T]$.
"$\supseteq$" Let $x \in p[T]$. Then

$$\begin{aligned} x \in p[T] \quad &\Leftrightarrow \quad \exists u \in \lambda^\omega \ (x, u) \in [T] \\ &\Leftrightarrow \quad \exists u \in \lambda^\omega \ \forall i \in \omega \ (x|i, u|i) \in T \\ &\Leftrightarrow \quad \exists u \in \lambda^\omega \ \forall i \in \omega \ \exists y_i \in A \text{ such that for all } n \leq i \\ &\qquad\quad y_i(n) = x(n) \ \wedge \ \varphi_n(y_i) = u(n) \end{aligned}$$

So $(x|i, u|i) = (y_i|i, (\varphi_0(y_i), \ldots, \varphi_{i-1}(y_i))$ for all $i < \omega$. Thus the sequence of the y_i converges against x and $\varphi_n(y_i) = u(n)$ for all $i > n$. Since (φ_n) is a λ-semi-scale we have $x \in A$. □

2.4 Wellfounded trees

We call a tree T on some set X **wellfounded** if $[T] = \emptyset$. This comes from the fact that for such a tree the relation $\supset$ of proper extension of finite sequences is wellfounded. A **rank function for a tree** T on X is any mapping

$$\rho : X^{<\omega} \longrightarrow \mathrm{Ord}$$

such that ρ is $\supset$-$<$ orderpreserving, i.e., if s, t are in T and $t \supset s$ then $\rho(t) < \rho(s)$.

So if we have a wellfounded tree T we can thus define a canonical rank function as on any wellfounded relation by:

$$\begin{aligned} \rho_T : X^{<\omega} \quad &\longrightarrow \quad \mathrm{Ord} \\ s \quad &\longmapsto \quad \sup\{\rho(s^\frown x) + 1 \mid s^\frown x \in T\} \end{aligned}$$

there we adopt the usefull convention that $\sup(\emptyset) = 0$. If X is of cardinality κ one can show that $\rho_T(s) < \kappa^+$ for all $s \in X^{<\omega}$.

On the other hand it is clear that if we have some rank function ρ on T, the tree is wellfounded. This is because since under **DC** being wellfounded is equivalent to the nonexistence of infinite descending chains. So if an infinite branch $f = (x_0, x_1, x_2, \ldots)$ would exist in T we would get an infinite descending chain of ordinals

$$\rho(x_0) > \rho(x_0, x_1) > \rho(x_0, x_1, x_2) \ldots$$

Since these results are so very helpful in its application we put them down as a theorem. See [Mosc80, 2D.1].

Theorem 2.4.1. *A tree T on a set X is wellfounded if and only if it admits a rank function. If $card(X) = \kappa$ and T is wellfounded then ρ_T is a rank function with range in κ^+.*

We introduce one more notation. For a tree T on $\omega \times \kappa$ and $x \in \omega^\omega$ define:

$$T(x) = \{(\xi_0, \xi_1, \ldots, \xi_{n-1}) \mid (x|n, (\xi_0, \xi_1, \ldots, \xi_{n-1})) \in T\}$$

With this the following lemma is trivial:

Lemma 2.4.2. *Let $A \subseteq \mathcal{N}$ be λ-Suslin as witnessed by a tree T. Then $x \in A$ iff $T(x)$ is not wellfounded.*

2.5 λ-Borel sets

In the next chapter we will introduce the Borel hierarchy. But we define the Borel sets and in generalization the λ-Borel sets here since we will see that κ-Suslin sets, where κ is a cardinal, are κ^{++}-Borel sets of the Baire space.

Definition 2.5.1. Let $(X, \mathcal{T})$ be a topological space. A subsets A of X is called a **Borel set** if A is an element of the smallest class of subsets of X which contains all open sets and is closed under complements and countable unions. We denote the class of Borel sets of X by $\boldsymbol{\mathcal{B}}(X, \mathcal{T})$ or just $\boldsymbol{\mathcal{B}}(X)$ if it is clear which topology of the space we consider.
A subset A of X is called a **λ-Borel set** if A is an element of the smallest class of subsets of X which contain all open sets and is closed under complements and (wellordered) unions of length less than λ. We denote the class of the λ-Borel sets of X by $\boldsymbol{\mathcal{B}}_\lambda(X)$.

Remark 2.5.2. With the above notion the Borel sets of a topological space X are exactly the ω_1-Borel sets of X. Obviously the open, closed, G_δ and F_σ subsets of X are Borel sets.

Before we prove the result about the κ-Suslin sets we state a generalization of the famous Lusin Separation Theorem. In modern literature the **Lusin Separation Theorem** is stated in the following form:

Theorem 2.5.3. *Let $(X,\mathcal{T})$ be a Polish space and A, A' be two disjoint analytic sets. Then there exists a Borel set B that separates A from A', i.e., $A \subseteq B$ and $A' \cap B = \emptyset$.*

A proof can for example be found in [Kech95, Theorem 14.7]. We have seen in the discussion of Definition 2.3.1 that the analytic sets of the Baire space are exactly the ω-Suslin sets and Borel sets are by definition ω_1-Borel sets. So we can read the Lusin Separation Theorem for the Baire space as follows:

Two disjoint ω-Suslin sets can be separated by an ω_1-Borel set.

We state now a generalization of this. A proof by contradiction as well as a constructive one for this Strong Separation Theorem can be found in [Mosc80, 2.E.1].

Theorem 2.5.4. *Let κ be an infinite cardinal. Let $A, B \subseteq \mathcal{N}$ be κ-Suslin and $A \cap B = \emptyset$. Then there exists a κ^+-Borel set C which separates A from B, i.e., $A \subseteq C$ and $B \cap C = \emptyset$.*

The following corollary is now trivial.

Corollary 2.5.5. *If $A \subseteq \mathcal{N}$ and $\mathcal{N} \setminus A$ are κ-Suslin, then $A \in \boldsymbol{\mathcal{B}}_{\kappa^+}(\mathcal{N})$.*

Proof. Since A is the only set that separates A from $\mathcal{N} \setminus A$ we are done with the above Theorem 2.5.4 □

In general this result is not true if just the subset A is κ-Suslin but not its complement. But we can then prove that A is κ^{++}-Borel.

Theorem 2.5.6. *If $A \subseteq \mathcal{N}$ is κ-Suslin, then $A \in \boldsymbol{\mathcal{B}}_{\kappa^{++}}(\mathcal{N})$.*

Proof. Let T be a tree on $\omega \times \kappa$ such that $A = p[T]$. For each $\lambda < \kappa^+$ and each $s \in \kappa^{<\omega}$ define now

$$A_s^\lambda = \{x \in \omega^\omega \mid \rho_{T(x)}(s) \leq \lambda\}$$

We prove by induction over λ that each of these sets are κ^+-Borel.
$\lambda = 0:\ A_s^0 = \bigcap_{\xi<\kappa}\{x \mid (x|n+1, s^\frown\xi) \notin T\} = \bigcap_{\xi<\kappa}\bigcup_{(x|n+1,s^\frown\xi)\notin T} N_{x|n+1}$
if s is of length n. Then A_s^0 is the intersection of less than κ^+ many finite unions of open sets, therefore κ^+-Borel.
Proof:

$$\begin{aligned} x \in A_s^0 &\Leftrightarrow \rho_{T(x)}(s) = 0 \\ &\Leftrightarrow \forall\xi < \kappa\, s^\frown\xi \notin T(x) \\ &\Leftrightarrow \forall\xi < \kappa\, (x|n+1, s^\frown\xi) \notin T \end{aligned}$$

$\lambda > 0:\ A_s^\lambda = \bigcap_{\xi<\kappa}\bigcup_{\xi<\lambda} A_{s^\frown\xi}^\xi$
Proof:

$$\begin{aligned} x \in A_s^\lambda &\Leftrightarrow \sup\{\rho_{T(x)}(s) + 1 \mid s^\frown\xi \in T(x)\} \leq \lambda \\ &\Leftrightarrow \forall\xi < \kappa\exists\eta < \lambda\, [s^\frown\xi \in T(x) \Leftarrow \rho_{T(x)}(s^\frown\xi) \leq \eta] \\ &\Leftrightarrow \forall\xi < \kappa\exists\eta < \lambda\, [\rho_{T(x)}(s^\frown\xi) \leq \eta] \\ &\Leftrightarrow \forall\xi < \kappa\exists\eta < \lambda\, (x \in A_{s^\frown\xi}^\eta) \\ &\Leftrightarrow x \in \bigcap_{\xi<\kappa}\bigcup_{\eta<\lambda} A_{s^\frown\xi}^\eta \end{aligned}$$

Claim: $\mathcal{N} \setminus A = \bigcup_{\lambda<\kappa^+} A^\lambda_\emptyset$
Proof:

$$\begin{aligned} x \notin A &\Leftrightarrow T(x) \text{ is wellfounded} \\ &\Leftrightarrow \rho_{T(x)}(\emptyset) \text{ is defined} \\ &\Leftrightarrow \rho_{T(x)}(\emptyset) < \kappa^+ \\ &\Leftrightarrow \exists\lambda < \kappa^+ \rho_{T(x)} \leq \lambda \\ &\Leftrightarrow \exists\lambda < \kappa^+ x \in A^\lambda_\emptyset \\ &\Leftrightarrow x \in \bigcup_{\lambda<\kappa^+} A^\lambda_\emptyset \end{aligned}$$

So A is as a complement of an κ^{++}-Borel set in $\mathcal{B}_{\kappa^{++}}$ □

We can strengthen the statement from the above Theorem if κ is a cardinal of cofinality greater than ω. First we repeat the notion of cofinality and notions related to it.

Definition 2.5.7. Let λ be a limit ordinal. A subset $S \subseteq \lambda$ is **unbounded** or **cofinal** in λ if for every $\xi < \lambda$ exists an $\eta \in S$ such that $\xi < \eta$. We define the **cofinality** of λ by

$$\mathrm{cf}(\lambda) = \min\{\overline{\overline{S}} \mid S \text{ is cofinal in } \lambda\}$$

A function $f : \xi \longrightarrow \lambda$ for $\xi \leq \lambda$ is called a **cofinal function** if the set $f[\xi]$ is cofinal in λ.
A cardinal κ is **regular** if $\mathrm{cf}(\kappa) = \kappa$.

Theorem 2.5.8. *If $A \subseteq \mathcal{N}$ is κ-Suslin with κ a cardinal of cofinality greater ω, then $A \in \boldsymbol{\mathcal{B}}_{\kappa^+}$.*

Proof. Let T be a tree on $\omega \times \kappa$ such that $A = p[T]$. For $\xi < \kappa$ and $x \in \mathcal{N}$ let $T^\xi(x) = \{s \in T(x) \mid \forall\alpha \in s\ \alpha < \xi\}$
(1) $T(x)$ is not wellfounded $\Leftrightarrow \exists\xi < \kappa \quad (T^\xi(x)$ is not wellfounded)
Proof: "$\Rightarrow$" Since $T(x)$ is not wellfounded there exists $f \in \kappa^\omega$ such that for all $n \in \omega$ $f|n \in T(x)$. Assume now that for all $\xi < \kappa$ the tree $T^\xi(x)$ is wellfounded. In particullar for all $\xi < \kappa$ the infinite branch f is not in $[T^\xi(x)]$. That means that for all $\xi < \kappa$ there exists $n < \omega$ such that $f(n) \geq \xi$. But then $f[\omega]$ is a cofinal set of length ω in κ and that contradicts the assumption $cf(\kappa) > \omega$.
" $\Leftarrow$" If there is a $f \in [T^\xi(x)]$ then $f \in [T(x)]$ q.e.d. (1)
Now let for $\xi < \kappa$ $\quad A_\xi = p[T^\xi]$. Since $\xi < \kappa$ we know that all A_ξ are κ^*-Suslin with $\kappa^* < \kappa$. Therefore $\kappa^{*++} \leq \kappa^+$ and from Theorem 2.5.6 we get that $A_\xi \in \boldsymbol{\mathcal{B}}_{\kappa^{*++}} \subseteq \boldsymbol{\mathcal{B}}_{\kappa^+}$.
By the above we have

$$x \in A \Leftrightarrow T(x) \text{ not wellfounded} \Leftrightarrow \exists\xi < \kappa\ (T^\xi(x) \text{ not wellfounded})$$

and therefore $A = \bigcup_{\xi<\kappa} A_\xi \in \boldsymbol{\mathcal{B}}_{\kappa^+}$. □

Chapter 3

The Borel and the projective hierarchy

In this chapter we will recall very briefly some of the basic definitions and properties of the Borel and the projective hierarchy together with its effective analogs. Proofs and more details can be found in an introctuary book on decriptive set theory, for example in [Mosc80] or [Kech95].

3.1 The Borel and the projective hierarchy

We will first introduce the notions of pointclasses.

Definition 3.1.1. We call Γ a **pointclass** if Γ is a collection of subsets of Polish spaces. A **pointset** is then just a set of this class. For a pointset A of a pointclass Γ we write $A \in \Gamma$ or say A is a Γ set. If X is a Polish space and Γ a pointclass we denote by $\Gamma(X)$ the pointsets of Γ which are subsets of X.
The **dual pointclass** $\check{\Gamma}$ for a pointclass Γ is defined by $\check{\Gamma} = \{A \mid X \setminus A \in \Gamma(X) \text{ for some Polish space } X\}$.
For each pointclass Γ the **ambiguous** part of Γ is the class $\Delta = \Gamma \cap \check{\Gamma}$.

We denote for example the class of Borel sets in Polish spaces (as introduced it in 2.5.1) by $\boldsymbol{\mathcal{B}}$ and this stands for the class

$$\boldsymbol{\mathcal{B}} = \{A \mid A \subseteq X \text{ for some Polish space } X \text{ and } A \text{ is a Borel set in } X\}.$$

For some Polish space X the set $\boldsymbol{\mathcal{B}}(X)$ consists of the Borel sets of X (for example $\boldsymbol{\mathcal{B}}(\mathcal{N})$ is the collection of all Borel sets of the Baire space $\mathcal{N}$). So the pointclass $\boldsymbol{\mathcal{B}}$ is the union of all $\boldsymbol{\mathcal{B}}(X)$ for X a Polish space. We could define pointclasses for other categories too, for example for the category of metrizable spaces, but we are here just interested in Polish spaces.

We define now the pointclasses of the Borel hierarchy by recursion on the ordinals.

Definition 3.1.2. Let A be a subset of some Polish space X. The **Borel hierarchy** of X is defined as follows.

$$\begin{array}{rcl}
A \in \mathbf{\Sigma}^0_1(X) & \Leftrightarrow & A \text{ is open in } X \\
A \in \mathbf{\Pi}^0_1(X) & \Leftrightarrow & A \text{ is closed in } X \\
A \in \mathbf{\Sigma}^0_\alpha(X) & \Leftrightarrow & A = \bigcup_{n\in\omega} A_n \text{ where } A_n \in \mathbf{\Pi}^0_{\beta_n}(X) \text{ for some } \beta_n < \alpha \\
A \in \mathbf{\Pi}^0_\alpha(X) & \Leftrightarrow & A \text{ is the complement of an } \mathbf{\Sigma}^0_\alpha(X) \text{ set in } X \\
A \in \mathbf{\Delta}^0_\alpha(X) & \Leftrightarrow & A \in \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)
\end{array}$$

For a Polish space X this forms indeed a hierarchy, that means, $\mathbf{\Sigma}^0_\alpha(X) \subseteq \mathbf{\Sigma}^0_{\alpha+1}(X)$ and similar for $\mathbf{\Pi}^0_\alpha(X)$ for $\alpha \in On$. We state this and other main properties in the next theorem. For proofs see for example [Kech95, II.11.B] or [Mosc80, 1.B; 1.F].

Theorem 3.1.3. *Let X be a Polish space. Then we have we following picture of inclusions:*

$$\begin{array}{ccccccccccc}
& & \mathbf{\Sigma}^0_1(X) & & & & \mathbf{\Sigma}^0_2(X) & & & & \\
& \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \\
\mathbf{\Delta}^0_1(X) & & & & \mathbf{\Delta}^0_2(X) & & & & \mathbf{\Delta}^0_3(X) & & \ldots \\
& \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & \\
& & \mathbf{\Pi}^0_1(X) & & & & \mathbf{\Pi}^0_2(X) & & & &
\end{array}$$

The union of all $\mathbf{\Sigma}^0_\alpha(X)$ is the collection of all Borel sets of X, so $\boldsymbol{\mathcal{B}}(X) = \bigcup_{\alpha\in\mathrm{Ord}} \mathbf{\Sigma}^0_\alpha(X)$. If X is an uncountable Polish space $\mathbf{\Sigma}^0_\alpha(X) \not\subseteq \mathbf{\Pi}^0_\alpha(X)$ for all $\alpha < \omega_1$, so we have proper inclusions in the above picture.

Furthermore, using **AC** *implies $\mathbf{\Sigma}^0_{\omega_1}(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X)$ and for $\alpha > \omega_1$ we have $\mathbf{\Sigma}^0_\alpha(X) = \mathbf{\Sigma}^0_{\omega_1}(X)$. From this it follows immediately that under* **AC** *we get $\boldsymbol{\mathcal{B}}(X) = \mathbf{\Sigma}^0_{\omega_1}(X)$.*

This last theorem thus justifies the name Borel hierarchy. We write boldface letters for this pointclasses to distinguish them from the arithmetical hierarchy we define in the next section. Sometimes, pointclasses closed under continous preimages are called **boldface pointclasses** (cf. for example [Andr??]). The just defined $\mathbf{\Sigma}^0_\alpha$ pointclasses are indeed closed under continuous preimages. The following theorem states the most interesting closure properties, see [Mosc80, 1C.2].

Theorem 3.1.4. *For a Polish space X the class $\mathbf{\Sigma}^0_\alpha(X)$ is closed under countable unions and finite intersections for all α. The pointclass $\mathbf{\Sigma}^0_\alpha$ is closed under continuous preimages for all α, i.e., the continuous preimage of an $\mathbf{\Sigma}^0_\alpha$ set is again an $\mathbf{\Sigma}^0_\alpha$ set.*
The class $\mathbf{\Pi}^0_\alpha(X)$ is closed under finite intersections and countable unions for all α. The pointclass $\mathbf{\Pi}^0_\alpha$ is closed under continuous preimages.
The ambiguous pointclass $\mathbf{\Delta}^0_\alpha$ is closed under finite unions and intersections, under continuous preimages and under complements.

Before we define now the projective hierarchy we will take a closer look at the analytic sets since they form the first level of the projective hierarchy. We introduced analytic sets of the Baire space as projections of closed sets of $\mathcal{N} \times \mathcal{N}$ and were able to characterize them as the ω-Suslin sets in the last section.

Historically these sets were discovered by Suslin who found a mistake in a paper of Lebesgue [Lebe05]. Lebesgue claimed that a projection of a Boreel set is again a Borel set. Suslin found out that the class of projections of Borel sets is strictly larger than the class of Borel sets. The following theorem gives a characterization of the analytic sets.

Proposition 3.1.5. *Let $(X, \mathcal{T})$ be a Polish space, $A \subseteq X$. Then the following are equivalent:*

(1) A is the continuous image of a function $f : \mathcal{N} \longrightarrow X$.

(2) $A = \mathrm{proj}_X[C]$ where $C \subseteq X \times \mathcal{N}, C$ closed.

(3) $A = \mathrm{proj}_X[B]$ where $B \subseteq X \times Y$ is a Borel set, Y is a Polish space.

(4) A is the continuous image of a Borel set of a Polish space .

Proof. (1) $\Rightarrow$ (2): Let $A = f[\mathcal{N}]$ where $f : \mathcal{N} \longrightarrow X$ is continuous. Then $\mathrm{graph}(f) := \{(f(x), x) \mid x \in \mathcal{N}\}$ is closed in $X \times \mathcal{N}$ and $A = \mathrm{proj}_X[\mathrm{graph}(f)]$.
(2) $\Rightarrow$ (3): trivial.
(3) $\Rightarrow$ (4): proj_X is a continuous mapping.
(4) $\Rightarrow$ (1): see 6.1.6. □

We postpone the last part of the proof until we have the characterization of Borel sets by a finer topology since we can then prove the missing part of this theorem very easily. Finally we write down the definition of the analytic sets in Polish spaces.

Definition 3.1.6. A set A in a Polish space X is called an **analytic set** if A is the projection of a Borel set in a Polish space $X \times Y$, where Y is a Polish space.

We already mentioned that the analytic subsets of the Baire space are exactly the ω-Suslin sets. This follows immediately from the above Proposition 3.1.5 and Proposition 2.1.3. Since this is so important we put this down as a theorem.

Theorem 3.1.7. *A subset A of the Baire space $\mathcal{N}$ is analytic iff A is ω-Suslin.*

Following Suslin, the analytic sets form a larger class of sets then the Borel sets. We will give a proof later (see 3.1.11 and 3.1.14). From the above characterization one can easily prove that the projection of an analytic set is again an analytic set. But if we take the dual class of the class of the analytic sets and apply projection we get a larger class than the class of the analytic sets. Iterating this process we get the projective hierarchy.

Definition 3.1.8. Let A be a subset of some Polish space X. We define the **projective hierarchy** of X by recursion on ω:

$$\begin{array}{rcl}
A \in \boldsymbol{\Sigma}^1_0(X) & \Leftrightarrow & A \in \boldsymbol{\Sigma}^0_1(X) \\
A \in \boldsymbol{\Pi}^1_0(X) & \Leftrightarrow & A \in \boldsymbol{\Pi}^0_1(X) \\
A \in \boldsymbol{\Sigma}^1_{n+1}(X) & \Leftrightarrow & A = \mathrm{proj}_X[B] \text{ where } B \in \boldsymbol{\Pi}^1_n(X \times \mathcal{N}) \\
A \in \boldsymbol{\Pi}^1_{n+1}(X) & \Leftrightarrow & X \setminus A \in \boldsymbol{\Sigma}^1_{n+1}(X) \\
A \in \boldsymbol{\Delta}^1_n(X) & \Leftrightarrow & A \in \boldsymbol{\Sigma}^1_n(X) \cap \boldsymbol{\Pi}^1_n(X)
\end{array}$$

We call a subset P of some Polish space a **projective set** if $P \in \boldsymbol{\Sigma}^1_n$ for some $n \in \omega$.

So with this notation the analytic sets are the $\boldsymbol{\Sigma}^1_1$ sets. In analogy to the Theorems 3.1.3 and 3.1.4 we state now theorems about the hierarchy that form the projective sets and the main closure properties of the projective sets.

Theorem 3.1.9. *Let X be a Polish space. Then the following picture of inclusions hold:*

$$\begin{array}{ccccccccccc}
 & & \boldsymbol{\Sigma}^1_1(X) & & & & \boldsymbol{\Sigma}^1_2(X) & & & & \\
 & \subseteq & & \subseteq & & \subseteq & & \subseteq & & \subseteq & \\
\boldsymbol{\Delta}^1_1(X) & & & & \boldsymbol{\Delta}^1_2(X) & & & & \boldsymbol{\Delta}^1_3(X) & & \cdots \\
 & \subseteq & & \subseteq & & \subseteq & & \subseteq & & \subseteq & \\
 & & \boldsymbol{\Pi}^1_1(X) & & & & \boldsymbol{\Pi}^1_2(X) & & & &
\end{array}$$

Note that we defined the projective sets just for integers and that by definition the union of all $\boldsymbol{\Sigma}^1_n$ sets is called the class of projective sets. For uncountable Polish spaces we have as with the sets of the Borel hierarchy proper inclusions in the above picture. To prove this, one uses the concept of universal sets. We come back to this after we state the closure properties.

Theorem 3.1.10. *For all $n \in \omega$ the class $\boldsymbol{\Sigma}^1_n$ is closed under countable intersections and unions, under continuous preimages and continuous images. The class $\boldsymbol{\Pi}^1_n$ is closed under countable unions and intersections and under continuous preimages. The class $\boldsymbol{\Delta}^1_n$ is closed under countable unions and intersections, under continuous preimages and under complents.*

It remains now to prove that for uncountable Polish spaces we have indeed a proper hierarchy and that the class of analytic sets is really larger than the class of Borel sets. For the latter we first prove that for a Polish space X we have $\boldsymbol{\mathcal{B}}(X) = \boldsymbol{\Delta}^1_1(X)$. We are done if we show afterwards that $\boldsymbol{\Sigma}^1_n(X) \not\subseteq \boldsymbol{\Pi}^1_n(X)$ for $n \in \omega$ if X is uncountable. Because then we have in particular that $\boldsymbol{\Sigma}^1_1(X)$ is a proper extension of $\boldsymbol{\Delta}^1_1(X) = \boldsymbol{\mathcal{B}}(X)$. And we also proved the fact about the proper hierarchy with this.

Theorem 3.1.11. *Let X be a Polish space. Then $\boldsymbol{\mathcal{B}}(X) = \boldsymbol{\Delta}^1_1(X)$.*

Proof. Let first $A \subseteq X$ be a Borel set. Taking the identity mapping between X we could see A as the continuous image of a Borel set. Therefore $A \in \boldsymbol{\Sigma}^1_1(X)$. Since Borel sets are closed under complements $X \setminus A$ is also a Borel set and therefore also in $\boldsymbol{\Sigma}^1_1(X)$. This implies $A \in \boldsymbol{\Pi}^1_1(X)$ and therefore $A \in \boldsymbol{\Delta}^1_1(X)$.

For the converse we use again the Lusin Separation Theorem 2.5.3. Let A be in $\mathbf{\Delta}^1_1(X)$. Then both A and its complement $X \setminus A$ are analytic sets. So by Theorem 2.5.3 A and $X \setminus A$ are separated by a Borel set and the only possible set that can separate A and $X \setminus A$ is the set A. Therefore A is a Borel set. □

We now introduce the notion of universal sets to prove that the projective hierarchy for uncountable Polish spaces is proper.

Definition 3.1.12. Let Γ be a pointclass of Polish spaces and let X be a Polish space. For Y another Polish space we call $U \subseteq Y \times X$ a Y**-universal set** for $\Gamma(X)$ if

- $U \in \Gamma(Y \times X)$
- $\{U_y \mid y \in Y\} = \Gamma(X)$, where $U_y = \{x \mid (y,x) \in U\}$

Universal sets exist for the classes of the projective hierarchy and also for the classes of the Borel hierarchy. For a proof see [Mosc80, 1D.2, 1E.3]. We state the result here only for the projective classes.

Theorem 3.1.13. *For every Polish space X and every uncountable Polish space Y exists an Y-universal set for $\mathbf{\Sigma}^1_n(X)$ and similar for $\mathbf{\Pi}^1_n(X)$ for all $n \in \omega$.*

With this theorem it is now easy to prove that the projective hierarchy is a proper hierarchy. The same proof applies for the classes $\mathbf{\Sigma}^0_\alpha$ of the Borel hierarchy for $\alpha < \omega_1$.

Proposition 3.1.14. *Let X be an uncountable Polish space. Then $\mathbf{\Sigma}^1_n(X) \neq \mathbf{\Pi}^1_n(X)$ for all $n \in \omega$. In particular this implies that $\mathbf{\Delta}^1_n(X) \subset \mathbf{\Sigma}^1_n(X)$ for all $n \in \omega$.*

Proof. Assume towards a contradiction that $\mathbf{\Sigma}^1_n(X) = \mathbf{\Pi}^1_n(X)$. Let U be an X-universal set for $\mathbf{\Sigma}^1_n(X)$. Therefore $U \in \mathbf{\Sigma}^1_n(X \times X)$. The function

$$\begin{array}{rcl} f : X & \longrightarrow & X \times X \\ x & \longmapsto & (x,x) \end{array}$$

is obviously continuous. Since the class $\mathbf{\Sigma}^1_n$ is closed under continuous preimages the set

$$\{x \mid (x,x) \in U\} = f^{-1}[U]$$

is in $\mathbf{\Sigma}^1_n(X)$. By our assumption this set is also in $\mathbf{\Pi}^1_n(X)$. So its complement $\{x \mid (x,x) \notin U\}$ is in $\mathbf{\Sigma}^1_n(X)$. But since U is an X-universal set there exists an $x_o \in X$ such that

$$\{x \mid (x,x) \notin U\} = \{x \mid (x,x_0) \in U\}$$

Considering $x = x_0$ leads now to a contradiction. □

3.2 The effective hierarchies

Considering the Borel and the projective hierarchy it seems reasonable that if we compare two levels of a hierarchy we say that the sets from the higher level of the hierarchy have greater complexity than the sets of the lower level since we had to apply operations like taking unions or intersections or even projections. In the language of set theory taking intersections is nothing else than applying the $\forall$-quantifier. So a natural way for a different approach to decide the complexity of a subset (for example of the Baire space or also from the discrete topological space ω) is to consider the complexity of the formula in the language of set theory that defines the set (and we want to decide the complexity of a formula by the number of quantifiers). We do this now by defining the arithmetical and analytical hierarchy. The study of the classes from these hierarchies is called the effective descriptive set theory. Classically this effective theory has its origins in recursion theory. We do not want to go in this area here, see for example [Mosc80, Ch 3] or [MaKe80, Ch 6].

It is not obvious that these new to define hierarchies have something to do with the Borel or the projective hierarchy but there is indeed a very close relation. So can the classes of the analytical hierarchy together with its relativized versions (we will introduce this in the upcoming section) be seen as a ramification of the corresponding classes of the projective hierarchy. A similar result applies for the arithmetical hierarchy and the pointclasses from the Borel hierarchy of finite order.

For the effective theory we restrict ourselves to product spaces of the form $\omega^r \times (\omega^\omega)^k$ and follow here the outline in [Kana97, sec. 12]. A different approach (by recursion theory) and in a more general context can be found in [Mosc80, Ch3].

Let $\mathcal{A} = (\omega, \omega^\omega, \mathrm{ap}, +, \cdot, \exp, <, 0, 1)$ be the structure with two domains ω and ω^ω. ap is the function

$$\begin{aligned} \mathrm{ap} : \omega^\omega \times \omega &\longrightarrow \omega \\ (x, m) &\longmapsto x(m) \end{aligned}$$

$+, \cdot$ are the usual arithmetic operations on ω, exp stands for the exponentation on ω. To distinguish the variables for the two domains our language contains variables $v_0^0, v_1^0, v_2^0, \ldots$ which stand for elements of ω and variables $v_0^1, v_1^1, v_2^1, \ldots$ which stand for elements of ω^ω. In addition we have the number quantifiers $\exists^0, \forall^0$ for the v_i^0 and the function quantifiers $\exists^1, \forall^1$ for the variables v_i^1. Terms and formulas of our language are defined in the obvious way. By terms for numbers we understand the smallest class of words which contains $0, 1, v_0^0, v_1^0, v_2^0, \ldots$ and is closed under $+, \cdot, \exp$ and ap. For any such term τ and any formula φ we write $(\exists^0 v_i^0 < \tau)\varphi$ for $\exists^0 v_i^0 (v_i^0 < \tau \wedge \varphi)$ and $(\forall^0 v_i^0 < \tau)\varphi$ for $\forall^0 v_i^0 (v_i^0 < \tau \rightarrow \varphi)$. These are the bounded quantifiers.

We consider now subsets A of $\omega^r \times (\omega^\omega)^k$ and will also see this A as a relation, that means we write interchangebly $(m_0, \ldots, m_{r-1}, x_0, \ldots, x_{k-1}) \in A$ or $A(m_0, \ldots, m_{r-1}, x_0, \ldots, x_{k-1})$.

A set $A \subseteq \omega^r \times (\omega^\omega)^k$ is **definable in** $\mathcal{A}$ by a formula φ iff $(m_0, \dots, m_{r-1}, x_0, \dots, x_{k-1}) \in A \Leftrightarrow \mathcal{A} \models \varphi[m_0, \dots, m_{r-1}, x_0, \dots, x_{k-1}]$.
A is Δ^0_0 in $\mathcal{A}$ iff A is definable by a formula whose only quantifiers are bounded. We can now define the **arithmetical hierarchy**.

Definition 3.2.1. Let A be a subset from some $\omega^r \times (\omega^\omega)^k$. For $n \in \omega$ set

$$\begin{aligned} A \in \Sigma^0_n &\quad\Leftrightarrow\quad \forall \mathbf{w}(\mathbf{w} \in A \leftrightarrow \exists^0 m_1 \forall^0 m_2 \dots Q m_n R(m_1, \dots, m_n, \mathbf{w})) \\ A \in \Pi^0_n &\quad\Leftrightarrow\quad \forall \mathbf{w}(\mathbf{w} \in A \leftrightarrow \forall^0 m_1 \exists^0 m_2 \dots Q m_n R(m_1, \dots, m_n, \mathbf{w})) \end{aligned}$$

where $R \subseteq \omega^{r+n} \times (\omega^\omega)^k$ is Δ^0_0 and Q is $\exists^0$ if n is odd and $\forall^0$ if n is even for the Σ^0_n case and vice versa for the Π^0_n case. A is called **arithmetical** if $A \in \bigcup_n \Sigma^0_n$. The ambiguous pointclasses are defined as before by $\Delta^0_n = \Sigma^0_n \cap \Pi^0_n$. A set A in Δ^0_1 is called recursive.

It can be shown that A is arithmetical iff A is definable by a formula without function quantifiers. A proof for this and proofs for the following are carried out in full detail in [Stei98].

Proposition 3.2.2. *(a)For all $n \in \omega$ the following holds:*
The complement of a Σ^0_n set is a Π^0_n set. The classes Σ^0_n and Π^0_n are closed under finite unions and intersections. For a set of the form $\omega^r \times (\omega^\omega)^k$ there exist only countable many subsets in Σ^0_n and only countable many in Π^0_n.
(b) The Σ^1_n and Π^1_n sets form a hierarchy, we get the following picture of inclusions:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & \Sigma^0_1 \arrow[dr, phantom, sloped, "\subseteq"] & & \Sigma^0_2 \arrow[dr, phantom, sloped, "\subseteq"] & & {} \\
\Delta^0_1 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \Delta^0_2 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \Delta^0_3 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & \dots \\
 & \Pi^0_1 \arrow[ur, phantom, sloped, "\subseteq"] & & \Pi^0_2 \arrow[ur, phantom, sloped, "\subseteq"] & & {}
\end{tikzcd}$$

Example 3.2.3. The basic sets of the Baire space are Σ^0_1 sets since for a finite sequence $s = (s_0, s_1, \dots, s_{n-1})$ of integers the set N_s is defined by the following formula:

$$x \in N_s \;\Leftrightarrow\; \mathrm{ap}(x, 0) = s_0 \wedge \mathrm{ap}(x, 1) = s_1 \wedge \dots \mathrm{ap}(x, n-1) = s_{n-1}$$

We call the collection of all the sets definable in $\mathcal{A}$ the class of **analytical sets**. By shifting quantifiers and using various coding maps we can classify the analytical sets in the **analytical hierarchy**:

Definition 3.2.4. Let $\Sigma^1_0 = \Sigma^0_1$ and $\Pi^1_0 = \Pi^0_1$. For $n > 0$ define

$$\begin{aligned} A \in \Sigma^1_n &\quad\Leftrightarrow\quad \forall \mathbf{w}(\mathbf{w} \in A \leftrightarrow \exists^1 x_1 \forall^1 x_2 \dots Q x_n R(\mathbf{w}, x_1, \dots, x_n)) \\ A \in \Pi^1_n &\quad\Leftrightarrow\quad \forall \mathbf{w}(\mathbf{w} \in A \leftrightarrow \forall^1 x_1 \exists^1 x_2 \dots Q x_n R(\mathbf{w}, x_1, \dots, x_n)) \end{aligned}$$

for some arithmetical $R \subseteq \omega^r \times (\omega^\omega)^{k+n}$ and Q is $\exists^1$ if n is odd and $\forall^1$ if n is even in the Σ^1_n case and vice versa in the Π^1_n case.
Define also $\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$.

We collect some main properties in the next proposition.

Proposition 3.2.5. *(a)For all $n \in \omega$ the following holds:*
The complement of a Σ^1_n set is a Π^1_n set. The classes Σ^1_n and Π^1_n are closed under finite unions and intersections. For a set of the form $\omega^r \times (\omega^\omega)^k$ there exist only countable many subsets in Σ^1_n and only countable many in Π^1_n.
(b) The Σ^1_n and Π^1_n sets form a hierarchy, we get the following picture of inclusions:

$$\begin{array}{ccccccccccc}
 & & \Sigma^1_1 & & & & \Sigma^1_2 & & & & \\
 & \subseteq & & \subseteq & & \subseteq & & \subseteq & & \subseteq & \\
\Delta^1_1 & & & & \Delta^1_2 & & & & \Delta^1_3 & & \cdots \\
 & \subseteq & & \subseteq & & \subseteq & & \subseteq & & \subseteq & \\
 & & \Pi^1_1 & & & & \Pi^1_2 & & & &
\end{array}$$

(c) A set A is analytical iff A is in some Σ^1_n.

We already mentioned that there is a deep connection between the just defined "lightface" hierarchies and the "boldface" hierarchies before. For this we have to consider the lightface classes relativized to some parameter a of ω^ω.

For $a \in \omega^\omega$ consider the structure

$$\mathcal{A}(a) = (\omega, \omega^\omega, \mathrm{ap}, +, \cdot, \exp, <, 0, 1, a)$$

A set $A \subseteq \omega^r \times (\omega^\omega)^k$ is $\Delta^0_0(a)$ if it can be defined by a formula in $\mathcal{A}(a)$. Starting with this definition we can obtain in the same way as before the classes $\Sigma^0_n(a), \Pi^0_n(a), \Delta^0_n(a), \Sigma^1_n(a), \Pi^1_n(a), \Delta^1_n(a)$. For $A \in \Sigma^0_1(a) \cap \Pi^0_1(a)$ we say A is recursive in a and so on. Most results, as for example the above facts about the hierarchies hold for the relativized version by relativizing everything to its parameter.

It is clear that $\Sigma^0_n \subseteq \Sigma^0_n(a), \Pi^0_n \subseteq \Pi^0_n(a), \Sigma^1_n \subseteq \Sigma^1_n(a)$ and $\Pi^1_n \subseteq \Pi^1_n(a)$ for all $a \in \omega^\omega$ and all $n \in \omega$ since a set definable in the structure $\mathcal{A}$ by a formula φ is also definable in the structure $\mathcal{A}(a)$ by the same formula φ where the parameter a just does not occur. Furthermore it is clear that for a set $\omega^r \times (\omega^\omega)^k$ only countable many subsets are in $\Sigma^1_n(a)$ since our language for the structure $\mathcal{A}(a)$ is finite, thus there are only countable many formulas. Analogous results hold for the classes $\Sigma^0_n(a), \Pi^0_n(a)$ and $\Pi^1_n(a)$.

We have seen that the boldface hierarchies were proper hierarchies. This is also true for the lightface hierarchies defined here and the relativized versions of it. Proofs can be obtained easily if we have the existence of universal sets. It is quite similar to the proof of Proposition 3.1.14 but note that the lightface classes are not closed under continuous preimages. But they are still closed under preimages of recursive functions and this is enough to finish the proof as before. For the notion of recursive functions and the proof of the following proposition see [Mosc80, 3.F].

Proposition 3.2.6. *For each set X of the form $\omega^r \times (\omega^\omega)^k$ and for each $n \in \omega$ exists a Y universal set for $\Sigma^1_n(X)$ with Y a product of multiples of ω and ω^ω. The same holds for Σ^0_n, Π^0_n and Π^1_n and the relativized classes.*

This implies that the arithmetical and analytical hierarchies (and its relativized versions) are hierarchies of proper inclusions.

The connection between the arithmetical hierarchy and the Borel hierarchy of finite order as well as between the projective hierarchy and the analytical hierqarchy is now the following:

Proposition 3.2.7. *Let $A \subseteq (\omega^\omega)^k$ and $0 < n \in \omega$. Then*

(a) $A \in \mathbf{\Sigma}^0_n$ iff $A \in \Sigma^0_n(a)$ for some $a \in \omega^\omega$

(b) $A \in \mathbf{\Sigma}^1_n$ iff $A \in \Sigma^1_n(a)$ for some $a \in \omega^\omega$

Analogous results for $\mathbf{\Pi}^0_n$ and $\mathbf{\Pi}^1_n$.

By this Proposition 3.2.7 the analytic sets are the union of the classes $\Sigma^1_1(a)$. The analytic sets of the Baire space were exactly the ω-Suslin sets. One could ask if we can distinguish which trees lead to a representation of an $\Sigma^1_1(a)$ set, $a \in \omega^\omega$, of the Baire space. The answer is yes but for this we can not avoid to introduce some of the coding functions necessary for a "normal form" of the Σ^1_n sets. To code finite sequences of natural numbers consider the following function

$$\begin{array}{rcl} \langle\ \rangle : \omega^{<\omega} & \longrightarrow & \omega \\ s = (s(0), \ldots, s(n-1)) & \longmapsto & \langle s \rangle = p_0^{s(0)+1} \cdots p_{n-1}^{s(n-1)+1} \end{array}$$

where p_i is the ith prime number.
If we are interested in just an initial segment of an $x \in \omega^\omega$ this can also be coded by a natural number using the above function:

$$\begin{array}{lcl} \bar{\ }: \omega^\omega \times \omega & \longrightarrow \omega & \\ \quad (x, m) & \longmapsto & \overline{x}(m) = \langle x|m \rangle = \langle x(0), \ldots, x(m-1) \rangle \end{array}$$

This function is Δ^0_0. For $\mathbf{w} = (m_0, \ldots, m_{r-1}, x_0, \ldots, x_{k-1}) \in \omega^r \times (\omega^\omega)^k$ and $n \in \omega$ set $\overline{\mathbf{w}}(n) = (m_0 \ldots, m_{r-1}, \overline{x_0}(n), \ldots, \overline{x_{k-1}}(n))$.

Proposition 3.2.8. *Let $A \subseteq \omega^r \times (\omega^\omega)^k$ be a $\Sigma^1_n(a)$ set for $a \in \omega^\omega$. Let $0 < n \in \omega$.*
For n even there exists an $\Delta^0_0(a)$ set $R \subseteq \omega^{r+k+n+1}$, such that

$$\mathbf{w} \in A \Leftrightarrow \exists^1 x_1 \ldots \forall^1 x_n \exists^0 m R(m, \overline{\mathbf{w}}(m), \overline{x_1}(m), \ldots, \overline{x_n}(m)).$$

For n odd there exists an $\Delta^0_0(a)$ set $R \subseteq \omega^{r+k+n+1}$ such that

$$\mathbf{w} \in A \Leftrightarrow \exists^1 x_1 \ldots \exists^1 x_n \forall^0 m R(m, \overline{\mathbf{w}}(m), \overline{x_1}(m), \ldots, \overline{x_n}(m)).$$

Similar results can be obtained for $\Pi^1_n(a)$ sets by negation.

It turns out that $A \subseteq \omega^\omega$ is a $\Sigma^1_n(a)$ set for $a \in \omega^\omega$ if an only if A is ω-Suslin with trees T recursive in a. By this we understand that the set of the codes of the sequences of T is recursive in a. To be exact we define:

Definition 3.2.9. A **tree** T on $\omega \times \omega$ is called **recursive in** a if the set $\langle T \rangle = \{(\langle s \rangle, \langle t \rangle) \mid (s, t) \in T\}$ is recursive in a.

So the result for the tree representation of $\Sigma^1_1(a)$ sets is the following.

Proposition 3.2.10. *Let $A \subseteq \omega^\omega, a \in \omega^\omega$. A is $\Sigma^1_1(a)$ iff there is a tree T on $\omega \times \omega$ recursive in a such that $A = p[T]$.*

Proof. Assume we have such a tree representation of A. Then

$$\begin{aligned} x \in A &\Leftrightarrow x \in p[T] \\ &\Leftrightarrow \exists^1 y (x,y) \in [T] \\ &\Leftrightarrow \exists^1 y \forall^0 n (x|n, y|n) \in T \\ &\Leftrightarrow \exists^1 y \forall^0 n \langle T \rangle (\langle x|n \rangle, \langle y|n \rangle) \end{aligned}$$

So A is $\Sigma^1_1(a)$.
Let now A be a $\Sigma^1_1(a)$ set. By Proposition 3.2.8 there exists an $\Delta^0_0(a)$ set $R \subseteq \omega^3$ such that

$$x \in A \Leftrightarrow \exists^1 y \forall^0 m R(m, \overline{x}(m), \overline{y}(m))$$

We define now a tree recursive in a by

$$\begin{aligned} (s,t) \in T &\Leftrightarrow \forall^0 p < \text{length}(s) R(p, \langle s(0), \ldots, s(p) \rangle, \langle t(o), \ldots, t(p) \rangle) \\ &\Leftrightarrow \exists^0 n (n = \text{length}(s)) \forall^0 p < n R(p, \langle s(0), \ldots, s(p) \rangle, \langle t(o), \ldots, t(p) \rangle) \\ &\Leftrightarrow \forall^0 n (n = \text{length}(s)) \forall^0 p < n R(p, \langle s(0), \ldots, s(p) \rangle, \langle t(o), \ldots, t(p) \rangle) \end{aligned}$$

The projection of the infinite sequences of this tree is indeed the set A:

$$\begin{aligned} x \in p[T] &\Leftrightarrow \exists^1 y (x,y) \in [T] \\ &\Leftrightarrow \exists^1 y \forall^0 n (x|n, y|n) \in T \\ &\Leftrightarrow \exists^1 y \forall^0 n \forall^0 p < n R(p, \overline{x}(p), \overline{y}(p)) \\ &\Leftrightarrow \exists^1 y \forall^0 p R(p, \overline{x}(p), \overline{y}(p)) \end{aligned}$$

□

Chapter 4

Games and (Axioms of) Determinacy

For the characterization of the $\mathbf{\Sigma}^1_n$ sets for $n > 1$ by finer topologies the theory **ZF** + **DC** is not strong enough. Even taking the full axiom of choice will not be of help. So we will consider other additional axioms, namely the axiom of projective determinacy (**PD**) where we consider games on integers and the much stronger axiom of determinacy of games on reals ($\mathbf{AD}_{\mathbb{R}}$). The axiom of determinacy (**AD**) will also be of importance. Even though **AD** contradicts the axiom of choice it is quite common in descriptive set theory since it implies a lot of nice properties of the reals and one can draw interesting conclusions out of it sometimes even for a model of set theory in which **AC** holds. Philipp Rohde gives in his Diplomarbeit an overview also about other determinacy axioms, see [Rohd01].

The foundation for these axioms is the notion of a two person game that we will introduce in the first section. The prototype of such a game is a game on integers. But we will also consider games on reals and ordinals. Also Polish spaces can be characterized by games. We will introduce this in the second section here. The game will then be a game on open subsets of some Polish space.

4.1 Games and determinacy

We inroduce first games on integers and the notion of a strategy.

Definition 4.1.1. (a) For a subset $A \subseteq \mathcal{N}$, called the **payoff set**, the two person **game** $\boldsymbol{G}_A$ is defined in the following way: The two players take turns in playing integers

$$\begin{array}{cccccc} \mathrm{I} & n_0 & & n_2 & & \dots \\ \mathrm{II} & & n_1 & & n_3 & \dots \end{array}$$

After ω moves the game is over and player I wins if the sequence $x = (n_i)_{i\in\omega}$ is in A. Otherwise II wins.

(b) A **strategy** for player I is a tree σ on ω which tells player I which move to make in every round of the game. That is, σ is a subtree of the full tree on ω with the following properties:

(i) σ is nonempty

(ii) if $(n_0, n_1, \dots, n_{2k}) \in \sigma, k \in \omega$, then $(n_0, n_1, \dots, n_{2k}, m) \in \sigma$ for all $m \in \omega$

(iii) if $(n_0, n_1, \dots, n_{2k-1}) \in \sigma, k \in \omega$ (for $k = 0$ this is the empty sequence), there exists a unique $m \in \omega$ such that $(n_0, n_1, \dots, n_{2k-1}, m) \in \sigma$.

Player I follows the strategy σ if he plays in his $2k$-th move the unique integer such that the finite sequence played so far is a member of the tree σ. We denote this unique integer by $\sigma * s$ if $s \in \omega^{2k-1}$ is the sequence of all the integers played before.

The strategy σ is called a **winning strategy for player I** if he wins every run of the game by following σ. Similarly, one defines the notion of a strategy and winning strategy for player II.

(c) The game G_A is **determined** if one of the players has a winning strategy.

Closely related to the subject of strategies is the concept of quasi-strategies. A quasi-strategy for player I is a tree as it is for a strategy but instead of giving player I a unique element to play following the strategy it gives him a set of possible answers in every stage of the game. So the definition is the following:

Definition 4.1.2. Let A be a subset of $\mathcal{N}$ and G_A be a game as in the definition above. A **quasi-strategy** for player I is a tree on ω with the following properties:

(i) σ is nonempty

(ii) if $(n_0, n_1, \dots, n_{2k}) \in \sigma, k \in \omega$, then $(n_0, n_1, \dots, n_{2k}, m) \in \sigma$ for all $m \in \omega$

(iii) if $(n_0, n_1, \dots, n_{2k-1}) \in \sigma, k \in \omega$ (for $k = 0$ this is the empty sequence), there exist integers $m \in \omega$ such that $(n_0, n_1, \dots, n_{2k-1}, m) \in \sigma$.

Player I follows the quasi-strategy σ if he plays in his $2k$-th move an integer such that the finite sequence played so far is a member of the tree σ.

A quasi-strategy σ is a **winning quasi-strategy for player I** if player I wins every run of the game by following σ. Similarly, one defines the notion of a quasi-strategy or a winning quasi-strategy for player II.

The game G_A is is **quasi-determined** if one of the players has a winning quasi-strategy.

Obviously it depends on the subset A of $\mathcal{N}$ if a game is (quasi-)determined or not. So one says that a subset $A \subseteq \mathcal{N}$ is (quasi-)determined if one means that the associated game G_A determined. Furthermore, it is also obvious that determined games exist.

For example taking A as the whole set $\mathcal{N}$ or just taking away finitely many points will lead easily to a winning strategy for player I. The question is now whether pointsets from certain pointclasses are determined. David Gale and

Frank Stewart proved in [GaSt53] that all open and all closed sets are determined. The proof uses **DC** but one can show in **ZF** that all open and closed sets of the Baire space are quasi-determined. It is pretty obvious that under **DC** we can always reduce a quasi-strategy for games of length ω to a strategy. So under **ZF + DC** the open and closed sets are determined. It was proven shortly after the Gale-Stewart Theorem that also $\mathbf{\Sigma}^0_2$ and $\mathbf{\Pi}^0_2$ sets are determined (cf. [Wolf55]). Using **ZF+ AC** Donald Martin even proved in [Mart75] that all sets of the Borel hierarchy are determined.

But not all pointsets are determined. Already in their 1953 paper, Gale and Stewart mentioned that under **AC** nondetermined subsets of the Baire space exist. Despite this fact (and knowing it will contradict **AC**) the Polish mathematicians Jan Mycielski and Hugo Steinhaus suggested in [MySt62] the **Axiom of determinacy** that asserts that all subsets of the Baire space are determined.

Definition 4.1.3. [**Axiom of determinacy (AD)**] For all $A \subseteq \mathcal{N}$ the game G_A is determined.

In the next chapter we will introduce the scale property and the projective ordinals. We will prove some results about it under the Axiom **AD**. Since we are mainly interested in pointclasses of the projective hierarchy it suffices for some of these results to work under the weaker assumption that just sets of the projective hierarchy of the Baire space are determined. The axiom that asserts this property is the Axiom of projective determinacy:

Definition 4.1.4. [**Axiom of projective determinacy (PD)**] For all $A \in \mathbf{\Sigma}^1_n(\mathcal{N}), n \in \omega$, the game G_A is determined.

It is straightforward how to describe two person games of length ω on arbitrary sets X. For a subset A of X^ω we define games G^X_A as above but instead of playing elements from ω the two players pick elements from X. The strategies will then be trees on X and winning strategies as well as determined sets of X^ω are described as above. Important for us will be games on reals. In such a game each player has to play elements of the Baire space and the payoff sets will then be subsets of $\mathcal{N}^\omega$. The axiom that all payoffs sets of $\mathcal{N}^\omega$ are determined for games of reals is much stronger than **AD** and it is denoted by $\mathbf{AD}_\mathbb{R}$:

Definition 4.1.5. [$\mathbf{AD}_\mathbb{R}$] For all $A \subseteq \mathcal{N}^\omega$ the game $G^\mathbb{R}_A$ is determined.

The axiom $\mathbf{AD}_\mathbb{R}$ implies the axiom **AD**. This is an easy result, see [Rohd01, 3.1].

A slightly different game on open subsets of a topological space will be introduced in the next chapter when we characterize Polish spaces by strong Choquet games.

4.2 Polish spaces as strong Choquet spaces

We start by defining the Choquet game.

Definition 4.2.1. Let X be a nonempty topological space. The **Choquet game** $G_{\mathrm{Ch}}(X, \mathcal{T})$ on X is defined as follows: Players I and II take turns in playing nonempty open subsets of X

$$\begin{array}{lcccccc} \mathrm{I} & U_0 & & U_1 & & \dots & \\ \mathrm{II} & & V_0 & & V_1 & \dots & , \end{array}$$

such that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \dots$

We say II wins this run of the game if $\bigcap_n V_n = \bigcap_n U_n \neq \emptyset$. Otherwise I wins.

Strategies and winning strategies for Choquet games are defined now as trees on open subsets of the Polish space as before. For our purpose, the strong Choquet game is more important. It is similar to the Choquet game but in addition to the Choquet game player I is required to play a point $x_n \in U_n$ on every turn and then player II must play $V_n \subseteq U_n$ with $x_n \in V_n$. So the definition is the following.

Definition 4.2.2. Let X be a nonempty topological space. The **strong Choquet game** $G_{\mathrm{sCh}}(X, \mathcal{T})$ on X is defined as follows: Players I and II take turns in playing nonempty open subsets of X and player I in addition a point in his open subset

$$\begin{array}{lcccccc} \mathrm{I} & U_0, x_0 & & U_1, x_1 & & \dots & \\ \mathrm{II} & & V_0 & & V_1 & \dots & , \end{array}$$

such that $U_0 \supseteq V_0 \supseteq \dots, x_n \in U_n, x_n \in V_n$ for $n \in \omega$. We say II wins this run of the game if $\bigcap_n V_n = \bigcap_n U_n \neq \emptyset$. Otherwise I wins.

An appropriate tree on the product set of open subsets of the Polish space X and points in X can be viewed as a strategy where the information of the extra point for player II is of no interest.

The Choquet game on a topological space X is determined if one of the players has a winning strategy. If player II has a winning strategy we will call the topological space a Choquet space:

Definition 4.2.3. A topological space X is called a **(strong) Choquet space** if player II has a winning strategy for the associated (strong) Choquet game $G_{\mathrm{Ch}}(X, \mathcal{T}), (G_{\mathrm{sCh}}(X, \mathcal{T}))$.

An example for strong Choquet spaces are the completely metrizable spaces.

Proposition 4.2.4. *A nonempty, completely metrizable space is a strong Choquet space.*

Proof. Let $(X, \mathcal{T})$ be a nonempty completely metrizable space, d a compatible complete metric on X. We define a winning strategy σ for player II by induction. If $(U_0, x_0, V_0, \dots, U_n, x_n)$ is a legal round in the game $G_{\mathrm{sCh}}(X, \mathcal{T})$, then choose an open ball V_n from $\{B_{\frac{1}{n+i+1}}(x_n) \mid i \in \omega\}$ such that $\mathrm{cl}_{\mathcal{T}}(V_n) \subseteq U_n$ (for example the least i such that this holds). Then $\bigcap_n U_n = \bigcap \mathrm{cl}_{\mathcal{T}}(V_n)$. For every

n the sequence $(x_n, x_{n+1}, \ldots)$ lies completely in $\mathrm{cl}_{\mathcal{T}}(V_n)$ and, since the diameter of the V_n gets arbitrarily small, is a Cauchy sequence. Thus this sequence converges in X and the limit point is in $\mathrm{cl}_{\mathcal{T}}(V_n)$ since this is a closed set. Since $\lim_{k\in\omega} x_k = \lim_{k\in\omega} x_{n+k}$ for every n, we have this limit point in every $\mathrm{cl}_{\mathcal{T}}(V_n)$. Thus $\lim_{k\in\omega} x_k \in \bigcap_n \mathrm{cl}_{\mathcal{T}}(V_n)$. □

Putting together this result with Lemma 1.3, a Polish space has the following properties.

Proposition 4.2.5. *Every Polish space is a second countable, regular, strong Choquet space which is Hausdorff.*

We will prove now that, if we assume in addition **AC**, the converse is also true. For this we show first the converse of Proposition 4.2.4 under **AC** that every separable, metrizable, strong Choquet space is complete. This will lead to a characterization of Polish spaces as strong Choquet spaces.

First we give two general lemmas, the first one about trees, the second a purely topological one.

Definition 4.2.6. Let T be a tree on a set A. T is called **finite splitting** if for every $s \in T$ there are at most finitely many $a \in A$ with $s^\frown a \in T$.

Lemma 4.2.7 (König's Lemma). *Let T be a finite splitting tree on a set A. Then $[T] \neq \emptyset$ iff T is infinite.*

Proof. If $[T] \neq \emptyset$ the tree cannot be finite.

Now let conversely T be infinite. We will inductively pick x_i at every level of the tree, such that the infinite sequence (x_i) is in $[T]$. Pick first an $x_o \in A$ such that the tree $T_{x_0} = \{s \in T \mid s \supseteq x_0\}$ is infinite. This is possible since we have only finitely many sequences of length 1, but the full tree is infinite. With the same argument we pick x_1 such that $(x_0, x_1) \in T_{x_0}$ and $T_{(x_0,x_1)} = \{s \in T_{x_0} \mid s \supseteq (x_0, x_1)\}$ is infinite. By iterating these process, we get an infinite branch in T. □

Lemma 4.2.8. *Let (Y, d) be a separable metric space. Let $\mathcal{U}$ be a family of nonempty open sets in Y. Then $\mathcal{U}$ has a* **point-finite refinement** *$\mathcal{V}$, i.e., $\mathcal{V}$ is a family of nonempty open sets with $\bigcup\mathcal{U} = \bigcup\mathcal{V}, \forall V \in \mathcal{V} \exists U \in \mathcal{U}\ (V \subseteq U)$ and $\forall y \in Y\ (\{V \in \mathcal{V} \mid y \in V\}$ is finite). More over, given $\varepsilon > 0$ we can also assume that $\mathrm{diam}(V) < \varepsilon$ for all $V \in \mathcal{V}$.*

Proof. Denote the induced topology of Y by $\mathcal{T}$. Since Y is second countable, let (U_n) be a sequence of open sets such that $\bigcup_n U_n = \bigcup\mathcal{U}$ and forall n exists an $U \in \mathcal{U}(U_n \subseteq U)$. Furthermore, given $\varepsilon > 0$ we can always assume that $\mathrm{diam}(U_n) < \varepsilon$. For example, fix a countable dense subset D of Y and take the U_n's to be the open balls around the points of $\bigcup\mathcal{U} \cap D$ which lie in some U of $\mathcal{U}$ and have rational radius smaller ε. (cf. the proof of Lemma 1.2).

Let next $U_n = \bigcup\limits_{p\in\omega} U_n^{(p)}$ with $U_n^{(p)}$ open, $U_n^{(p)} \subseteq U_n^{(p+1)}$ and $\mathrm{cl}_{\mathcal{T}}(U_n^{(p)}) \subseteq U_n$ for every $p \in \omega$. Put

$$V_m = U_m \setminus \bigcup_{n<m} \mathrm{cl}_{\mathcal{T}}(U_n^{(m)}) = U_m \cap \sim \bigcup_{n<m} \mathrm{cl}_{\mathcal{T}}(U_n^{(m)}) = U_m \cap \bigcap_{n<m} \sim \mathrm{cl}_{\mathcal{T}}(U_n^{(m)})$$

open, where $\sim A$ denotes the complement of a set A in Y.

(1) $\bigcup_n V_n = \bigcup_n U_n$:

Cleary for every m we have $V_m \subseteq U_m$. Let $x \in \bigcup_{n\in\omega} U_n$ and m the least integer with $x \in U_m$. Then $x \in U_m \setminus \bigcup_{n<m} \mathrm{cl}_\mathcal{T} U_n^{(m)} = V_m$ by the choice of m.

(2) For all $y \in Y$ there are only finitely many V_m which contain y:

Let $x \in U = \bigcup \mathcal{U}$. Then $x \in U_n$ for an n and then $x \in U_n^{(p)}$ for some p. So $x \notin V_m$ if $m > p, n$.

Let $\mathcal{V} = \{V_n \mid V_n \neq \emptyset\}$. □

Theorem 4.2.9 (AC). *Let X be a nonempty separable metrizable strong Choquet space, $\hat{X}$ a Polish space and X a subspace of $\hat{X}$. Then X is G_δ in $\hat{X}$.*

Proof. Fix a compatible complete metric d for $\hat{X}$ and a winning strategy σ for player II in the strong Choquet game $G_{\mathrm{sCh}}(X)$.

Claim: There exists a tree S on $X \times \mathcal{P}(X) \times \mathcal{P}(\hat{X})$ with the following properties: If $((x_o, V_0, \hat{V}_0), \dots, (x_n, V_n, \hat{V}_n)) \in S$, then for $0 \leq i \leq n$ we have V_i is open in X, $\hat{V}_i$ is open in $\hat{X}$, $x_i \in \hat{V}_{i-1}$ $(\hat{V}_{-1} = \hat{X})$,$x_i \in V_i$, $\hat{V}_i \cap X \subseteq V_i$, $\hat{V}_i \subseteq \hat{V}_{i-1}$ and $(X, x_0), V_0, (\hat{V}_0 \cap X, x_1), V_1, \dots (\hat{V}_{n-1}, x_n), V_n, \hat{V}_n$ is a legal run of the game where II follows σ. Additionallay, if $s = ((x_0, V_0, \hat{V}_0), \dots, (x_{n-1}, V_{n-1}, \hat{V}_{n-1})) \in S$, $\hat{\mathcal{V}}_s = \{\hat{V}_n \mid s^\frown(x_n, V_n, \hat{V}_n) \in S\}$, then $X \cap \hat{V}_{n-1} \subseteq \bigcup \hat{\mathcal{V}}_s$,diam$\hat{V}_n < 2^{-n}$ for all $\hat{V}_n \in \hat{\mathcal{V}}_s$ and for every $\hat{x} \in \hat{X}$ there are at most finitely many $(x_n, V_n, \hat{V}_n)$ with $s^\frown(x_n, V_n \hat{V}_n) \in S$ such that $\hat{x} \in \hat{V}_n$.

Proof: We construct a tree by induction on the length of the sequences. Let $s = ((x_0, V_0, \hat{V}_0), \dots, (x_{n-1}, V_{n-1}, \hat{V}_{n-1}))$ be in S such that all properties hold (s may be the empty sequence). Let $\hat{\mathcal{V}}_s = \{\hat{V} \mid \hat{V}$ is open in $\hat{X}$ and $\hat{V} \subseteq \hat{V}_{n-1}$ and $\exists x_n \in \hat{V}_{n-1} \cap X$ such that $\hat{V} \cap X \subseteq \sigma*(x_0, X, V_0, \dots, x{-}n, \hat{V}_{n-1} \cap X)\}$. Let $\hat{\mathcal{V}}_s^*$ be a point-finite refinement such that diam$(\hat{V}^*) < 2^{-n}$ for every $\hat{V}^* \in \hat{\mathcal{V}}_s^*$. By the axiom of choice choose now for every $\hat{V}^*$ an $x_n(\hat{V}^*) \in \hat{V}_{n-1} \cap X$ such that $\hat{V}^* \cap X \subseteq \sigma * (x_0, X, \dots, x_n(\hat{V}^*), \hat{V}_{n-1} \cap X)$. Then put $s^\frown(x_n(\hat{V}^*), \sigma * (x_o, X, \dots, x_n(\hat{V}^*), \hat{V}_{n-1} \cap X), \hat{V}^*))$ in S for all $\hat{V}^* \in \hat{\mathcal{V}}_s^*$. One can easily prove that the so constructed tree has all the properties. For example to see that $X \cap \hat{V}_{n-1} \subseteq \bigcup \hat{\mathcal{V}}_s^*$, note that we put in neighborhoods for every point of $X \cap \hat{V}_{n-1}$.

q.e.d. Claim

Fix a tree with all these conditions and let

$$W_n = \bigcup \{\hat{V}_n \mid ((x_0, V_0, \hat{V}_0), \dots, (x_n, V_n, \hat{V}_n)) \in S\}.$$

Then W_n is open and, using $X \cap \hat{V}_{n-1} \subseteq \bigcup \hat{\mathcal{V}}_s$, one can prove by an easy induction that $X \subseteq W_n$. It remains to show that $\bigcap_n W_n \subseteq X$.

Let $\hat{x} \in \bigcap_n W_n$. Consider the subtree $S_{\hat{x}}$ of S consisting of all sequences $((x_0, V_0, \hat{V}_0), \dots, (x_n, V_n, \hat{V}_n)) \in S$ for which $\hat{x} \in \hat{V}_n$. This is a tree since $\hat{x} \in \hat{V}_n \subseteq \hat{V}_i$ for all $i < n$. Since $\hat{x} \in \bigcap_n W_n$, $S_{\hat{x}}$ is infinite. By the preceding conditions on S it is also finite splitting. So, by König's Lemma, $[S_{\hat{x}}] \neq \emptyset$. Say $((x_0, V_0, \hat{V}_0), (x_1, V_1, \hat{V}_1), (x_2, V_2, \hat{V}_2), \dots) \in [S_{\hat{x}}]$. Then $(X, x_0), V_0, x_1, (\hat{V}_0 \cap$

$X, x_1), V_1, (\hat{V}_1, x_2), V_2, \ldots$is a run of G_X^s compatible with σ, so $\bigcap_n \hat{V}_n \cap X \neq \emptyset$. In particular there is a point of X in $\bigcap_n \hat{V}_n$ and by construction $\hat{x} \in \bigcap_n \hat{V}_n$. But these two points must coincide with each other since $\operatorname{diam}(\hat{V}_n) < 2^{-n}$. Thus $\hat{x} \in X$. □

Given a second countable metrizable space X we can consider the **completion** $\hat{X}$, that is, a second countable complete metrizable space $\hat{X}$ such that X is a subspace of $\hat{X}$ and X is dense in $\hat{X}$. Such a completion exists for every metrizable space.

Theorem 4.2.10. *Let (X, d) be a metric space. Then there exists a unique, up to isometry, completion $(\hat{X}, \hat{d})$ of (X, d). If X is separable, the completion $\hat{X}$ is also separable. In particular, a completion of a separable metric space is a Polish space.*

A proof for this theorem can be found in [Kura66, Ch. III, § 33, VII] where this theorem is called Hausdorff Theorem since Hausdorff proved it in [Haus65, p. 135]. We have already seen in Theorem 1.14 that G_δ subsets of Polish spaces are again Polish. So X in the above Theorem 4.2.9 is a Polish space. Together with the Hausdorff Theorem 4.2.10 we thus know that a separable metrizable strong Choquet space is a Polish space.

Furthermore by Lemma 1.3 a metrizable space is a regular T1 space. To get the different characterization of a Polish space we will state now Urysohn's Metrization Theorem that asserts the converse for second countable topological spaces.

Theorem 4.2.11 (Urysohn Metrization Theorem). *Let X be a second countable topological space. Then X is metrizable iff X is T1 and regular.*

A proof for this theorem can, for example, be found in the books of the Polish topologists R. Engelking [Enge68, Ch.4 §2, Theorem 4] or K. Kuratowski [Kura66, Ch.2, §22, II, Theorem 1].

If we put now together all these results, we get, by using AC, the following characterisation of a Polish space. Note, that we did not use AC to prove that a Polish space is strong Choquet, T1 and regular. This is only required for the converse.

Theorem 4.2.12 (AC). [Choquet] *A nonempty, second countable topological space is Polish iff it is T1, regular and strong Choquet.*

This is the characterization of Polish spaces we will mainly use for our characterization of the projective sets.

Chapter 5

The scale property and projective ordinals

In Section 2.3 we introduced norms and scales and mentioned that these concepts get more interesting if we examine norms (and scales) of a certain complexity, that is, roughly speaking, the associated prewellorderings should be in certain pointclasses (for the exact defintion see Definitions 5.1.1 and 5.1.10). The pointclasses we consider will be the pointclasses that occur in the projective hierarchy. So we will define $\mathbf{\Gamma}$-norms and $\mathbf{\Gamma}$-scales for pointclasses $\mathbf{\Gamma}$ from the projective hierarchy and state properties of these notions mainly under the axiom **PD**. The reason for considering **PD** here is that one of the great assets of PD is that one can show that a lot of pointsets in the projective hierarchy admit $\mathbf{\Gamma}$-scales. We also introduce a bound for the length of such a $\mathbf{\Gamma}$-norm. This will be the projective ordinals $\boldsymbol{\delta}^1_n$.

We proved in Theorem 2.3.7 that the pointsets of the Baire space that admit λ-scales are λ-Suslin sets. So the results under **PD** lead to a lot of examples of λ-Suslin sets where λ is an ordinal related to the projective ordinals. The goal of the first section is to prove that $\mathbf{\Sigma}^1_n$ sets are such λ-Suslin sets.

In the second section we will take a closer look at the projective ordinals. It will turn out that these ordinals are under the axiom **AD** in fact regular successor cardinals.

5.1 The prewellordering and scale properties under PD

Definition 5.1.1. Let $\mathbf{\Gamma}$ be a pointclass. Let X be a Polish space and $A \subseteq X$. A norm $\varphi : A \longrightarrow \mathrm{Ord}$ is called a **$\mathbf{\Gamma}$-norm** if there are relations $\leq^{\mathbf{\Gamma}}_\varphi, \leq^{\check{\mathbf{\Gamma}}}_\varphi \subseteq X \times X$ in $\mathbf{\Gamma}, \check{\mathbf{\Gamma}}$ respectively such that for every y we have

$$y \in A \Rightarrow \forall x[x \in A \wedge \varphi(x) \leq \varphi(y) \Leftrightarrow x \leq^{\mathbf{\Gamma}}_\varphi y \Leftrightarrow x \leq^{\check{\mathbf{\Gamma}}}_\varphi y]$$

A pointclass $\mathbf{\Gamma}$ has the **prewellordering property** (or is **normed**) if eyery pointset in $\mathbf{\Gamma}$ admits a $\mathbf{\Gamma}$-norm.

Since we are here only interested in projective sets we will only consider pointclasses $\mathbf{\Gamma}$ that occur in the projective hierarchy. For this reason we denoted in the above definition and will denote in the following all pointclasses with boldface letters. Of course in general this definition applies not only for boldface pointclasses if we understand by this pointclasses closed under continuous preimages.

Notice that for a set $A \in \mathbf{\Gamma}$ (where $\mathbf{\Gamma}$ is $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$) the defining property for a norm φ being a $\mathbf{\Gamma}$-norm is stronger than requiring that the associated prewellordering $\leq_\varphi$ is in $\mathbf{\Gamma}$ but weaker than insisting that $\leq_\varphi$ is in $\mathbf{\Delta}$. On the other hand the definition implies that a $\mathbf{\Gamma}$-norm φ on $A \in \mathbf{\Delta}$ is already a $\mathbf{\Delta}$-norm, since intersecting the two relations $\leq^{\mathbf{\Gamma}}_\varphi, \leq^{\check{\mathbf{\Gamma}}}_\varphi$ with A gives the prewellordering $\leq_\varphi$ and this is therefore in $\mathbf{\Delta}$ and can serve as $\leq^{\mathbf{\Gamma}}_\varphi, \leq^{\check{\mathbf{\Gamma}}}_\varphi$. Despite the simplicity of this argument we put this down as a Proposition since we will use this fact more often.

Proposition 5.1.2. *Let $\mathbf{\Gamma}$ be $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$. Every $\mathbf{\Gamma}$-norm on a pointset $A \in \mathbf{\Delta}$ is a $\mathbf{\Delta}$-norm.*

Proof. Let φ be a $\mathbf{\Gamma}$-norm on a $\mathbf{\Delta}$ set $A \subseteq X$ and let $\leq^{\mathbf{\Gamma}}_\varphi, \leq^{\check{\mathbf{\Gamma}}}_\varphi$ be two relations in $\mathbf{\Gamma}, \check{\mathbf{\Gamma}}$ respectively with the defining properties for φ being a $\mathbf{\Gamma}$-norm. We want to show that $\leq_\varphi = \leq^{\mathbf{\Gamma}}_\varphi \cap A \times A = \leq^{\check{\mathbf{\Gamma}}}_\varphi \cap A \times A$ and has also the defining property.

We first prove that $\leq^{\mathbf{\Gamma}}_\varphi \cap A \times A = \leq_\varphi = \leq^{\check{\mathbf{\Gamma}}}_\varphi \cap A \times A$:
"$\subseteq$" Let $(x,y) \in \leq^{\mathbf{\Gamma}}_\varphi \cap A \times A$. Then $(x,y) \in A \times A$ and $\varphi(x) \leq \varphi(y)$. Thus $(x,y) \in \leq_\varphi$.
"$\supseteq$" Let $(x,y) \in \leq_\varphi$. Then $x \in A, y \in A$ and $\varphi(x) \leq \varphi(y)$. Therefore $(x,y) \in \leq^{\mathbf{\Gamma}}_\varphi \cap A \times A$.

The proof for $\leq^{\check{\mathbf{\Gamma}}}_\varphi$ is exactly the same. So $\leq_\varphi \in \mathbf{\Delta}$.

Next we show that $\leq_\varphi$ has indeed the defining property. For this let $y \in A, x \in X$. We have to show

$$x \in A \wedge \varphi(x) \leq \varphi(y) \;\Leftrightarrow\; (x,y) \in \leq^{\mathbf{\Gamma}}_\varphi \cap A \times A$$

"$\Rightarrow$" $x \in A \wedge \varphi(x) \leq \varphi(y) \;\Rightarrow\; (x,y)n \in \leq^{\mathbf{\Gamma}}_\varphi \wedge (x,y) \in A \times A$
$\Rightarrow\; (x,y) \in \leq^{\mathbf{\Gamma}}_\varphi \cap A \times A$
"$\Leftarrow$" $(x,y) \in \leq^{\mathbf{\Gamma}}_\varphi \cap A \times A \;\Rightarrow\; x \in A \wedge \varphi(x) \leq \varphi(y)$
Analogous for $\leq^{\check{\mathbf{\Gamma}}}_\varphi$.

So $\leq_\varphi \in \mathbf{\Delta}$ and has the defining property for φ being a $\mathbf{\Delta}$-norm. □

Even if in general it is not true that a $\mathbf{\Gamma}$-norm on a pointset $A \in \mathbf{\Gamma}$ is in $\mathbf{\Delta}$, this holds for initial segments of the associated prewellordering:

Lemma 5.1.3. *Let $\mathbf{\Gamma}$ be $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ and let $\varphi : A \longrightarrow |\leq_\varphi|$ be a regular $\mathbf{\Gamma}$-norm on some pointset $A \in \mathbf{\Gamma}$. Then for $\alpha < |\leq_\varphi|$ the sets $A^\alpha = \{x \mid \varphi(x) \leq \alpha\}$ and $A^{<\alpha} = \{x \mid \varphi(x) < \alpha\}$, initial segments of the prewellordering $\leq_\varphi$, are in $\mathbf{\Delta}$. In particular, $A = \bigcup_{\alpha < |\leq_\varphi|} A^\alpha$ with each A^α in $\mathbf{\Delta}$.*

Proof. The norm φ on A is a surjective mapping. Choose for $\alpha < |\leq_\varphi|$ some y in A such that $\varphi(y) = \alpha$. Then

$$\begin{aligned} x \in A^\alpha &\Leftrightarrow x \leq_\varphi^{\mathbf{\Gamma}} y \\ &\Leftrightarrow x \leq_\varphi^{\check{\mathbf{\Gamma}}} y \end{aligned}$$

Similar for $A^{<\alpha}$:

$$\begin{aligned} x \in A^{<\alpha} &\Leftrightarrow x \leq_\varphi^{\mathbf{\Gamma}} y \wedge \neg y \leq_\varphi^{\check{\mathbf{\Gamma}}} x \\ &\Leftrightarrow x \leq_\varphi^{\check{\mathbf{\Gamma}}} y \wedge \neg y \leq_\varphi^{\mathbf{\Gamma}} x \end{aligned}$$

□

There are two other relations associated to a norm φ on a subset A of some Polish space X that will be of special interest. We extend the prewellordering $\leq_\varphi$ to a relation to all of X by putting all points from $X \setminus A$ above all the points from A. This gives us the relations $\leq_\varphi^*, <_\varphi^*$ defined by:

$$\begin{aligned} x \leq_\varphi^* y &\Leftrightarrow x \in A \wedge [y \notin A \vee \varphi(x) \leq \varphi(y)] \\ x <_\varphi^* y &\Leftrightarrow x \in A \wedge [y \notin A \vee \varphi(x) < \varphi(y)] \end{aligned}$$

Proposition 5.1.4. *Let $\mathbf{\Gamma}$ be $\mathbf{\Sigma}_n^1$ or $\mathbf{\Pi}_n^1$ and let φ be a norm on some A in $\mathbf{\Gamma}$. Then φ is a $\mathbf{\Gamma}$-norm iff the relations $\leq_\varphi^*, <_\varphi^*$ are both in $\mathbf{\Gamma}$.*

Proof. Let φ be a $\mathbf{\Gamma}$-norm on A. Let $\leq_\varphi^{\mathbf{\Gamma}}, \leq_\varphi^{\check{\mathbf{\Gamma}}}$ be two relations with the defining conditions for φ being a $\mathbf{\Gamma}$-norm.
(1) $x \leq_\varphi^* y \quad \Leftrightarrow \quad x \in A \wedge [x \leq_\varphi^{\mathbf{\Gamma}} y \vee \neg y \leq_\varphi^{\check{\mathbf{\Gamma}}} x]$

Proof: "⇒ " Let $x \leq_\varphi^* y$. Then $x \in A$. If $y \in A$ then $\varphi(x) \leq \varphi(y)$, so $x \leq_\varphi^{\mathbf{\Gamma}} y$. If $y \notin A$ we want to show that $\neg y \leq_\varphi^{\check{\mathbf{\Gamma}}} x$. But $y \leq_\varphi^{\check{\mathbf{\Gamma}}} x$ implies $y \in A$. So this would lead to a contradiction.

"⇐" Let $x \in A$ and $x \leq_\varphi^{\mathbf{\Gamma}} y \vee \neg y \leq_\varphi^{\check{\mathbf{\Gamma}}} x$.

Case 1: $y \in A$. If $x \leq_\varphi^{\mathbf{\Gamma}} y$ then $\varphi(x) \leq \varphi(y)$ and we are done. If $\neg y \leq_\varphi^{\check{\mathbf{\Gamma}}} y \Leftrightarrow \neg y \in A \vee \neg \varphi(y) \leq \varphi(x)$. Since we have $y \in A$ we must have $\neg \varphi(y) \leq \varphi(x)$. Since φ is a norm on A it must be that $\varphi(y) > \varphi(x)$, thus $x \leq_\varphi^* y$.

Case 2: $y \notin A$ implies by definition of $\leq_\varphi^*$ that $x \leq_\varphi^* y$. q.e.d.(1)

(1) proves that $\leq_\varphi^*$ is indeed a relation in $\mathbf{\Gamma}$. The upcoming (2) proves it for the relation $<_\varphi^*$.

(2) $x <_\varphi^* y \Leftrightarrow x \in A \wedge \neg y \leq_\varphi^{\check{\mathbf{\Gamma}}} x$

Proof: " ⇒" Let $x <_\varphi^* y$. Then $x \in A$. If $y \notin A$ and would have $y \leq_\varphi^{\check{\mathbf{\Gamma}}} x$ this would lead to a contradiction since $y \leq_\varphi^{\check{\mathbf{\Gamma}}} x$ implies $y \in A$. If $y \in A$ and $\varphi(x) < \varphi(y)$ we have $x <_\varphi^{\check{\mathbf{\Gamma}}}$, so $\neq y \leq_\varphi^{\check{\mathbf{\Gamma}}} x$.

"⇐" Same as in the proof of (1). q.e.d.(2)

Let for the converse $\leq_\varphi^*, <_\varphi^*$ be in $\mathbf{\Gamma}$. Define the relations $\leq_\varphi^{\mathbf{\Gamma}}, \leq_\varphi^{\check{\mathbf{\Gamma}}}$ by

$$\begin{aligned} x \leq_\varphi^{\mathbf{\Gamma}} y &\Leftrightarrow x \leq_\varphi^* y \\ x \leq_\varphi^{\check{\mathbf{\Gamma}}} y &\Leftrightarrow \neg y <_\varphi^* x \end{aligned}$$

By this definition $\leq_\varphi^{\mathbf{\Gamma}}$ is in $\mathbf{\Gamma}$ and $\leq_\varphi^{\check{\mathbf{\Gamma}}}$ is in $\check{\mathbf{\Gamma}}$. Let $y \in A$. Then

$$x \leq_\varphi^{\mathbf{\Gamma}} y \;\Leftrightarrow\; x \leq_\varphi^* y \;\Leftrightarrow\; x \in A \wedge \varphi(x) \leq \varphi(y)$$

Thus $\leq_\varphi^{\mathbf{\Gamma}}$ has the wanted property.

Now for $\leq_\varphi^{\check{\mathbf{\Gamma}}}$. Let $y \in A$. If $x \in A$ and $\varphi(x) \leq \varphi(y)$, then $x \leq_\varphi^* y$, so $\neg\, y <_\varphi^* x$. Suppose for the converse that we have $\neg\, y <_\varphi^* x$. Assume $x \notin A$, then $y <_\varphi^* x$ since $y \in A$. A contradiction. So $x \in A$. Therfore $x \leq_\varphi^* y$ and this implies $\varphi(x) \leq \varphi(y)$. This proves that $\leq_\varphi^{\check{\mathbf{\Gamma}}}$ has the defining property for φ being a $\mathbf{\Gamma}$-norm. □

Of course we are now interested in pointclasses of the projective hierarchy which are normed. It is known that $\mathbf{\Pi}_1^1$ and $\mathbf{\Sigma}_2^1$ are normed classes (cf.[Mosc80, 4B.2, 4B.3]). One of the great assets of **PD** is that under **PD** for each of the projective classes, the class has or does not have the prewellordering property. This result is due to Moschovakis and proved by his "First Periodicity Theorem" [Mosc80, 6B.1].

Theorem 5.1.5 (PD). *For all $n \geq 0$ the following holds: $\mathbf{\Pi}_{2n+1}^1$ and $\mathbf{\Sigma}_{2n+2}^1$ have the prewellordering property and $\mathbf{\Sigma}_{2n+1}^1$ and $\mathbf{\Pi}_{2n+2}^1$ do not have the prewellordering property.*

Next we will define the projective ordinals. They serve as an upper bound for the length of a $\mathbf{\Gamma}$-norm on a set in $\mathbf{\Gamma}$. It will turn out later that they will be the length of the basis for the topology we define on the $\mathbf{\Sigma}_n^1$ sets.

Definition 5.1.6. For all $n \geq 1$ the **projective ordinals** $\boldsymbol{\delta}_n^1$ are defined as: $\boldsymbol{\delta}_n^1 = \sup\{\alpha \mid \alpha \text{ is the length of a } \mathbf{\Delta}_n^1 \text{ prewellordering of } \mathcal{N}\}$

We will give first some basic facts about the projective ordinals.

Proposition 5.1.7. *Let $\mathbf{\Gamma}$ be $\mathbf{\Sigma}_n^1$ or $\mathbf{\Pi}_n^1$ for $n \geq 1$.*
(a) $\boldsymbol{\delta}_n^1$ is a limit ordinal that is not attained by a $\mathbf{\Delta}_n^1$ prewellordering of $\mathcal{N}$.
(b) Every $\mathbf{\Delta}_n^1$-norm on a $\mathbf{\Delta}_n^1$ set has length less than $\boldsymbol{\delta}_n^1$.
(c) Every $\mathbf{\Gamma}$-norm on a $\mathbf{\Gamma}$ set has length less or equal $\boldsymbol{\delta}_n^1$.
(d) For every $\alpha < \boldsymbol{\delta}_n^1$ there exists a $\mathbf{\Delta}_n^1$ prewellordering of $\mathcal{N}$ of length α.
(e) $\mathrm{cf}(\boldsymbol{\delta}_n^1) > \omega$

Proof. (a) Assume $\boldsymbol{\delta}_n^1$ is a successor ordinal. This implies in particular that there is a prewellordering $\leq$ of $\mathcal{N}$ of length $\boldsymbol{\delta}_n^1$. Let φ be the associated rank function. Since $\boldsymbol{\delta}_n^1 \geq \omega$ (for example $x \leq y \Leftrightarrow x(0) \leq y(0)$ is a $\mathbf{\Delta}_1^1$ prewellordering of length ω) we have the following bijection

$$\begin{aligned} f : \boldsymbol{\delta}_n^1 &\longrightarrow \boldsymbol{\delta}_n^1 + 1 \\ \alpha &\longmapsto \begin{cases} \boldsymbol{\delta}_n^1 & \text{if } \alpha = 0 \\ \alpha - 1 & \text{if } 0 < \alpha < \omega \\ \alpha & \text{if } \alpha \geq \omega \end{cases} \end{aligned}$$

Now $f \circ \varphi : \mathcal{N} \longrightarrow \boldsymbol{\delta}^1_n + 1$ is a regular norm. Pick an $a \in \mathcal{N}$ such that $\varphi(a) = 0$. Then the prewellordrering $\leq_{f\circ\varphi}$ is given by

$$\begin{aligned} x \leq_{f\circ\varphi} y \quad \Leftrightarrow \quad & (x \leq y \wedge y \leq x) \\ & \vee (y \leq a \wedge a \leq y) \\ & \vee \neg(x \leq a \wedge a \leq x \wedge y \leq a \wedge a \leq y) \;\rightarrow\; x \leq y \end{aligned}$$

So we just defined a $\boldsymbol{\Delta}^1_n$ prewellordering of $\mathcal{N}$ of length $\boldsymbol{\delta}^1_n + 1$. This contradicts our assumption and tells us furthermore that $\boldsymbol{\delta}^1_n$ is not attained by a $\boldsymbol{\Delta}^1_n$ prewellordering of $\mathcal{N}$.

(b) We show first that by Theorem 2.2.3 it is enough to consider a $\boldsymbol{\Delta}^1_n$ subset of $\mathcal{N}$. Let X be a Polish space and $A \subseteq X$ be a $\boldsymbol{\Delta}^1_n$ subset of X together with a $\boldsymbol{\Delta}^1_n$ norm φ. There exists by 2.2.3 a continuous bijection b between a closed subset of $\mathcal{N}$ and the Polish X and we can use this bijection to pull back the $\boldsymbol{\Delta}^1_n$ prewellordering $\leq_\varphi$ of A to a $\boldsymbol{\Delta}^1_n$ prewellordering of the same lenght of the $\boldsymbol{\Delta}^1_n$ subset $b^{-1}[A]$ of $\mathcal{N}$ since the pointclass $\boldsymbol{\Delta}^1_n$ is closed under continuous preimages.

So let $\varphi : A \longrightarrow \alpha$ be a $\boldsymbol{\Delta}^1_n$-norm on $A \subseteq \mathcal{N}$. If $A = \mathcal{N}$ we are done with (a). Otherwise consider the $\boldsymbol{\Delta}^1_n$ prewellordering $\leq_\varphi$ of A. Define then a prewellordering $\leq$ of $\mathcal{N}$ by

$$x \leq y \;\Leftrightarrow\; x \leq_\varphi y \,\vee\, y \notin A$$

This prewellordering is $\boldsymbol{\Delta}^1_n$ and has length $\alpha + 1$. Thus $\alpha < \boldsymbol{\delta}^1_n$ by (a).

(c) Let A be a $\boldsymbol{\Gamma}$ set and φ be a regular $\boldsymbol{\Gamma}$-norm. By Lemma 5.1.3 the sets A^α for $\alpha < |\varphi|$ are in $\boldsymbol{\Delta}^1_n$. Intersecting $\leq_\varphi$ with A^α gives us a $\boldsymbol{\Delta}^1_n$-norm on A^α. Thus by (b), α has to be less than δ^1_n. Since $|\varphi| = \sup_{\alpha<|\varphi|} \alpha$ we have $|\varphi| \leq \boldsymbol{\delta}^1_n$.

(d)Let $\alpha < \boldsymbol{\delta}^1_n$. Then there exists an ordinal $\beta > \alpha$ and a $\boldsymbol{\Delta}^1_n$ prewellordering on $\mathcal{N}$ of length β (by the definition of the projective ordinals). Define now a prewellordering $\leq_\alpha$ on $\mathcal{N}$ by

$$x \leq_\alpha y \Leftrightarrow (x, y) \in \leq \cap \mathcal{N}^{<\alpha} \times \mathcal{N}^{<\alpha} \vee \neg x \in \mathcal{N}^{<\alpha}$$

there $\mathcal{N}^{<\alpha} = \{x \mid \varphi(x) < \alpha\}$.

From Lemma 5.1.3 we know that $\mathcal{N}^{<\alpha}$ is in $\boldsymbol{\Delta}^1_n$. Thus $\leq_\alpha$ is a $\boldsymbol{\Delta}^1_n$ prewellordering with regular associated norm

$$\begin{aligned} \varphi_\alpha : \mathcal{N} &\longrightarrow \alpha \\ x &\longmapsto \begin{cases} 0 & \text{if } x \notin \mathcal{N}^{<\alpha} \\ \varphi(x) & \text{otherwise} \end{cases} \end{aligned}$$

Thus the length of $\leq_\alpha$ equals α.

(e) Let $(\alpha_i)_{i\in\omega}$ be a sequence of ordinals $< \boldsymbol{\delta}^1_n$. Let $\leq_i$ be a $\boldsymbol{\Delta}^1_n$ prewellordering of $\mathcal{N}$ with $|\leq_i| = \alpha_i$. Consider the following two homeomorphisms

$$\begin{aligned} \pi_i : \mathcal{N} &\longrightarrow N_{(i)} \\ x &\longmapsto (i)^\frown x \end{aligned}$$

and

$$\begin{array}{rcl} \sigma : \mathcal{N} & \longrightarrow & \sum_{i\in\omega} N_{(i)} \\ x & \longmapsto & x \end{array}$$

where we understand by $\sum_{i\in\omega} N_{(i)}$ the topological sum of the Polish spaces $N_{(i)}$ which are disjoint by definition. The mapping π_i carries the prewellordering $\leq_i$ to the prewellordering $\leq_i^{\pi_i}$ of $N_{(i)}$. Putting together these prewellorderings of all the $N_{(i)}$ we get a prewellordering of $\sum_i N_{(i)}$ by

$$\begin{array}{rcl} x \leq y & \Leftrightarrow & x \in N_{(i)} \wedge y \in N_{(i)} \wedge x \leq_i^{\pi_i} y \\ & & \vee\, (x \in N_{(i)} \wedge y \in N_{(j)} \wedge i < j) \end{array}$$

This is a prewellordering of length $\sum_{i\in\omega} \alpha_i$. Also $\leq$ is in $\boldsymbol{\Delta}^1_n$ since

$$\leq = \bigcup_{i\in\omega} \leq_i^{\pi_i} \cup \bigcup_{i<j} N_{(i)} \times N_{(j)}$$

Pulling back this prewellordering $\leq$ to $\mathcal{N}$ with the homeomorphism σ gives us then a $\boldsymbol{\Delta}^1_n$ prewellordering of $\mathcal{N}$ of length $\sum_{i\in\omega} \alpha_i$. Thus $\sup \alpha_i \leq \sum_{i\in\omega} \alpha_i < \boldsymbol{\delta}^1_n$. □

The results from this last Proposition 5.1.7 are pretty much all we know about the projective ordinals under the axioms **ZF** + **DC**. And even if we work in addition under the assumption of **PD** we are not able to prove a lot more. This looks different if we assume the theory **ZF** + **DC** + **AD** and we will come back to this in the next section.

Under classical set theory the only result of interest left to prove is the calculation of $\boldsymbol{\delta}^1_1$. For this we state now the Kunen-Martin Theorem, which is fundamental for all of the rest of this chapter. A detailed proof using the notion of a good semiscale can be found in [Mosc80, 2G.2].

Theorem 5.1.8. *Let $\preceq \subseteq \mathcal{N} \times \mathcal{N}$ be a wellfounded relation. If $\preceq$ is κ-Suslin, then $|\preceq| < \kappa^+$.*

With this Theorem 5.1.8 it is now easy to prove that $\boldsymbol{\delta}^1_1 = \omega_1$.

Proposition 5.1.9. $\boldsymbol{\delta}^1_1 = \omega_1$

Proof. Let $\preceq$ be a $\boldsymbol{\Delta}^1_1$ prewellordering of $\mathcal{N}$. Then the relation $\preceq$ is in particular in $\boldsymbol{\Sigma}^1_1$ and therefore ω-Suslin by Theorem 3.1.7. So the length of the prewellordering is less than ω_1 by the Kunen-Martin Theorem 5.1.8. Therefore $\boldsymbol{\delta}^1_1 \leq \omega_1$. We proved on the other hand in Proposition 5.1.7(e) that $\boldsymbol{\delta}^1_1$ has cofinality greater than ω. Since this is not possible for ordinals below ω_1 we conclude that $\boldsymbol{\delta}^1_1 = \omega_1$. □

Similar to $\boldsymbol{\Gamma}$-norms we define now $\boldsymbol{\Gamma}$-scales.

Definition 5.1.10. For a pointclass $\mathbf{\Gamma}$ we call a scale $(\varphi_n)_{n\in\omega}$ a **$\mathbf{\Gamma}$-scale** if the following two relations are in $\mathbf{\Gamma}$:

$$\begin{array}{rcl} S(n,x,y) & \Leftrightarrow & x \leq^*_{\varphi_n} y \\ T(n,x,y) & \Leftrightarrow & x <^*_{\varphi_n} y \end{array}$$

A pointclass $\mathbf{\Gamma}$ has the **scale property** or is **scaled** if every pointset in $\mathbf{\Gamma}$ admits a $\mathbf{\Gamma}$-scale.

In particular this definition implies that all norms in a $\mathbf{\Gamma}$-scale are $\mathbf{\Gamma}$-norms. So if for example a $\mathbf{\Delta}^1_n$-scale on a $\mathbf{\Delta}^1_n$ set $A \subseteq \mathcal{N}$ exists, we thus know that this scale is a $\boldsymbol{\delta}^1_n$-scale and by Theorem 2.3.7 the set A is $\boldsymbol{\delta}^1_n$-Suslin. Similar results hold for the pointclasses $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$. We give a result below. So we will get a whole class of examples for $\boldsymbol{\delta}^1_n$-Suslin sets if we know which pointclasses are scaled. The answer under **PD** gives us Moschovakis "Second Periodicity Theorem", see [Mosc80, 6C].

Theorem 5.1.11 (PD). *The pointclasses $\mathbf{\Pi}^1_{2n+1}$ and $\mathbf{\Sigma}^1_{2n+2}$ are scaled for all $n \geq 0$.*

Using now Theorem 2.3.7 and Proposition 2.3.2 we can view $\mathbf{\Sigma}^1_n$ sets as λ-Suslin sets:

Theorem 5.1.12 (PD). *For all $n \geq 0$ the following holds:*
(i) Every $\mathbf{\Sigma}^1_{2n+2}$ set is $\boldsymbol{\delta}^1_{2n+1}$-Suslin.
(ii) Every $\mathbf{\Sigma}^1_{2n+1}$ set A is $\kappa_{2n+1}(A)$-Suslin for a cardinal $\kappa_{2n+1}(A) < \boldsymbol{\delta}^1_{2n+1}$.

Proof. (i) By Proposition 2.3.2 it is enough to prove that each $\mathbf{\Pi}^1_{2n+1}$ set is $\boldsymbol{\delta}^1_{2n+1}$-Suslin since the $\mathbf{\Sigma}^1_{2n+2}$ sets are by definition projections of $\mathbf{\Pi}^1_{2n+1}$ sets. But by the "Second Periodictiy Theorem" 5.1.11 we know that each $\mathbf{\Pi}^1_{2n+1}$ set has a $\mathbf{\Pi}^1_{2n+1}$-scale. All the norms in this scale are $\mathbf{\Pi}^1_{2n+1}$-norms and thus have length less or equal than $\boldsymbol{\delta}^1_n$ by Proposition 5.1.7(c). So all $\mathbf{\Pi}^1_{2n+1}$ sets admit $\boldsymbol{\delta}^1_{2n+1}$-scales and thus Theorem 2.3.7 implies that all $\mathbf{\Pi}^1_{2n+1}$ sets are $\boldsymbol{\delta}^1_{2n+1}$-Suslin.

(ii) Let A be a $\mathbf{\Sigma}^1_{2n+1}$ set and $B \in \mathbf{\Pi}^1_{2n}$ such that $A = p[B]$. Since $B \in \mathbf{\Delta}^1_{2n+1}$ there exists by Theorem 5.1.11 a $\mathbf{\Pi}^1_{2n+1}$-scale $(\varphi_i)_{i\in\omega}$ on B. Each φ_i is a $\mathbf{\Delta}^1_{2n+1}$-norm on B, so by Proposition 5.1.7(b) has length less than $\boldsymbol{\delta}^1_{2n+1}$. The length of the scale is $\sup_{i\in\omega} |\leq_{\varphi_i}|$ and since $\mathrm{cf}(\boldsymbol{\delta}^1_{2n+1}) > \omega$ by Proposition 5.1.7(e) the sequence $(|\leq_{\varphi_i}|)_{i\in\omega}$ is bounded below $\boldsymbol{\delta}^1_{2n+1}$. Hence there is a cardinal $\kappa_{2n+1}(A) < \boldsymbol{\delta}^1_{2n+1}$ such that $|\leq_{\varphi_i}| \leq \kappa_{2n+1}(A)$ for all $i \in \omega$. Thus $(\varphi_i)_{i\in\omega}$ is a $\kappa_{2n+1}(A)$-scale on B. By Theorem 2.3.7 we thus know that B is $\kappa_{2n+1}(A)$-Suslin and therefore also A by Proposition 2.3.2. $\square$

We close this section by stating a result about the length of a $\mathbf{\Pi}^1_n$ norm under the assumption **PD**. In Proposition 5.1.7 we proved that the length of such a norm on a set in $\mathbf{\Pi}^1_n$ is less or equal to $\boldsymbol{\delta}^1_n$. In fact there are $\mathbf{\Pi}^1_n$ sets with $\mathbf{\Pi}^1_n$-norms with length equal to $\boldsymbol{\delta}^1_n$. These are the $\mathbf{\Pi}^1_n$-complete sets and we define this notion next.

For the upcoming the pointclasses $\mathbf{\Gamma}$ should always stand for $\mathbf{\Sigma}^1_n(\mathcal{N})$ or $\mathbf{\Pi}^1_n(\mathcal{N})$ for $n \geq 1$.

Definition 5.1.13. Let $A, B \subseteq \mathcal{N}$. A is called **(Wadge-)reducible to** B, $A \leq_W B$, if there exists a continuous function $f : \mathcal{N} \longrightarrow \mathcal{N}$ such that $f^{-1}[B] = A$.
We say A is **$\boldsymbol{\Gamma}$-complete** if $A \in \boldsymbol{\Gamma}$ and all $B \in \boldsymbol{\Gamma}$ are reducible to A.

The following theorem will turn out to be very helpful to us at various stages in the rest of this paper. A proof can be found in [Mosc70, Theorem 8.1], using facts from recursion theory.

Theorem 5.1.14 (PD). *If φ is a $\boldsymbol{\Pi}^1_n$-norm on a $\boldsymbol{\Pi}^1_n$-complete set, then the prewellordering $\leq_\varphi$ has length $\boldsymbol{\delta}^1_n$.*

Of course it arises now the question if $\boldsymbol{\Gamma}$-complete sets exist? Since we will apply Theorem 5.1.14 mainly under the assumption of **AD** in the next section, the following theorem implies a result of interest in the context of complete sets.

Theorem 5.1.15 (AD, Wadge's Lemma). *Let $A, B \subseteq \mathcal{N}$. Then either $A \leq_W B$ or $B \leq_W \mathcal{N} \setminus A$.*

Proof. Consider the Wadge game WG(A, B)

$$\begin{array}{cccccc} \mathrm{I} & x(0) & & x(1) & & \dots \\ \mathrm{II} & & y(0) & & y(1) & \dots \end{array}$$

where I and II play integers and II wins if $(x \in A \leftrightarrow y \in B)$. Since we are working under AD this game is determined.
Assume II has a winning strategy τ. If I plays x we denote the element played by II following his strategy τ by $x * \tau$. So we have $x \in A \leftrightarrow x * \tau \in B$. We can obviously view τ as a monotone mapping between the full trees on ω. By Proposition 2.1.5 the function

$$\begin{aligned} f_\tau : \mathcal{N} &\longrightarrow \mathcal{N} \\ x &\longmapsto x * \tau \end{aligned}$$

is continuous and by the property of τ we have $f_\tau^{-1}[B] = A$. So $A \leq_W B$.
If I has a winning strategy σ one can show with the same argument that $B \leq_W \mathcal{N} \setminus A$. □

Corollary 5.1.16 (AD). *Every set in $\boldsymbol{\Gamma} \setminus \boldsymbol{\Delta}$ is $\boldsymbol{\Gamma}$-complete.*

Proof. Let $A \in \boldsymbol{\Gamma} \setminus \boldsymbol{\Delta}$ and $B \in \boldsymbol{\Gamma}$. From Wadge's Lemma we have $B \leq_W A$ or $A \leq_W \mathcal{N} \setminus B$. But $A \leq_W \mathcal{N} \setminus B$ leads to a contradiction since then A is the preimage of some $\check{\boldsymbol{\Gamma}}$-set and therefore also in $\check{\boldsymbol{\Gamma}}$ (since both $\boldsymbol{\Sigma}^1_n$ and $\boldsymbol{\Pi}^1_n$ are closed under continuous preimages). □

We conclude from this Corollary 5.1.16 and Theorem 5.1.14 that under the assumption of **AD** all $\boldsymbol{\Pi}^1_n$-norms on a set in $\boldsymbol{\Pi}^1_n \setminus \boldsymbol{\Delta}^1_n$ has length $\boldsymbol{\delta}^1_n$. One could expect that a similar result is true for the complete $\boldsymbol{\Sigma}^1_n$ sets, but we will show in Theorem 5.2.8 that this does not hold.

5.2 Projective ordinals under AD

The projective ordinals turned out to be very important for the results of the last section. But even working under **PD** does not give us a lot of information about the projective ordinals. The picture looks completely different if we assume **AD**. We will prove here that under **AD** the projective ordinals are regular successor cardinals. Crucial for a proof of this is the very powerful "Coding Lemma" by Moschovakis that holds under **AD** and which we will state first.

We mentioned before that **AD** contradicts **AC**. The Coding Lemma allows us now to use some sort of choice for (a subset of) the powerset of any set Y if we have a function from an ordinal λ, that can be coded by a wellfounded relation (or more exact by the associated rank function), to the powerset of Y. Furthermore the Coding Lemma assures that if λ is coded by an $\mathbf{\Sigma}^1_n$ wellfounded relation the choice set (or rather the codes for the choice set, see the exact definition below) is also in $\mathbf{\Sigma}^1_n$. The definition of such a choice set is the following:

We can restrict ourselves for our purpose to spaces of the form $\omega^k \times (\omega^\omega)^\ell$. Let X be such a space and $<$ be a strict wellfounded relation on some subset S of X. Let $\rho : S \twoheadrightarrow \lambda$ be the associated rank function. So the elements of S can be seen as codes for ordinals below λ. Let Y be another space and $f : \lambda^n \longrightarrow \mathcal{P}(Y)$ be any function. A **choice set for** f is a subset C of $X^m \times Y$ such that the following holds

(i) $(x_0, \ldots, x_{m-1}, y) \in C \;\Rightarrow\; x_0, \ldots, x_{m-1} \in S \wedge y \in f(\rho(x_0), \ldots, \rho(x_{m-1}))$

(ii) $f(\xi_0, \ldots, \xi_{m-1}) \neq \emptyset \;\Rightarrow\; \exists x_0 \ldots \exists x_{m-1} \exists y [\rho(x_0) = \xi_0 \wedge \ldots \rho(x_{m-1}) = x_{m-1} \wedge y \in f(x_0, \ldots, x_{m-1}) \wedge (x_0, \ldots, x_{m-1}, y) \in C]$

Theorem 5.2.1 (Coding Lemma I). *Assume* **AD**. *Let $m, n \in \omega$. Let $< \subseteq X \times X$ be a strict wellfounded relation in $\mathbf{\Sigma}^1_n$ of length λ. Then for every $f : \lambda^m \longrightarrow \mathcal{P}(Y)$ there exists a choice set in $\mathbf{\Sigma}^1_n$.*

For a proof see [Mosc80, 7D.5]. Important to us will be the following Corollary, which Moschovakis calls "Coding Lemma II" (see [Mosc80, 7D.6]). It tells us that the set of codes of each subset of an ordinal λ which is coded by an $\mathbf{\Delta}^1_n$ prewellordering on the reals is also in $\mathbf{\Delta}^1_n$. So we consider now more generally prewellorderings $\leq_0, \ldots, \leq_{m-1}$ on subsets $S_0, \ldots, S_{m-1}$ of spaces $X_0, \ldots, X_{m-1}$ respectively with associated regular norms $\rho_0 : S_0 \twoheadrightarrow \lambda_0, \ldots, \rho_{m-1} : S_{m-1} \twoheadrightarrow \lambda_{m-1}$. For any $A \subseteq \lambda_0 \times \ldots \times \lambda_{m-1}$ set

$$\mathrm{Code}(A; \leq_0, \ldots, \leq_{m-1}) = \{(x_0, \ldots, x_{m-1}) \mid (\rho_0(x_0), \ldots, \rho_{n-1}(x_{m-1})) \in A\}.$$

Corollary 5.2.2 (Coding Lemma II). *Assume* **AD**. *Let $m, n \in \omega$. Let $\leq_0, \ldots, \leq_{m-1}$ be prewellorderings with lengths $\lambda_0, \ldots, \lambda_{m-1}$ on $S_0 \subseteq X_0, \ldots, S_{n-1} \subseteq X_{m-1}$ such that $\leq_0, \ldots, \leq_{m-1} \in \mathbf{\Delta}^1_n$. Then for every $A \subseteq \lambda_0 \times \ldots \times \lambda_{m-1}$ the set* $\mathrm{Code}(A; \leq_0, \ldots, \leq_{m-1})$ *is in $\mathbf{\Delta}^1_n$.*

Proof. Let $\leq$ be the lexicographic ordering on $X = X_0 \times \ldots \times X_{m-1}$ induced by the prewellorderings $\leq_0, \ldots, \leq_{m-1}$ and let $<$ be its strict part. For simplicity we write now $x_i \sim_i x'_i$ for $x_i \leq_i x'_i \wedge x'_i \leq_i x_i$ for $0 \leq i \leq m-1$. So we have

$$\begin{aligned}&(x_0, \ldots, x_{m-1}) < (x'_0, \ldots, x'_{m-1}) \Leftrightarrow\\ &x_0 <_i x'_0\\ &\vee (x_0 \sim_0 x'_0 \wedge x_1 <_1 x'_1)\\ &\vee (x_0 \sim_0 x'_0 \wedge \ldots x_{m-2} \sim_{m-2} x'_{m-2} \wedge x_{m-1} <_{m-1} x'_{m-1})\end{aligned}$$

and therefore $< \in \mathbf{\Delta}^1_n$.
Consider also the lexicographical ordering on $\lambda_0 \times \ldots \times \lambda_{m-1}$ and let $\langle\ \rangle : \lambda_0 \times \ldots \times \lambda_{m-1} \longrightarrow \lambda$ be the isomorphism of this ordering to its order-type. Then the associated regular norm ρ of $<$ is given by $\rho(x_0, \ldots, x_{m-1}) = \langle \rho_1(x_1), \ldots, \rho_n(x_m)\rangle$. Let now

$$\begin{aligned} f : \lambda &\longrightarrow \mathcal{P}(\omega)\\ \langle \xi_0, \ldots, \xi_{m-1}\rangle &\longmapsto \begin{cases}\{1\} & \text{if } (\xi_0, \ldots, \xi_{m-1}) \in A\\ \{0\} & \text{if } (x_0, \ldots, \xi_{m-1}) \notin A\end{cases}\end{aligned}$$

Let $C \subseteq X \times \omega$ be a choice set for f in $\mathbf{\Sigma}^1_n$. We claim

$$\begin{aligned}&(x_0, \ldots, x_{m-1}) \in \mathrm{Code}(A; \leq_1, \ldots, \leq_{m-1})\\ &\Leftrightarrow\ \exists x'_0 \ldots \exists x'_{m-1}[x_0 \sim_0 x'_0 \wedge \ldots \wedge x_{m-1} \sim_{m-1} x'_{m-1} \wedge (x'_0, \ldots, x'_{m-1}, 1) \in C]\end{aligned}$$

Proof of claim:
"$\Rightarrow$"

$$\begin{aligned}&(x_0, \ldots, x_m) \in \mathrm{Code}(A; \leq_0, \ldots, \leq_{m-1})\\ \Leftrightarrow&(\rho_0(x_0), \ldots, \rho_{m-1}(x_{m-1})) \in A\\ \Leftrightarrow&f(\langle \rho_0(x_0), \ldots, \rho_{m-1}(x_{m-1})\rangle) = \{1\}\\ \Rightarrow&\exists x'_0 \ldots \exists x'_{m-1} \exists y x_0 \sim_0 x'_0 \wedge \ldots \wedge x_{m-1} \sim_{m-1} x'_{m-1}\\ &\wedge y \in \{1\} \wedge (x_0, \ldots, x_{m-1}, y) \in C\\ &\text{since } \langle\ \rangle \text{ is a bijection and by (ii) of the definition of a choice set}\\ \Rightarrow&\exists x'_0 \ldots \exists x'_{m-1} x_0 \sim_0 x'_0 \wedge \ldots \wedge x_{m-1} \sim_{m-1} x'_{m-1} \wedge (x'_0, \ldots, x'_{m-1}, 1) \in C\\ &\text{since 1 is the only element in } \{1\}\end{aligned}$$

"$\Leftarrow$"

$$\begin{aligned}&\exists x'_0 \ldots \exists x'_{m-1} x_0 \sim_0 x'_0 \wedge \ldots \wedge x_{m-1} \sim_{m-1} x'_{m-1} \wedge (x'_0, \ldots, x'_{m-1}, 1) \in C\\ \Rightarrow&1 \in f(\langle \rho_0(x'_0), \ldots, \rho_{m-1}(x'_{m-1})\rangle) \text{ by (i) of the definition of a choice set}\\ \Rightarrow&f(\langle \rho_0(x'_0), \ldots, \rho_{m-1}(x'_{m-1})\rangle) = f(\langle \rho_0(x_0), \ldots, \rho_{m-1}(x_{m-1})\rangle) = \{1\}\\ \Rightarrow&(x_0, \ldots, x_{m-1}) \in \mathit{Code}(A; \leq_0, \ldots, \leq_{m-1})\end{aligned}$$

This proves that $\mathrm{Code}(A; \leq_0, \ldots, \leq_{m-1}) \in \mathbf{\Sigma}^1_n$. Similary we prove that the complement of $\mathrm{Code}(A; \leq_0, \ldots, \leq_{m-1})$ is in $\mathbf{\Sigma}^1_n$ by showing

$$\begin{aligned}&(x_0, \ldots, x_{m-1}) \notin \mathrm{Code}(A; \leq_1, \ldots, \leq_{m-1})\\ &\Leftrightarrow\ \exists x'_0 \ldots \exists x'_{m-1}[x_0 \sim_0 x'_0 \wedge \ldots \wedge x_{m-1} \sim_{m-1} x'_{m-1} \wedge (x'_0, \ldots, x'_{m-1}, 0) \in C]\end{aligned}$$

This proves that $\mathrm{Code}(A; \leq_0, \ldots, \leq_{m-1})$ is indeed in $\mathbf{\Delta}^1_n$. □

Now we are able to prove that the projective ordinals are cardinals.

Theorem 5.2.3 (AD). *For all $n \geq 1$, $\boldsymbol{\delta}^1_n$ is a cardinal.*

Proof. Assume this is not true. Then let $\xi < \boldsymbol{\delta}^1_n$ and $\leq$ be a prewellordering of $\mathcal{N}$ of length ξ and $f : \xi \longrightarrow \boldsymbol{\delta}^1_n$ be a bijection. Let ρ be the associated regular norm for $\leq$. Define the following relation $<^*$ on ξ by

$$\eta <^* \vartheta \;\Leftrightarrow\; f(\eta) < f(\vartheta)$$

Thus $<^*$ is a wellordering of ξ of ordertype $\boldsymbol{\delta}^1_n$. From the above Corollary 5.2.2 we have $\mathrm{Code}(<^*; \leq, \leq) \in \boldsymbol{\Delta}^1_n$.
But

$$\begin{aligned}\mathrm{Code}(<^*; \leq, \leq) &= \{(x_1, x_2) \in \mathcal{N}^2 \mid \varphi(x_1) <^* \varphi(x_2)\} \\ &= \{(x_1, x_2) \mid f(\varphi(x_1)) < f(\varphi(x_2))\}\end{aligned}$$

is a prewellordering of $\mathcal{N}$ of length $\boldsymbol{\delta}^1_n$ which contradicts 5.1.7(b). □

To prove now that the projective ordinals are successor cardinals we have to examine more closely the relations between pointsets from the projective hierarchy and κ-Suslin sets (cf. Theorem 2.3.7 and Theorem 5.1.12) as well as between such pointsets and the κ-Borel sets (cf. Section 2.5) under the axiom **AD**. In particular, we will prove a genaralization of Theorem 3.1.11 in which we show that $\boldsymbol{\Delta}^1_{2n+1} = \boldsymbol{\mathcal{B}}_{\boldsymbol{\delta}^1_{2n+1}}$. We proved in Theorem 3.1.11 that the $\boldsymbol{\Delta}^1_1$ subsets of $\mathcal{N}$ are exactly the Borel sets of the Baire space. By definition we call Borel sets also ω_1-Borel sets and $\omega_1 = \boldsymbol{\delta}^1_1$ by Proposition 5.1.9. So we can restate Theorem 3.1.11 as

$$\boldsymbol{\mathcal{B}}_{\boldsymbol{\delta}^1_1} = \boldsymbol{\Delta}^1_1.$$

This statement remains true under **AD** if we replace the lower 1 by any odd integer.

Theorem 5.2.4 (AD). $\boldsymbol{\mathcal{B}}_{\boldsymbol{\delta}^1_{2n+1}}(\mathcal{N}) = \boldsymbol{\Delta}^1_{2n+1}(\mathcal{N})$ *for $n \geq 1$.*

Proof. "$\supseteq$" Let $A \in \boldsymbol{\Delta}^1_{2n+1}$. The $\mathcal{N} \setminus A \in \boldsymbol{\Delta}^1_{2n+1}$ and by Theorem 5.1.12 there is a cardinal $\kappa < \boldsymbol{\delta}^1_{2n+1}$ such that A and $\mathcal{N} \setminus A$ are κ-Suslin. By Corollary 2.5.5 $A \in \boldsymbol{\mathcal{B}}_{\kappa^+} \subseteq \boldsymbol{\mathcal{B}}_{\boldsymbol{\delta}^1_{2n+1}}$.
"$\subseteq$" It suffices to show that $\boldsymbol{\Delta}^1_{2n+1}$ is closed under unions of length strictly smaller than $\boldsymbol{\delta}^1_{2n+1}$. Assume towards a contradiction that there is a $\vartheta < \boldsymbol{\delta}^1_{2n+1}$ minimal such that a sequence $(A_\xi)_{\xi<\vartheta}$ with $A_\xi \in \boldsymbol{\Delta}^1_{2n+1}$ for $\xi < \vartheta$ exists and $A = \bigcup_{\xi<\vartheta} A_\xi \notin \boldsymbol{\Delta}^1_{2n+1}$. Since $\boldsymbol{\Delta}^1_{2n+1}$ is closed under countable unions ϑ has to be uncountable and obviously be a limit ordinal. Without loss of generality we can assume that for all $\xi < \eta < \vartheta$, we have that $A_\xi \subseteq A_\eta$ and $A_\lambda = \bigcup_{\xi<\lambda} A_\xi$ if λ is a limit ordinal smaller than ϑ.
(1) A is in $\boldsymbol{\Sigma}^1_{2n+1}$.

Proof: Let $\leq$ be a $\boldsymbol{\Delta}^1_{2n+1}$ prewellordering of $\mathcal{N}$ of length ϑ and φ be the associated regular norm. Consider now the following mapping:

$$\begin{aligned} f : \vartheta &\longrightarrow \mathcal{P}(\mathcal{N}) \\ \xi &\longmapsto \{z \mid z \text{ is a } \boldsymbol{\Delta}^1_{2n+1}\text{-code for } A_\xi\}\end{aligned}$$

By a $\mathbf{\Delta}^1_{2n+1}$-code we mean the following: Let W be a $\mathcal{N}$-universal set for $\mathbf{\Sigma}^1_{2n+1}(\mathcal{N})$, let V be a $\mathcal{N}$-universal set for $\mathbf{\Pi}^1_{2n+1}(\mathcal{N})$ and let $\langle\,\rangle$ be a homeomorphism between $\mathcal{N}$ and $\mathcal{N}\times\mathcal{N}$. If $\langle z\rangle = (z_1, z_2)$ and $W_{z_1} = V_{z_2}$ we denote this set by D_z and say z is a code for this $\mathbf{\Delta}^1_{2n+1}$ set.

Let C now be a choice set for f in $\mathbf{\Sigma}^1_{2n+1}$ (that exists by the Coding Lemma 5.2.1). Then

$$x \in A \;\Leftrightarrow\; \exists y \exists z[(y,z) \in C \wedge x \in D_z]$$

"$\Rightarrow$" Let $x \in A$. Then there is an $\xi < \vartheta$ such that $x \in A_\xi$. Since W, V are universal sets there exists a code $z \in \mathcal{N}$ such that $A_\xi = D_z$. So $f(\xi) \neq \emptyset$. Thus there exists an $y \in \mathcal{N}$ and $z \in \mathcal{N}$ such that $\varphi(y) = \xi$ and $z \in f(\xi)$ and $(y,z) \in C$ by definition of the choice set. But $z \in f(\xi)$ implies $D_z = A_\xi$.
"$\Leftarrow$" Now let y, z be such that $(y,z) \in C \wedge x \in D_z$. By definition of a choice set $z \in f(\varphi(y))$ where $\varphi(y)$ is some ordinal less than ϑ. By definition of f, z codes then the set $A_{\varphi(y)}$. So $x \in A_{\varphi(y)}$, in particular, $x \in A$.
This proves that A is a $\mathbf{\Sigma}^1_{2n+1}$ set. q.e.d. (1)

Since A is not in $\mathbf{\Delta}^1_{2n+1}$, we know by Corollary 5.1.16 that A is $\mathbf{\Sigma}^1_{2n+1}$-complete. We get now a contradiction to the prewellordering Theorem 5.1.5 by defining a $\mathbf{\Sigma}^1_{2n+1}$-norm on A. Because then we get a $\mathbf{\Sigma}^1_{2n+1}$ prewellordering for every $\mathbf{\Sigma}^1_{2n+1}$ subset B of $\mathcal{N}$ by transfering the prewellordering of A to B with a continuous function witnessing $B \leq_W A$.
Define the norm ψ on A by

$$\begin{array}{rcl} \psi : A & \longrightarrow & \vartheta \\ x & \longmapsto & \text{the minimal } \xi \text{ such that } x \in A_{\xi+1} \setminus A_\xi \end{array}$$

(2) ψ induces a $\mathbf{\Sigma}^1_{2n+1}$ prewellordering on A.

Proof: We use the characterization of Proposition 5.1.4.

$$\begin{array}{rcl} x \leq^*_\psi y & \Leftrightarrow & \exists \xi < \vartheta\, [x \in A_{\xi+1} \setminus A_\xi \wedge y \notin A_\xi] \\ x <^*_\psi y & \Leftrightarrow & \exists \xi < \vartheta\, [x \in A_{\xi+1} \setminus A_\xi \wedge y \notin A_{\xi+1}] \end{array}$$

Therefore $\leq^*_\psi$ and $<^*_\psi$ are unions of less than ϑ many $\mathbf{\Delta}^1_{2n+1}$ sets. With the same argument as in (1) one shows that $\leq^*_\psi$ and $<^*_\psi$ are in $\mathbf{\Sigma}^1_{2n+1}$. □

We can now prove that the projective ordinals are successor cardinals. We recollect before the results from section 2.6 about the relation between κ-Suslin sets and κ^{++}-Borel sets as well as κ^+-Borel sets. We proved there that a κ-Suslin subset of the Baire space is κ^{++}-Borel and if κ is of cofinality greater than ω then the κ-Suslin set is even a κ^+-Borel set. First we show that the $\boldsymbol{\delta}^1_n$'s are successor cardinals if n is odd.

Theorem 5.2.5 (AD). *For all $n \geq 0$, $\boldsymbol{\delta}^1_{2n+1} = \kappa^+_{2n+1}$ where κ_{2n+1} is a cardinal of cofinality ω.*

Proof. Let $\kappa_{2n+1} < \boldsymbol{\delta}^1_{2n+1}$ be the smallest cardinal such that all $\mathbf{\Sigma}^1_{2n+1}$-sets are κ_{2n+1}-Suslin. (Such a κ_{2n+1} exists, cf. 5.1.12.)
(1) $\kappa^+_{2n+1} = \boldsymbol{\delta}^1_{2n+1}$

Proof: Assume $\kappa_{2n+1}^{++} \leq \boldsymbol{\delta}_{2n+1}^1$. Since every $\boldsymbol{\Sigma}_{2n+1}^1$-set is κ_{2n+1}^1-Suslin, using Theorem 2.5.6 and Theorem 5.2.4 we get $\boldsymbol{\Sigma}_{2n+1}^1 \subseteq \boldsymbol{\mathcal{B}}_{\kappa_{2n+1}^{++}} \subseteq \boldsymbol{\mathcal{B}}_{\boldsymbol{\delta}_{2n+1}^1} = \boldsymbol{\Delta}_{2n+1}^1$, a contradiction. q.e.d.(1)

(2) $\mathrm{cf}(\kappa_{2n+1}) = \omega$

Proof: Assume $\mathrm{cf}(\kappa_{2n+1}) > \omega$. Using theorem 2.5.8 we get $\boldsymbol{\Sigma}_{2n+1}^1 \subseteq \boldsymbol{\mathcal{B}}_{\kappa_{2n+1}^{+}} = \boldsymbol{\mathcal{B}}_{\boldsymbol{\delta}_{2n+1}^1} = \boldsymbol{\Delta}_{2n+1}^1$, a contradiction. q.e.d.(2)

□

An application of Theorem 5.1.14 and the Kunen-Martin Theorem 5.1.8 for the converse proves now that the $\boldsymbol{\delta}_{2n+2}^1$'s are the successors of the $\boldsymbol{\delta}_{2n+1}^1$'s.

Theorem 5.2.6 (AD). *For all $n \geq 0$, $(\boldsymbol{\delta}_{2n+1}^1)^+ = \boldsymbol{\delta}_{2n+2}^1$.*

Proof. "$\leq$" Let φ be a $\boldsymbol{\Pi}_{2n+1}^1$-norm on a $\boldsymbol{\Pi}_{2n+1}^1$-complete set. By theorem 5.1.14 the length of φ is $\boldsymbol{\delta}_{2n+1}^1$. Thus there exists a $\boldsymbol{\Delta}_{2n+2}^1$ prewellordering of $\mathcal{N}$ of length $\boldsymbol{\delta}_{2n+1}^1$(induced by the prewellordering on the $\boldsymbol{\Pi}_{2n+1}^1$-complete set). So we have $\boldsymbol{\delta}_{2n+1}^1 < \boldsymbol{\delta}_{2n+2}^1$ and since the projective ordinals are cardinals we get $(\boldsymbol{\delta}_{2n+1}^1)^+ \leq \boldsymbol{\delta}_{2n+2}^1$

"$\geq$" Let $\leq$ be a prewellordering of $\mathbb{R}$ with $\leq \in \boldsymbol{\Delta}_{2n+2}^1 \subseteq \boldsymbol{\Sigma}_{2n+2}^1$. It follows from theorem 5.1.12 that $\leq$ is $\boldsymbol{\delta}_{2n+1}^1$-Suslin. By the Kunen-Martin theorem we have $|\leq| < (\boldsymbol{\delta}_{2n+1}^1)^+$. Thus $\boldsymbol{\delta}_{2n+1}^1 \leq (\boldsymbol{\delta}_{2n+1}^1)^+$. □

From this last Theorem 5.2.6 it is clear that for all odd integers n we have $\boldsymbol{\delta}_n^1 < \boldsymbol{\delta}_{n+1}^1$. For the even integers this follows from the fact that the projective ordinals are of cofinality greater than ω and Theorem 5.2.5.

Theorem 5.2.7 (AD). *For all $n \geq 1$, $\boldsymbol{\delta}_n^1 < \boldsymbol{\delta}_{n+1}^1$.*

Proof. For all odd integers this follows from Theorem 5.2.6. Let $n = 2m$ be even. Assume $\boldsymbol{\delta}_{2m}^1 = \boldsymbol{\delta}_{2m+1}^1$. Using Theorem 5.2.5 and Theorem 5.2.6 we get $\boldsymbol{\delta}_{2m+1}^1 = \kappa_{2m+1}^+ = \boldsymbol{\delta}_{2m}^1 = (\boldsymbol{\delta}_{2m-1}^1)^+$. Therefore we have $\boldsymbol{\delta}_{2m}^1 = \kappa_{2m+1}$ but this can not be true since κ_{2m+1} has cofinality ω and $\mathrm{cf}(\boldsymbol{\delta}_{2m}^1) > \omega$ by Proposition 5.1.7. □

We already mentioned that we can not prove a result similar to Theorem 5.1.14 for the pointclasses $\boldsymbol{\Sigma}_n^1$. Under **AD** a simple application of the Kunen-Martin Theorem 5.1.8 even proves that all $\boldsymbol{\Sigma}_n^1$ prewellorderings or even $\boldsymbol{\Sigma}_n^1$ wellfounded relations have length less than $\boldsymbol{\delta}_n^1$.

Theorem 5.2.8. *For all $n \geq 1$,*

$$\boldsymbol{\delta}_n^1 = \{\xi \mid \xi \textit{ is the length of a } \boldsymbol{\Sigma}_n^1 \textit{ wellfounded relation }\}.$$

In particular has any $\boldsymbol{\Sigma}_n^1$ wellfonded relation length less than $\boldsymbol{\delta}_n^1$.

Proof. Since every $\boldsymbol{\Delta}_n^1$ prewellordering is a $\boldsymbol{\Sigma}_n^1$ wellfounded relation there is nothing to prove for the "$\leq$"-direction.

So let $\prec$ be a $\boldsymbol{\Sigma}_n^1$ wellfounded relation. For n even $\prec$ is $\boldsymbol{\delta}_{n-1}^1$-Suslin by Theorem 5.1.12 and therefore, by the Kunen-Martin Theorem, the length of $\prec$ is less than $(\boldsymbol{\delta}_{n-1}^1)^+$ and this equals $\boldsymbol{\delta}_n^1$ by Theorem 5.2.6.

For n odd $\prec$ is κ_n-Suslin with $\kappa_n < \boldsymbol{\delta}_n^1$ (again by Theorem 5.1.12) and so $|\prec| < \kappa_n^+ \leq \boldsymbol{\delta}_n^1$ by Theorem 5.1.8 □

We finish this chapter by showing that all projective ordinals are regular cardinals. For the proof we have again to rely on the Coding Lemma 5.2.1.

Theorem 5.2.9 (AD). *For all $n \geq 1$, $\boldsymbol{\delta}^1_n$ is regular.*

Proof. Assume towards a contradiction that there is a cofinal mapping $g : \lambda \longrightarrow \boldsymbol{\delta}^1_n$ for some $\lambda < \boldsymbol{\delta}^1_n$. Let $\leq$ be a $\boldsymbol{\Delta}^1_n$ prewellordering on $\mathcal{N}$ of length λ with associated canonical norm φ. Let $U \subseteq \mathcal{N}^3$ be a universal set for $\boldsymbol{\Sigma}^1_n(\mathcal{N} \times \mathcal{N})$. We will define a $\boldsymbol{\Sigma}^1_n$-wellfounded relation $\prec$ on $\mathcal{N}^3$ of length greater or equal $\boldsymbol{\delta}^1_n$. But this contradicts our last Theorem 5.2.8.

Consider first the following function:

$$\begin{array}{rcl} f : \lambda & \longrightarrow & \mathcal{P}(\mathcal{N}) \\ \xi & \longmapsto & \{x \mid U_x \text{ is a } \boldsymbol{\Sigma}^1_n\text{-wellfounded relation of length } g(\xi)\} \end{array}$$

Note that f is defined since there exists for all $\xi < \lambda$ a $\boldsymbol{\Delta}^1_n$-prewellordering of length $f(\xi)$. Let $C \subseteq \mathcal{N} \times \mathcal{N}$ be a choice set (such a choice set exists Theorem 5.2.1) for f in $\boldsymbol{\Sigma}^1_n$ and define the relation $\prec$ on $\mathcal{N}^3$ by:

$$(x, y, z) \prec (x', y', z') \;\Leftrightarrow\; x = x' \wedge y = y' \wedge (x, y) \in C \wedge (z, z') \in U_y$$

Obviously this relation is $\boldsymbol{\Sigma}^1_n$. And $\prec$ is also wellfounded, because if we assume that there is an infinite descending chain $(x_0, y_0, z_0), (x_1, y_1, z_1), \ldots$ with respect to $\prec$ we have $x := x_0 = x_1 = \ldots, y := y_0 = y_1 = \ldots$ and $z_0, z_1, \ldots$ is an infinite descending chain with respect to U_y, but since $(x, y) \in C$, i.e. $y \in f(\varphi(x))$, we know that U_y is a wellfounded relation and has therefore now infinite descending chains.

For all $\xi < \lambda$ there exists now an embedding

$$\begin{array}{rcl} (\mathcal{N}, U_y) & \longrightarrow & (\mathcal{N}^3, \prec) \\ z & \longmapsto & (x, y, z) \end{array}$$

with $\varphi(x) = \xi$ and $(y, x) \in C$.

Hence we have $g(\xi) = |U_y| \leq |\prec|$ for all $\xi < \lambda$. Since g was a cofinal mapping we have $|\prec| \geq \boldsymbol{\delta}^1_n$ and we arrived at the contradiction. $\square$

Part II

Characterization of projective sets by finer topologies

In this second part we come now to the main objective of this work, the characterization of the projective sets by finer topologies.

In Chapter 1 we will prove the classical results about a characterization of Borel sets in Polish spaces.

Theorem 1. *Let $(X, \mathcal{T})$ be a Polish space. A subset A of X is a Borel set iff there exists a finer topology t on A (i.e., $t \supseteq \mathcal{T}|A$) such that (A, t) is a Polish space.*

This is the prototype of results we will prove here. For the whole Chapter 6 the theory **ZF+DC** will be sufficient. Recall that we proved under these axioms in Proposition 4.2.5 that every Polish space is a second countable, regular, strong Choquet space with the separation property T1. We proceed in Chapter 6 by a characterization of the analytic sets:

Theorem 2. *Let $(X, \mathcal{T})$ be a Polish space. A subset A of X is analytic iff there exists a finer topology t on A such that (A, t) is a second countable, strong Choquet space.*

Trivially, a finer topology t of a Polish topology $\mathcal{T}$ remains Hausdorff, so in particular T1. So the only property we have to drop is that the finer topology is not regular any more.

For classes of a higher level we have to drop additional properties. We start in chapter 2 by proving that we do not get anywhere by dropping the strong Choquet property. So the only property that remains to be considered is the second countable property.

This will lead to the general characterization of projective sets. The idea is to imitate the proofs of Theorem 2.

Crucial for a construction of the finer topology in the analytic case is that $\mathbf{\Sigma}^1_n$ sets are ω-Suslin. If we would have Suslin representations of $\mathbf{\Sigma}^1_n$ sets for $n > 1$ we could pretty much imidiately construct a finer topology for any $\mathbf{\Sigma}^1_n$ set by the same idea as in the case of the analytic sets. By Theorem 5.1.12 the additional axiom **PD** gives us the Suslin representation for each $\mathbf{\Sigma}^1_n$ set. So the first main result in Chapter 7 will be under the theory **ZF + DC + PD** the construction of a finer topology for each $\mathbf{\Sigma}^1_n$ set such that this finer topology has a basis of length less than $\boldsymbol{\delta}^1_n$ and is strong Choquet.

Theorem 3 (ZF+DC+PD). *Let $(X, \mathcal{T})$ be a Polish space. Then there exists for every subset A of X a finer topology t on A which has a basis of length less than $\boldsymbol{\delta}^1_n$ and is strong Choquet.*

The converse can not hold under **ZF+DC+PD** by a result from Donald Martin and John Steel. They proved in [MaSt89] that in a **ZFC** model with infinitely many Woodin cardinals[1] **PD** holds. By the usual methods of forcing[2]

[1]For a definition of Woodin cardinals see for example [Kana97, p. 360]. Woodin proved that the Theory **ZF + AD** is equiconsistent to the theory **ZFC**+ *there are infinitely many Woodin cardinals.* Since we are working here under **ZF + AD** we may as well assume that there are models of **ZFC** with infinitely many Woodin cardinals.

[2]An introduction to forcing is given in [Kune80].

we get a generic extension in which the Continuum Hypothesis is true. Joel David Hamkins and Hugh Woodin showed in [HaWo00] that after small forcing a cardinal κ is Woodin iff it was Woodin in the ground model. So the generic extension of the Martin-Steel Model is a model of **ZFC+CH+PD**.

In this model all projective ordinals have the same cardinality ω_1. So if we construct for some $n \geq 1$ by the above result a finer topology for a subset A in $\mathbf{\Sigma}^1_{n+1}(\mathcal{N}) \setminus \mathbf{\Delta}^1_{n+1}$ (and such a set exists by Proposition 3.1.14) the converse of Theorem 3 in such a Martin Steel Model would imply that $A \in \mathbf{\Sigma}^1_n(\mathcal{N})$ and therefore in $\mathbf{\Delta}^1_{n+1}(\mathcal{N})$. But this contradicts the assumption that A was not in $\mathbf{\Delta}^1_{n+1}(\mathcal{N})$.

So for the converse of Theorem 3 we have to assume that the projective ordinals are all ordinals of different cardinality. This holds under **ZF+DC+AD**, so we could hope to prove the converse under this axioms. Unfortunately we are not able to give such a proof and have to assume the much stronger axiom $\mathbf{AD}_{\mathbb{R}}$ for the following characterization of projective sets by finer topologies:

Theorem 4 (ZF+DC+AD$_{\mathbb{R}}$). *Let $(X, \mathcal{T})$ be a Polish space. A subsets A of X is a $\mathbf{\Sigma}^1_n$ set iff there exists a finer topology t on A such that t has a basis of length less than $\boldsymbol{\delta}^1_n$ and t is strong Choquet.*

We actually need not really the determinacy of games on reals but rather the result that every set of reals has a scale. But, by a result of Woodin, this is, under the assumption **ZF + DC**, equivalent to $\mathbf{AD}_{\mathbb{R}}$. (This result is quoted in [Kana97, Theorem 32.23].)

Chapter 6

Characterization of Borel and analytic sets by finer topologies

6.1 Borel sets

We start now by showing that a finer Polish topology t on a Borel set in a Polish space $(X, \mathcal{T})$ exists . In the first lemma we do this just for closed sets, so we enlarge for a closed set C of X the topology $\mathcal{T}$ to a Polish topology $\mathcal{T}_C$ such that C is open (and closed) with respect to this topology. The relative topology $\mathcal{T}_C|C$ is then a finer Polish topology on C.

Lemma 6.1.1. *Let $(X, \mathcal{T})$ be a Polish space, let $C \subseteq X$ be closed. Let $\mathcal{T}_C$ be the topology generated by $\mathcal{T} \cup \{C\}$, that is, $\mathcal{T} \cup \{U \cap C \mid U \in \mathcal{T}\}$ is a basis of $\mathcal{T}_C$. Then $\mathcal{T}_C$ is a Polish topology, C is open and closed with respect to $\mathcal{T}_C$ and $\boldsymbol{\mathcal{B}}(X, \mathcal{T}_C) = \boldsymbol{\mathcal{B}}(X, \mathcal{T})$.*

Proof. Consider the following mapping:

$$\begin{array}{rcl} \mathrm{id} : (X, \mathcal{T}_C) & \longrightarrow & (C, \mathcal{T}|C) \oplus (X \setminus C, \mathcal{T}|(X \setminus C)) \\ x & \longmapsto & x \end{array}$$

By Theorem 1.14 and Proposition 1.13 the closed set C and the open set $X \setminus C$ are Polish spaces, and by Theorem 1.7 is the sum of this two spaces again a Polish space. To prove that $\mathcal{T}_C$ is a Polish topology it is therefore enough to show that id is an homeomorphism. id is obviously a bijection.
(1) id is continuous.

Proof: Let V be an open set in $C \oplus (X \setminus C)$. By definition of the topological sum $V \cap C$ is open in C with respect to $\mathcal{T}|C$, i.e., there exists an open set $U_1 \in \mathcal{T}$ such that $C \cap V = C \cap U_1$. Then

$$\mathrm{id}^{-1}(V \cap C) = C \cap V = C \cap U_1 \in \mathcal{T}_C.$$

On the other hand there must be a $U_2 \in \mathcal{T}$ such that

$$(X \setminus C) \cap V = (X \setminus C) \cap U_2,$$

and since $X \setminus C$ is open with respect to $\mathcal{T}$ we have

$$\mathrm{id}^{-1}((X \setminus C) \cap V) = (X \setminus C) \cap V = (X \setminus C) \cap U_2 \in \mathcal{T} \subseteq \mathcal{T}_C.$$

Thus

$$\mathrm{id}^{-1}(V) = (C \cap U_1) \cup ((X \setminus C) \cap U_2) \in \mathcal{T}_C.$$

q.e.d. (1)

(2) id is open.

Proof: Let U be an open set with respect to $\mathcal{T}_C$. So

$$U = \bigcup_i U_i \cup \bigcup_j (U_j \cap C)$$

for open sets $U_i, U_j \in \mathcal{T}$. Then

$$\mathrm{id}(U) \cap C = \bigcup_i (U_i \cap C) \cup \bigcup_j (U_j \cap C) = \bigcup_{k=i,j} U_k \cap C$$

is open in C and by the same argument $\mathrm{id}(U) \cap (X \setminus C)$ is open in $X \setminus C$. Thus $\mathrm{id}(U)$ is open. q.e.d (2)

So, $\mathcal{T}_C$ is a Polish topology on X. Now C is open and closed with respect to the new topology by definition of $\mathcal{T}_C$.

It is clear that $\boldsymbol{\mathcal{B}}(X, \mathcal{T}) \subseteq \boldsymbol{\mathcal{B}}(X, \mathcal{T}_C)$. To prove the converse it suffices to show that $C \cap U$ is in $\boldsymbol{\mathcal{B}}(X, \mathcal{T})$ for every $U \in \mathcal{T}$. But every open set U is in $\boldsymbol{\mathcal{B}}(X, \mathcal{T})$ and C is as a complement of an open set in $\boldsymbol{\mathcal{B}}(X, \mathcal{T})$, therefore $C \cap U$ is in $\boldsymbol{\mathcal{B}}(X, \mathcal{T})$ for every open set $U \in \mathcal{T}$. $\square$

The next lemma asserts that if we have a sequence of finer Polish topologies $\mathcal{T}_n$ on a Polish space $(X, \mathcal{T})$, then the topology generated by the union of all the open sets from the $\mathcal{T}_n$ is again a Polish topology on X.

Lemma 6.1.2. *Let $(X, \mathcal{T})$ be a Polish space, $(\mathcal{T}_n)_{n \in \omega}$ be a sequence of Polish topologies on X with $\mathcal{T} \subseteq \mathcal{T}_n$ for all $n \in \omega$. Then $\mathcal{T}_\infty$ is Polish where $\mathcal{T}_\infty$ is the topology generated by $\bigcup_{n \in \omega} \mathcal{T}_n$. If $\mathcal{T}_n \subseteq \boldsymbol{\mathcal{B}}(X, \mathcal{T})$, then $\boldsymbol{\mathcal{B}}(X, \mathcal{T}_\infty) = \boldsymbol{\mathcal{B}}(X, \mathcal{T})$.*

Proof. Let $X_n = (X, \mathcal{T}_n)$ for $n \in \omega$. Consider the map

$$\begin{array}{rccc} \varphi : & X & \longrightarrow & \prod_{n \in \omega} X_n \\ & x & \longmapsto & (x, x, x, \dots) \end{array}$$

where $\prod_{n \in \omega} X_n$ stands for the topological product of the spaces X_n.

(1) $\varphi[X]$ is closed in $\prod_{n \in \omega} X_n$.

Proof: Let $(x_n)_{n \in \omega} \notin \varphi[X]$. Then there exists an $i < \omega$ such that $x_i \neq x_{i+1}$. Let U be an open neighborhood of x_i in X and V be an open neighborhood of x_{i+1} in X with $U \cap V = \emptyset$ (note that X is a Hausdorff space). By our assumption is $U \in \mathcal{T}_i, V \in \mathcal{T}_{i+1}$. Therefore we have $(x_n)_{n \in \omega} \in \prod_n W_n \subseteq \prod_n X_n \setminus \varphi[X]$ with $W_i = U, W_{i+1} = V$ and $W_j = X_j$ for $j \neq i, i+1$. Thus $\varphi[X]$ is closed in $\prod_{n \in \omega} X_n$. q.e.d. (1)

(2) φ is an homeomorphism from $(X, \mathcal{T}_\infty)$ to $\varphi[X]$.

Proof: It is clear that φ is a bijection.

The mapping φ is continuous, since for $U_{i_k} \in \mathcal{T}_{i_k}$, $1 \leq k \leq n$, the preimage of $\prod_{n\in\omega} V_n$ with $V_{i_k} = U_{i_k}$ for $1 \leq k \leq n$, $V_n = X_n$ otherwise is the intersection of the U_{i_k}, so

$$\varphi^{-1}\left[\prod_{n\in\omega} V_n\right] = \bigcap_{j=1}^{n} U_{i_j} \in \mathcal{T}_\infty.$$

φ is open: Let $\{U_i^{(n)} \mid i \in \omega\}$ be a basis for $\mathcal{T}_n$. Then $\{U_i^{(n)} \mid i \in \omega, n \in \omega\}$ is a subbasis for $\mathcal{T}_\infty$. And so we get

$$\varphi\left[\bigcap_{j=1}^{k} U_{i_j}^{(n_j)}\right] = \prod_{n\in\omega} V_n \cap \varphi[X]$$

where $V_n = U_{i_j}^{(n_j)}$ for $n = n_j$, $V_n = X_n$ otherwise. q.e.d. (2)

By (1), (2) and Theorem 1.7 as well as Theorem 1.14 the space $(X, \mathcal{T}_\infty)$ is a Polish space.

The fact about the Borel sets is clear since with $\mathcal{T}_n \subseteq \boldsymbol{\mathcal{B}}(X, \mathcal{T})$ we have $\mathcal{T}_\infty \subseteq \boldsymbol{\mathcal{B}}(X, \mathcal{T})$ and therefore $\boldsymbol{\mathcal{B}}(X, \mathcal{T}_\infty) \subseteq \boldsymbol{\mathcal{B}}(X, \mathcal{T})$. The converse inclusion holds trivially. □

We can now put together this two lemmas to prove the existence of a finer Polish topology on every Borel set in a Polish space.

Theorem 6.1.3. *Let $(X, \mathcal{T})$ be a Polish space, $A \subseteq X$ be a Borel set. Then there exists a Polish topology $\mathcal{T}_A \supseteq \mathcal{T}$ such that A is open and closed with respect to $\mathcal{T}_A$ and $\mathcal{B}(\mathcal{T}_A) = \mathcal{B}(\mathcal{T})$.*

Proof. Let $S = \{A \subseteq X \mid$ there exists a Polish topology $\mathcal{T}_A \supseteq \mathcal{T}$ such that A is open and closed and $\mathcal{B}(\mathcal{T}_A) = \mathcal{B}(\mathcal{T})\}$. It suffices to show that S is closed under complements and countable unions if we show that $\mathcal{T} \subseteq S$ (since then $\boldsymbol{\mathcal{B}}(X, \mathcal{T}) \subseteq S$). But by 6.1.1, all open and all closed sets are in S, so $\mathcal{T} \subseteq S$.

(1) S is closed under complements, since for $A \in S$ the topology $\mathcal{T}_A$ witnesses that $X \setminus A$ is in S as well.

(2) S is also closed under countable unions. Let for this $(A_n)_{n\in\omega}$ be a sequence in S and let $\mathcal{T}_{A_n} = \mathcal{T}_n, \mathcal{T}_\infty$ like in the above Lemma 6.1.2. Then $A = \bigcup_{n\in\omega} A_n$ is open with respect to $\mathcal{T}_\infty$. By 6.1.1 there exists an $\mathcal{T}_A \supseteq \mathcal{T}_\infty \supseteq \mathcal{T}$ Polish such that A is open and closed and $\boldsymbol{\mathcal{B}}(X, \mathcal{T}_A) = \boldsymbol{\mathcal{B}}(X, \mathcal{T}_\infty) = \boldsymbol{\mathcal{B}}(X, \mathcal{T})$. Therefore $\bigcup_{n\in\omega} A_n \in S$. □

The following corollary states now the above Theorem 6.1.3 in the way we need it for our charcterization of the Borel sets.

Corollary 6.1.4. *Let $(X, \mathcal{T})$ be a Polish space. For every Borel set $A \subseteq X$ exists a finer Polish topology t on A.*

Proof. Let $A \subseteq X$ be a Borel set. By Theorem 6.1.3 there exists a finer topology $\mathcal{T}_A$ on X such that $(X, \mathcal{T}_A)$ is a Polish space and A is closed and open with respect to $\mathcal{T}_A$. So the restriction of $\mathcal{T}_A$ to A is a Polish topology on A by Theorem 1.14. □

The following theorem is a nice application of Theorem 6.1.3 that readily implies the proof of the missing part of Proposition 3.1.5 about the different characterizations of analytic sets. It asserts that Borel sets in a Polish space can be seen as continuous images of the Baire space.

Theorem 6.1.5. *Let $(X, \mathcal{T})$ be a Polish space, $A \subseteq X$ a Borel set. Then there exists a closed subset $F \subseteq \mathcal{N}$ and a continuous bijection $f : F \longrightarrow A$. If $A \neq \emptyset$ there is a continuous surjection $G : \mathcal{N} \longrightarrow A$ extending f.*

Proof. Enlarge by Theorem 6.1.3 the topology $\mathcal{T}$ of X to a Polish topology $\mathcal{T}_A$ in which A is closed and open. Then there exists by Theorem 2.2.3 a closed $F \subseteq \mathcal{N}$ and a bijection $f : F \longrightarrow A$ continuous for $\mathcal{T}_A | A$. Since $\mathcal{T} \subseteq \mathcal{T}_A$ we have $f : F \longrightarrow A$ is continuous for $\mathcal{T}$ as well. The second assertion follows from 2.1.7. □

In Proposition 3.1.5 we characterized an analytic set as a continuous image of the baire space as well as a continous image of a Borel set. But we have not proved this yet. The proof is now easy. We first repeat the proposition.

Proposition 6.1.6. *Let $(X, \mathcal{T})$ be a Polish space, $A \subseteq X$. Then the following are equivalent:*

(1) A is the continuous image of a function $f : \mathcal{N} \longrightarrow X$.

(2) $A = \mathrm{proj}_X[C]$ where $C \subseteq X \times \mathcal{N}, C$ closed.

(3) $A = \mathrm{proj}_X[B]$ where $B \subseteq X \times Y$ is a Borel set, Y is a Polish space.

(4) A is the continuous image of a Borel set of a Polish space.

Proof. Comparison with the proof of Proposition 3.1.5 tells us that it remains to show that (4) $\Rightarrow$ (1):
Let $h : Y \longrightarrow X$ be a continuous mapping from a Polish space Y to X and let B be a Borel set in Y such that $h[B] = A$. By Theorem 6.1.5 there exists a continuous surjection $g : \mathcal{N} \longrightarrow B$. Then obviously the mapping $g^* : \mathcal{N} \longrightarrow Y$ defined by $g^*(x) = g(x)$ for $x \in \mathcal{N}$ is a continuous mapping $g^*[\mathcal{N}] = B$. But now the composition $h \circ g^*$ is a continuos function from $\mathcal{N}$ to X such that $h \circ g^*[\mathcal{N}] = A$. □

We proved by Theorem 3.1.11 and Theorem 3.1.14 that the class of analytic sets in an uncountable Polish space is larger than the class of the Borel sets in such a space. The above characterization of analytic sets thus implies that the continuous image of a Borel set is in general not a Borel set. But we will prove now that the image of a Borel set of a continuous injection is again a Borel set. This implies the converse of Theorem 6.1.3. Because given a Polish space $(X, \mathcal{T})$ and a finer topology t on X such that a set A is closed and open with respect to t we can consider the identity mapping between (X,t) and $(X,\mathcal{T})$. This mapping is continuous since t is finer than $\mathcal{T}$ and the image of the Borel set A in (X,t) equals A in $(X,\mathcal{T})$ and is therefore also Borel with respect to $\mathcal{T}$.

To prove that the image of a Borel set under a continuous injection is again Borel we construct now a Lusin scheme (cf. Definition 2.2.1 and Proposition 2.2.2). The construction makes again use of the classical Lusin Separation Theorem 2.5.3 for analytic sets.

For the construction of the upcoming Lusin scheme we need separation for a whole sequence of disjoint analytic sets. We get this by recursion out of the Lusin Separation Theorem 2.5.3 and prove this in the following lemma.

Lemma 6.1.7. *Let $(A_n)_{n\in\omega}$ be a sequence of pairwise disjoint analytic sets in a Polish space. Then there are pairwise disjoint Borel sets B_n with $B_n \supseteq A_n$ for all $n \in \omega$.*

Proof. Let $(A_n)_{n\in\omega}$ be a sequence of disjoint analytic sets. We define now the B_n by recursion.

Let B_0 be the Borel set that separates A_0 from $\bigcup_{n>0} A_n$ (such a set exists by Theorem 2.5.3).

If $B_0,\dots,B_n$ are defined such that B_i separates A_i from $\bigcup_{j<i} B_i \cup \bigcup_{j>i} A_j$ for all $0 \le i \le n$, let B_{n+1} be a Borel set that separates A_{n+1} from the analytic set $\bigcup_{i<n} B_i \cup \bigcup_{j>n+1} A_j$.

By this definition we get pairwise disjoint Borel sets B_n such that $B_n \supseteq A_n$ for all $n \in \omega$. □

Now we can prove that the image of a continuous injection of a Borel set is again a Borel set.

Theorem 6.1.8 (Lusin-Suslin). *Let X, Y be Polish spaces and $f : X \longrightarrow Y$ be continuous. If $A \subseteq X$ is Borel and $f|A$ is injective, then $f[A]$ is Borel.*

Proof. Without loss of generality we can assume $X = \mathcal{N}$ and $A \subseteq \mathcal{N}$ is closed. (By Theorem 2.2.3 there exists a closed $F \subseteq \mathcal{N}$ and a continuous bijection $b : F \longrightarrow A$ that can be extended to a continuous surjection $g : \mathcal{N} \longrightarrow A$. But then $f \circ g : \mathcal{N} \longrightarrow Y$ is continuous, $f \circ g|F$ is injective and $f \circ g[F] = f[A]$.)

Let $\mathcal{T}$ be the topology of Y. Let $B_s = f[A \cap N_s]$ for $s \in \omega^{<\omega}$. Since $f|A$ is injective, $(B_s)_{s\in\omega^{<\omega}}$ is a Lusin scheme where $B_\emptyset = f[A], B_s = \bigcup_{n\in\omega} B_{s^\frown n}$ and B_s is analytic. By Lemma 6.1.7 we find a Lusin scheme B'_s where B'_s is Borel such that $B'_\emptyset = Y, B_s \subseteq B'_s$. We finally define by recursion on length(s) Borel sets B^*_s such that $(B^*_s)_{s\in\omega^{<\omega}}$ is also a Lusin scheme:

$$\begin{aligned} B^*_\emptyset &= Y \\ B^*_{(n_0)} &= B'_{(n_0)} \cap \mathrm{cl}_{\mathcal{T}}(B_{(n_0)}) \\ B^*_{(n_0,\dots,n_k)} &= B'_{(n_0,\dots,n_k)} \cap B^*_{(n_0,\dots,n_{k-1})} \cap \mathrm{cl}_{\mathcal{T}}(B_{(n_0,\dots,n_k)}) \end{aligned}$$

(1) For all $k \in \omega$ we have $B_{(n_0,\dots,n_k)} \subseteq B^*_{(n_0,\dots,n_k)} \subseteq \mathrm{cl}_{\mathcal{T}}(B_{(n_0,\dots,n_k)})$

Proof: By induction on k. The second inclusion is clear by the definition of the B^*_s.

$k=0$: $B_{(n_0)} \subseteq B'_{(n_0)}$ and $B_{(n_0)} \subseteq \mathrm{cl}_{\mathcal{T}}(B_{(n_0)})$, so we are done.
Let us assume the assumption is proved for $k-1, k \geq 1$. Then
$B_{(n_0,\ldots,n_k)} \subseteq B'_{(n_0,\ldots,n_k)}$ by the definition of B'
$B_{(n_0,\ldots,n_k)} \subseteq \mathrm{cl}_{\mathcal{T}}(B_{(n_0,\ldots,n_k)})$ and
$B_{(n_0,\ldots,n_k)} \subseteq B_{(n_0,\ldots,n_{k-1})} \subseteq B^*_{(n_0,\ldots,n_{k-1})}$ by the assumption. q.e.d. (1)

(2)$f[A] = \bigcap_{k\in\omega} \bigcup_{s\in\omega^k} B^*_s$

Proof: Let $x \in f[A]$. Then there exists an $a \in A$ with $f(a) = x$, so $x \in \bigcap_{k\in\omega} B_{a|k}$ and thus $x \in \bigcap_{k\in\omega} B^*_{a|k} \subseteq \bigcap_{k\in\omega} \bigcup_{s\in\omega^\omega} B^*_s$.
For the converse let $x \in \bigcap_{k\in\omega} \bigcup_{s\in\omega^\omega} B^*_s$. Then there is a unique $a \in \mathcal{N}$ such that $x \in \bigcap_{k\in\omega} B^*_{a|k}$ (note that the sets B^*_s form a Lusin scheme). Then also $x \in \bigcap_{k\in\omega} \mathrm{cl}_{\mathcal{T}}(B_{a|k})$. So in particular $B_{a|k} \neq \emptyset$ for all k and thus $A \cap N_{a|k} \neq \emptyset$ for all k, which means $a \in A$ since A is closed. So $f(a) \in \bigcap_{k\in\omega} B_{a|k}$. We claim that $f(a) = x$. Otherwise by the continuity of f there is an open neighborhood $N_{a|k_0}$ of a with $f[N_{a|k_0}] \subseteq U$ where U is open such that $x \notin \mathrm{cl}_{\mathcal{T}}(U)$. But then $x \notin \mathrm{cl}_{\mathcal{T}}(f[N_{a|k_0}]) \supseteq \mathrm{cl}_{\mathcal{T}}(B_{a|k_0})$, a contradiction. q.e.d.(2)

□

With this result we can easily finish our characterization of Borel sets. The converse of Corollary 6.1.4 is no more than a corollary to this last Theorem 6.1.8

Corollary 6.1.9. *Let $(X, \mathcal{T})$ be a Polish space and A a subset of X such that there exists a finer topology t on A such that (A, t) is Polish. Then A is a Borel set in $(X, \mathcal{T})$.*

Proof. Consider the identity mapping from (A, t) into $(X, \mathcal{T})$. Since t is finer than $\mathcal{T}|A$ this mapping is continuous and it is obviously an injection. So by Theorem 6.1.8 A is in $\boldsymbol{\mathcal{B}}(X, \mathcal{T})$. □

We finish this section by stating the characterization of Borel sets by finer topologies as it is witnessed by Corollary 6.1.4 and Corollary 6.1.9.

Theorem 6.1.10. *Let $(X, \mathcal{T})$ be a Polish space. A subset A of X is a Borel set in $(X, \mathcal{T})$ iff there exists a finer toplogy t on A (,i.e., $t \supseteq \mathcal{T}|A$) such that (A, t) is a Polish space.*

6.2 Analytic sets

Our next task is to construct a finer topology for each analytic pointset of a Polish space such that the topology is second countable and strong Choquet. By finer we understand again finer as the restriction of the topology of the Polish space to the analytic subset. It is sufficient to find such finer topologies for the analytic subsets of the Baire space by the following general argument:

Remark 6.2.1. *To prove that for $n \in \omega$ each $\boldsymbol{\Sigma}^1_n$ subset A of a Polish space $(X, \mathcal{T})$ has a topology t such that*

1. $t \supseteq \mathcal{T}|A$

2. *t has a basis of length a cardinal κ*

3. *t is strong Choquet*

it suffices to prove that each $\mathbf{\Sigma}^1_n$ subset of the Baire space $\mathcal{N}$ has a topology with these properties.

Proof. Let A be a $\mathbf{\Sigma}^1_n$ subset of a Polish space $(X, \mathcal{T})$. By Theorem 2.2.3 there exists a closed set C in $\mathcal{N}$ and a continuous bijection $b : C \longrightarrow X$. Since $\mathbf{\Sigma}^1_n$ sets are closed under continuous preimages (Theorem 3.1.10) the set $b^{-1}[A]$ is $\mathbf{\Sigma}^1_n$ in C and also in $\mathcal{N}$. Now the finer topology (or just a basis of it) of this set can be transferred by the bijection b into the set A. It is clear that all the properties of the topology on $b^{-1}[A]$ are then properties of this transferred topology since this is a one-to-one transfer. $\square$

We will proceed by constructing a basis for such a topology of an analytic set A in the Baire space and check then all the properties of the so constructed topology. A basis $\mathcal{B}$ for a topology on a set A is characterized by the properties that the intersection of two members of $\mathcal{B}$ can be written as the union of members of $\mathcal{B}$ and that the union of all members of $\mathcal{B}$ equals the whole set A.

Since analytic sets are closed under finite intersections the set of all analytic subsets of A would be a candidate for such a basis. This may lead to a desired topology but the length of this basis is very large. Under **AC**, this basis has for the most analytic sets the length of the continuum. Therefore such a topology will never lead to a characterization of the analytic sets by finer topologies since we can easily define topologies with this properties for any subset of the Baire space. So we are interested in a basis with a length as short as possible. Since our topology should be finer than the topology of the Baire space the basis must at least have length ω.

By Proposition 3.2.7 we know that each $\mathbf{\Sigma}^1_1$ subset of the Baire space is in $\Sigma^1_1(a)$ for a real a. Consider $a \in \omega^\omega$ such that $A \in \Sigma^1_1(a)$. This set $\Sigma^1_1(a)$ is countable and contains all basic open sets as well as A. Furthermore, $\Sigma^1_1(a)$ is closed under finite intersections by Proposition 3.2.5(a). So a natural candidate for a basis of the finer topology on A would be the set of all subsets of A which are in $\Sigma^1_1(a)$. The only thing to check for this topology is the strong Choquet property.

We will prove below that this topology has indeed the strong Choquet property. This fact makes this topology also interesting for other works in descriptive set theory, see for example [HKeL90]. In the paper of Harrington, Kechris, and Louveau the topology where the Σ^1_1 sets of $\mathcal{N}$ serve as a basis is called **Gandy-Harrington topology**. We consider here a relativized version of it. The proof that the Gandy-Harrington topology is strong Choquet can also be found in [HKeL90].

Crucial for the proof that the Gandy-Harrington topology is strong Choquet is the tree representation from Proposition 3.2.10. Before we start with the proof we remind on a notation connected with trees. In generalization of Definition

2.3.6 we define for a tree T on $\omega\times\omega$ and $(s,t)\in T$ the subtree of the compatible sequences of T by

$$T_{(s,t)}=\{(s',t')\in T\mid (s',t')\subseteq(s,t)\vee(s',t')\supseteq(s,t)\}.$$

It is clear that if T is recursive in some a then $T_{(s,t)}$ is recursive in a.

Theorem 6.2.2. *Let $(X,\mathcal{T})$ be a Polish space. Let $A\in\mathbf{\Sigma}^1_1(X)$. Then there exists a finer topology t on A such that t is second countable and strong Choquet.*

Proof. By Remark 6.2.1 we can assume $X=\mathcal{N}$.

Let $\mathcal{B}_t=\{B\mid B\subseteq A$ and B is $\Sigma^1_1(a)\}$. Since the intersection of two $\Sigma^1_1(a)$ sets is again $\Sigma^1_1(a)$ by Proposition 3.2.5 and since $\bigcup\mathcal{B}_t=A$ $(A\in\mathcal{B}_t)$ the set $\mathcal{B}_t$ serves as a basis for a topology. Let t be the topology on A generated by $\mathcal{B}_t$. It is clear that this topology refines the relative topology of the Baire space on A, since the basis open sets in $\mathcal{N}$ are Σ^0_1 (cf. Example 3.2.3). It is also clear that $\mathcal{B}_t$ is countable since $\Sigma^1_1(a)$ is countable (cf. the discussion below Proposition 3.2.5).

It remains to show that t is strong Choquet. We will describe a winning strategy for II in the strong Choquet game in (A,t):

(i) Suppose I starts by playing (x_0,U_0). Then let $A_0\in\Sigma^1_1(a)$ such that $x_0\in A_0\subseteq U_0$ and let T_0 be a tree recursive in a such that $A_0=p[T_0]$. Since $x_0\in A_0$ there is an $y_0\in\mathcal{N}$ such that $(x_0,y_0)\in T_0$. (y_0 is a witness for x_0 being in $p[T_0]$) Now let $s_0=x_0|1, t^0_0=y_0|1$. The tree $(T_0)_{(s_0,t^0_0)}$ is recursive in a. Let player II play $V_0=p[(T_0)_{(s_0,t^0_o)}]$. This set is $\Sigma^1_1(a)$, $x_0\in V_0$ and $V_0\subseteq A_0\subseteq U_0$.

(ii) Let I's next move be (x_1,U_1) with $x_1\in U_1\subseteq V_0$

- Since $x_1\in V_0$ there exists a witness $y'_0\in\mathcal{N}$ such that $(x_1,y'_0)\in[(T_0)_{(s_0,t^0_0)}]$. Set $s_1=x_1|2, t^0_1=y'_0|2$. Then $s_0\subseteq s_1, t^0_0\subseteq t^0_1$. $(T_0)_{(s_1,t^0_1)}$ is again a tree recursive in a and $x_1\in p[(T_0)_{(s_1,t^0_1)}]\subseteq V_0$.

- Let $A_1\in\Sigma^1_1(a)$ such that $x_1\in A_1\subseteq U_1$ and let T_1 be a tree recursive in a such that $p[T_1]=A_1$. Since $x_1\in A_1$ there is a witness $y_1\in\omega^\omega$ such that $(x_1,y_1)\in[T_1]$. Set $t^1_0=y_1|1$. Then $x_1\in p[(T_1)_{(s_0,t^1_0)}]\subseteq U_1$.

Player II answers this move from player I by playing
$V_1=p[(T_0)_{(s_1,t^0_1)}]\cap p[(T_1)_{(s_0,t^1_0)}]$.

Proceeding this way, when I plays $(x_0,U_0),(x_1,U_1),\ldots$ II produces $V_0,V_1,\ldots$ with $U_0\supseteq V_0\supseteq U_1\supseteq V_1\supseteq\ldots, x_n\in V_n$ and moreover one defines for each n a recursive tree T_n with $x_n\in A_n=p[T_n]\subseteq U_n$ and sequences $s_0\subseteq s_1\subseteq s_2\subseteq\ldots, t^n_0\subseteq t^n_1\subseteq\ldots$ with $(s_k,t^n_k)\in T_n$ such that for each k the finite sequences s_k,t^n_k have length $k+1$ and $V_k=p[(T_0)_{s_k,t^0_k)}]\cap p[(T_1)_{(s_{k-1},t^1_{k-1}}]\cap\ldots\cap p[(T_k)_{(s_0,t^k_o)}]$.

By this construction we get indeed a winning strategy for player II. Let $x=\bigcup_{k\in\omega}s_k\in\omega^\omega$. We claim that $x\in\bigcap A_n=\bigcap V_n$. So player II wins the strong Choquet gamec since the intersection of the open sets he played is not empty. To prove the claim consider $A_n=p[T_n]$. Let $y_n=\bigcup_{k\in\omega}t^n_k$. We have $(s_k,t^n_k)\in T_n$ for all k. Therefore $(x,y_n)\in[T_n]$, so $x\in p[T_n]=A_n$. □

Obviously our version of the Gandy-Harrington topology is Hausdorff since it is a refinement of a Hausdorff topology. We have seen in Proposition 4.2.5 that every Polish space is a second countable, *regular*, strong Choquet space with the Hausdorff property. So we only property we had to drop for our finer topology is the property that the topology is regular. The following remark asserts that the (relativized) Gandy-Harrington topology is indeed not regular (otherwise we would have made a mistake).

Remark 6.2.3. *The (relativized) Gandy-Harrington topology is not regular.*

Proof. Let t be the topology on $\mathcal{N}$ where all $\Sigma_1^1(a)$ sets serve as a basis for an $a \in \omega^\omega$. By Proposition 3.2.6 and Proposition 3.1.14 there exists a $\Pi_1^1(a)$ set P in $\mathcal{N}$ which is not $\mathbf{\Sigma}_1^1$. With respect to the topology t this set P is closed.

Assume towards a contradiction that t is regular. So for every point $x \notin P$ exists a open neighborhood V of x such that the closure of V does not intersect P. Without loss of generality we can choose basic open sets for these open neighborhoods. Since the topology t is second countable this are only countable many sets. The countable union of the closures of these sets is in $\mathbf{\Pi}_1^1$ by Theorem 3.1.10 and equals $\mathcal{N} \setminus P$. Therefore P as a complement of an $\mathbf{\Pi}_1^1$ set is $\mathbf{\Sigma}_1^1$, but this contradicts our choice of P. □

To get now a characterisation of the analytic sets we will prove the converse of Theorem 6.2.2. It will be neccessary for the proof that player II has a winning strategy in the strong Choquet game in which he plays just basic open sets and the diameter of his basic open set in his n-th move is less than $\frac{1}{n+1}$. The following lemma asserts that player II has indeed such a strategy for the considered strong Choquet spaces.

Lemma 6.2.4. *Let $(X, \mathcal{T})$ be a Polish space and $A \subseteq X$. If there exists a topology t on A such that $t \supseteq \mathcal{T}|A$ and (A, t) is a strong Choquet space, then player II has a winning strategy in the strong Choquet space $G_{\mathrm{sCh}}(A, t)$ by which he plays just basic open sets from t with diameter less than $\frac{1}{n+1}$ in his n-th move for all $n \in \omega$.*

Proof. Let σ be a winning strategy for II in the strong Choquet game $G_{\mathrm{sCh}}(A, t)$. We define first a winning strategy σ' out of σ in which the diameter of the sets he has to play in the n-th move is less than $\frac{1}{n+1}$. This strategy σ' is defined in the following way:

$$\sigma' * ((U_o, x_0), V_0, \ldots, (U_n, x_n)) = \sigma * ((U_0, x_0), V_0, \ldots, (U_n \cap B_{\frac{1}{n+1}}(x_n), x_n))$$

This strategy has obviously the desired property and is a winning strategy.

Given such a winning strategy σ' we will now define by recursion a strategy σ'' such that player II always plays t basic open sets. For this we will always consider two runs of the strong Choquet game $G_{\mathrm{sCh}}(A, t)$. One run R' in which II follows σ' and another run R'' in which we define the new strategy σ''. Assume player I starts in the game $G_{\mathrm{sCh}}(A, t)$ by playing (U_0, x_0) and II answers following σ' by an open set V_0. Choose now an t basic open set B_0 such that

$x_0 \in B_0$ and $B_0 \subseteq V_0$. Define $\sigma'' * ((U_0, x_0)) = B_0$. Let (U_1, x_1) be the answer by player I to the t basic open set played by player II. To define σ'' for this sequence consider in the run R' the following first two moves by each player

$$\begin{array}{lcccc} \mathrm{I} & (U_0, x_0) & & (U_1, x_1) & \\ \mathrm{II} & & V_0 & & V_1 \end{array}$$

where player II followed σ'. Choose for strategy σ'' an t basic open set B_1 such that $x_1 \in B_1$ and $B_1 \subseteq V_1$. So in the run R'' the game until now looks as follows:

$$\begin{array}{lcccc} \mathrm{I} & (U_0, x_0) & & (U_1, x_1) & \\ \mathrm{II} & & B_0 & & B_1 \end{array}$$

Proceeding this way we consider now the answer by player I in run R'' as his next move in the run R' and choose an t basic open set in the open set player II plays following his winning strategy σ' in R'. So the strategy σ'' is defined by recursion as follows. If $((U_0, x_o), B_0, (U_1, x_1), B_1, \ldots, (U_n, x_n))$ is a sequence played in R'' then choose an t basic open set B_n such that $x_n \in B_n$ and

$$\begin{aligned} B_n \subseteq \sigma' * \quad & ((U_0, x_0), \sigma' * ((U_0, x_0)), (U_1, x_1), \\ & \sigma' * ((U_0, x_0), \sigma' * ((U_0, x_0), (U_1, x_1))), \ldots, (U_n, x_n)). \end{aligned}$$

Let $\sigma'' * ((U_0, x_o), B_0, (U_1, x_1), B_1, \ldots, (U_n, x_n)) = B_n$.

It is now easy to see that σ'' is indeed a winning strategy for player II. Because player II wins the run R' since he followed his winning strategy σ'. Therefore $\bigcap_{n\in\omega} U_n \neq \emptyset$. But then player II has also won the run R'' since the outcome is also $\bigcap_{n\in\omega} U_n$.

By construction the winning strategy σ'' has now both of the required properties of the lemma. $\square$

We can now prove the converse of Theorem 6.2.2 and finish our characterization of analytic sets by finer topologies.

Theorem 6.2.5. *Let $(X, \mathcal{T})$ be a Polish space, $A \subseteq X$ and there is a topology t on A such that*

- *$t \supseteq \mathcal{T}|A$*
- *t is second countable*
- *t is strong Choquet.*

Then A is a $\boldsymbol{\Sigma}^1_1$ set in X with respect to $\mathcal{T}$.

Proof. Let $\mathcal{B} = \{B_i \mid i \in \omega\}$ be a basis for $(X, \mathcal{T})$, d be a complete compatible metric for this space. Let $\mathcal{C} = \{C_i \mid i \in \omega\}$ be a basis for (A, t). Fix further a winning strategy for player II in the strong Choquet game $G_{\mathrm{sCh}}(A, t)$ which chooses in the n-th move a set $C_i \in \mathcal{C}$ with $\mathrm{diam}(C_i) < \frac{1}{n+1}$.

We start by defining a tree T on $\omega \times (\omega \times A \times \omega)$ in the following way:

$$((i_0, j_0, x_0, k_0), \ldots, (i_{n-1}, j_{n-1}, x_{n-1}, k_{n-1})) \in T \Leftrightarrow$$

(i) $\text{diam}(B_{i_m}) < \frac{1}{m+1}$ for all $m < n$

(ii) $\text{cl}_{\mathcal{T}}(B_{i_{m+1}}) \subseteq B_{i_m}$ for all $m < n$

(iii) $((C_{j_0}, x_0), C_{k_0}, (C_{j_1}, x_1), C_{k_1}, \ldots, (C_{j_{n-1}}, x_{n-1}), C_{k_{n-1}})$ is an initial segment of a run in the strong Choquet game in which II follows his strategy σ

(iv) $B_{i_m} \cap C_{k_m} \neq \emptyset$ for all $m < n$

For a countable subset $Q \subseteq A$ the tree $T^Q = T \cap (w \times (\omega \times Q \times \omega))^{<\omega}$ is a countable tree. By using bijections between ω and Q and between ω^3 and ω we can view this tree as a tree on $\omega \times \omega$.

Then

$$p[T^Q] = \{u \in \omega^\omega \mid \exists v \in (\omega \times Q \times \omega)^\omega \ (u, v) \in [T^Q]\}$$

is a $\mathbf{\Sigma}^1_1$ set by Theorem 3.1.7 and

$$P_{T^Q} = \{x \in X \mid \exists u \in p[T^Q] \wedge x \in \bigcap_m B_{u(m)}\}$$

is a $\mathbf{\Sigma}^1_1$ set in X since

$$x \in P_{T^Q} \Leftrightarrow \exists u (u \in p[T^Q] \wedge \forall m \, x \in B_{u(m)}).$$

We will finish the proof now by constructing a countable Q such that $P_{T^Q} = A$. That P_{T^Q} is a subset of A is easy to see for any countable Q. We start by proving this.

(1) $P_{T^Q} \subseteq A$ for all countable $Q \subseteq A$.

Proof: Let $x \in P_{T^Q}$ witnessed by $x \in \bigcap_m B_{i_m}$ and $C_{j_0}, x_0, C_{k_0}, \ldots$. By construction of the tree and of σ the set $\bigcap_m C_{k_m}$ has exactly one member in A, let us say $\bigcap_m C_{k_m} = \{a\}$. We claim that $x = a$. Assume $x \neq a$. Then $d(x,a) > 0$, say $d(x,a) = \varepsilon$. Let $m \in \omega$ be large enough such that $\frac{1}{m} < \frac{\varepsilon}{2}$. By our definitions above $\text{diam}(B_{i_m}) < \frac{\varepsilon}{2}, \text{diam}(C_{k_m}) < \frac{\varepsilon}{2}$. Since $B_{i_m} \cap C_{k_m} \neq \emptyset$ there exists an $z \in B_{i_m} \cap C_{k_m}$. But now we have

$$d(x,a) \leq d(x,z) + d(z,a) \leq \text{diam}(B_{i_m}) + \text{diam}(C_{k_m}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \epsilon$$

This is a contradiction. q.e.d. (1)

It remains now to find a countable $Q \subseteq A$ such that $A \subseteq P_{T^Q}$. For a proof of $A \subseteq P_{T^Q}$ we have to find for each $x \in A$ an infinite sequence through T^Q that witnesses $x \in P_{T^Q}$. It will turn out that an Q with the following property will be proper to construct such infinite sequences.

(2) There exists a countable $Q \subseteq A$ with the following property:

For every $s \in T^Q$ and every $i, j, k \in \omega$ the following holds. If there is an

$a \in A$ such that $s^\frown(i,j,a,k) \in T$, then there exists an $\overline{a} \in Q$ such that $s^\frown(i,j,\overline{a},k) \in T$.

Proof: Define by recursion on ω some Q_n.
$Q_0 = \emptyset$. Assume now for $n > 0$ a countable Q_n is defined such that for every $s \in T^{Q_{n-1}}(Q_{-1} = \emptyset)$ and for every $i,j,k \in \omega$ we have that if there exists an $a \in A$ such that $s^\frown(i,j,a,k) \in T$ then there is an $\overline{a} \in Q_n$ such that $S^\frown(i,j,\overline{a},k) \in T^{Q_n}$. Since Q_n is countable the tree T^{Q_n} is countable. Consider now for every $s \in T^{Q_n}$ and every $i,j,k \in \omega$ the set $M^n_{s,i,j,k} = \{a \in A \mid s^\frown(i,j,a,k) \in T\}$. There are only countable many sets of these form. Using AC_ω we can choose one point in any of these sets $M^n_{s,i,j,k}$ and call the set of the chosen points Q'_{n+1}. Set $Q_{n+1} = Q_n \cup Q'_{n+1}$. This set is a countable by construction. Finally set $Q = \bigcup_n Q_n$. Q is countable (here we use again AC_ω). Q has now the requested property. A finite sequence $s \in T^Q$ must allready be in some Q_n, so $s \in T^{Q_n}$. If there are $i,j,k \in \omega$ and $a \in A$ such that $s^\frown(i,j,a,k) \in T$ then $a \in M^n_{s,i,j,k}$. So there is an $\overline{a} \in Q_{n+1} \subseteq Q$ such that $s^\frown(i,j,\overline{a},k) \in T^Q$. q.e.d. (2)

Fix such an Q. The property of (2) suffices now to prove that for such an Q our set A equals the $\mathbf{\Sigma}^1_1$ set P_{T^Q}.

(3) $A \subseteq P_{T^Q}$

Proof: Let $x \in A$. We construct by recursion on the length of a sequence an infinite sequence $s = (\mu,\tau,\vec{y},\rho)$ in the tree T^Q such that for $s_n = (\mu_n,\tau_n,\vec{y_n},\rho_n) \in T^Q$ we have $x \in B_{\mu_n(n-1)} \cap C_{\rho_n(n-1)}$.
Let s_0 be the empty sequence. Assume $s_n = (\mu_n,\tau_n,\vec{y_n},\rho_n)$ is given with the above property. The sequence $(\tau_n,\vec{y_n},\rho_n)$ describes the first $n-1$ moves in the strong Choquet game. Let player I's next move be x. By our assumption on s_n the point x is in $C_{\rho_n(n-1)}$. Assume also player I plays an basic open set C_p with $x \in C_p \subseteq C_{\rho_n(n-1)}$ and player II answers by playing an C_q following his strategy σ. Furthermore let B_r be a $\mathcal{T}$-neighborhood of x with diameter less than $\frac{1}{n+1}$. Now $s^*_n = (\mu_n{}^\frown r, \tau_n{}^\frown p, \vec{y_n}{}^\frown x, \rho_n{}^\frown q)$ is a sequence in T. By our choice of Q there exists an $z \in Q$ such that $s_{n+1} = (\mu_n{}^\frown r, \tau_n{}^\frown p, \vec{y_n}{}^\frown z, \rho_n{}^\frown q) \in T^Q$ and $x \in B_{\mu_n{}^\frown r(n)} \cap C_{\rho_n{}^\frown q(n)} = B_r \cap C_q$. This finishes our construction of s.
By construction of s we have $\forall n \quad x \in B_{\mu_n(n-1)} \cap C_{\rho_n(n-1)}$. So in particular $x \in \bigcap_m B_{\mu(m)}$. So s is a witness for x being in P_{T^Q}. q.e.d. (3)

So by (1) and (3) we have $A = P_{T^Q}$. And since P_{T^Q} is $\mathbf{\Sigma}^1_1$ we proved that $A \in \mathbf{\Sigma}^1_1$. □

Chapter 7

Characterization of projective sets by finer topologies

We are now interested in results similar to that of Chapter 6 for higher classes of the projective hierarchy. So we have to consider additional ways of weakening the topological conditions in our space. We mentioned that a finer topology of a Hausdorff space will always remain Hausdorff and we already dropped the regularity. One could ask what happens if the weaken the strong Choquet property to the Choquet property. The next proposition shows that this leads nowhere.

Proposition 7.1. *Let $(X, \mathcal{T})$ be a Polish space and A an arbitrary subset of X. Then there exists a topology t on A such that t is finer than $\mathcal{T}$ and t is regular, second countable and Choquet.*

Proof. Let $\mathcal{B}$ be a basis for $(A, \mathcal{T}|A)$. Let $\mathcal{C}$ be the closure of $\mathcal{B}$ under complements and finite intersections. Pick a point x_C in each nonempty $C \in \mathcal{C}$.
Now let $\mathcal{D} = \mathcal{C} \cup \{\{x_C\} \mid C \in \mathcal{C}\}$ be the countable basis for the topology t. Since the basis consists of clopen sets t is regular. The isolated points are dense in t, so player II wins the Choquet game in his first move by playing one of the x_C's. $\square$

By this Proposition 7.1 the only topological condition that remains to be considered is the second countable condition. As described in the introduction to Part 2 we will now characterize $\boldsymbol{\Sigma}^1_n$ sets by finer topologies with bases of length less than the projective ordinals $\boldsymbol{\delta}^1_n$. In section 7.1 we will under the theory **ZF+DC+PD** construct such a finer strong Choquet topology for each $\boldsymbol{\Sigma}^1_n$ subset of a Polish space. We mentioned that the characterization can not hold in this theory and therefore we will work for the converse under the axioms **ZF+DC+AD**$_{\mathbb{R}}$. The proof of the converse has some technical difficulties. In particular will the length of the basis be coded by certain scales. In section 2 of this chapter we will introduce the notion of a scale coding and notions related to it that will be neccessary for the proof. In section 3 we will finaly finish our characterization of the projective sets by finer topologies.

7.1 Finer topologies on $\mathbf{\Sigma}^1_n$ sets

In this short section we will see that for each $\mathbf{\Sigma}^1_n$ set exists a finer strong Choquet topology with a basis of length less than the projective ordinal $\boldsymbol{\delta}^1_n$. This is a pretty straightforward generalisation of the construction for the finer topology for $\mathbf{\Sigma}^1_1$ sets as we introduced it in the proof of Theorem 6.2.2.

Assume **ZF+DC+PD** for this section. We already constructed a topology for $\mathbf{\Sigma}^1_1$ sets in Theorem 6.2.2. Crucial was the ω-Suslin property of the $\mathbf{\Sigma}^1_1$ subsets of $\mathcal{N}$. Under **PD** we proved in Theorem 5.1.12 that each $\mathbf{\Sigma}^1_n$ subset of the Baire space is κ-Suslin for a cardinal κ which is as an ordinal less than $\boldsymbol{\delta}^1_n$. Comparison with the proof of Theorem 6.2.2 gives us directly an idea how to define now a finer topology for an $\mathbf{\Sigma}^1_1$ set.

Theorem 7.1.1 (PD). *Let $(X, \mathcal{T})$ be a Polish space. For $n \geq 1$ let $A \in \mathbf{\Sigma}^1_n(X)$. Then there exists a finer topology t on A such that t is strong Choquet and has a basis of length a cardinal less than $\boldsymbol{\delta}^1_n$ (less as an ordinal, not necessarily less in cardinality).*

Proof. Let $n \geq 1$. By Remark 6.2.1 we can assume that A is a $\mathbf{\Sigma}^1_n$ subset of the Baire space $\mathcal{N}$. By Theorem 5.1.12 there exists a cardinal κ such that κ is less than $\boldsymbol{\delta}^1_n$ as an ordinal and a tree T on $\omega \times \kappa$ such that $A = p[T]$. Fix such an κ and a tree T.

As in Theorem 6.2.2 we will define a basis for our finer topology t. In Definition 2.3.6 we defined for $s \in T$ the subtree T_s consisting of all sequences compatible with s as

$$T_s = \{t \in T \mid t \text{ is compatible with } s\} = \{t \in T \mid t \subseteq s \vee s \subseteq t\}.$$

Let $\mathcal{A} = \{p[T_s] \mid s \in T\}$. Then $\mathcal{A}$ is a set of cardinality κ. Let $\mathcal{B}$ be the closure of $\mathcal{A}$ and all the $\mathcal{T}$-basic open sets of A under finite intersections, i.e., the intersection of all sets that contain $\mathcal{A}$ and all $\mathcal{T}|A$ basic open sets and are closed under finite intersections. The cardinality of $\mathcal{B}$ is also κ. Let $\mathcal{B}$ serve as a basis for our topology t.

This so defined topology t has now by definition a basis of length less than $\boldsymbol{\delta}^1_n$ and is finer than $\mathcal{T}|A$ since it contains all basic open sets from $\mathcal{T}|A$. So it remains to show that this topology t is strong Choquet. We do this as before in the $\mathbf{\Sigma}^1_1$ case by describing a winning strategy for II.

Assume player I starts by playing (x_0, U_0). Then choose a basic open set of the form $p[T_{r_0^0}] \cap p[T_{r_0^1}] \cap \ldots \cap p[T_{r_0^{m_0}}] \cap N_{u_0}, u_0 \in \omega^{<\omega}$, such that this set is a subset of U_0 and contains the point x_0. We want to make sure our set is not just a $\mathcal{T}$-basic open set, so intersect the basic set with $p[T]$ if necessary. Since $x_0 \in p[T_{r_0^i}], 0 \leq i \leq m_0$, there exists an $\eta_0^i \in \kappa^\omega$ such that $(x_0, \eta_0^i) \in T_{r_0^i}$. Set $s_0 = x_0|1, t_0^{0,i} = \eta_0^i|1$. Then $(x_0, \eta_0^i) \in (T_{r_0^i})_{(s_0, t_0^{0,i})}$ (Of course this operation really only applies here if r_0^i is the empty sequence). Let II play

$$V_0 = p[(T_{r_0^0})_{(s_0, t_0^{0,0})}] \cap \ldots \cap p[(T_{r_0^{m_0}})_{(s_0, t_0^{0,m_0})}] \cap N_{u_0}$$

Let player I's answer be (x_1, U_1) with $x_1 \in U_1 \subseteq V_0$.
Since $x_1 \in V_0$ there exists for $0 \leq i \leq m_0$ an $\overline{\eta}_0^i \in \kappa^\omega$ such that $(x_1, \overline{\eta}_0^i) \in$

$(T_{r_0^i})_{(s_0,t_0^{0,i})}$. Set $s_1 = x_1|2, t_1^{0,i} = \overline{\eta}_0^i|2$. Then $s_0 \subseteq s_1, t_0^{0,i} \subseteq t_1^{0,i}$ and $x_1 \in p[(T_{r_0^i})_{(s_1,t_0^{0,i})}]$.

Choose now again a basic set $p[T_{r_1^0}] \cap p[T_{r_1^1}] \cap \ldots \cap p[T_{r_1^{m_1}}] \cap N_{u_1}$ such that this is a subset of U_1 and contains x_1. Let η_1^i be in κ^ω for $0 \le i \le m_1$ such that $(x_1, \eta_1^i) \in T_{r_1^i}$. Set $t_0^{1,i} = \eta_1^i|1$. Then $x_1 \in p[(T_{r_1^i})_{(s_0,t_0^{1,m_1})}]$. In particular $x_1 \in p[(T_{r_1^0})_{(s_0,t_0^{1,0})}] \cap \ldots \cap p[(T_{r_1^{m_1}})_{(s_0,t_0^{1,m_1})}] \cap N_{u_1}$. Now

$$\begin{aligned} V_1 &= p[(T_{r_0^0})_{(s_1,t_1^{0,1})}] \cap \ldots \cap p[(T_{r_0^{m_0}})_{(s_0,t_0^{0,m_0})}] \cap N_{u_0} \cap p[(T_{r_1^0})_{(s_0,t_0^{1,0})}] \cap \ldots \\ &\cap p[(T_{r_1^{m_1}})_{(s_0,t_0^{1,m_1})}] \cap N_{u_1} \end{aligned}$$

is a legal move for player II.

Proceeding this way, when I plays $(x_0, U_0), (x_1, U_1), \ldots$ II produces $V_0, V_1, \ldots$ with $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \ldots, x_n \in V_n$ and moreover one defines for each n basic sets A_n with $x_n \in A_n = p[T_{r_n^0}] \cap \ldots \cap p[T_{r_n^{m_n}}] \cap N_{u_n} \subseteq U_n$ and sequences $s_0 \subseteq s_1 \subseteq s_2 \ldots, t_o^{n,0} \subseteq t_1^{n,0} \subseteq t_2^{n,0} \ldots, \ldots, t_0^{n,m_n} \subseteq t_1^{n,m_n} \subseteq t_2^{n,m_n} \subseteq \ldots$ with $(s_k, t_k^{n,i}) \in T_{r_n^i}, 0 \le i \le m_n$, such that for each k the finite sequences $s_k, t_k^{n,i}$ have length $k+1$ and

$$\begin{aligned} V_k &= p[(T_{r_0^0})_{(s_k,t_k^{0,0})}] \cap \ldots \cap p[(T_{r_0^{m_0}})] \cap N_{u_0} \\ &\cap p[(T_{r_1^0})_{(s_{k-1},t_{k-1}^{1,0})}] \cap \ldots \cap p[(T_{r_1^{m_1}})_{(s_{k-1},t_{k-1}^{1,m_1})}] \cap N_{u_1} \\ &\cap \ldots \\ &\cap p[(T_{r_k^0})_{(s_0,t_0^{k,0})}] \cap \ldots \cap p[(T_{r_k^{m_k}})_{(s_0,t_0^{k,m_k})}] \cap N_{u_k} \end{aligned}$$

To prove now that this so defined strategy is indeed a winning strategy we have to prove that $\bigcap_n V_n \neq \emptyset$. But this intersection contains a point, namely the point $x = \bigcup_k s_k$.

Claim: $x \in \bigcap_n A_n = \bigcap_n V_n$

Proof: Consider $A_n = p[T_{r_n^0}] \cap \ldots \cap p[T_{r_n^{m_n}}] \cap N_{u_n}$. Let $0 \le i \le m_n$. Let $\eta_n^i = \bigcup_k t_k^{n,i}$. We have $(s_k, t_k^{n,i}) \in T_{r_n^i}$ for all k. Therefore $(x, \eta_n^i) \in [T_{r_n^i}]$ and thus $x \in p[T_{r_n^i}]$. It remains to show that $x \in N_{u_n}$. For this let S be the full tree on ω^ω. For a sequence $s \in \omega^{<\omega}$ we have $[S_s] = N_s$ our basic set in the Baire space. It suffices to show now that $s_k \in S_{u_n}$ for every k. Note that $s_n = x_n|n+1$. Since $x_n \in N_{u_n}$ we have s_n and u_n compatible, therefore $s_k \in S_{u_n}$ for $k \le n$. Assume towards a contradiction that for $k > n$ the sequence s_k is not in S_{u_n}. This implies $x_k \notin N_{u_n}$. In particular, $x_k \notin V_n$, but $x_k \in V_k \subseteq V_n$, a contradiction. q.e.d. Claim

□

Obviously this proof applies for $\mathbf{\Sigma}^1_1$ sets without assuming **PD**. We introduced the Gandy-Harrington topology for $\mathbf{\Sigma}^1_1$ sets in Theorem 6.2.2 since this topology is somewhat more natural. The rest of this paper is devoted to the proof of the converse of Theorem 7.1.1.

7.2 Reliable ordinals

It will be necessary for the proof of the converse of Theorem 7.1.1 that we can code the length of our basis for the finer topology not only by some norm, but by a scale. We will define the notion of such a **scale-coding** next. An ordinal that admits a scale-coding on some subset of the Baire space will be called **reliable**.

Definition 7.2.1. (i) A scale $(\varphi_i)_{i\in\omega}$ on some subset $W \subseteq \mathcal{N}$ is called a **scale-coding** for some ordinal λ if φ_0 is a surjection on λ and the length of the other norms φ_n are less or equal to λ for all $n \geq 1$.
(ii) An ordinal λ is called **reliable** if λ admits a scale-coding. For some point-class $\mathbf{\Gamma}$ we call λ **$\mathbf{\Gamma}$-reliable** if it admits a scale-coding by some $\mathbf{\Gamma}$ scale on a set in $\mathbf{\Gamma}$.

We already mentioned in the Introduction to Part 2 that the characterization of the projective sets by topologies of length less than the projective ordinals can only hold, if the projective ordinals have distinguished cardinality. This is true under **AD** as we proved in Theorem 5.2.5 and Theorem 5.2.6 together with Theorem 5.2.7. In particular these results assert that the projective ordinals are successor cardinals. In view of Theorem 7.1.1 and Theorem 4 we have to consider the predecessors of the projective ordinals since this will be the lengths of the bases. So there should be reliable ordinals with cardinality of these cardinals. Our proof of Theorem 4 requires that such ordinals have to be even $\mathbf{\Delta}^1_n$-reliable. We will prove now that such ordinals indeed exist.

Proposition 7.2.2 (PD). *$\boldsymbol{\delta}^1_{2n+1}$ is $\mathbf{\Delta}^1_{2n+2}$-reliable for all $n \geq 0$.*

Proof. Let $W \subseteq \mathcal{N}$ be a complete $\mathbf{\Pi}^1_{2n+1}$ set and $(\varphi_i)_{i\in\omega}$ a regular $\mathbf{\Pi}^1_{2n+1}$ scale on W (this exists by the second periodicity Theorem). By Theorem 5.1.14 each φ_i has length $\boldsymbol{\delta}^1_{2n+1}$. Therefore φ_0 is a surjection on $\boldsymbol{\delta}^1_{2n+1}$. Since $W \in \mathbf{\Delta}^1_{2n+2}$ and $(\varphi_i)_{i\in\omega}$ is obviously a $\mathbf{\Delta}^1_{2n+2}$-scale we are done. □

So the odd projective ordinals are reliable in the needed sense. We can not prove that the predecessor of any odd projective ordinal $\boldsymbol{\delta}^1_{2n+1}$ is $\mathbf{\Delta}^1_{2n+1}$ reliable. But under the assumption of **AD** the set of all $\mathbf{\Delta}^1_{2n+1}$-reliable ordinals less than $\boldsymbol{\delta}^1_{2n+1}$ is unbounded. So there exists an ordinal with the cardinality of the predecessor of $\boldsymbol{\delta}^1_{2n+1}$ that is $\mathbf{\Delta}^1_{2n+1}$-reliable and this will be sufficient for our purpose.

Proposition 7.2.3 (AD). *The set of $\mathbf{\Delta}^1_{2n+1}$-reliable ordinals less than $\boldsymbol{\delta}^1_{2n+1}$ is unbounded in $\boldsymbol{\delta}^1_{2n+1}$ for all $n \geq 0$.*

Proof. Let $\xi_0 < \boldsymbol{\delta}^1_{2n+1}$. Let $(\varphi_i)_{i\in\omega}$ be a regular $\mathbf{\Pi}^1_{2n+1}$-scale on a complete $\mathbf{\Pi}^1_{2n+1}$-set $P \subseteq \mathcal{N}$.
Set $P_{\xi_0} = \{x \in \mathcal{N} \mid \forall i\, \varphi_i(x) \leq \xi_0\} = \bigcap_i P_i^{\xi_0} \in \mathbf{\Delta}^1_{2n+1}$,
where $P_i^{\xi_0} = \{x \in \mathcal{N} \mid \varphi_i(x) \leq \xi_0\}$ and this set is in $\mathbf{\Delta}^1_{2n+1}$ by Lemma 5.1.3.
(1) $(\varphi_i|P_{\xi_0})_{i\in\omega}$ is a $\mathbf{\Delta}^1_{2n+1}$-scale on P_{ξ_0}.
Proof: Let $(x_k)_{k\in\omega}$ be a sequence in P_{ξ_0} converging against some point $x \in \mathcal{N}$

and $\varphi_i(x_k)$ converges against some $\lambda_i < \xi_0$ for all $i \in \omega$. Then $x \in W$ and $\varphi_i(x) \le \lambda_i < \xi_0$ for all $i \in \omega$. Therefore $x \in P_i^{\xi_0}$ for all i, thus $x \in P_{\xi_0}$. Since $P_{\xi_0} \in \mathbf{\Delta}^1_{2n+1}$ we know from Theorem 5.1.2 that $(\varphi_i|P_{\xi_0})_{i\in\omega}$ is a $\mathbf{\Delta}^1_{2n+1}$-scale. q.e.d.(1)
Define now by recursion an increasing sequence of ξ_i in the following way:
Let $\xi_i < \boldsymbol{\delta}^1_{2n+1}$ be given. For $\alpha < \xi_i$ such that $\alpha \notin \operatorname{ran}(\varphi_0|P_{\xi_i})$ let ξ_i^α be minimal with the property that there exists an $x \in P_{\xi_i^\alpha}$ with $\varphi_0(x) = \alpha$. Let $\xi_{i+1} = \sup\{\xi_i^\alpha \mid \alpha < \xi_i \wedge \alpha \notin \operatorname{ran}(\varphi_0|P_{\xi_i})\}$. Since $\boldsymbol{\delta}^1_{2n+1}$ is regular we have $\xi_{i+1} < \boldsymbol{\delta}^1_{2n+1}$.
Let $\xi_\omega = \sup\{\xi_i \mid i \in \omega\}$. Then $\xi_\omega < \boldsymbol{\delta}^1_{2n+1}$ because of the regularity of $\boldsymbol{\delta}^1_{2n+1}$.
As in (1) we have that $(\varphi_i|P_{\xi_\omega})_{i\in\omega}$ is a $\mathbf{\Delta}^1_{2n+1}$-scale on P_{ξ_ω}. Furthermore $ran(\varphi_0|P_{\xi_\omega}) = \xi_\omega$, since for $\alpha < \xi_\omega$ there exists an $i \in \omega$ such that $\alpha < \xi_i$. If there is an $x \in P_{\xi_i} \subseteq P_{\xi_\omega}$ such that $\varphi_0(x) = \alpha$ we are done. Otherwise there exists by construction of the ξ_i some $x \in P_{\xi_{i+1}} \subseteq P_{\xi_\omega}$ with $\varphi_0(x) = \alpha$. □

Corollary 7.2.4. *There exists a $\mathbf{\Delta}^1_{2n+1}$-reliable ordinal less than $\boldsymbol{\delta}^1_{2n+1}$ of cardinality the predecessor of $\boldsymbol{\delta}^1_{2n+1}$.*

The following notions and results in connection with reliable ordinals will also be necessary for the proof of Theorem 4.

We fix now for a reliable ordinal λ a scale-coding $(\varphi_i)_{i\in\omega}$ on $W \subseteq \mathcal{N}$.

Definition 7.2.5. Let S be a countable subset of λ.
Let ξ be in S. The set S is called **ξ-honest** if there exists an $w \in W$ such that $\varphi_0(w) = \xi$ and $\varphi_n(w) \in S$ for all $n \in \omega$.
S is called **honest** if S is ξ-honest for all ξ in S.

The following Theorem we will be crucial in the proof of our main Theorem 4. We remind here on the bijection between ω^ω and $(\omega^\omega)^\omega$ we used in the proof of Lemma 2.1.2:
Let $\langle\, , \rangle : \omega \times \omega \longrightarrow \omega$ be a bijection such that $\langle i, 0\rangle \le i$ and $\langle i, k\rangle < \langle i, l\rangle$ for all i and $k < l$. Then define

$$\begin{array}{rcl} (\ ) : \omega^\omega & \longrightarrow & (\omega^\omega)^\omega \\ x & \longmapsto & ((x)_i)_{i\in\omega} \end{array}$$

where $(x)_i(m) = x(\langle i, m\rangle)$.

One last notion is necessary. A function $F : X^\omega \longrightarrow Y^\omega$ is called a **Lipschitz function** if it is already defined on the initial segments of each element of X^ω, i.e., the function F is also defined on $X^{<\omega}$ and forall $x \in X$ and forall $n \in \omega$ we have $F(x|n) = F(x)|n$.

Theorem 7.2.6. *Let $(\varphi_i)_i$ be a scale-coding on $W \subseteq \mathcal{N}$ for λ.*
(i) There exists a Lipschitz function $F : \lambda^\omega \longrightarrow \mathcal{N}$ such that range$(F) \subseteq W$ and for $f \in \lambda^\omega$ the following holds:

$$\{f(0), f(1), \ldots\} \textit{ is } f(0) - \textit{honest } \Rightarrow \varphi_0(F(f)) = f(0)$$

(ii) There exists a Lipschitz function $F : \lambda^\omega \longrightarrow \mathcal{N}$ such that range$(F) \subseteq \{x \mid \forall n (x)_n \in W\}$ and for $f \in \lambda^\omega$ the following holds:

$$\{f(0), f(1), \ldots\} \textit{ is honest } \Rightarrow \forall n\ \varphi_0((F(f))_n) = f(n)$$

Proof. (i) Let T be the tree on $\omega \times \lambda$ associated to the scale $(\varphi_i)_i$ on W, i.e.

$$\begin{aligned} & ((k_o, \ldots, k_n), (\xi_0, \ldots, \xi_n)) \in T \\ \Leftrightarrow \quad & \exists x \in W \text{ such that } x(i) = k_i \text{ and } \varphi_i(x) = \xi_i \text{ for } i \leq n \end{aligned}$$

Consider now the following game on λ

$$\begin{array}{llllll} \mathrm{I} & f(0) & & f(1) & & \ldots \\ \mathrm{II} & & w(0), h(0) & & w(1), h(1) & \ldots \end{array}$$

where $f(i), h(i) \in \lambda$ and $w(i) \in \omega$ for all $i \in \omega$.
II wins the game if

$$(w, h) \in [T_{f(0)}] \wedge \forall v[v \in p[T_{f(0)} | \{f(0), f(1), \ldots\}] \Rightarrow \varphi_0(v) \leq \varphi_0(w)]$$

where $T_{f(0)}$ is the subtree of T where each branch s starts with $(n_0, f(0))$ for some $n_0 \in \omega$ and $T_{f(0)} | \{f(0), f(1), \ldots\}$ is the subtree of $T_{f(0)}$ where for a sequence $s = (r, t)$ of length n we have $t(i) \in \{f(0), f(1), \ldots\}$ for all $i < n$.

Claim: II has a winning strategy for this game

Proof: Let I start by playing $f(0)$. Then II chooses an $w \in W$ such that $\varphi_0(w) = f(0)$ and plays on his n-th move $w(n), h(n) = \varphi_n(w)$.
Then we have obviously $(w, h) \in [T_{f(0)}]$. If $v \in p[T_{f(0)} | \{f(0), f(1), \ldots\}]$ then there exists by construction of the tree a sequence (y_i) converging against v such that $\varphi_0(y_i) = f(0)$ for all i. (cf the proof of Theorem 2.3.7, "$\supseteq$") Since $(\varphi_i)_i$ is a scale we have $\varphi_0(v) \leq f(0) = \varphi_0(w)$. q.e.d. Claim

Let τ be a winning strategy for II. Define now the function F by

$$F(f) = w \Leftrightarrow f, w, h \text{ is a run in the game where II follows his strategy } \tau$$

This function has the required properties. Let $F(f) = w$. Since II played w following his strategy τ this means $w \in p[T_{f(0)}] \subseteq p[T] = W$, thus $range(F) \subseteq W$. Let now $\{f(0), f(1), \ldots\}$ be $f(0)$-honest. We have to show $\varphi_0(F(f)) = f(0)$. Since the set $\{f(0), f(1), \ldots\}$ is $f(0)$-honest there exists an $x \in W$ such that $\varphi_0(x) = f(0)$ and $\varphi_i(x) = f(k)$ for an $k \in \omega$. This x is in $p[T_{f(0)} | \{f(0), f(1), \ldots\}]$. Since τ is a winning strategy we have $f(0) = \varphi_0(x) \leq \varphi_0(w) = \varphi_0(F(f))$. On the other hand one shows as in (1) that $\varphi_0(w) \leq f(0)$. This proves everything.

(ii) The idea is to transfer the tree from (i) by the function ( ) to annother tree and then imitate the proof of (i). So we define a tree T on $\omega \times \lambda$ by

$$\begin{aligned} & ((k_0, \ldots, k_n), (\xi_0, \ldots, \xi_n)) \in T \\ \Leftrightarrow \quad & \exists x \in \mathcal{N} \text{ such that } \forall i (x)_i \in W \\ & \text{and if } \langle i, j \rangle = m \text{ then } k_m = (x)_i(j) \text{ and } \xi_m = \varphi_j((x)_i) \end{aligned}$$

For $f \in \lambda^\omega$ let

$$T_f = \{(t, r) \in T \mid \text{ for } l_i = \langle i, 0 \rangle < length(t, r) : r(l_i) = f(i)\}$$

and

$$T_f^i = \{(t,r) \in T_f \mid r(\langle i,k\rangle) \in \{f(0), f(1), \dots\} \forall k \in \omega\}.$$

Consider now the the following game on λ

$$\begin{array}{llllll} \mathrm{I} & f(0) & & f(1) & & \dots \\ \mathrm{II} & & w(0), h(0) & & w(1), h(1) & \dots \end{array}$$

where $f(i), h(i) \in \lambda$ and $w(i) \in \omega$ for all $i \in \omega$.

$$\begin{aligned} \text{II wins} \quad \Leftrightarrow \quad & (x,h) \in [T_f] \\ & \wedge \forall v \forall i [v \in p[T_f^i] \Rightarrow \varphi_0((v)_i) \leq \varphi_0((x)_i)] \end{aligned}$$

Now we can do the same as in part (i).

Claim: II has a winning strategy

Proof: We define again a winning strategy for player II. For every $f(i)$ player I plays II chooses an $w_i \in W$ such that $\varphi_0(w_i) = f(i)$ (since φ_0 is a surjection onto λ such a w_i exists). Player II wins by playing $x(\langle i,j\rangle) = w_i(j)$ and $h(\langle i,j\rangle) = \varphi_j(w_i)$. Since $\langle i,j\rangle > i$ player II has at any time allready the necessary information.

Now $(x,h) \in [T_f]$ since $(x)_i = w_i \in W$ and for $\langle i,j\rangle = m$ we have $x(m) = (x)_i(j) = w_i(j)$ and $h(m) = \varphi_j((x)_i)$. Furthermore $h(\langle i,0\rangle) = \varphi_0(w_i) = f(i)$.

Let now $v \in \mathcal{N}, i \in \omega$ and $v \in p[T_f^i]$. That means

$$\begin{aligned} & \exists u \in \lambda^\omega (v,u) \in [T_f^i] \\ \Leftrightarrow \quad & \exists u \in \lambda^\omega \forall l \in \omega (v|l, u|l) \in T_f^i \\ \Leftrightarrow \quad & \exists u \in \lambda^\omega \forall l \in \omega \exists y_l \in \mathcal{N} \text{ such that } \forall n (y_l)_n \in W \\ & \text{ and if } \langle n,j\rangle = m \text{ then } v(m) = (y_l)_n(j) \text{ and } u(m) = \varphi_j((y_l)_n) \end{aligned}$$

Thus in particular the sequence $(y_l)_i$ converges (in l) against $(v)_i$ and $\varphi_0((y_l)_i) = f(i)$ for all l. Since $(\varphi_i)_i$ is a scale we have $(v)_i \in W$ and $\varphi_0((v)_i) \leq f(i) = \varphi_0(w)$. q.e.d. Claim

Fix now a winning strategy τ for player II and define as above $F(f) = x$ if player II answers to I's play f by x, h. Note that we used in (1) to show that $(v)_i \in W$ and $\varphi_0((v)_i) \leq f(i)$ just the fact that $v \in p[T_f]$. So we can prove as in (1) that $(F(f))_i = (x)_i$ is in W for all i and $\varphi_0((F(f))_i) \leq f(i)$ since τ being a winning strategy for II implies $x \in p[T_f]$.

Let now $S = \{f(0), f(1), \dots\}$ be honest. Let $i \in \omega$. Since S is $f(k)$-honest for $k \in \omega$ there exists an w_k with $\varphi_0(w_k) = f(k)$ and $\varphi_l(w_k) = f(m)$ for some $m \in \omega$ and all $l \in \omega$. Let $v \in \mathcal{N}$ be defined by $v(\langle k,n\rangle) = w_k(n)$. This $v \in p[T_f^i]$. Since τ is a winning strategy we have $f(i) = \varphi_0((v)_i) \leq \varphi_0((x)_i) = \varphi_0((F(f))_i)$. Thus $\varphi_0((F(f))_i) = f(i)$.

□

Now we can finally start with the proof of Theorem 4.

7.3 Proof of Theorem 4

To prove Theorem 4 and get the characterization of projective sets by finer topologies we have to prove the following theorem:

Theorem 7.3.1 ($\mathbf{AD}_{\mathbb{R}}$). *Let $(X,\mathcal{T})$ be a Polish space and let $A \subseteq X$. If there exists a finer topology t on A such that t is strong Choquet and has a basis of length less than $\boldsymbol{\delta}^1_n$, then $A \in \boldsymbol{\Sigma}^1_n$.*

Proof. We work now under **ZF+DC+** $\mathbf{AD}_{\mathbb{R}}$.
Fix the following objects:

- Let $X, \mathcal{T}, A, t, n$ be given.
- Let $\mathcal{B} = \{B_i \mid i \in \omega\}$ be a basis for $(X,\mathcal{T})$.
- Let d be a complete metric on X which induces the topology on $(X,\mathcal{T})$.
- Let κ be a $\boldsymbol{\Delta}^1_n$-reliable ordinal with cardinality the predecessor of $\boldsymbol{\delta}^1_n$.[1]
- Let $\mathcal{C} = \{C_\xi \mid \xi < \kappa\}$ be a basis for (A,t).
- Let σ be a winning strategy for player II in the strong Choquet game $G_{\mathrm{sCh}}(A,t)$ which chooses basic sets from $\mathcal{C}$ of diameter less or equal $\frac{1}{i}$ in the i-th move.
- Let $W \subseteq \mathcal{N}$ be a $\boldsymbol{\Delta}^1_n$ set and $\varphi_i : W \longrightarrow \kappa$ be a $\boldsymbol{\Delta}^1_n$-scale on W with $ran(\varphi_0) = \kappa$.
- Let $F : \kappa^\omega \longrightarrow \omega^\omega$ be a Lipschitz function with the properties from Theorem 7.2.6.

We start the proof by defining a game G.

A game G

We define G in the following way:

$$\begin{array}{lccccc} \mathrm{I} & \alpha_0,\xi_0,x_0 & & \alpha_1,\xi_1,x_1 & & \ldots \\ \mathrm{II} & & \beta_0,\eta_0 & & \beta_1,\eta_1 & \ldots \end{array}$$

where $\alpha_i, \xi_i, \beta_i, \eta_i \in \kappa$ and $x_i \in A$ for $i \in \omega$.

The players must obey the following **rule** $\mathcal{R}$:
The players must play such that the finite initial segments of

$$\begin{array}{lccccc} \mathrm{I} & (C_{\xi_0},x_0) & & (C_{\xi_1},x_1) & & \ldots \\ \mathrm{II} & & C_{\eta_0} & & C_{\eta_1} & \ldots \end{array} \qquad (*)$$

[1] Note that we just proved for n even that κ is a cardinal. We do not know this for n odd. Nevertheless we denote contrary to our usual notation this ordinal by κ.

are legal moves in the strong Choquet game for (A,t) with $\mathrm{diam}(C_{\eta_i}) < \frac{1}{i}$. The first player to fail loses.

The **payoff set** is the following:
Let us assume no player violates the rule. Let $f = (\alpha_0, \xi_0, \beta_0, \eta_0, \alpha_1, \xi_1, \beta_1, \eta_1, \dots) \in \kappa^\omega$. Set $\hat{\eta}_i(f) = \varphi_0((F(f))_{4i+3})$. Let finally

$$P = \{f \in \kappa^\omega \mid \exists x \in A \forall m \in \omega x \in C_{\hat{\eta}_m(f)}\}.$$

II wins the run of the game $\Leftrightarrow f \in P$.

The following remark turns out to be very important.

Remark 7.3.2. The definition of F (see Theorem 7.2.6(ii) for the properties of F) implies that if f is honest, then

$$\hat{\eta}_i(f) = \varphi_0((F(f))_{4i+3} = f(4i+3) = \eta_i.$$

Hence if f is honest, then $f \in P \Leftrightarrow$ II wins the round of the strong Choquet game $(*)$.

We proceed now with the key lemma of this proof. We will show that player II has a winning quasi-strategy in this game G independent of the points from A played by player I. This lemma is the only part of the proof that requires the axiom $\mathbf{AD}_{\mathbb{R}}$.

Lemma 7.3.3. *Player II has a winning quasi-strategy τ independent of the points x_i played by I in the following sense:*
Let

$$\begin{aligned} s &= ((\alpha_0, \xi_0, x_0), (\beta_0, \eta_0), \dots, (\alpha_{m-1}, \xi_{m-1}, x_{m-1})) \text{ and} \\ s' &= ((\alpha_0, \xi_0, x'_0), (\beta_0, \eta_0), \dots, (\alpha_{m-1}, \xi_{m-1}, x'_{m-1})) \end{aligned}$$

be two positions in G which are legal and consistent with τ. Let $(\beta_{m-1}, \eta_{m-1}) \in \kappa \times \kappa$. If $x_{m-1}, x'_{m-1} \in C_{\eta_{m-1}}$ and $s^\frown(\beta_{m-1}, \eta_{m-1})$ is consistent with τ, then $s'^\frown(\beta_{m-1}, \eta_{m-1})$ is also consistent with τ.

Proof. We will prove this lemma in several steps. Our first aim is to see that our payoff set $P \subseteq \kappa^\omega$ is λ-Suslin[2] for some ordinal λ. Since we are working under $\mathbf{AD}_{\mathbb{R}}$ (and this is equivalent to the fact that every subset of the reals admits scales) we will prove first that P can be seen as the preimage of a subset R of the reals under the function F. The subset R is then λ-Suslin for some ordinal λ by $\mathbf{AD}_{\mathbb{R}}$ and we can apply the Lipschitz function F to transfer a tree on $\omega \times \lambda$ that witnesses the Suslin representation for R to a tree on $\kappa \times \lambda$ that witnesses that P is λ-Suslin.

(1)There exists a subset $R \subseteq \mathcal{N}$ such that $F^{-1}[R] = P$

[2]Note that we defined being λ-Suslin just for subsets of ω^ω but the generalization for arbitrary sets of the form X^ω for any set X is straightforward.

Proof: Obviously the only candidate for such an R is $F[P]$. So we have to show that $F^{-1}[F[P]] = P$.
"$\supseteq$" clear
"$\subseteq$" Let $g \in F^{-1}[F[P]]$. Then there is an $f \in P$ such that $F(f) = F(g)$. This implies that for all $i \in \omega$ we have $\hat{\eta}_i(f) = \varphi_0((F(f))_{4i+3}) = \varphi_0((F(g))_{4i+3}) = \hat{\eta}_i(g)$. Since $f \in P$ there exists an $x \in A$ such that $x \in \bigcap_{i\in\omega} C_{\hat{\eta}_i(f)}$. But $\bigcap_{i\in\omega} C_{\hat{\eta}_i(f)} = \bigcap_{i\in\omega} C_{\hat{\eta}_i(g)}$. Therefore $g \in P$ by definition of P. q.e.d.(1)

$AD_{\mathbb{R}}$ implies that every set of reals admits a scale. So in particular there is a scale for R and by Theorem 2.3.7 R is Suslin. Let T_R be a tree on $\omega \times \lambda$ for some ordinal λ such that $R = p[T_R]$.

Using the fact that F is a Lipschitz function we get a tree representation for P in the following way. Let a tree T^* on $\kappa \times \lambda$ be given by

$$\begin{aligned} &((\xi_0, \dots, \xi_{n-1}), (\zeta_0, \dots, \zeta_{n-1})) \in T^* \\ \Leftrightarrow\ &(F(\xi_0, \dots, \xi_{n-1}), (\zeta_0, \dots, \zeta_{n-1})) \in T_R \end{aligned}$$

(2) $p[T^*] = P$

Proof:

$$\begin{aligned} & \overline{\xi} \in p[T^*] \\ \Leftrightarrow\quad & \exists \overline{\zeta} \in \lambda^\omega\ (\overline{\xi}, \overline{\zeta}) \in [T^*] \\ \Leftrightarrow\quad & \exists \overline{\zeta} \in \lambda^\omega \forall k((\xi_0, \dots, \xi_k), (\zeta_0, \dots, \zeta_k)) \in T^* \\ \Leftrightarrow\quad & \exists \overline{\zeta} \forall k(F(\xi_0, \dots, \xi_k), \zeta_0, \dots, \zeta_k)) \in T_R \\ \Leftrightarrow\quad & \exists \overline{\zeta} \in \lambda^\omega (F(\overline{\xi}), \overline{\zeta}) \in [T_R] \\ \Leftrightarrow\quad & F(\overline{\xi}) \in p[T_R] = R \\ \Leftrightarrow\quad & \overline{\xi} \in F^{-1}[R] \\ \Leftrightarrow\quad & \overline{\xi} \in P \end{aligned}$$

q.e.d. (2)

The Suslin representation of the payoff set P does not suffice to prove the determinacy of the game G, but there is a technique of homogenizing a tree T^* on $\kappa \times \lambda$ with the help of a strong partition cardinal[3] $\mu > \max\{\kappa, \lambda\}$ that will imply the needed result[4]. This technique is due to Kechris, Kleinberg, Moschovakis, Woodin ([KKMW81]) and is described in detail in Philipp Rohde's thesis [Rohd01]. Therefore we shall only sketch the following argument and point to the corresponding proofs in Rohde's thesis.

First of all we have to quote an important theorem from the paper of Kechris, Kleinberg, Moschovakis and Woodin:

[3]A strong partition cardinal is a cardinal κ such that for all functions $f : [\kappa]^\kappa \to 2$ there is a subset $H \subseteq \kappa$ with cardinality κ such that $f \restriction [H]^\kappa$ is constant. For more on strong partition cardinals, cf. [Kana97] p. 432.

[4]A definition of homogeneous trees and the general idea how to apply this for determinacy results can be found in [MaSt89], in particular see their Theorem 2.3. The following aproach here is slightly different.

Theorem 7.3.4 (AD). *For each $\kappa < \Theta$ there is a μ such that $\kappa < \mu < \Theta$ and μ is a strong partition cardinal*[5].

For a proof see [KKMW81, Theorem 1.1]. We look at the tree T^* on $\kappa \times \lambda$ and find a strong partition cardinal $\mu > \max\{\kappa, \lambda\}$ according to Theorem 7.3.4[6]. Following the outline in [Rohd01] we can assign an ordinal $\pi(s)$ to each $s \in \kappa^{<\omega}$ and attach a μ-complete ultrafilter $\mathcal{U}_s$ on $[\mu]^{\pi(s)}$ to s in a way such that the system $(\mathcal{U}_s)_{s\in\kappa^{<\omega}}$ becomes a homogeneous system of ultrafilters.[7]. The homogenization of T^* is done in Satz (5.15) of [Rohd01].

With the homogenized tree $(T^*, (\mathcal{U}_s)_{s\in\kappa^{<\omega}})$ in mind, we can define an auxiliary game G':

In the game G' player I and player II play as in the game G, so in particular they have to follow the rule $\mathcal{R}$, but in addition, player II plays an object f_n in round n such that the following holds:[8]

If in round n of the game, before player II plays, the players have produced a sequence

$$t_n := ((\alpha_0, \xi_0, x_0), (\beta_0, \eta_0, f_0), \dots, (\alpha_n, \xi_n, x_n)),$$

and we let

$$\hat{t}_n := ((\alpha_0, \xi_0), (\beta_0, \eta_0), \dots, (\alpha_n, \xi_n)),$$

then $f_n \in [\mu]^{\pi(\hat{t}_n)}$ and $f_{n-1} \subseteq f_n$.

The payoff of this game G' is the same as in G, the additional object f_i only adds to the rules.

It can be seen that the game G' is an open game, hence quasi-determined (the proof is Behauptung 1 of Satz (5.16) in [Rohd01]), so either player I or player II has a winning quasi-strategy in this game. In fact, if player II has a winning quasi-strategy, then the maximal quasistrategy $\tau_{\max}$ (moving to non-losing positions) is winning and this winning quasi-strategy is independent of the points in A played by player I in the sense of this key lemma:

(3) The maximal winning quasi-strategy $\tau_{\max}$ has the following property:
Let

$$\begin{aligned} t &= ((\alpha_0, \xi_0, x_0), (\beta_0, \eta_0, f_0), \dots, (\alpha_{m-1}, \xi_{m-1}, x_{m-1})) \text{ and} \\ t' &= ((\alpha_0, \xi_0, x'_0), (\beta_0, \eta_0, f_0), \dots, (\alpha_{m-1}, \xi_{m-1}, x'_{m-1})) \end{aligned}$$

be two positions in G' which are legal and consistent with $\tau_{\max}$. Let $(\beta_{m-1}, \eta_{m-1}, f_{m-1})$ be such that if $x_{m-1}, x'_{m-1} \in C_{\eta_{m-1}}$ and $t^\frown(\beta_{m-1}, \eta_{m-1}, f_{m-1})$ is consistent with $\tau_{\max}$, then $t'^\frown(\beta_{m-1}, \eta_{m-1}, f_{m-1})$ is also consistent with $\tau_{\max}$.

[5] Θ is the supremum of all the lengths of prewellorderings of the Baire space.

[6] $\kappa < \Theta$ since it is the length of $\mathbf{\Delta}^1_n$ prewellordering, $\lambda < \Theta$ since λ came from a scale of a subset of $\mathcal{N}$

[7] The definition of $\pi(s)$ is Definition (5.11) in [Rohd01]

[8] For the definition of the f_n's and for the following, cf. the proof of Theorem 5.16 in Rohdes thesis.

Proof:Let t and t' be as in the statement of (3) and $(\beta_{m-1}, \eta_{m-1}, f_{m-1})$ be an answer for II following $\tau_{\max}$. Then $t^\frown(\beta_{m-1}, \eta_{m-1}, f_{m-1})$ is a winning position for II by definition of $\tau_{\max}$, i.e., a position such that player I has no winning strategy from this position. (That such a quasi-strategy $\tau_{\max}$ is a winning quasi-strategy for II in a open game see the proof of the Gale-Stewart Theorem for example in [Kana97, Proposition 27.1].)

Assume towards a contradiction that $t'^\frown(\beta_{m-1}, \eta_{m-1}, f_{m-1})$ is no winning position for II. Then player I has a winning strategy from this position on. Player II does not lose by violating any rule if he plays $(\beta_{m-1}, \eta_{m-1}, f_{m-1})$ in round m, so player I really has to play following a winning strategy to win the run of the game that starts with $t'^\frown(\beta_{m-1}, \eta_{m-1}, f_{m-1})$. So the outcome of this run is an element not in P. If player I would use this strategy from the position $t^\frown(\beta_{m-1}, \eta_{m-1}, f_{m-1})$ on he would also produce an outcome not in P. And he has not violated any rule since the elements from A played in the beginning initial segment t play no role in his upcoming moves (this is so by definition of the strong Choquet game). So player I would have a winning strategy for the run starting with $t^\frown(\beta_{m-1}, \eta_{m-1}, f_{m-1})$. But this contradicts the assumption that player II followed his winning quasi-strategy $\tau_{\max}$. q.e.d.(3)

Because of the homogeneity of the ultrafilter system, being a winning quasi-strategy for G' transfers now to the game G as follows:

1. Suppose that player II has a winning quasi-strategy in G'. Then we can see every quasi-strategy as a quasi-strategy in the game G by forgetting the f_i-moves. Clearly, this quasi-strategy is still winning.

2. Suppose that player I has a winning quasi-strategy in G'. Then we can construct a winning quasi-strategy for player I in the game G. This claim uses the homogeneity of the ultrafilter system and is the proof of Behauptung 2 in Satz (5.16) in [Rohd01].[9]

So we have proved that the game G is quasi-determined and, even more, that if player II has a quasi-winning strategy he has a winning quasi-strategy with the demanded property (by (3) and the way player II gets his winning quasi-strategy for G out of the winning quasi-strategy for G'). In order to finish the proof of this key lemma now we have to show that player I cannot have a winning quasi-strategy in G.

Assume towards a contradiction that he does have a winning quasi-strategy in G and let $\hat{\tau}$ be such a winning quasi-strategy for player I in G. Note that if we use a surjection from ω^ω onto X (cf. Theorem 2.2.3) for a coding of the Polish space X by the reals, φ_0 as a coding of the ordinals less than κ by $W \subseteq \mathcal{N}$ and if we identify the t-basic open sets C_ξ with ξ we can view both G and the strong Choquet game for (A, t) as being games on the reals. With this in mind

[9] Note that Rohde's game $G_\alpha(A)$ does not have real moves, but the real moves do not matter for the construction of the quasi-strategy for player I. All we have to worry about is the simulation of the moves f_i for player II that do not occur in the game G but are necessary to apply the given quasi-strategy.

the following claim makes sense.

(4) There exists a countable subset Z of ω^ω such that

(a) The set of ordinals less than κ with codes in Z is honest

(b.1) Every position in G consistent with $\hat{\tau}$ with all moves from Z has an extension consistent with $\hat{\tau}$ and all moves from Z.

(b.2) Every position in $G_{sCh}(A,t)$ is consistent with σ and all moves from Z has an extension consistent with σ and all moves from Z.

Proof: We can view the winning quasi-strategy $\hat{\tau}$ and the winning strategy σ as trees on $\kappa \times \kappa \times A \times \kappa \times \kappa$ and $\kappa \times A \times \kappa$ respectively. We want to define Z by recursion.

Let Z_0 be the emptyset and let Z_i countable be defined. To get Z_{i+1} consider the tree $\hat{\tau}|Z_i$. That of course should be $\hat{\tau}$ restricted to the elements coded by Z_i. Let S be the set of all finite branches in the countable tree $\hat{\tau}|Z_i$. For $s \in S$ let

$$s^\frown = \{(\alpha, \xi, x, \beta, \eta) \in \kappa \times \kappa \times A \times \kappa \times \kappa \mid s^\frown(\alpha, \xi, x, \beta, \eta) \in \hat{\tau}\}$$

By $\mathbf{AC}_\omega$, we can choose for each $s \in S$ one element from $s^\frown$ and let S^* be the set of all chosen elements.

We do the same with the tree $\sigma|Z_i$ and get a countable set R^*.

The third set we consider is $T^* = \bigcup_{z \in Z_i \cap W} \{\varphi_i(z) \mid i \in \omega\}$.

Applying $\mathbf{AC}_\omega$, we get three countable subsets $\overline{R^*}, \overline{S^*}, \overline{T^*}$ of reals coding R^*, S^*, T^*. Let now $Z_{i+1} = Z_i \cup \overline{R^*} \cup \overline{S^*} \cup \overline{T^*}$.

Set $Z = \bigcup_{i \in \omega} Z_i$.

It is now easy to see that this Z has the demanded properties. For (a) let α be an ordinal coded by some $w \in Z \cap W$. Then $w \in Z_i$ for some i. By the definition of Z $\varphi_k(w)$ is coded for all k in $Z_{i+1} \subseteq Z$. For (b)(1) let s be a position in G consistent with $\hat{\tau}$ with all elements in s from Z. Since s is a finite branch there are only finitely many elements in s. So there is a Z_i for some i such that $s \in \hat{\tau}|Z_i$. Now s has a proper extension with elements in Z consistent with $\hat{\tau}$ since we added exactly such extensions in Z_{i+1}. The same argument holds for (b)(ii). q.e.d.(4)

Fix an Z as in (4).

Then it is clear that there exists a run of G such that

(i) all moves are in Z (again, i.e. all moves are coded in Z)

(ii) this round is consistent with I's winning quasi-strategy $\hat{\tau}$ for G

(iii) The ξ_i's, x_i's and η_i's are consistent with II's winning strategy σ for the strong Choquet game $G_{(A,t)}$.

(iv) $\beta_0, \beta_1, \beta_2, \ldots$ is an enumeration of the ordinals with codes in Z.

By (ii) I wins this run of G. II does not lose this run by violating rule $\mathcal{R}$ since he follows σ by (iii). So the outcome of this run is an f that is not in P. f is honest, since f consists of all ordinals coded by Z (by (iv), putting in the $\beta_0, \beta_1, \ldots$) and this set is honest by construction of Z. Hence Remark 7.3.2 implies that I wins the strong Choquet game $G_{(A,t)}$. But this contradicts (iii). $\square$

The tree T_0

Let τ be a winning quasi-strategy for II as in the above Lemma 7.3.3. τ is essentially a tree on $(\kappa \times \kappa \times A \times \kappa \times \kappa)$. Let us call this tree T_0 and we assume all positions in T_0 legal, that is, if I loses by violating the rule $\mathcal{R}$ we remove this branch from the tree.

By the key lemma the points of A in this tree play no role for our purpose, so we remove this points and get a tree T_1 on κ^4:

The tree T_1

Define a tree T_1 on κ^4 by

$$
\begin{aligned}
((\alpha_0, \xi_0, \beta_0, \eta_0), \ldots, (\alpha_{m-1}, \xi_{m-1}, \beta_{m-1}, \eta_{m-1})) &\in T_1 \\
\Leftrightarrow \exists x_0, \ldots x_{m-1} &\in A \text{ such that} \\
((\alpha_0, \xi_0, x_0, \beta_0, \eta_0), \ldots, (\alpha_{m-1}, \xi_{m-1}, x_{m-1}, \beta_{m-1}, \eta_{m-1})) &\in T_0
\end{aligned}
$$

The important Remark 7.3.2 implies the following property of T_1.

Lemma 7.3.5. *Let $f = (\alpha_0, \xi_0, \beta_0, \eta_0, \alpha_1, \xi_1, \beta_1, \eta_1, \ldots)$ be an infinite branch in T_1. If f is honest, then $\bigcap_i C_{\eta_i}$ contains a point $x_f \in A$.*

Proof. Let f be given. We want first find some $x_0, x_1, \ldots$ in A such that $g = (\alpha_0, \xi_0, x_0, \beta_0, \eta_0, \alpha_1, \xi_1, x_1, \beta_1, \eta_1, \ldots)$ is an infinite branch through T_0. We define the x_i by induction.
Let $x_0, x_1, \ldots, x_{n-1}$ be defined such that

$$(\alpha_0, \xi_0, x_0, \beta_0, \eta_0, \ldots, \alpha_{n-1}, \xi_{n-1}, x_{n-1}, \beta_{n-1}, \eta_{n-1}) \in T_0.$$

Since f is an infinite branch in T_1 there exists $x'_0, \ldots, x'_n$ such that

$$(\alpha_0, \xi_0, x'_0, \beta_0, \eta_0, \ldots, \alpha_{n-1}, \xi_{n-1}, x'_{n-1}, \beta_{n-1}, \eta_{n-1}, \alpha_n, \xi_n, x'_n, \beta_n, \eta_n) \in T_0.$$

Now

$$s = (\alpha_0, \xi_0, x_0, \beta_0, \eta_0, \ldots, \alpha_{n-1}, \xi_{n-1}, x_{n-1}, \beta_{n-1}, \eta_{n-1}, \alpha_n, \xi_n, x'_n)$$

is a legal move in G consistent with τ and

$$
\begin{aligned}
s'^\wedge(\beta_n, \eta_n) \quad = \quad & (\alpha_0, \xi_0, x'_0, \beta_0, \eta_0, \ldots, \\
& \ldots, \alpha_{n-1}, \xi_{n-1}, x'_{n-1}, \beta_{n-1}, \eta_{n-1}, \alpha_n, \xi_n, x'_n, \beta_n, \eta_n)
\end{aligned}
$$

is a legal move in G consistent with τ since it is a sequence in T_0. By the property of τ we have $s^\wedge(\beta_n, \eta_n)$ is a legal move in G consistent with τ. So

define x_n to be x'_n.
This definition assures that $g = (\alpha_0, \xi_0, x_0, \beta_0, \eta_0, \alpha_1, \xi_1, x_1, \beta_1, \eta_1, \ldots)$ is an infinite branch through T_0. Since τ is a winning strategy for II g is the outcome of a round in G in which II wins. So $f \in P$. By the remark to the definition of the game G II wins the strong Choquet game, so $\bigcap_i C_{\eta_i} \neq \emptyset$. $\square$

Finally we will define with the help of T_1 a tree T on $\omega \times \kappa^4$ that will lead to a definition of a $\mathbf{\Sigma}^1_n$ set A'. We will see that A' equals A and finish in this way the proof of Theorem 4.

The tree T
Let T be the following tree on $\omega \times \kappa^4$:
$((i_0, \alpha_0, \xi_0, \beta_0, \eta_0), \ldots, (i_{m-1}, \alpha_{m-1}, \xi_{m-1}, \beta_{m-1}, \eta_{m-1})) \in T \Leftrightarrow$

(i) For all k, $\mathrm{diam}(B_{i_k}) < \frac{1}{k}$

(ii) For all k, $\overline{B_{i_{k+1}}} \subseteq B_{i_k}$

(iii) $((\alpha_0, \xi_0, \beta_0, \eta_0), \ldots, (\alpha_{m-1}, \xi_{m-1}, \beta_{m-1}, \eta_{m-1})) \in T_1$

(iv) For all k, $B_{i_k} \cap C_{\eta_k} \neq \emptyset$

The definition of the set A' is now the following:

The set A'
Define $A' \subseteq X$ by

$$\begin{aligned} x \in A' \quad \Leftrightarrow \quad & \exists y \in \omega^\omega \exists \overline{\alpha}, \overline{\xi}, \overline{\beta}, \overline{\eta} \in \kappa^\omega \\ & [(y, \overline{\alpha}, \overline{\xi}, \overline{\beta}, \overline{\eta}) \in T \text{ and } x \in \bigcap_m B_{y(m)} \\ & \text{ and } \{\overline{\alpha}(m), \overline{\xi}(m), \overline{\beta}(m), \overline{\eta}(m) \mid m \in \omega\} \text{ is honest }] \end{aligned}$$

We claim that A' is a $\mathbf{\Sigma}^1_n$ set. To see this we want to use the Coding Lemma 5.2.2.

Lemma 7.3.6. *A' is in $\mathbf{\Sigma}^1_n$.*

Proof. We prove first that the tree T is $\mathbf{\Delta}^1_n$-in-the-codes[10].
Define $\mathrm{Code}(T^m, \leq_{\varphi_0})$ the following way:

$$\begin{aligned} (y(0), \ldots, y(m-1), (x_0)_0, \ldots, (x_0)_{m-1}, \ldots, (x_3)_0, \ldots, (x_3)_{m-1}) \\ \in \mathrm{Code}(T^m, \leq_{\varphi_0}) \\ \Leftrightarrow \\ [(y(0), \varphi_0((x_0)_0), \varphi_0((x_1)_0), \varphi_0((x_2)_0), \varphi_0((x_3)_0)), \\ \ldots, \\ ((y(m-1), \varphi_0((x_0)_{m-1}), \varphi_0((x_1)_{m-1}), \varphi_0((x_2)_{m-1}), \varphi_0((x_3)_{m-1}))] \\ \in T \cap (\omega \times \kappa^4)^m. \end{aligned}$$

[10] The notion of a tree being $\mathbf{\Gamma}$-in-the-codes is by no means a standard definition. The definition here seems to us the most natural to apply the coding Lemma to it.

By Corollary 5.2.2 this set is $\mathbf{\Delta}^1_n$. So if we define that T is $\mathbf{\Delta}^1_n$-in-the-codes should stand for the fact that the union of all the $\mathrm{Code}(T^m, \leq_{\varphi_0})$ is in $\mathbf{\Delta}^1_n$ we have just shown that T is $\mathbf{\Delta}^1_n$-in-the-codes.

Now we can rewrite the defining formula for A':

$$\begin{array}{rcl} x \in A' & \Leftrightarrow & \exists y \in \omega^\omega \, \exists x_0, x_1, x_2, x_3 \in \omega^\omega \\ & \wedge & \forall k[(x_0)_k \in W \wedge (x_1)_k \in W \wedge (x_2)_k \in W \wedge (x_3)_k \in W] \\ & \wedge & \forall m (y(0), \ldots, y(m-1), (x_0)_0, \ldots, (x_0)_{m-1}, \ldots, \\ & & (x_3)_0, \ldots, (x_3)_{m-1}) \in \mathrm{Code}(T^m \leq_{\varphi_0}) \\ & \wedge & \forall m \, x \in B_{y(m)} \\ & \wedge & \forall k \exists w \in W \forall i \forall j (w \in A^{\varphi_0((x_0)_j)}_{\leq \varphi_i} \wedge (x_0)_j \in A^{\varphi_0(w)}_{\leq \varphi_i}) \\ & \wedge & \forall k \exists w \in W \forall i \forall j (w \in A^{\varphi_0((x_1)_j)}_{\leq \varphi_i} \wedge (x_1)_j \in A^{\varphi_0(w)}_{\leq \varphi_i}) \\ & \wedge & \forall k \exists w \in W \forall i \forall j (w \in A^{\varphi_0((x_2)_j)}_{\leq \varphi_i} \wedge (x_2)_j \in A^{\varphi_0(w)}_{\leq \varphi_i}) \\ & \wedge & \forall k \exists w \in W \forall i \forall j (w \in A^{\varphi_0((x_3)_j)}_{\leq \varphi_i} \wedge (x_3)_j \in A^{\varphi_0(w)}_{\leq \varphi_i}) \end{array}$$

where $A^{\varphi_0(w)}_{\leq \varphi_i}$ and $A^{\varphi_0((x_\ell)_j)}_{\leq \varphi_i}$ for $\ell = 0, 1, 2, 3$ are initial segments of the prewellordering $\leq_{\varphi_i}$ which are in $\mathbf{\Delta}^1_n$ following Lemma 5.1.3.

From this formula we see that A' is indeed in $\mathbf{\Sigma}^1_n$. □

So we can finish the proof if we show that $A = A'$.

$A' \subseteq A$

Let $x \in A'$. Let $y, \overline{\alpha}, \overline{\xi}, \overline{\beta}, \overline{\eta} \in \omega^\omega \times (\kappa^\omega)^4$ witness that $x \in A'$. Let

$$f = (\alpha_0, \xi_0, \beta_0, \eta_0, \alpha_1, \xi_1, \beta_0, \eta_0, \ldots).$$

Then $(y, f) \in [T]$ and f is honest. By definition of T we have $f \in [T_1]$. So by Lemma 7.3.5 there exists an $x_f \in A$ such that $x_f \in \bigcap_m C_{\eta_m}$. Since x is the only point in $\bigcap_m B_{y(m)}$ (by (i) of the definition of T) it suffices to show that $x_f \in \bigcap_m B_{y(m)}$ because then $x = x_f \in A$.

Claim: $x_f \in \bigcap_m B_{y(m)}$
Proof: Assume not. So there is an $m \in \omega$ with $x_f \notin B_{y(m)}$. Therefore $d(x_f, B_{y(m)}) > 0$, let us say $d(x_f, B_{y(m)}) = \varepsilon > 0$. But now there exists an $k > m$ with $\mathrm{diam}(C_{\eta_k}) < \frac{\varepsilon}{4}$ (because of rule $\mathcal{R}$ in the definition of G) and $x_f \in C_{\eta_k}$. Also $\mathrm{diam}(B_{y(k)}) < \frac{\varepsilon}{4}$ by (i) of the definition of T and $B_{y(k)} \subseteq B_{y(m)}$. By (iv) of the definition there is an $z \in B_{y(k)} \cap C_{\eta_k}$. Since $z, x_f \in C_{\eta_k}$ we have $d(z, x_f) < \frac{\varepsilon}{4}$. But also $z \in B_{y(k)} \subseteq B_{y(m)}$ and hence $d(x_f, B_{y(m)}) = \inf\{d(z', x_f) \mid z' \in B_{y(m)}\} \leq d(z, x_f) < \frac{\varepsilon}{4}$. This contradicts $d(B_{y(m)}, x_f) > \varepsilon$. q.e.d. Claim

This proves that $A' \subseteq A$.

$A \subseteq A'$

Let $x \in A$. Let $h : \omega^\omega \twoheadrightarrow A$ be a coding of A by the reals. [11]

Let Z be a countable subset of ω^ω such that

1. there is an $\overline{x} \in Z$ with $h(\overline{x}) = x$
2. the set of ordinals less than κ with codes in Z is honest
3. $\exists \nu < \kappa$ such that $x \in C_\nu$ and ν has a code in Z
4. every position in G consistent with τ with all moves from elements coded from Z has an extension consistent with τ and with all moves from elements coded from Z

To prove the existence of such a set we define by recursion countable sets Z_i for $i \in \omega$ (using AC_ω in every other step of the construction) and the take Z to be the union of all Z_i. To make an easy thing not look to complicated (by jumping back and forth between the "coded game" and G) note that if there is a countable set of ordinals less than κ one can get by AC_ω a countable subset of W coding these ordinals. Simultaneously one can get for a countable subset of A a countable set of reals coding the elements of the subset throug h.

Let $\overline{x} \in \omega^\omega$ such that $h(\overline{x}) = x \in A$ and let $\overline{y} \in W$ such that $\varphi_0(\overline{y}) = \nu$ and $x \in C_\nu$. Set $Z_0 = \{\overline{x}, \overline{y}\}$.

Let now Z_i countable be given for an $i \in \omega$. We want to define Z_{i+1}. Let $T_0 \upharpoonright Z_i$ be the tree on $\kappa \times \kappa \times A \times \kappa \times \kappa$ restricted to elements coded by Z_i. Consider in $T_0 \upharpoonright Z_i$ the countable set of all finite sequences $s \in T_0 \upharpoonright Z_i$ which have no proper extension. Let s be such a finite sequence and let $s^\wedge = \{(\alpha, \xi, x, \beta, \eta) \in \kappa \times \kappa \times A \times \kappa \times \kappa \mid s^\wedge(\alpha, \xi, x, \beta, \eta) \in T_0\}$. By AC_ω we find a countable set $\tilde{Z}_{i+1}$ such that for all such s there is a proper extension of s from $s^\wedge$ in $\tilde{Z}_{i+1}$. Again by AC_ω and the above remark there is a countable set Z'_{i+1} of reals coding these elements. Choose further codes for the ordinals $\varphi_k(\overline{z})$ for $\overline{z} \in Z_i \cap W, k \in \omega$ and let M_i be the set of these codes. Then let $Z_{i+1} = Z_i \cup M_i \cup Z'_{i+1}$. Z_{i+1} is countable. Set $Z = \bigcup_{i \in \omega} Z_i$.

By definition of Z_0 1. and 3. are satisfied. If $\overline{w} \in Z_i$ for some $i \in \omega$ then for $\varphi_k(\overline{w})$ there is a code in $M_i \subseteq Z_{i+1}$ for all $k \in \omega$. Hence the set of ordinals less than κ with codes in Z is honest. If s is a position in G consistent with τ and all moves are in Z then there is an $i \in \omega$ such that $s \in T_0 \upharpoonright Z_i$ and there is an extension so $s \in T_0 \upharpoonright Z_{i+1} \subseteq T_0 \upharpoonright Z$. So this extension is also consistent with τ.

Using such an Z there is a run of the game G such that

(i) all moves are in Z (that is, coded by Z)

(ii) the run is consistent with II's winning quasi-strategy τ for G

(iii) $x_m = x$ for all m (so player I always plays the same element $x \in A$)

[11] such a coding exists by Theorem 2.2.3

(iv) $\xi_0 = \nu, \xi_{m+1} = \eta_{m+1}$

(v) $\alpha_0, \alpha_1, \ldots$ is an enumeration of the ordinals less than κ with codes in Z

Since such a run $g = (\alpha_0, \xi_0, x, \beta_0, \eta_0, \alpha_1, \xi_1, x, \beta_1, \eta_1, \ldots)$ of G is consistent with τ, we know that g is an infinite branch in T_0. By definition of T_1 the sequence $f = (\alpha_0, \xi_0, \beta_0, \eta_0, \alpha_1, \xi_1, \beta_1, \eta_1, \ldots)$ is an infinite branch through T_1.

Using this f we want to get an infinite branch in T. Property (iii) in the definition of T is already satisfied. Now let $i_0, i_1, \ldots$ be such that $x \in \bigcap_m B_{i_m}$ and (i) and (ii) of the definition of T holds. Since II wins the run g of G all C_{η_m} are legal moves of II and therefore are moves in the strong Choquet game. So $x \in C_{\eta_m}$ for all m. This implies $B_{i_m} \cap C_{\eta_m} \neq \emptyset$ for all m and therefore (iv) in the definition of T holds. So $(i_0, \alpha_0, \xi_0, \beta_0, \eta_0, i_1, \alpha_1, \xi_1, \beta_1, \eta_1, \ldots) \in [T]$.

To show now that $x \in A'$ it remains (by the definition of A') to show that $\{\alpha_m, x_m, \beta_m, \eta_m \mid m \in \omega\}$ is honest. But all this elements were chosen in Z and the α_m are all ordinals less than κ coded by Z and this set is honest by the construction of Z. $\square$

Together with Theorem 7.1.1 we have now proved the main Theorem 4 under the assumption of **ZF+DC+AD**$_{\mathbb{R}}$. The assumption that *every set of reals has a scale* is essential for the proof of the key lemma, Lemma 7.3.3, in our proof of Theorem 7.3.1. So it seems, unfortunately, not possible to proof the main Theorem 4 under the weaker assumption of **ZF+DC+AD** in this fashion. But, as a compensation, Becker suggests that this proof of the Theorem generalizes to pointclasses beyond the projective hierarchy which are scaled and projective-like. For further remarks and results we could not cover here we refer to the notes of Howard Becker, [Beck91] and [Beck92].

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