Contents

Introduction

In many cases the eigenvalues of linear operators, are of interest, as they may help to solve differential equations or give a more accessible representation of an operator.

To begin with consider the finite dimensional Hilbert space $\mathbb{C}^n$. Then every normal operator N can be written in the form:

$$N = \sum_{k \in \sigma(N)} \lambda_k P_k,$$

where $\sigma(N)$ is the spectrum of N and P_k is the orthogonal projection of the eigenspace corresponding to λ_k. For $\omega \subset \sigma(N)$ consider the projection-valued set function $E(\omega) := \sum_{\sigma(N)} \delta_{\lambda_k}(\omega) P_k$ with δ_{λ_k} being the dirac measure at the point λ_k. Then the equality above becomes

$$N = \int_{\sigma(N)} \lambda \; dE,$$

where the integral sign represents a finite sum.

The spectral theorem for unbounded, normal operators shows that this representation can be extended to a possibly infinite-dimensional Hilbert space H and bounded or unbounded operators on H.

On an infinte dimensional Hilbert space H the spectrum of a bounded operator N is not necessarily finite and neither does the spectrum only contain eigenvalues. However, the spectral theorem for *bounded*, normal operators shows that the expression above makes sense for an unique spectral measure E on the Borel sets of $\sigma(N)$. The theorem will be proved in chapter 3 and in particular, we will give sense to the expression $\int_{\sigma(N)} \lambda \; dE(\lambda)$, where E is a spectral measure.

The main tool to show the existence of E will be the Gelfand transform, especially the Gelfand-Naimark theorem. Chapter one and two develop the necessary theory to prove it, along with some important properties of spectra in Banach algebras.

As an application we obtain some results on the eigenvalues of normal operators and the existence of square roots for positive operators.

An important class of linear operators are the differential ones, for example acting on functions from $\mathbb{R}^n$ to C. The spectral theorem for bounded operators cannot be applied to them, as they are neither defined, nor bounded on a Hilbert space, such as $L^2(\mathbb{R})$. However, they are well-defined on sub spaces of $L^2(\mathbb{R})$.

The spectral theorem for *unbounded*, normal operators will be stated for such operators defined on subspaces of H and it will give the same representation as above.

Finally we apply the spectral theorem for unbounded, normal operators to the Multiplication operator M_{x^2} and the Laplacian $\Delta := -\frac{\partial^2}{\partial x^2}$ on $L^2(\mathbb{R})$.

1 Spectra and maximal Ideals in Banach algebras

1.1 The spectrum in a Banach algebra

Definition 1.1.1. Let B be a complex vector space. Then we call B a *complex algebra*, if there exists a multiplication on B, such that the follwoing are satisfied for all x,y,$z \in B$ and $\alpha \in \mathbb{C}$:

(1) $x(yz) = (xy)z$.
(2) $x(y+z) = xy + xz, \qquad (y+z)x = yx + zx$.
(3) $\alpha(yz) = (\alpha x)y = x(\alpha y)$.

If B is additionally a Banach space such that

$$\|xy\| \leq \|x\|\|y\|$$

for all x,$y \in B$ and there exists $e \in B$ with $ex = x = xe$ and $\|e\| = 1$
for all $x \in B$, we call B a *Banach algebra*.
B is called *commutative*, if $xy = yx$ holds for all $x, y \in B$.
An element $x \in B$ is *invertible*, if there exists $x^{-1} \in B$, such that $xx^{-1} = e = x^{-1}x$.

Remark 1.1.2. If $\|x\| < 1$, the series $\sum_{n=0}^{\infty} x^n$ converges in B and we see that $e - x$ is invertible, as $(e-x)\sum_{n=0}^{\infty} x^n = e = \big(\sum_{n=0}^{\infty} x^n\big)(e-x)$.

Examples 1.1.3.
(1) Let K be compact and $C(K)$ the complex vector space of all continuous functions from K to $\mathbb{C}$ with the supremum norm. Then $C(K)$ is a commutative Banach algebra with addition and multiplication defined pointwise.
(2) Let H be a Banach space and $B(H)$ the set of all linear and bounded operators $T : H \to H$. Define the addition on $B(H)$ pointwise and the multiplication to be the composition. Then $B(H)$ is a Banach algebra with norm

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|.$$

Definition 1.1.4. Let B be a Banach algebra. For fix $x \in B$

$$r(x) = \{\lambda \in \mathbb{C} \mid (\lambda e - x)^{-1} \in B\}$$

is called the *resolvent set* of x. Its complement $\sigma(x)$ is called the *spectrum* of x and $\rho(x) = \sup\{|\lambda| \mid \lambda \in \sigma(x)\}$ is defined to be the *spectral radius* of x.

In many cases the spectrum of an element in an Banach algebra is of interest. For example, in the Banach algebra $B(H)$ the spectrum of an operator contains its eigenvalues. We observe the following important property of the spectrum:

Theorem 1.1.5. *Suppose $x \in B$ and B is a Banach algebra. Then*

(1) *$\sigma(x)$ is non-empty and compact.*
(2) *$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$.*

Proof. (1) We begin with some general observations. If $\|x\| < 1$, the series $-\sum_{n=0}^{\infty} x^n$ is the inverse of $x-e$ by remark (1.1.2). For $y,z \in B$, y invertible and $\|z\| < \|y^{-1}\|^{-1}$ this gives:

$$(z-y)^{-1} = (y(y^{-1}z-e))^{-1} = (y^{-1}z-e)^{-1}y^{-1} = -\sum_{n=0}^{\infty}(y^{-1}z)^n y^{-1} \tag{1.1}$$

For $\|z\| < |\lambda|$ we obtain $\lambda \in r(z)$, as

$$(z-\lambda e)^{-1} = -\lambda^{-1}\sum_{n=0}^{\infty}\lambda^{-n}z^n. \tag{1.2}$$

In particular

$$\|(z-\lambda e)^{-1}\| \leq \frac{1}{|\lambda|(1-\frac{\|z\|}{|\lambda|})} = \frac{1}{|\lambda|-\|z\|} \tag{1.3}$$

holds. Therefore $\lim_{|\lambda|\to\infty}\|(x-\lambda e)^{-1}\| = 0$ and $\rho(x) \leq \|x\|$.

To see that $r(x)$ is open let $\mu \in r(x)$ and $\lambda \in \mathbb{C}$ with $|\lambda-\mu| < \|(x-\mu e)^{-1}\|^{-1}$. For $z = (\lambda-\mu)e$ and $y = x-\mu e$ (1.1) shows:

$$(x-\lambda e)^{-1} = -((\lambda-\mu)e-(x-\mu e))^{-1} = \sum_{n=0}^{\infty}(\lambda-\mu)^n(x-\mu e)^{-n-1}. \tag{1.4}$$

Hence $\sigma(x)$ is compact as a bounded and closed set.

The identity (1.4) shows that the map $\lambda \mapsto (x-\lambda e)^{-1}$ is B-locally analytic on $r(x)$. This means for every $\mu \in r(x)$ exists a neighborhood, in which the mapping can be written as a power series centered at μ and with coefficients in B.

Suppose $\sigma(x) = \emptyset$ and take an arbitrary continuous, linear functional $\Lambda : B \to \mathbb{C}$. The function $\mathbb{C} \to \mathbb{C}$, $\lambda \mapsto \Lambda((x-\lambda e)^{-1})$ is then locally analytic on $r(x) = \mathbb{C}$ and therefore entire. By Liouville's Theorem ([5], Chapter 10, 10.23) and (1.3) we see that $\Lambda((x-\lambda e)^{-1}) = 0$ for every Λ in the dual space B' of B. The Hahn-Banach theorem ([5], Chapter 5, 5.16) implies $(x-\lambda e)^{-1} = 0$, which is a contradiction and $\sigma(x) \neq \emptyset$ is therefore proved.

(2) Fix $x \in B$ and let Λ be a continuous linear functional on B. Define the function

$$f_\Lambda(\zeta) = \Lambda((x-\zeta^{-1}e)^{-1}) = -\zeta\sum_{n=0}^{\infty}\zeta^n\Lambda(x^n)$$

for $\zeta \in \mathbb{C}$, $0 < |\zeta| < \rho(x)^{-1}$. It is analytic by (1.4) and using (1.3) f_Λ can be analytically extended by setting $f_\Lambda(0) = 0$. The extension is continuous and using the Laurent-expansion ([1], Chapter 5, 5.2) at 0, we see that f_Λ has a powerseries representation around 0.

The second equality holds for all $|\zeta| < \|x\|^{-1}$ by (1.1). However, the function is analytic on $\mathcal{B}(0, \rho(x)^{-1})$ and has a unique power series representation on $\mathcal{B}(0, \rho(x)^{-1})$.Therefore the second equality holds for all $|\zeta| < \rho(x)^{-1}$ by uniqueness of the power series representation.

Fix ζ with $|\zeta| < \rho(x)^{-1}$. Then the series $-\sum_{n=0}^{\infty} \zeta^{n+1}\Lambda(x^n)$ converges for every Λ in the dual space B' and the sequence $(\zeta^n \Lambda(x)^n)_{n\in\mathbb{N}}$ is therefore bounded. For all $n \in \mathbb{N}$ consider the functions

$$\zeta^n(\cdot)(x^n) : B' \to \mathbb{C} \qquad \Lambda \to \zeta^n\Lambda(x^n)$$

with operator norm $|\zeta|^n\|x^n\|$. Applying the Banach-Steinhaus theorem ([4], Chapter 2, 2.6) to $\{\zeta^n(\cdot)(x^n)\}_{n\in\mathbb{N}}$ shows that there exists M_ζ with $|\zeta|^n\|x^n\| < M_\zeta$ for all $n \in \mathbb{N}$. Consequently

$$\|x^n\|^{1/n} \leq \zeta^{-1} M_\zeta^{1/n} \qquad \text{and} \qquad \limsup_{n\to\infty} \|x^n\|^{1/n} \leq \zeta^{-1}$$

for every $|\zeta| < \rho(x)^{-1}$ and therefore

$$\limsup_{n\to\infty} \|x^n\|^{1/n} \leq \rho(x) \tag{1.5}$$

For $\lambda^n \in r(x^n)$ the inverse of $x - \lambda e$ is given by $(x^n - \lambda^n e)^{-1}(x^{n-1} + \lambda x^{n-2} + \cdots + \lambda^{n-1}e)$ and therefore $\lambda \in r(x)$. Hence $\sigma(x)^n \subseteq \sigma(x^n)$, $\rho(x)^n \leq r(x^n) \leq \|x^n\|$ and therefore $\rho(x) \leq \|x^n\|^{1/n}$. Together with (1.5) this proves

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n}.$$

□

Theorem 1.1.6. (Gelfand-Mazur) *Suppose B is a Banach algebra with every $x \in B$ being invertible. Then B is isomorphic to $\mathbb{C}$.*

Proof. Since $\sigma(x)$ is non-empty for $x \in B$ by (1.1.5), there exists a $\lambda \in \mathbb{C}$ with $(\lambda e - x)$ is not invertible. By assumption $(\lambda e - x) = 0$, hence $\lambda e = x$. □

1.2 Homomorphisms and Maximal Ideals

We now introduce the *Gelfand-space* Δ of a Banach algebra, which will be necessary to define the Gelfand Transform and state the theorem of Gelfand-Naimark.

Definition 1.2.1. ***(Gelfand space)*** Consider a Banach algebra B and $\phi : B \to \mathbb{C}$ a linear functional on B. Then ϕ is called a *complex homomorphism*, if

$$\phi(xy) = \phi(x)\phi(y)$$

for all x,$y \in B$ and $\phi \neq 0$.

The set Δ of all complex homomorphisms is called the *Gelfand space* of B.

Proposition 1.2.2. *Let B be a complex Banach algebra, $x \in B$ and ϕ a complex homomorphism on B. Then:*

(1) *If x is invertible, then $\phi(x) \neq 0$. In particular $\phi(e) = 1$.*
(2) $\|\phi\| := \sup_{\|x\| \leq 1} \|\phi(x)\| = 1$

Proof. (1) Let $x \in B$ with $\phi(x) \neq 0$. Then $\phi(x) = \phi(xe) = \phi(x)\phi(e)$ and $\phi(e) = 1$. If $x \in B$ is invertible, $\phi(x) \neq 0$ as $1 = \phi(e) = \phi(x)\phi(x^{-1})$ and (1) holds.
(2) Let $\|x\| < 1$ and $|\alpha| \geq 1$. Then $(e - \alpha^{-1}x)$ is invertible by remark (1.1.2) and

$$1 - \alpha^{-1}\phi(x) = \phi(1 - \alpha^{-1}x) \neq 0$$

implies $\phi(x) \neq \alpha$ and therefore $|\phi(x)| < 1$. If $\|x\| = 1$ take $t > 1$ and observe that $|\phi(x)| = t|\phi(\frac{x}{t})| < t$. Now $\|\phi\| = 1$ as $\phi(x) < t$ holds for all $t > 1$ and $\phi(e) = 1$. ☐

The homomorphisms of a commutative Banach algebra B have a close and important connection to the maximal ideals, which will be obtained in proposition (1.2.5).

Definition 1.2.3. Let B be a Banach algebra and $I \subseteq B$ a Subspace. Then I is called an *Ideal*, if $a \in I$, $x \in B$ implies $ax \in I$ and $xa \in I$. I is a *proper* Ideal, if $0 \subsetneq I \subsetneq B$ and it is *maximal* , if it is proper and not contained in another proper Ideal.

Remark 1.2.4. In a commutative Banach algebra B every maximal ideal M is closed and every proper ideal lies in a maximal ideal. ([4], Chapter 11, 11.3)

Proposition 1.2.5. *Let B be a commutative Banach algebra and $M \subset B$ a maximal ideal in B. Then the following hold:*

(1) *Every M is the nullspace of some complex homomorphism ϕ on B.*
(2) *The nullspace of any ϕ is a maximal Ideal in B.*
(3) $\phi(x) = 0$ *for some complex Homomorphism ϕ if and only if x is not invertible.*
(4) $\lambda \in \sigma(x) \Leftrightarrow \phi(x) = \lambda$ *for some complex homomorphism ϕ.*

Proof. (1) As M is closed, B/M is a Banach space ([4], Chapter 1, 1.41). The mapping $\pi(x) = [x]$ is a continuous homomorphism between the algebras B and B/M and its nullspace is M. As M is a maximal ideal, the quotient algebra is a field, particularly every non-zero element in B/M is invertible and B/M is isomorphic to $\mathbb{C}$ by theorem (1.1.6). Composing the corresponding isomorphism ψ with π gives a complex homomorphism $\psi \circ \pi : B \to \mathbb{C}$ with $\ker \psi \circ \pi = M$.

For (2) we use the fact, that an ideal M is maximal in a ring R, if R/M is a field. As $\phi : B \to \mathbb{C}$ is onto, $B/\ker\phi$ is isomorphic to $\mathbb{C}$ and $\ker\phi$ is therefore maximal.

(3) In (1.2.2) it was already shown that x invertible implies $\phi(x) \neq 0$ and therefore $x \notin J$ for any maximal Ideal J. If x is not invertible, then $\{bx \mid b \in B\}$ is a proper ideal, hence contained in a maximal one.

(4) $\lambda \in \sigma(x) \Leftrightarrow (\lambda e - x)$ not invertible $\Leftrightarrow \phi(\lambda e - x) = 0$ for some ϕ by (c). ☐

2 The Gelfand Transform and Involution

2.1 The Gelfand Transform

In this chapter the Gelfand transform on a Banach algebra will be introduced. The central statement is the Gelfand-Naimark theorem, which allows an identification of a B^*-algebra with the continuous functions on its Gelfand space. It will be vital for the proof of the spectral theorem for bounded, normal operators.

Definition 2.1.1. Any x in a Banach algebra B induces a mapping

$$\widehat{x} : \Delta \to \mathbb{C}, \qquad \widehat{x}(h) = h(x),$$

which is called the *Gelfand Transform of x* .

To define a topology on Δ, we firstly consider the dual B' of B with the so-called *weak*-topology*. We will see that the restriction of the weak*-topology to the Gelfand space space Δ of B turns Δ into a compact Hausdorff space.

Definition 2.1.2. Every $x \in B$ defines a linear functional

$$f_x : B' \to \mathbb{C}, \qquad f_x(\Lambda) = \Lambda(x)$$

The *weak*-topology* is the smallest topology on B' that makes every f_x continuous. Hence it is the smallest topology that contains $f_x^{-1}(U)$ for every $x \in B$, $U \subseteq \mathbb{C}$ open.

The *Gelfand topology* on Δ is the restriction of the weak*-topology to Δ.

Remark 2.1.3. (a) The set $\{f_x\}_{x\in B}$ separates points on B', as $f_x(\Lambda) = f_x(\Lambda')$ for all x implies that $\Lambda = \Lambda'$. The weak*-topology is therefore a Hausdorff topology.

(b) The Gelfand topology is the smallest topology that makes every $\widehat{x} : \Delta \to \mathbb{C}$ continuous:

If $U \subseteq \mathbb{C}$ is open, $\widehat{x}^{-1}(U) = f_x^{-1}(U) \cap \Delta$, hence open in Δ and $\widehat{x}$ is continuous with respect to the Gelfand topology. The weak*-topology is the smallest topology containing $\{f_x^{-1}(U)\}_{x\in B}$ for all $U \in \mathbb{C}$, hence the Gelfand topology is the smallest topology containing $\{f_x^{-1}(U) \cap \Delta\}_{x\in B}$, for all $U \subseteq \mathbb{C}$. But $f_x^{-1}(U) \cap \Delta = \widehat{x}^{-1}(U)$ and therefore the Gelfand topology is the smallest topology, that makes every $\widehat{x}$ continuous.

Theorem 2.1.4. (Alaoglu) *Let B be a Banach Space and $\mathcal{U}$ the closed unit ball of B' centered at 0. Then $\mathcal{U}$ is compact in the weak*-topology of B'.*

The proof is omitted, but can be found in ([2], Chapter IV, 1.4).

Proposition 2.1.5. Δ *is compact with respect to the Gelfand*-topology.*

Proof. Every $\phi \in \Delta$ has norm 1 by proposition (1.2.2). Therefore Δ lies $\mathcal{U}$ and by theorem (2.1.4) and the definition of the Gelfand space it is enough to show that Δ is weak*-closed.

For $x \in B$ the mapping $B' \to \mathbb{C}$, $\Lambda \mapsto \Lambda(x)$ is weak*-continuous. Hence

$$B' \to \mathbb{C} \times \mathbb{C} \times \mathbb{C}, \qquad \Lambda \mapsto (\Lambda(xy), \Lambda(x), \Lambda(y))$$

is weak*-continuous for all $x, y \in B$ and so is

$$B' \to \mathbb{C},\ \Lambda \mapsto \Lambda(xy) - \Lambda(x)\Lambda(y).$$

Therefore

$$M_{x,y} := \{\Lambda \in B' \mid \Lambda(xy) - \Lambda(x) - \Lambda(y) = 0\}$$

is weak*-closed for all x,$y \in B$ as the preimage of 0 under a weak*-continuous map.

Now $\Delta \cup \{0\} = \bigcap_{x,y\in B} M_{x,y}$ and $M_0 = \{\Lambda \mid \Lambda(e) = 1\}$ are weak*-closed, which implies that $\Delta = M_0 \cap \left(\bigcap_{x,y\in B} M_{x,y}\right)$ is weak*-closed. □

Definition 2.1.6. ***(Gelfand transform)*** On a commutative Banach algebra B the mapping

$$\widehat{\cdot} : B \to C(\Delta), \qquad x \mapsto \widehat{x}$$

is called the *Gelfand Transform.*

It is a continuous algebra homomorphism, as $\phi \leq 1$ for every $\phi \in \Delta$.

Note that for every x, $\widehat{x}(\Delta) = \sigma(x)$ and therefore $\|\widehat{x}\|_\infty = r(x) \leq \|x\|$.

2.2 Involutions

So far the Gelfand Transform is not necessarily injective, however under further conditions it will become a surjective isometry. We therefore introduce the Involution:

Definition 2.2.1. ***(Involution)*** On a Banach space B a mapping $^* : B \to B$ is called an *involution*, if it satisfies

(1) $\cdot^*$ is conjugate-linear.
(2) $(xy)^* = y^*x^*$, for all $x, y \in B$.
(3) $x^{**} = x$, for all $x \in B$.

$x \in B$ is called *normal*, if $xx^* = x^*x$ and *self-adjoint*, if $x = x^*$. A Subalgebra of B is called *normal*, if it is commutative and closed under involution.

A Banach algebra with an involution, that satisfies

$$\|xx^*\| = \|x\|^2 \tag{2.1}$$

for all $x \in B$ is called a B^**-algebra.*

Remark 2.2.2. In a B^*-algebra $\|x\|^2 = \|xx^*\| \leq \|x\|\|x^*\|$ holds, hence $\|x\| \leq \|x^*\|$ and as $x^{**} = x$ we obtain

$$\|x\| = \|x^*\|.$$

for all $x \in B$. The involution is therefore an isometry on B^*-algebras. It also follows that

$$\|xx^*\| = \|x\|\|x^*\|.$$

Examples 2.2.3.
(1) The Banach algebra $C(K)$ is a B^*-algebra with complex conjugation being the involution.
(2) If H is a Hilbert space, we will show in chapter 3 that $B(H)$ is a B^*-algebra. The involution $\cdot^*$ on $B(H)$ maps every $T \in B(H)$ on its Hilbert space adjoint.

Proposition 2.2.4. *In any B^*-algebra B, the following hold:*

(1) *The unit e is self-adjoint.*
(2) *If $x \in B$ is normal, then $r(x) = \|x\|$.*
(3) *If $x \in B$ is self-adjoint, then $\sigma(x) \subset \mathbb{R}$.*

Proof. (1) $e^* = ee^*$ shows that e is self-adjoint, as xx^* is self-adjoint for any $x \in B$.
For (2) observe, that for normal x in a B^*-algebra

$$\|x\|^4 = \|xx^*\|^2 = \|xx^*(xx^*)^*\| = \|xx^*x^*x\| = \|x^2(x^2)^*\| = \|x^2\|^2$$

holds, hence $\|x^2\| = \|x\|^2$. As x is normal, so is x^n for any $n \in \mathbb{N}$. Hence by induction

$$\|x^{2^{n+1}}\| = \|x^{2^n}\|^2 = \|x\|^{2^{n+1}}$$

is valid for any $n \in \mathbb{N}$ and the spectral radius formula (1.1.5) gives $r(x) = \|x\|$.
(3) If x is self-adjoint and $\lambda \in \sigma(x) \setminus \mathbb{R}$, $\tilde{x} := x - \mathrm{Re}(\lambda)e$ is selfadjoint and $\mathrm{Im}(\lambda)\mathrm{i} \in \sigma(\tilde{x})$. Now for $t > 0$, $(\mathrm{Im}(\lambda) + t)\mathrm{i} \in \sigma(\tilde{x} + \mathrm{i}te)$ and

$$(\mathrm{Im}(\lambda) + t)^2 \leq \|\tilde{x} + \mathrm{i}te\|^2 = \|(\tilde{x} + \mathrm{i}te)(\tilde{x} - \mathrm{i}te)\| = \|\tilde{x}^2 + t^2e\| \leq, \|\tilde{x}\|^2 + t^2$$

which is a contradiction for $t > \|\frac{\tilde{x}^2}{2}\|$. □

We can now state the theorem of Gelfand-Naimark.

Theorem 2.2.5. (Gelfand-Naimark) *Let B be a commutative B^*-algebra with Gelfand space Δ. Then the Gelfand transform $\widehat{\cdot}: B \to C(\Delta)$ is a surjective, isometric $*$-isomorphism meaning that for all $x \in B$*

$$\widehat{x^*}(\phi) = \overline{\widehat{x}(\phi)} \tag{2.2}$$

In particular, $x \in B$ is self-adjoint if and only if its Gelfand transform is real-valued.

Proof. In definition (2.1.6) we saw that the Gelfand transform is a homomorphism and $\|\widehat{x}\|_\infty = r(x)$. In commutative B^*-algebras every element is normal, hence $\|x\| = r(x)$ by theorem (2.2.4) and $\widehat{\cdot}: B \to \mathbb{C}$ is thereforean isometry.

To show that $\widehat{\cdot}$ is surjective we use the Stone-Weierstrass theorem ([2],Chapter III, 1.4). $\widehat{B}$ also contains $\widehat{e}$, the constant 1-function in $C(\Delta)$ and exactly as in remark (2.1.3) we see that $\widehat{B}$ seperates points on Δ. To see that $\widehat{B}$ is closed under complex conjugation, note first that $x + x^*$ and $i(x - x^*)$ are self-adjoint for every $x \in B$. Hence we can write

$$\widehat{x} = \widehat{\left(\frac{x+x^*}{2} + \frac{x-x^*}{2}\right)} = \widehat{\left(\frac{x+x^*}{2}\right)} - \mathrm{i}\,\widehat{\left(\mathrm{i}\frac{x-x^*}{2}\right)}, \tag{2.3}$$

$$\widehat{x^*} = \widehat{\left(\frac{x^*+x}{2} + \frac{x^*-x}{2}\right)} = \widehat{\left(\frac{x+x^*}{2}\right)} + \mathrm{i}\,\widehat{\left(\mathrm{i}\frac{x-x^*}{2}\right)}. \tag{2.4}$$

As $\frac{x+x^*}{2}$ and $\mathrm{i}\frac{x-x^*}{2}$ are self-adjoint, theorem (2.2.4) shows that their Gelfand transforms are real-valued and so

$$\overline{\widehat{x}} = \widehat{x^*}. \tag{2.5}$$

Hence $\widehat{B}$ is closed under conjugation and therefore dense in $C(\Delta)$.

The image $\widehat{B}$ of B is a closed sub-algebra of $C(\Delta)$, as it is the isometric image of a complete Banach algebra, hence $\widehat{B} = C(\Delta)$.

In theorem (2.2.4) it was proved, that a self-adjoint x has a real spectrum, hence $\widehat{x}$ is real-valued. If $\widehat{x}$ is real-valued (2.5) shows that $\widehat{x} = \overline{\widehat{x}} = \widehat{x^*}$ and so $x = x^*$, as the Gelfand-Transforms is one-to-one. □

The next lemma establishes a connection between the Gelfand space of certain B^*-algebra and the spectrum of a specific element in those algebras, which will be used in the spectral theorem.

Lemma 2.2.6. *Let B be a B^*-algebra with an $x \in B$, such that the Polynomials $P(x, x^*)$ are dense in B. Then*

$$\widehat{x} : \Delta \to \sigma(x)$$

is a homeomorphism.

Proof. $\widehat{x}$ is a continuous function from the compact space Δ onto the Hausdorff-space $\sigma(x)$. It is therefore enough to show, that $\widehat{x}$ is one-to-one. Suppose $\widehat{x}(h_1) = \widehat{x}(h_2)$ for h_1, $h_2 \in \Delta$. This means $h_1(x) = h_2(x)$ and by theorem (2.2.5) $h_1(x^*) = h_2(x^*)$. Now for every polynomial P, $h_1(P(x, x^*)) = h_2(P(x, x^*))$ as h_1, h_2 are homomorphisms. The polynomials $P(x, x^*)$ are dense in B and therefore $h_1 = h_2$ by continuity. Hence x^* is one-to-one. □

Remark 2.2.7. In the following chapter we will focus on the B^*-algebra $B(H)$. To be able to apply the above theorems, we will have to restrict ourselves to closed, normal sub-algebras of $B(H)$. The fact that the spectrum of $T \in B(H)$ in such sub-algebras is the same as in $B(H)$ is not immediate. It will be omitted here and can be found in ([4], Chapter 11, 11.29).

3 The spectral theorem for bounded, normal operators

3.1 Properties of $B(H)$

After the general observations about Banach algebras this chapter will deal with the algebra $B(H)$ of bounded, linear operators on a Hilbert space. The spectral theorem for bounded, normal operators will be stated for the normal elements in $B(H)$.

Definition 3.1.1. Suppose H is a topological vector space. Then H is called a *Hilbert space*, if there exists an inner product

$$(\cdot, \cdot) : H \times H \to \mathbb{C}$$

and H is complete with respect to the norm $\|x\| := \sqrt{(x, x)}$.

The Banach algebra of all bounded linear functions on a Hilbert space will be denoted by $B(H)$.

The three following statements will be useful throughout the next chapters. The proofs will be omitted, but can be found in ([4], Chapter 12, 12.6, 12.7, 12.8).

Proposition 3.1.2. *Consider a sequence $\{x_n\}_{n \in \mathbb{N}}$ of pairwise orthogonal vectors in a Hilbert space H. Then the following are equivalent:*

(1) $\sum_{n=0}^{\infty} x_n$ *converges in H.*
(2) $\sum_{n=0}^{\infty} \|x_n\|^2 < \infty$.
(3) $\sum_{n=0}^{\infty} (x_n, y)$ *converges for every $y \in H$.*

Proposition 3.1.3. *Let $T \in B(H)$ with $(Tx, x) = 0$ for every $x \in H$. Then $T = 0$.*

Lemma 3.1.4. *Let $f : H \times H \to \mathbb{C}$ be a sesquilinear and bounded, meaning that*

$$C = \sup \{|f(x, y)| \mid \|x\| = \|y\| = 1\} < \infty.$$

Then there exists an unique $S \in B(H)$ with $\|S\| = C$ and

$$f(x, y) = (x, Sy) \qquad \text{for all } x, y \in H. \tag{3.1}$$

3.2 Adjoints and normal operators

We wish to define an involution on $B(H)$ and therefore observe the following: The mapping

$$H \times H \to \mathbb{C}, \qquad (x, y) \mapsto (Tx, y)$$

is sesquilinear and letting $y = \frac{Tx}{\|Tx\|}$, we see that

$$\|T\| = \sup \{|(Tx, y)| \mid \|x\| = \|y\| = 1\}.$$

Now by Lemma (3.1.4) there exists an unique $T^* \in B(H)$ such that for all $x, y \in H$

$$(Tx, y) = (x, T^*y) \tag{3.2}$$

and $\|T\| = \|T^*\|$.

Definition 3.2.1. The operator $T^* \in B(H)$, such that

$$(Tx, y) = (x, T^*y)$$

is called the *adjoint* of T. In fact, $\cdot^* : B \to B$ is an involution and B is a Banach algebra ([4], Chapter 12, 12.9).

For the normal operators in $B(H)$ the following properties hold:

Proposition 3.2.2. *Let $T \in B(H)$. Then T is normal if and only if*

$$\|Tx\| = \|T^*x\| \tag{3.3}$$

for every $x \in H$. If T is normal, the following hold:

(1) $\mathcal{N}(T) = \mathcal{N}(T^*)$
(2) *If $Tx = \lambda x$ for $\lambda \in \mathbb{C}$, $x \in H$, then $T^*x = \overline{\lambda} x$.*
(3) *The eigenspaces of distinct eigenvalues λ_1, λ_2 of T are orthogonal.*

Proof. The first equivalence follows from the equalities

$$(Tx, Tx) = (T^*Tx, x) \qquad \text{and} \qquad (T^*x, T^*x) = (TT^*x, x).$$

and (1) is a direct consequence.

Applying (1) to $T - \lambda I$ shows (2) and the equality

$$\lambda_1(x, y) = (Tx, y) = (x, T^*y) = (x, \overline{\lambda_2} y) = \lambda_2(x, y)$$

proves (3) as $\lambda_1 - \lambda_2 \neq 0$. □

Example 3.2.3. Consider $\mathbb{C}^n$ with the standard inner product and a normal mapping $T : \mathbb{C}^n \to \mathbb{C}^n$. Then there exists an orthonormal basis of eigenvectors $\{x_1, \dots, x_n\}$ of T and orthogonal projections $\{P_k\}_{k \in \{1,\dots,n\}}$ from $\mathbb{C}^n$ onto the corresponding eigenspaces of x_k. Hence, we can write

$$T = \sum_k \lambda_k P_k,$$

where λ_k is the eigenvalue of x_k. Every P_k is an orthogonal projection, hence self-adjoint and the ranges $\mathcal{R}(P_k)$ are pairwise orthogonal by proposition (3.2.2)

We conclude the section with a commutativity theorem for normal operators. The proof can be found in ([4], Chapter 12, 12.16).

Theorem 3.2.4. (Fudglede) *Let $T, N \in B(H)$ and N be normal. If $TN = NT$, then*

$$TN^* = N^*T$$

3.3 Spectral measures

One important tool to state the spectral theorems will be spectral measures, which are operator-valued set functions. It makes sense to call them spectral *measures*, as they share many properties with ordinary measures.

Definition 3.3.1. We call an operator $P \in B(H)$ a *projection*, if $PP = P$ holds.

Remark 3.3.2. Let $P, Q \in B(H)$ be self-adjoint projections. Then:

(1) $\mathcal{R}(P) = \mathcal{N}(P)^{\perp}$.
(2) $PQ = 0$ if and only if $\mathcal{R}(P) \perp \mathcal{R}(Q)$.

Both statements are shown in ([4], Chapter 12, 12.4).

Definition 3.3.3. ***(Spectral measure)*** Consider a set Ω with a locally compact Hausdorff topology and $\mathcal{A}$ the corresponding σ-algebra. A map

$$E : \mathcal{A} \to B(H)$$

is called a *spectral measure*, if it satisfies the following properties:

(1) $E(\omega)$ is a self-adjoint projection for every $\omega \in \mathcal{A}$.
(2) $E(\emptyset) = 0$ and $E(\Omega) = I$.
(3) $E(\omega_1 \cup \omega_2) = E(\omega_1) + E(\omega_2)$ for $\omega_1 \cap \omega_2 = \emptyset$.
(4) $E(\omega_1 \cap \omega_2) = E(\omega_1)E(\omega_2)$.
(5) The set function $E_{x,y}(\omega) := (E(\omega)x, y)$ is a complex, regular Borel measure on Ω for every $x, y \in H$.

Note that the projections $E(\omega)$ commute with each other by (4) and remark (3.3.2).

Example 3.3.4. We continue example (3.2.3): Let $\Omega = \sigma(T)$ and define:

$$E_T(\omega) = \sum_k \delta_{\lambda_k}(\omega) P_k,$$

,where $\omega \subset \sigma(T)$ and $\delta_{\lambda_k}(\cdot)$ denotes the dirac measure at the point λ_k.

Then E_T is a spectral measure: Remark (3.3.2) shows that $E_T(\omega)$ is a self-adjoint projection, as the P_k are self-adjoint projections with pairwise orthogonal ranges. Finite additivity and multiplicativity follow from the properties of δ_{λ_k} and the fact that the P_k have pairwise orthogonal ranges. Now fix $x, y \in \mathbb{C}^n$ and observe that

$$(E_T(\omega)x, y) = \sum_k \delta_{\lambda_k}(\omega)(P_k x, y).$$

As every δ_{λ_k} is a measure, so is $E_{T;x,y}$. E_T is therefore a spectral measure on $\sigma(T)$.

Proposition 3.3.5. *Let E be a spectral measure on $(\Omega, \mathcal{A})$ and take an arbitrary $x \in H$. Then:*

(1) *The map $\omega \mapsto E(\omega)x$ is countably additive for every $\omega \in \mathcal{A}$.*
(2) *For $x, y \in H$, $|E|_{x,y} \leq \|x\|\|y\|$.*

Proof. (1) Let $\{\omega_n\}_{n\in\mathbb{N}}$ be pairwise disjoint and $\omega = \bigcup_{n=0}^{\infty} \omega_n$. Then $E(\omega_n)E(\omega_m) = 0$ for $n \neq m$ by (4) of definition (3.3.3). Hence remark (3.3.2) shows that all $E(\omega_n)$ have pairwise orthogonal ranges. Now for all $y \in H$

$$\sum_{n=0}^{\infty} (E(\omega_n)x, y) = (E(\omega)x, y)$$

as $E_{x,y}$ is a measure and $\sum_{n=0}^{\infty} E(\omega_n)x = E(\omega)x$ follows from proposition (3.1.2).

(2) Consider a partition $\{\omega_n\}_{n\in\mathbb{N}}$ of Ω and choose complex numbers α_n such that $|E_{x,y}(\omega_n)| = \alpha_n E_{x,y}(\omega_n)$ and $|\alpha_n| = 1$. The projections $E(\omega_n)$ have pairwise orthogonal ranges and therefore:

$$\begin{aligned}
\sum_{n=1}^{N} |E_{x,y}(\omega_n)| &= \sum_{n=1}^{N} (\alpha_n E(\omega_n)x, y) = \sum_{n=1}^{N} (\alpha_n E(\omega_n)x, E(\omega_n)y) \\
&= \left(\sum_{n=1}^{N} \alpha_n E(\omega_n)x, \sum_{n=1}^{N} E(\omega_n)y \right) \\
&\leq \left\| \sum_{n=1}^{N} \alpha_n (E(\omega_n)x \right\| \left\| \sum_{n=1}^{N} (E(\omega_n)y \right\|.
\end{aligned}$$

Since $\alpha_n E(\omega_n)x \perp \alpha_m E(\omega_m)x$ for $n \neq m$, we have

$$\begin{aligned}
\left\| \sum_{n=1}^{N} \alpha_n E(\omega_n)x \right\|^2 &= \sum_{n=1}^{N} \|\alpha_n E(\omega_n)x\|^2 = \sum_{n=1}^{N} \|(E(\omega_n)x\|^2 \\
&= \left\| \sum_{n=1}^{N} (E(\omega_n)(x) \right\|^2 = \left\| E\left(\bigcup_{n=1}^{N} \omega_n \right) x \right\|^2 \leq \|E(\Omega)x\|^2.
\end{aligned}$$

The same argument holds for $\sum_{n=1}^{N} (E(\omega_n)y$ and we obtain

$$\sum_{n=1}^{\infty} |E_{x,y}(\omega_n)| \leq \|x\|\|y\|.$$

Since the partition is arbitrary, $|E_{x,y}| \leq \|x\|\|y\|$ holds.

In particular $E_{x,x} \leq \|x\|^2$ and $(E(\omega)x, x) = (E(\omega)x, E(\omega)x) = \|(E(\omega)x)\|^2$ shows that $E_{x,x}$ is a positive measure. □

Proposition 3.3.6. *Let E be a spectral measure on $(\Omega, \mathcal{A})$ and consider $\{\omega_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$. Then the following hold:*

(1) *$E(\omega_n) = 0$ for all $\{\omega_n\}_{n\in\mathbb{N}}$ implies $E(\bigcup_{n=0}^{\infty} \omega_n) = 0$.*
(2) *If $E(\omega_2) = 0$ and $\omega_1 \subseteq \omega_2$, then $E(\omega_1) = 0$.*

Proof. (1) Using (1) and (5) of definition (3.3.3) shows that

$$\|E\Big(\bigcup_{n=0}^{\infty} \omega_n\Big)x\|^2 = (E\Big(\bigcup_{n=0}^{\infty} \omega_n\Big)x, x) = \sum_{n=0}^{\infty} (E(\omega_n)x, x) = \sum_{n=0}^{\infty} \|E(\omega_n)x\|^2$$

holds for any $x \in H$.

(2) follows from

$$E(\omega_1) = E(\omega_2 \cap \omega_1) = E(\omega_1)E(\omega_2) = 0.$$

□

Proposition 3.3.7. *Let E be a spectral measure on $(\Omega, \mathcal{A})$ and consider another set Ω' with a σ-algebra $\mathcal{A}'$. Suppose $h : \Omega \to \Omega'$ is measurable. Then*

$$E' : \mathcal{A}' \to B(H), \qquad E'(\omega') := E(h^{-1}\{\omega'\})$$

is a spectral measure and for a measurable $f : \Omega \to \mathbb{C}$, $x, y \in H$

$$\int f \; dE'_{x,y} = \int (f \circ h) \; dE_{x,y},$$

holds, if one of the two integrals exists.

The proof can be found in ([4], Chapter 13, 13.28).

The Gelfand-Naimark theorem (2.2.5) shows the existence of an isometric *-isomorphism between a normal, closed subalgebra of $B(H)$ and the continuous functions on its corresponding Gelfand space. To extend this isomorphism to bounded functions we introduce the Banach algebra $L^\infty(E)$.

Definition 3.3.8. Consider a spectral measure E on a set Ω and let $f : \Omega \to \mathbb{C}$ be measurable. The topology of $\mathbb{C}$ is generated by countably many discs $\mathcal{B}_n$ and let $V := \bigcup(\mathcal{B}_n)$ be the union of those $\mathcal{B}_n$ with $E(f^{-1}(\mathcal{B}_n) = 0$. Then $E(f^{-1}(V)) = 0$ by proposition (3.3.6).

The set $R_f = V^c$ is called the *essential range* of f and $\|f\|_{ess} := \sup\{|\lambda| \mid \lambda \in R_f\}$ the *essential supremum* of f.

We say f is *essentially bounded*, if $\|f\|_{ess} < \infty$.

Proposition 3.3.9. *Consider the Banach algebra B of all bounded functions on Ω with the supremum norm. Then*

$$M := \{f \mid \|f\|_{ess} = 0\}$$

is a closed ideal in B and we define $L^\infty(E)$ to be the Banach algebra B/M.

In $L^\infty(E)$ the following hold:

(1) $\|f\|_{ess} = \|[f]\| := \inf\{\|f-h\| \mid h \in M\}$.
(2) $R_f = \sigma([f])$.

Proof. We proof first that M is closed: Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in M, that converges in B to some $f \in B$. We define $N_f = \{p \in \Omega \mid f(p) = 0\}$ and observe:

$$N_f \supset \bigcap_{n=0}^{\infty} N_{f_n}$$

Therefore $E(N_f) = I$ by proposition (3.3.6), which implies $f \in M$. Proving that M is a subspace, works the same way and B/M is a Banach algebra (See ([4], Chapter 1, 1.41)).

For (1) fix $f \in B$ and define

$$g(p) = \begin{cases} f(p) & \text{if } f(p) \notin R_f \\ 0 & \text{if } f(p) \in R_f \end{cases}$$

Then $g \in M$, as $f^{-1}\{R_f\} \subseteq g^{1-}\{0\}$. Hence $E(\{g=0\}) = I$ by proposition (3.3.6). Now $\|f-g\| = \|f\|_{ess}$ and therefore $\|[f]\| \leq \|f\|_{ess}$.

For the other direction take $h \in M$ and observe

$$\|f\|_{es} \leq \|f-h\|_{es} + \|h\|_{ess} = \|f-h\|_{ess} \leq \|f-h\|,$$

,where we use $\|f+g\|_{ess} \leq \|f\|_{ess} + \|g\|_{ess}$ for $f, g \in L^{\infty}(E)$. Hence, $\|f\|_{ess} \leq \|[f]\|$.

(2) For the first inclusion suppose $\lambda \in R_f$, but $\lambda \notin \sigma([f])$. Then there exists $g \in B$, such that $g(\lambda - f) = 1$ E-almost surely. More precisley: There is an ω with $E(\omega) = 0$ and $g(p)(\lambda - f(p)) = 1$ on ω^c. Pick $\varepsilon > 0$, such that $\varepsilon\|g\|_{ess} < 1/2$ and consider $s \in \mathcal{B}_\varepsilon(\lambda)$. We will show that $E(f^{-1}\mathcal{B}_\varepsilon(\lambda)) = 0$.

For all $p \in \omega^c \cap g^{-1}\{R_g\}$:

$$g(p)(s - f(p)) = g(p)(\lambda - f(p)) + g(p)(s-\lambda) = 1 - g(p)(s-\lambda) \neq 0,$$

as $|\lambda - s| < \varepsilon$. Therefore

$$g(p)(s-f(p)) \neq 0 \qquad \text{and} \qquad f(p) \neq s.$$

We see that

$$f^{-1}\mathcal{B}_\varepsilon(\lambda) \cap \{\omega^c \cap g^{-1}\{R_g\}\} = \emptyset.$$

Hence $f^{-1}\mathcal{B}_\varepsilon(\lambda) \subseteq \omega \cup g^{-1}\{R_g\}^c$ and by proposition (3.3.6) $E(f^{-1}\mathcal{B}_\varepsilon(\lambda)) = 0$. Now $\lambda \notin R_f$, which is a contradiction.

For the other inclusion suppose $\lambda \in \sigma([f])$ and $\lambda \notin R_f$. Then there exists $\varepsilon > 0$ with $E(f^{-1}\mathcal{B}_\varepsilon(\lambda)) = 0$. Define $g : \Omega \to \mathbb{C}$ in the following way:

$$g(p) = \begin{cases} \frac{1}{\lambda - f(p)} & \text{if } f(p) \in R_f \\ 0 & \text{if } f(p) \notin R_f \end{cases}$$

Then $g \in B$, as $|\lambda - f(p)| > \varepsilon$ on $f^{-1}\{R_f\}$ and

$$(\lambda - f)g = 1 - \chi_{f^{-1}\{R_f^c\}},$$

where χ_ω denotes the characteristic function of a set $\omega \subseteq \Omega$. $E(f^{-1}\{R_f^c\}) = 0$ implies $\lambda - f$ is invertible in B/M and we get a contradiction as $\lambda \notin \sigma([f])$. $\square$

3.4 The spectral theorem

The spectral theorem is proved in two steps: Firstly we will focus on the integration of essentially bounded functions with respect to an arbitrary spectral measure, namely we will give sense to the expression $\int f dE$.

Secondly we will focus on a closed, normal sub algebra B of $B(H)$. Using the Gelfand Naimark theorem we will show that there is a specific spectral measure on the Gelfand space of B, such that:

$$T = \int_{\Delta} \widehat{T} dE$$

The spectral theorem will follow as a corollary.

Theorem 3.4.1. *Let E be a spectral measure on a set Ω. Then there exists an isometric *-isomorphism Φ from $L^\infty(E)$ onto a closed, normal sub algebra A of $B(H)$, such that*

$$(\Phi(f)x, y) = \int f dE_{x,y}, \tag{3.4}$$

for all x,$y \in H$, $f \in L^\infty(E)$ and

$$\|\Phi(x)\|^2 = \int |f|^2 dE_{x,x}. \tag{3.5}$$

For every $f \in L^\infty(E)$ we define

$$\int_\Omega f dE = \Phi(f).$$

Additionally: $S \in B(H)$ commutes with every $\Phi(f)$ if and only if S commutes with every projection $E(\omega)$.

Proof. We firstly define Φ on simple functions and extend it to $L^\infty(E)$ afterwards. Let $\{\omega_1, \ldots, \omega_n\}$ be a partition of Ω and $s = \sum_{i=0}^n \alpha_i \chi_{\omega_i}$. Then we define

$$\Phi(s) := \sum_{i=0}^n \alpha_i E(\omega_i). \tag{3.6}$$

Note that Φ is well-defined by the properties of spectral measures.

Consider another simple function t, $t = \sum_{j=0}^m \beta_j \chi_{\omega'_j}$ and observe that

$$\Phi(s)\Phi(t) = \sum_{i,j} \alpha_i \beta_j E(\omega_i) E(\omega'_j) = \sum_{i,j} \alpha_i \beta_j E(\omega_i \cap \omega'_j) = \Phi(st),$$

as $st = \sum_{i,j} \alpha_i \beta_j \chi_{\omega_i \cap \omega'_j}$. Proving $\Phi(\alpha s + t) = \alpha \Phi(s) + \Phi(t)$ works the same way.

As every $E(\omega)$ is self-adjoint

$$\Phi(s)^* = \sum_{i=0}^n \overline{\alpha_n} E(\omega_i) = \Phi(\overline{s}) \tag{3.7}$$

and we see that Φ has the properties of a *-isomorphism on simple functions.

Equation (3.6) implies that for all x,$y \in H$

$$(\Phi(s)x, y) = \sum_{i=0}^{n} \alpha_i (E(\omega_i)x, y) = \sum_{i=0}^{n} \alpha_i (E_{x,y}(\omega_i)) = \int_\Omega s dE_{x,y}. \tag{3.8}$$

By (3.7) $\Phi(s)^*\Phi(s) = \Phi(|s|^2)$ and therefore (3.8) implies

$$\|\Phi(s)x\|^2 = (\Phi(s)^*\Phi(s)x, x) = (\Phi(|s|^2)x, x) = \int_\Omega |s|^2 dE_{x,x}. \tag{3.9}$$

Hence (3.4) and (3.5) hold for simple functions.

Now (3.9) implies $\|\Phi(s)x\|^2 = \int |s|^2 dE_{x,x} \leq \|s\|_{ess}^2 \|x\|^2$ and therefore

$$\|\Phi\| \leq \|s\|_{ess}.$$

However, for $x \in \mathcal{R}(E(\omega_i))$ we observe that $\Phi(s)x = \alpha_i x$, as $E(\omega_i)E(\omega_j) = 0$,whenever $i \neq j$. For $|\alpha_i| = \|s\|_{ess}$ this shows, that $\|\Phi\| \geq \|s\|_{ess}$. Φ is therefore an isometry on simple functions.

Consider now $f \in L^\infty(E)$. Then there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ of simple functions that converges to f in $L^\infty(E)$, particularly $\{s_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $B(H)$. Hence, $\{\Phi(s_n)\}_{n\in\mathbb{N}}$ is a Cauchy-sequence and we define

$$\Phi(f) = \lim_{n\to\infty} \Phi(s_n). \tag{3.10}$$

Note that f does not depend on $\{s_n\}_{n\in\mathbb{N}}$.

We now proceed by extending the results on simple functions to any $f \in L^\infty(E)$. Observe firstl that Φ remains an isometry, namley

$$\|\Phi(f)\| = \|f\|_\infty. \tag{3.11}$$

Convergence in $L^\infty(E)$ means uniform convergence E-almost surely and therefore $E_{x,y}$-almost surely for any $x, y \in H$. Let $\{s_n\}_{n\in\mathbb{N}}$ be simple functions and $\lim_{n\to\infty} s_n = f$ in $L^\infty(E)$. Using that every $E_{x,y}$ has bounded total variation and applying the dominated covergence theorem implies that, (3.4) follows from (3.8). In the same way (3.9) implies (3.5).

Let $f, g \in L^\infty(E)$ and $\{s_n\}_{n\in\mathbb{N}}$, $\{t_n\}_{n\in\mathbb{N}}$ be simple functions approximating f and g in $L^\infty(E)$. Then

$$\Phi(fg) = \lim_{n\to\infty} \Phi(s_n t_n) = \lim_{n\to\infty} \Phi(s_n)\Phi(t_n) = \Phi(f)\Phi(g).$$

The equations $\Phi(\alpha f + g) = \alpha\Phi(f) + \Phi(g)$ and $\Phi(f)^* = \Phi(\overline{f})$ are proved in the same way and Φ is therefore an isometric *-isomorphism. Since A is the isometric image of a complete space, it is closed and normal, as $L^\infty(E)$ is normal.

Suppose $S \in B(H)$ commutes with every $E(\omega)$. Then S commutes with every $\sum_{i=0}^n \alpha_i E(\omega_i)$ and therefore with every $\Phi(f) \in A$ as $\Phi(f)$ can be approximated by the image of simple functions. □

For the following proof the Riesz representation theorem will be vital. It can be found in ([5], Chapter 6, 6.19).

Theorem 3.4.2. *Consider a closed, normal sub algebra $A \subseteq B(H)$, that contains the identity I and has Gelfand space Δ. Then:*

(1) *There exists an unique spectral measure E on the Borel subset of Δ such that*

$$T = \int_\Delta \widehat{T} dE \tag{3.12}$$

for every $T \in A$, where $\widehat{T}$ denotes the Gelfand transform of T. In particular, the isometric *-isomorphism Φ defined in theorem (3.4.1)

$$\Phi : L^\infty(E) \to B, \qquad \Phi(f) = \int_\Delta f dE \tag{3.13}$$

is an extension of the inverse of the Gelfand transform.

(2) *If $\omega \subseteq \Delta$ is open, then $E(\omega) \neq 0$.*

(3) *$S \in B(H)$ commutes with all $T \in A$ if and only if S commutes with all $E(\omega)$.*

Proof. Note first that A is a B^*-algebra as $B(H)$ is a B^*-algebra. By the Gelfand-Naimark theorem (2.2.5) $\widehat{\cdot} : A \to C(\Delta)$ is a *surjective* isometric *-isomorphism.

We begin with the proof of uniqueness: If (1) holds, then for all $x, y \in H$, $T \in A$

$$(Tx, y) = \int_\Delta \widehat{T} dE_{x,y}.$$

The mapping $\widehat{T} \to T \to (Tx, y)$ is a bounded linear functional on $C(\Delta)$ and $E_{x,y}$ is unique by the Riesz representation theorem. The uniqueness of the projections $E(\omega)$ follows from $E_{x,y}(\omega) = (E(\omega)x, y)$.

(1) The map $\widehat{T} \to (Tx, y)$ is a bounded linear functional with norm smaller than $\|x\|\|y\|$. Again by the Riesz-representation theorem there exists an unique measure $\mu_{x,y}$ satisfying

$$(Tx, y) = \int_\Delta \widehat{T} d\mu_{x,y} \tag{3.14}$$

for all $T \in B(H)$ and $|\mu_{x,y}| \leq \|x\|\|y\|$. For fixed $T \in B(H)$ the mapping

$$\mathbb{C} \times \mathbb{C} \to \mathbb{C}, \qquad (x, y) \to (Tx, y)$$

is a bounded, sesquilinear mapping. By equality (3.14) the same is true for

$$(x, y) \to \int_\Delta \widehat{T} d\mu_{x,y},$$

hence by the Gelfand-Naimark theorem for every $f \in C(\Delta)$.

The characteristic function χ_K of any compact set $K \subseteq \Delta$ is the pointwise limit of Urysohn functions $\{u_n\}_{n\in\mathbb{N}}$ associated with K. As $|\mu_{x,y}|$ has bounded total variation

$$\int_\Delta \chi_K d\mu_{x,y} = \lim_{n\to\infty} \int_\Delta u_n d\mu_{x,y}$$

holds and the left side is sesquilinear as every u_n lies in $C(\Delta)$. By the regularity of $\mu_{x,y}$ every simple function s on K is the L^1-limit of finite linear combinations of such χ_K. Hence $\int_\Delta s d\mu_{x,y}$ is sesquilinear for all simple functions and as every bounded f is the uniform limit of simple functions, the same holds for f in place of s.

Since $|\mu_{x,y}| \leq \|x\|\|y\|$, the sesquilinear map $(x,y) \to \int f d\mu_{x,y}$ is bounded. Proposition (3.1.4) shows that for all bounded f there exists a unique $\Phi(f) \in B(H)$ with

$$(\Phi(f)x, y) = \int_\Delta f d\mu_{x,y} \tag{3.15}$$

for all $x, y \in H$. Now Φ is linear and particularly (3.15) and (3.14) show that $\Phi(\widehat{T}) = T$.

We proceed by showing that Φ is a *-isomorphism.

To see that $\Phi(f)^* = \Phi(\overline{f})$ holds, we use that by the Gelfand-Naimark theorem T is selfadjoint if and only if $\widehat{T}$ is real-valued. For such T:

$$\int_\Delta \widehat{T} d\mu_{x,y} = (Tx, y) = (x, Ty) = \overline{\int_\Delta \widehat{T} d\mu_{y,x}}$$

and therefore $\mu_{x,y} = \overline{\mu_{y,x}}$ as all Urysohn-functions on Δ are real-valued. Hence

$$(\Phi(\bar{f})x, y) = \int_\Delta \bar{f} d\mu_{x,y} = \overline{\int_\Delta f d\mu_{y,x}} = (x, \Phi(f)y).$$

We will prove next that $\Phi(fg) = \Phi(f)\Phi(g)$ is valid for bounded f and g. Take $S, T \in A$ and observe that by the Gelfand-Naimark theorem $\widehat{ST} = \widehat{S}\widehat{T}$. Hence,

$$\int_\Delta \widehat{S}\widehat{T}\ d\mu_{x,y} = (STx, y) = \int_\Delta \widehat{S}\ d\mu_{Tx,y}$$

holds for every continuous function $\widehat{S}$ on Δ and therefore we can replace $\widehat{S}$ by any bounded f. (Use the same approximations as before.) Now for any $\widehat{T} \in C(\Delta)$ and any bounded f

$$\int_\Delta f\widehat{T}\ d\mu_{x,y} = \int_\Delta f\ d\mu_{Tx,y} = (\Phi(f)Tx, y) = \int_\Delta \widehat{T}\ d\mu_{x,\Phi(f)^*y}.$$

holds and again we can replace $\widehat{T}$ by an arbitrary bounded g. Hence,

$$(\Phi(fg)x, y) = \int_\Delta fg\ d\mu_{x,y} = \int_\Delta g\ d\mu_{x,\Phi(f)^*y} = (\Phi(g)x, \Phi(f)^*y) = (\Phi(f)\Phi(g)x, y).$$

and therefore $\Phi(fg) = \Phi(f)\Phi(g)$.

We now use Φ to define the desired spectral measure on Ω. As $E(\omega) = \int_\Delta \chi_\omega dE$ should hold, we define:

$$E(\omega) = \Phi(\chi_\omega) \tag{3.16}$$

for any Borel set ω.

Additivity and multiplicativity of E is follow from the properties of Φ. As Φ is an extension of the inverse of the Gelfand transform $E(\emptyset) = 0$ and $E(\Omega) = I$. Since $\Phi(\chi_\omega)$ is a real-valued, characteristic function $E(\omega)$ is a self-adjoint projection. Finally

$$(E(\omega)x, y) = (\Phi(\chi_\omega)x, y) = \int_\Delta \chi_\omega d\mu_{x,y} = \mu_{x,y}(\omega)$$

is a complex measure and E is a spectral measure.

Note that by the definition of E, Φ is exactly the *-isomorphism constructed in theorem (3.4.1). Therefore it is an isometry and (1) is proved.

Let ω be open. Then there exists a non-zero Urysohn function $\widehat{T}$ with support in ω. If $E(\omega) = 0$, $(Tx, y) = \int_\Delta \widehat{T} dE_{x,y}$ implies $T = 0$. Therefore $\widehat{T} = 0$, which is a contradiction. Hence $E(\omega) \neq 0$.

(2) Consider a Borel set $\omega \neq \emptyset$. Then there exists a non-zero Urysohn-function u with support in ω. By the Gelfand-Naimark theorem, there is a T in A, such that $\widehat{T} = u$. If $E(\omega) = 0$, then $T = 0$, as $(Tx, y) = \int \widehat{T}\, dE_{x,y}$ for all x, y in H. Hence $u = \widehat{T} = 0$, which is a contradiction.

(3) Take any $T \in A$ and fix $S \in B(H)$, a Borel set ω and $x, y \in H$. Consider the pair of equations

$$(STx, y) = \int_\Delta \widehat{T}\, dE_{x,S^*y}, \qquad (TSx, y) = \int_\Delta \widehat{T}\, dE_{Sx,y}$$

and

$$(SE(\omega)x, y) = E_{x,S^*y}, \qquad (E(\omega)Sx, y) = E_{Sx,y}.$$

If S commutes with every T, the first equations are equal. Hence the measures E_{x,S^*y} and $E_{Sx,y}$ are equal and using the second pair of equations we see that S commutes with every $E(\omega)$. The other direction follows by reversing the steps above. □

Theorem 3.4.3. (Spectral theorem) *Let T be a normal Operator in $B(H)$. Then there exists an unique spectral measure E on the Borel subsets of $\sigma(T)$, such that*

$$T = \int_{\sigma(T)} \lambda\, dE(\lambda).$$

Furthermore $S \in B(H)$ commutes with every projection $E(\omega)$ if and only if S commutes with T.

E is called the spectral decomposition of T.

Proof. Let B_T be the smallest, closed sub-algebra of $B(H)$, that contains I, T and T^*. As T is normal, so is B_T and Theorem (3.4.2) shows that there exists an unique spectral measure E' on the Gelfand space Δ of B_T, such that

$$T = \int_\Delta \widehat{T} \, dE'.$$

By Lemma (2.2.6) Δ and $\sigma(T)$ are homeomorphic and the existence of E follows now from proposition (3.3.7). To see uniqueness, observe that the polynomials $P(\lambda, \overline{\lambda})$ are dense in $C(\sigma(T))$ and by theorem (3.4.1)

$$P(T, T^*) = \int_\Delta P(\lambda, \bar{\lambda}) \, dE(\lambda).$$

Hence $\int f \, dE$ is uniquely determined for every $f \in C(\sigma(T))$ and the riesz representation theorem shows that $E_{x,y}$ is uniquely determined for all $x, y \in H$. It follows that every projection $E(\omega)$ is unique.

If $ST = TS$, then $ST^* = T^*S$ by theorem (3.2.4). Hence S commutes with every element of B_T and the equivalence follows from theorem (3.4.2). $\square$

Example 3.4.4. The normal mapping $T : \mathbb{C}^n \to \mathbb{C}^n$ has the spectral decomposition E_T defined in example (3.3.4): We only have to show that

$$(Tx, y) = \int_{\sigma(T)} \lambda \, dE_{x,y}$$

for all $x, y \in \mathbb{C}^n$, which holds

$$\int_{\sigma(T)} \lambda \, dE_{x,y}(\lambda) = \sum_{k=1}^{n} \lambda_k E_{x,y}(\{\lambda_k\}) = \sum_{k=1}^{n} \lambda_k (P_k x, y) = (Tx, y)$$

and we see that E_T is the spectral decomposition of T.

3.5 The symbolic calculus and some applications

For a fixed, normal operator $T \in B(H)$ and a bounded function f it will be convenient to denote $\int_{\sigma(T)} f \, dE$ by $f(T)$. In particular $T = \mathrm{id}(T)$ and $E(\omega) = \chi_\omega(T)$.

In the following, we will see some applications of the symbolic calculus. We start with a characterization of self-adjoint operators.

Theorem 3.5.1. *A normal $T \in B(H)$ is self-adjoint if and only if $\sigma(T) \subset \mathbb{R}$.*

Proof. Proposition (2.2.4) shows that self-adjoint elements have real spectra.

For the converse suppose that $\sigma(T) \subset \mathbb{R}$ and observe

$$T = \mathrm{id}(T) = \int_{\sigma(T)} \lambda \, dE(\lambda) = \int_{\sigma(T)} \bar{\lambda} \, dE(\lambda) = \overline{\mathrm{id}}(T) = (\mathrm{id}(T))^* = T^*.$$

$\square$

Next, we will use the symbolic calculus to obtain information about the eigenvalues of normal operators. In particular we will see how the eigenvalues can be characterized through the spectral decomposition.

Lemma 3.5.2. *Suppose $T \in B(H)$ is normal with spectral decomposition E and take an arbitrary $f \in C(\sigma(x))$. If $\omega_0 = f^{-1}(0)$, then:*

$$\mathcal{N}(f(T)) = \mathcal{R}(E(\omega_0)).$$

Proof. Consider the characteristic function χ_{ω_0} of ω_0. As $f\chi_0 = 0$, $f(T)\chi_{\omega_0}(T) = 0$ holds. But $\chi_{\omega_0}(T) = E(\omega_0)$ and therefore

$$\mathcal{R}(E_{\omega_0}) \subset \mathcal{N}(f(T)).$$

For the other inclusion define $\omega_n := \{\lambda \in \sigma(T) |\ 1/n \leq |f(\lambda)| \leq 1/(n-1)\}$ for every $n \in N$. (For $n = 1$, take all $\lambda \in \sigma(T)$ with $1 \leq |\lambda|$.) and let $\tilde{\omega}$ be the union of all ω_n. Then $\sigma(T)$ is the disjoint union of ω_0 and $\tilde{\omega}$.

On $\sigma(T)$ we define

$$f_n(\lambda) = \chi_{\omega_n} \frac{1}{f(\lambda)} \tag{3.17}$$

for every $n \in \mathbb{N}$. Every f_n is bounded and therefore $f_n(T)f(T) = E(\chi_{\omega_n})$.

Suppose $x \in \mathcal{N}(f(T))$. Then $E(\omega_n)x = 0$ for all n and therefore $E(\tilde{\omega})x = 0$ as the map $\omega \to E(\omega)x$ is countably additive. But $E(\omega_0) + E(\tilde{\omega}) = I$. Hence $E(\omega_0)x = x$ and $x \in \mathcal{R}(E(\omega_0))$. □

Proposition 3.5.3. *Let $T \in B(H)$ be a normal operator with spectral decomposition E and consider $\lambda_0 \in \sigma(T)$. Then :*

(1) *λ_0 is an eigenvalue of T if and only if $E(\{\lambda_0\}) \neq 0$.*

(2) *Every isolated $\lambda_0 \in \sigma(T)$ is an eigenvalue of T.*

Proof. (1) Consider the function $f(\lambda) = \lambda - \lambda_0$ on $\sigma(T)$. Using the notation above lemma (3.5.2) shows that

$$\mathcal{N}(T - \lambda_0 I) = \mathcal{N}(f(T)) = \mathcal{R}(E(\{\omega_0\}) = \mathcal{R}(E(\{\lambda_0\})),$$

which proves the equivalence.

(2) If λ_0 is isolated, it is open in $\sigma(T)$ and by theorem (3.4.2) $E(\{\lambda_0\}) \neq 0$. □

As a last application of the symbolic calculus, we show the existence of square roots for *positive* operators in $B(H)$.

Definition 3.5.4. $T \in B(H)$ is called *positive*, if $(Tx, x) \geq 0$ for every $x \in H$.

Lemma 3.5.5. *For $T \in B(H)$ the following are equivalent:*

(1) *T is positive.*

(2) *T is self-adjoint and $\sigma(T) \subset [0, \infty)$.*

Proof. We show first that (1) implies (2). As (Tx, x) is real-valued

$$(Tx, x) = (x, T^*x) = \overline{(T^*x, x)} = (T^*x, x)$$

holds for ever $x \in H$ and Proposition (3.1.3) shows that $T = T^*$. T is therefore self-adjoint and particularly $\sigma(T) \subseteq \mathbb{R}$.

To show that $\sigma(T)$ is non-negative, we use the fact that a normal operator $S \in B(H)$ is invertible if and only if there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in H$. The statement will not be proved here, but is proved in ([4], Chapter 12, 12.12).

Choose an arbitrary $\lambda > 0$ and observe that by (1)

$$\lambda\|x\|^2 = (\lambda x, x) \leq ((T + \lambda I)x, x) \leq \|T + \lambda I\|\|x\|^2.$$

That implies $T + \lambda I$ is invertible in $B(H)$. Hence $\sigma(T) \subseteq [0, \infty)$.

To see that (2) implies (1) let E be the spectral decomposition of T and observe that for any $x \in H$

$$(Tx, x) = \int_{\sigma(T)} \lambda \, dE_{x,x}(\lambda).$$

As $E_{x,x}$ is a positive measure and $\sigma(T) \subseteq [0, \infty)$ the integral is non-negative. Therfore $(Tx, x) \geq 0$. □

Proposition 3.5.6. *Let $T \in B(H)$ be positive. Then there exists an unique, positive $S \in B(H)$, such that $T = S^2$.*

Proof. Let B_T be the smallest closed, normal sub-algebra containing I and T. By the Gelfand-Naimark theorem (2.2.5) $\widehat{B_T} = C(\Delta)$, where Δ is the Gelfand space of B_T. $\widehat{T}$ maps Δ onto $\sigma(T)$ and therefore $\widehat{T} \geq 0$ by lemma (3.5.5). Hence there exists an unique continuous $\widehat{S_0} \geq 0$ on Δ, such that $\widehat{T} = \widehat{S_0}\widehat{S_0}$ holds. Now $T = {S_0}^2$ follows from the fact that the Gelfand transform is an isomorphism. S_0 is a positive operator by lemma (3.5.5) and because $\widehat{S_0} \geq 0$.

For the uniqueness suppose there exists another positive S, such that $T = (S^2)$. Let B_S be the smallest closed, normal sub-algebra containing I and S. Then $T \in B_S$, as $T = (S^2)$ and therefore $B_T \subset B_S$. Hence $S_0 \in B_S$ and by the uniqueness of $\widehat{S_0}$ we see that $S = S_0$. □

4 The spectral theorem for unbounded, normal operators

An important class of linear operators are the differential ones. However, the spectral theorem for bounded operators cannot be applied to them directly.

One problem is to choose a suitable domain for differential operators: Consider the Hilbert space $L^2(\mathbb{R})$ and the dense subspace $C_c^\infty(\mathbb{R})$. Every differential operator is well-defined on $C_c^\infty(\mathbb{R})$, but neither is $C_c^\infty(\mathbb{R})$ a Hilbert space, nor are differential operators necessarily bounded on $C_c^\infty(\mathbb{R})$.

However, we will see that the spectral theorem for *unbounded* operators can be applied to a large class of such differential operators, namely the *normal* ones.

4.1 Linear operators in H

Definition 4.1.1. Let H be a Hilbert space. Then T is called an operator *in H*, if T is defined on a subspace $\mathcal{D}(T)$ of H and T is linear from $\mathcal{D}(T)$ to H.

An operator S in H is an *extension* of T, if $\mathcal{D}(T) \subseteq \mathcal{D}(S)$ and $S \equiv T$ on $\mathcal{D}(T)$. We write $T \subset S$.

An operator T in H is *densely defined*, if $\mathcal{D}(T)$ is dense in H.

Example 4.1.2. (1) Consider the Hilbert space $L^2(\mathbb{R})$ and the dense subspace $C_c^\infty(\mathbb{R})$. Then the *Laplacian*

$$\Delta : C_c^\infty(\mathbb{R}) \to L^2(\mathbb{R}), \qquad \Delta(\varphi) := -\frac{\partial^2}{\partial x^2}(\varphi)$$

is a densely defined operator in $L^2(\mathbb{R})$.

A possible extension is $\tilde{\Delta}$ defined as the Laplacian on $C_c^2(\mathbb{R})$.

(2) For $f \in C(\mathbb{R})$ define the multiplication operator

$$M_f : C_c^\infty(\mathbb{R}) \to L^2(\mathbb{R}), \qquad \varphi \mapsto (x \mapsto f\varphi(x)).$$

Then M_f is another example of a densely defined operator in $L^2(\mathbb{R})$ and an extension is obtained by defining M_f on the domain $\mathcal{D}(M_f) := \{\varphi \in L^2(\mathbb{R}) \mid f\varphi \in L^2(\mathbb{R})\}$.

Remark 4.1.3. It is important to watch the domains when adding or composing opertors in H:

Consider the operator $\frac{\partial}{\partial x} : C_c^1(\mathbb{R}) \to L^2(\mathbb{R})$ in $L^2(\mathbb{R})$. Then the composition $\frac{\partial}{\partial x} \circ \frac{\partial}{\partial x}$ is not well-defined on $C_c^1(\mathbb{R})$. We have to restrict $\mathcal{D}(\frac{\partial}{\partial x} \circ \frac{\partial}{\partial x})$ to those $\varphi \in C_c^1(\mathbb{R})$, such that $\frac{\partial}{\partial x}(\varphi) \in C_c^1(\mathbb{R})$.

For T, S in H, we therefore define $T + S$ and $T \circ S$ on the following domains:

$$\mathcal{D}(T+S) = \mathcal{D}(T) \cap \mathcal{D}(S)$$
$$\mathcal{D}(T \circ S) = \{x \in \mathcal{D}(S) \mid Sx \in \mathcal{D}(T)\}$$

With this the associative laws hold for addition and composition. The distributive law holds for right-multiplication, but only in the form $TR + TS \subset T(R+S)$ for left-multiplication.

As in the bounded case, one can define an adjoint for densely defined operators.

Definition 4.1.4. Let T be a densely defined operator in H.
Define $\mathcal{D}(T^*)$ to be the set of all $y \in H$, such that the map $x \to (Tx, y)$ is continuous on $\mathcal{D}(T)$.
$\mathcal{D}(T^*)$ is a sub space of H and for $y \in \mathcal{D}(T^*)$ the map $x \to (Tx, y)$ can be extended continuously to H by the Hahn-Banach theorem ([5], Chapter 5, 5.16). The extension is unique, as $\mathcal{D}(T)$ is dense in H. Hence there exists an unique $T^*y \in H$, such that

$$(Tx, y) = (x, T^*y)$$

for all $x \in \mathcal{D}(T)$. T^* is a linear operator in H and it is called the *adjoint of* T in H.
Note that T^* is not necessarily densely defined, although T is.

Now that the adjoint of an operator in H is defined, we will turn to self-adjoint operators in H. It will be important to watch the domains of $\mathcal{D}(T)$ and $\mathcal{D}(T^*)$.

Definition 4.1.5. Suppose T is a densely defined operator in H with adjoint T^*. Then T is called *symmetric*, if $T \subset T^*$ or equivalently $\mathcal{D}(T) \subset \mathcal{D}(T^*)$ and $Ty = T^*y$ for every $y \in \mathcal{D}(T)$.
T is called *self-adjoint*, if $T = T^*$.

Example 4.1.6. We continue example (4.1.2):
(1) The Laplacian on $C_c^\infty(\mathbb{R})$ is symmetric, but not self-adjoint: For $\varphi, \psi \in \mathbb{C}_c^\infty(\mathbb{R})$

$$(\Delta\varphi, \psi)_{L^2} = -\int_{\mathbb{R}} \frac{\partial^2}{\partial x^2}(\varphi)\overline{\psi} d\lambda = -\int_{\mathbb{R}} \varphi \overline{\frac{\partial^2}{\partial x^2}(\psi)} d\lambda = (\varphi, \Delta\psi)_{L^2},$$

where the equality holds by applying integration by parts twice and the fact that $\varphi, \psi \in \mathbb{C}_c^\infty(\mathbb{R})$. Δ is not self-adjoint, as the equation holds for every $\varphi \in C_c^2(\mathbb{R})$.
(2) The adjoint of the multiplication operator M_f on $C_c^\infty(\mathbb{R})$ is $M_{\overline{f}}$ on some subspace including $C_c^\infty(\mathbb{R})$. This is directly obtained by the definition of the scalar product in $L^2(\mathbb{R})$. Hence M_f is symmetric on $C_c^\infty(\mathbb{R})$, if f is real-valued.
(3) M_{x^2} on $\mathcal{D}(M_{x^2}) := \{\varphi \in L^2(\mathbb{R}) \mid f\varphi \in L^2(\mathbb{R})\}$ is self-adjoint:
The function $x \mapsto x^2$ is real-valued on $\mathbb{R}$ and every $\phi \in \mathcal{D}(M_{x^2})$ lies in $\mathcal{D}(M_{x^2}^*)$. Hence M_{x^2} is symmetric and it remains to show that $\mathcal{D}(M_{x^2}^*) \subset \mathcal{D}(M_{x^2})$:
Suppose $\psi \in \mathcal{D}(M_{x^2}^*)$. Then the map

$$\mathcal{D}(M_{x^2}) \to \mathbb{C}, \qquad \varphi \mapsto (M_{x^2}(\varphi), \psi)_{L^2}$$

is continuous by definition of $\mathcal{D}(M^*_{x^2})$ and can therefore be continuously extended to $L^2(\mathbb{R})$. Hence there exists $C > 0$, such that for every $\varphi \in \mathcal{D}M_{x^2}$

$$|(M_{x^2}(\varphi), \psi)_{L^2}|^2 \leq C\|\varphi\|^2. \tag{4.1}$$

Let $\varphi(x) = x^2\psi(x)\chi_{[-n,n]}$. Then $\varphi \in L^2(\mathbb{R})$ and inequality (4.1) becomes

$$\left(\int_{\mathbb{R}} x^2|\psi(x)|^2\chi_{[-n,n]}\ dx\right)^2 \leq C\int_{\mathbb{R}} x^2|\psi(x)|^2\chi_{[-n,n]}\ dx.$$

Now for all $n \in \mathbb{N}$

$$\int_{\mathbb{R}} x^2|\psi(x)|^2\chi_{[-n,n]}\ dx \leq C$$

and by the monotone convergence theorem we see that $\psi \in \mathcal{D}(M_{x^2})$.

It will be of interest, if symmetric operators admit self-adjoint extensions. We therefore define the following:

Definition 4.1.7. A symmetric operator T in H is *maximally symmetric*, if for every symmetric S in H, $T \subset S$ implies $T = S$.

Proposition 4.1.8. *Every self-adjoint T in H is maximally symmetric.*

Proof. Suppose $T \subset S$ and S is symmetric. Then $S \subset S^* \subset T^* = T$, where we use that $S^* \subset T^*$, if $T \subset S$. Hence $S = T$. □

The following observation for the adjoints of two operators in H. The proof is omitted here, but can be found in ([4], Chapter 13, 13.2).

Proposition 4.1.9. *Let S, T be densely defined operators in H. Then*

$$T^*S^* \subset (ST)^*$$

and equality holds, if $S \in B(H)$.

4.2 The graph of an operator in H

Let H be a Hilbert space with an inner product $(\cdot,\cdot)$. Then $H \times H$ is a Hilbert space with the inner product:

$$\langle\cdot,\cdot\rangle : H \times H \to \mathbb{C}, \qquad \langle\{a, b\}, \{c, d\}\rangle = (a, c) + (b, d)$$

Consider the mapping $V : H \times H \to H \times H$, $V\{a, b\} = \{-b, a\}$. Then $V^2 = -I$ and $\langle V\{a, b\}, V\{c, d\}\rangle = \langle\{a, b\}, \{c, d\}\rangle$ hold. V is in particular an isometry.

Definition 4.2.1. Let T be a densely defined operator in H. Then

$$\mathcal{G}(T) := \{\{x, Tx\} \mid x \in \mathcal{D}(T)\} \subset H \times H$$

is called the *graph of T*.

T is called *closed*, if $\mathcal{G}(T)$ is closed in $H \times H$ and an operator S in H is called the *closure* of T, if $\overline{\mathcal{G}(T)} = \mathcal{G}(S)$. Note that $T \subset S$ is equivalent to $\mathcal{G}(T) \subseteq \mathcal{G}(S)$.

The graphs of T and T^* are related through V in the following way:

Proposition 4.2.2. *Let T be a densely defined operator in H. Then*

$$\mathcal{G}(T^*) = (V\mathcal{G}(T))^\perp,$$

where $(V\mathcal{G}(T))^\perp$ is the orthogonal complement of $V\mathcal{G}(T)$ in $H \times H$.

Proof. Let $\{y, z\} \in \mathcal{G}(T^*)$. Then $\{Tx, y\} = \{x, z\}$ and therefore $\langle\{-Tx, x\}, \{y, z\}\rangle = 0$ for each $x \in \mathcal{D}(T)$ by the definition of $\langle\cdot,\cdot\rangle$. But this implies $\{y, z\} \in (V\mathcal{G}(T))^\perp$, hence $\mathcal{G}(T^*) \subset (V\mathcal{G}(T))^\perp$.

The other inclusion follows directly from reversing the above steps. □

Corollary 4.2.3. *The adjoint T^* of a densely defined operator T in H is closed.*

Proof. By proposition (4.2.2) we see that $\mathcal{G}(T^*)$ is the orthogonal complement of a sub space of $H \times H$. Hence $\mathcal{G}(T^*)$ is closed. □

Example 4.2.4. In example (4.1.6) we have seen, that M_{x^2} is self-adjoint on $\mathcal{D}(M_{x^2})$, hence it is closed by corollary (4.2.3).

We will prove some properties of closed, densely defined operators. In particular the adjoint of such operators will be densely defined.

Proposition 4.2.5. *Let T be a closed, densely defined operator in H. Then*

$$H \times H = V\mathcal{G}(T) \oplus \mathcal{G}(T^*).$$

Proof. Since $\mathcal{G}(T)$ is closed, so is $V\mathcal{G}(T)$, as V is an isometry. Hence

$$H \times H = V\mathcal{G}(T) \oplus (V\mathcal{G}(T))^\perp$$

and now proposition (4.2.2) shows that $(V\mathcal{G}(T))^\perp = \mathcal{G}(T^*)$. □

Remark 4.2.6. As $V^2 = -I$ and V contains angles, proposition (4.2.5) shows also:

$$H \times H = \mathcal{G}(T) \oplus V\mathcal{G}(T^*).$$

Proposition 4.2.7. *For a closed and densely defined operator T in H the adjoint T^* is denseley defined and*

$$T = T^{**}.$$

Proof. By remark (4.2.6) $H \times H = \mathcal{G}(T) \oplus V\mathcal{G}(T^*)$.

Suppose $x \perp \mathcal{D}(T^*)$ and let $y \in \mathcal{D}(T^*)$ be arbitrary. Then

$$\langle\{0, x\}, \{-T^*y, y\}\rangle = (0, -T^*y) + (x, y) = 0.$$

Therefore $\{0, x\} \in (V\mathcal{G}(T^*))^\perp$ and $\{0, x\} \in \mathcal{G}(T)$. This implies $x = T(0) = 0$ and $\mathcal{D}(T^*)$ is dense in H. It now makes sense to define T^{**} and proposition (4.2) shows

$$H \times H = V\mathcal{G}(T^*) \oplus \mathcal{G}(T^{**}). \tag{4.2}$$

Comparing equation (4.2) with remark (4.2.6) gives $\mathcal{G}(T) = \mathcal{G}(T^{**})$ and therefore $T = T^{**}$. □

The next proposition, especially (2), will be vital to prove the spectral theorem.

Proposition 4.2.8. *Let T be a closed, densely defined operator in H. Consider the operator $Q = I + T^*T$ with domain $\mathcal{D}(Q) = \mathcal{D}(T^*T)$ in H. Then the follwoing hold:*

(1) *The operator $Q : \mathcal{D}(Q) \to H$ is bijective.*
(2) *There are operators $B, C \in B(H)$ with $\|B\| \leq 1$, $\|C\| \leq 1$, such that*
   (a) $C = TB$
   (b) $BQ \subset QB = I$
   Furthermore the operator B is self-adjoint and $\sigma(B) \subset [0, 1]$.
(3) *If $TT^* = T^*T$, then $CB = BC$ and $BT \subset C$.*

Proof. We proof (1) and (2) together: Note first, that for $y \in \mathcal{D}(Q)$

$$\|y\|^2 \leq (y, y) + (Ty, Ty) = (y, y) + (y, T^*Ty) = (y, Qy) \leq \|y\|\|Qy\| \tag{4.3}$$

as $Ty \in \mathcal{D}(T^*)$. Hence $\|y\| \leq \|Qy\|$ and Q is one-to-one.

Proposition (4.2.5) gives the decomposition $H \times H = V\mathcal{G}(T) \oplus \mathcal{G}(T^*)$. This implies that for every $x \in H$ there exist unique $Bx, Cx \in H$, such that:

$$\left\{ \begin{array}{c} 0 \\ x \end{array} \right\} = \left\{ \begin{array}{c} -TBx \\ Bx \end{array} \right\} + \left\{ \begin{array}{c} Cx \\ T^*Cx \end{array} \right\} \tag{4.4}$$

The vectors on the right-hand are orthogonal. B and C are therefore linear and

$$\|x\|^2 \geq \|Bx\|^2 + \|Cx\|^2$$

by the definition of the norm in $H \times H$. Hence $B, C \in B(H)$ and $\|B\| \leq 1$, $\|C\| \leq 1$.

The first component of equation (4.4) shows $C = TB$. Using this, the second component gives

$$x = Bx + T^*Cx = Bx + T^*TBx = QBx$$

and therefore $I = QB$. Hence Q is onto and B is one-to-one from H onto $\mathcal{D}(Q)$.

Suppose $y \in \mathcal{D}(Q)$ and take $x \in H$ such that $y = Bx$. Then

$$BQy = BQBx = Bx = y$$

and we see that $BQ \subset I$.

As Q is onto, we see that for every $x \in H$ there exists an $y \in \mathcal{D}(Q)$ satisfying $Qy = x$. Therefore

$$(Bx, x) = (BQy, Qy) = (y, Qy) \geq 0$$

by equation (4.3) and B is a positive operator. Hence $\sigma(B) \subset [0, 1]$, as $\|B\| \leq 1$ and (1) and (2) are proved.

(3) In (2) we saw, that $B(I + T^*T) \subset (I + T^*T)B = I$. If $T^*T = TT^*$, we obtain

$$BT = BT(I + T^*T)B = B(I + T^*T)TB \subset TB = C$$

and therefore $BC = B(TB) = (BT)B \subset CB$. As $B, C \in B(H)$, it follows that BC=CB and (3) is proved. ☐

Remark 4.2.9. The domain of the operator Q defined in proposition (4.2.8) is $D(T^*T)$. It turns out that $D(T^*T)$ is non-trivial. In fact, the following holds:

If T' is the restriction of T to $D(T^*T)$, then $\mathcal{G}(T')$ is dense in $\mathcal{G}(T)$ and T^*T is self-adjoint. For the proof see ([4], Chapter 13, 13.13).

Definition 4.2.10. A closed, densely defined operator N in H is called *normal*, if

$$N^*N = NN^*$$

holds. Particularly, $\mathcal{D}(N^*N) = \mathcal{D}(NN^*)$.

Note that every self-adjoint operator in H is normal.

The spectral theorem for unbounded operators will be stated for normal operators. They have the following properties, which are proved in ([4], chapter 13, 13.32).

Proposition 4.2.11. *Assume N is a normal operator in H. Then:*

(1) $\mathcal{D}(N) = \mathcal{D}(N^*)$
(2) $\|Nx\| = \|N^*x\|$ *for every* $x \in \mathcal{D}(N)$.
(3) N *is maximally normal.*

4.3 The spectral theorem for normal, unbounded operators

We begin with two lemmas, that will allow us to extend the results of theorem (3.4.1) and the spectral theorem for bounded operators to operators in H.

Lemma 4.3.1. *Let E be a spectral measure on $(\Omega, \mathcal{A})$ and $f : \Omega \to \mathbb{C}$ be measurable. Then*

$$\mathcal{D}_f := \left\{ x \in H \;\middle|\; \int |f|^2 \, dE_{x,x} < \infty \right\} \tag{4.5}$$

is a dense subspace of H. For $x, y \in H$

$$\int |f| \, d|E_{x,y}| \leq \|y\| \left(\int |f|^2 \, dE_{x,x} \right)^{1/2} \tag{4.6}$$

holds, where $|E_{x,y}|$ denotes the total variation of $E_{x,y}$.

Proof. Consider $x, y \in \mathcal{D}_f$, let $z = x + y$ and observe that

$$\|E(\omega)z\|^2 \leq (\|E(\omega)x\| + \|E(\omega)y\|)^2 \leq 2\|E(\omega)x\|^2 + 2\|E(\omega)y\|^2.$$

Therefore $E_{z,z}(\omega) \leq 2(E_{x,x}(\omega) + E_{y,y}(\omega))$, which implies $z \in \mathcal{D}_f$. Scalar multiplication follows similarly and $\mathcal{D}_f$ is a subspace.

To see that $\mathcal{D}_f$ is dense in H, define

$$\omega_n = \{p \in \Omega \mid |f(p)| < n\} \tag{4.7}$$

for all $n \in \mathbb{N}$. If $x \in \mathcal{R}(E(\omega_n))$,

$$E(\omega)x = E(\omega)E(\omega_n)x = E(\omega \cap \omega_n)x$$

holds and for every $\omega \in \mathcal{A}$, $E_{x,x}(\omega) = E_{x,x}(\omega \cap \omega_n)$. Hence $x \in \mathcal{D}_f$, as

$$\int_\Omega |f|^2 \; dE_{x,x} = \int_{\omega_n} |f|^2 \; dE_{x,x} \leq n^2 \|x\|^2.$$

Since $\Omega = \bigcup \omega_n$ by (4.7), we obtain $y = E(\bigcup \omega_n)y = \lim E(\omega_n)y$ for every $y \in H$ and $\mathcal{D}_f$ is dense in H, since every $E(\omega_n)y$ lies in $\mathcal{D}_f$.

For (4.6), let $x, y \in H$ and assume first that f is bounded. As a consequence of the Radon-Nikodym theorem ([5], Chapter 6, 6.9, 6.12) there exists a function u on Ω, such that $|u| = 1$ and $uf \; dE_{x,y} = |f| \; d|E_{x,y}|$.

Theorem (3.4.1) shows that for $\Phi(fu) = \int fu \; dE$

$$\|\Phi(fu)\|^2 = \int |fu|^2 \; dE_{x,x} = \int |f|^2 \; dE_{x,x}$$

and therefore

$$\int |f| \; d|E_{x,y}| = (\Phi(fu)x, y) \leq \|y\| \|\Phi(fu)x\| = \|y\| \left(\int |f|^2 \; dE_{x,x} \right)^{1/2}.$$

If f is unbounded, define the sequence $\{|f|\chi_n\}_n \in \mathbb{N}$, where χ_n is the charateristic function of $\{|f| < n\|\}$. Every $|f|\chi_n$ is bounded and the functions are monotonically increasing: The monotone convergence theorem shows therefore that inequality (4.6) holds for unbounded f. □

Lemma 4.3.2. *Let E be a spectral measure on $(\Omega, \mathcal{A})$ and f measurable and bounded on Ω. Consider $u, v \in H$ with $v = \Phi(f)u$. Then*

$$dE_{x,v} = \bar{f} \; dE_{x,v},$$

where $\Phi(f) = \int f \; dE$.

Proof. Let g be an arbitrary bounded function on Ω and $\Phi(g) {=} \int g \; dE$. Then

$$\int_\omega g \; dE_{x,v} = (\Phi(g)x, v) = (\Phi(g)x, \Phi(f)u) = (\Phi(\bar{f})\Phi(g)x, u) = \int_\omega g\bar{f} \; dE_{x,u}$$

and therefore $dE_{x,v} = \bar{f} dE_{x,u}$. □

We continue with an extension of theorem (3.4.1) for unbounded functions on the one hand and operators in H on the other.

Theorem 4.3.3. *Let E a spectral measure E on $(\Omega, \mathcal{A})$. Then:*

(1) *For every measurable $f : \Omega \to \mathbb{C}$ exists a densely defined, closed operator $\Phi(f)$ in H with domain $\mathcal{D}_f$. For $x \in \mathcal{D}_f$, $y \in H$*

$$(\Phi(f)x, y) = \int f dE_{x,y} \tag{4.8}$$

and furthermore

$$\|\Phi(f)x\|^2 = \int |f|^2 dE_{x,x}. \tag{4.9}$$

(2) *If f, g are measurable, then*

$$\Phi(f) + \Phi(g) \subset \Phi(f+g) \qquad \text{and} \qquad \Phi(f)\Phi(g) \subset \Phi(fg)$$

hold. For the second equality, we obtain additionally $\mathcal{D}(\Phi(f)\Phi(g)) = \mathcal{D}_g \cap \mathcal{D}_{fg}$. Hence, $\Phi(f)\Phi(g) = \Phi(fg)$ if and only if $\mathcal{D}_{fg} \subset \mathcal{D}_g$.

(3) *For measurable f,*

$$\Phi(f)^* = \Phi(\bar{f}) \tag{4.10}$$

and every $\Phi(f)$ is normal. More precisely: $\Phi(f)\Phi(f)^* = \Phi(|f|^2) = \Phi(f)^*\Phi(f)$.

Proof. (1) Pick $x \in \mathcal{D}_f$ and define

$$\Lambda : H \to \mathbb{C}, \qquad \Lambda(y) = \int_\Omega f \; dE_{x,y}.$$

Λ is well-defined by lemma (4.3.1) and it is conjugate-linear, as $E_{x,y}$ is conjugate-linear in y. Lemma (4.3.1) also shows that $\|\Lambda\| \leq (\int |f|^2 \; dE_{x,x})^{1/2}$. Hence, there exists an unique $\Phi(f)x \in H$ satisfying (4.8), ([4], Chapter 12, 12.5). Additionally $\Phi(f+g)x = \Phi(f)x + \Phi(g)x$ as the integral is linear and

$$\|\Phi(f)x\|^2 \leq \int |f|^2 \; dE_{x,x}. \tag{4.11}$$

$\Phi(f)$ is also linear, as (4.8) holds and $E_{x,y}$ is linear in x.

For bounded f, the definition of $\Phi(f)$, coincides with the one in theorem (3.4.1).

Consider the sequence $\{f\chi_n\}_{n\in\mathbb{N}}$, where $\chi_n := \chi_{\{|f|<n\}}$. Every $f\chi_n$ is bounded and therefore $\mathcal{D}(f - f\chi_n) = \mathcal{D}(f)$. By (4.11) and the dominated convergence theorem

$$\begin{aligned} \lim_{n\to\infty} \|\Phi(f)x - \Phi(f\chi_n)x\|^2 &= \lim_{n\to\infty} \|\Phi(f - f\chi_n)x\|^2 \\ &\leq \lim_{n\to\infty} \int |f(1-\chi_n)|^2 \; dE_{x,x} = 0. \end{aligned} \tag{4.12}$$

Since the $f\chi_n$ are bounded, they satisfy (4.9) by theorem (3.4.1) and (4.12) shows that (4.9) holds for unbounded f.

Now (1) is proved, except for the fact that all $\Phi(f)$ are closed. This will follow from (3), namely $\Phi(f)^* = \Phi(\bar{f})$.

(2) We will only prove the part concerning the multiplicative equality. The additive part follows similarly.

Suppose first that g is measurable and f is bounded, hence $\mathcal{D}_g \subset \mathcal{D}_{fg}$. Consider $x \in \mathcal{D}(g), u \in H$, such that $v = \Phi(\bar{f})v$. Then lemma (4.3.2) and theorem (3.4.1) imply

$$(\Phi(f)\Phi(g)x, u) = (\Phi(g)x, \Phi(\bar{f})u) = \int g\ dE_{x,v} = \int gf\ dE_{x,u} = (\Phi(fg)x, u).$$

Hence,

$$\Phi(f)\Phi(g)x = \Phi(fg)x \tag{4.13}$$

and for $y = \Phi(g)x$, we obtain

$$\int_\Omega |f|^2 dE_{y,y} = \|\Phi(f)y\|^2 = \|\Phi(f)\Phi(g)x\|^2 = \int_\Omega |fg|^2 dE_{x,x}.$$

If $x \in \mathcal{D}_g$, the equality

$$\int_\Omega |f|^2 dE_{y,y} = \int_\Omega |fg|^2 dE_{x,x} \tag{4.14}$$

remains valid for unbounded f, as it is true for every $|f|\chi_n$ and we can apply the monotone convergence theorem.

By definition $\mathcal{D}(\Phi(f)\Phi(g))$ consists of all $x \in \mathcal{D}_g$, such that $y = \Phi(g)x \in \mathcal{D}_f$. Now (4.14)implies $y \in \mathcal{D}_f$ if and only if $x \in \mathcal{D}_{fg}$. Hence

$$\mathcal{D}(\Phi(f)\Phi(g)) = \mathcal{D}_g \cap \mathcal{D}_{fg}.$$

Consider $x \in \mathcal{D}_g \cap \mathcal{D}_{fg}$ and let $y = \Phi(g)x$. Then

$$f\chi_n \to f \text{ in } L^2(E_{y,y}) \qquad \text{and} \qquad f\chi_n g \to fg \text{ in } L^2(E_{x,x}).$$

Therefore, by (4.9), $\Phi(f\chi_n)y \to \Phi(f)y$ and $\Phi((f\chi_n)g)x \to \Phi(fg)x$. Now (4.13) proves (2), as

$$\Phi(f)\Phi(g)x = \Phi(f)y = \lim_{n\to\infty} \Phi(f_n)y = \lim_{n\to\infty} \Phi(f_n)\Phi(g)x = \lim_{n\to\infty} \Phi(f_n g)x = \Phi(fg)x.$$

(3) Note that $\mathcal{D}_f = \mathcal{D}_{\bar{f}}$ and let $x, y \in \mathcal{D}_f$. By (4.12) and theorem (3.4.1)

$$(\Phi(f)x, y) = \lim_{n\to\infty} (\Phi(f\chi_n)x, y) = \lim_{n\to\infty} (x, \Phi(\overline{f\chi_n})y) = (x, \Phi(\overline{f})y)$$

holds and therefore

$$\Phi(\bar{f}) \subset \Phi(f)^*.$$

It remains to show that $\mathcal{D}(\Phi(f)^*) \subset \mathcal{D}_{\bar{f}}$. Let $w \in \mathcal{D}(\Phi(f)^*)$ and $v = \Phi(f)^* u$.

As χ_n is real-valued and bounded, $\Phi(\chi_n)$ is self-adjoint and we obtain $\Phi(f)\Phi(\chi_n) = \Phi(f\chi_n)$. Now by proposition (4.1.9)

$$\Phi(\chi_n)\Phi(f)^* = (\Phi(f)\Phi(\chi_n))^* = \Phi(f\chi_n)^* = \Phi(\overline{f\chi_n}).$$

Hence, $\Phi(\chi_n)v = \Phi(\overline{f\chi_n})u$. Therefore

$$\int_\Omega |f|^2 \, dE_{u,u} = \lim_{n\to\infty} \|\Phi(\overline{f\chi_n})u\|^2 = \lim_{n\to\infty} \|\Phi(\chi_n)v\|^2$$
$$= \lim_{n\to\infty} \int_\Omega \chi_n \, dE_{v,v} \leq E_{v,v}(\Omega) < \infty.$$

for all $n \in \mathbb{N}$ and $u \in \mathcal{D}_f$ by (4.9).

Every measure E_x, x is finite and therefore $\mathcal{D}_{f\bar{f} \subset \mathcal{D}_f}$. Hence (3) is proved, as

$$\Phi(f)\Phi(f)^* = \Phi(f)\Phi(\bar{f}) = \Phi(|f|^2).$$

□

Definition 4.3.4. Let T be an operator in H. Then the *resolvent set* $r(T) \subset \mathbb{C}$ consists of all $\lambda \in \mathbb{C}$ for which $T - \lambda I$ has an inverse in $B(H)$. More precisely, there an exists $S \in B(H)$ with $S(T - \lambda I) \subset (T - \lambda I)S = I$. The *spectrum* $\sigma(T)$ of T is the complement of $r(T)$ in $\mathbb{C}$.

Proposition 4.3.5. *Let E be a spectral measure on $(\Omega, \mathcal{A})$, $f : \Omega \to \mathbb{C}$ measurable and*

$$\omega_\lambda = \{p \in \Omega \mid f(p) = \lambda\}. \tag{4.15}$$

The following hold:

(1) *If $E(\omega_\lambda) \neq 0$, then $\Phi(f) - \lambda I$ is not one-to-one.*
(2) *If λ is in the essential range of f and $E(\omega_\lambda) = 0$, then $\Phi(f) - \lambda I$ is one-to-one and maps $\mathcal{D}_f$ onto a proper subspace of H.*
(3) *$\sigma(\Phi(f))$ is the essential range of f.*

Proof. We suppose $\lambda = 0$ without loss of generality.

(1) Let χ_0 be the characteristic function of ω_0. Then $f\chi_0 = 0$ and therefore $\Phi(f)\Phi(\chi_0) = \Phi(f)E(\omega_\lambda) = 0$.

As $E(\omega_\lambda) \neq 0$, there exists $x_0 \in \mathcal{R}(E(\omega_0)) \setminus \{0\}$. Now (1) follows from

$$\Phi(f)x_0 = \Phi(f)E(\omega_\lambda)x_0 = 0.$$

(2) We prove first that $\Phi(f)$ is one-to-one: Suppose $\Phi(f)x = 0$ for $x \in \mathcal{D}_f$. Then

$$\int_\Omega |f|^2 \, dE_{x,x} = \|\Phi(f)x\|^2 = 0.$$

The complex measure $E_{x,x}$ is positive and by assumption $|f| > 0$ $E_{x,x}$-a.e., as $\Phi(f)$ is not 0. Therefore $E_{x,x}(\Omega) = \|x\|^2 = 0$, hence $x = 0$.

For the second part consider the sets

$$\omega_n := \{p \in \Omega \mid |f(p)| < 1/n\}$$

for all $n \in \mathbb{N}$. Since 0 lies in the essential range of f, $E(\omega_n) \neq 0$ for every n and there are $x_n \in \mathcal{R}(E(\omega_n))$ with $\|x_n\| = 1$. Let χ_n be the characteristic function of ω_n and observe that by theorem (3.4.1)

$$\|\Phi(f)x_n\| = \|\Phi(f)E(\omega_n)x_n\| = \|\Phi(f\chi_n)x_n\| \leq \|\Phi(f\chi_n)\| = \|f\chi_n\|_\infty < 1/n.$$

Assume $\mathcal{R}(\Phi(f)) = H$. Since T is closed, so is T^{-1} and the closed graph theorem ([4], Chapter 2, 2.14, 2.15) shows that T^{-1} is continuous. But this is a contradiction, as $\|x_n\| = 1$, although $\lim \Phi(f)x_n = 0$.

(3) The essential range of f is contained in $\sigma(\Phi(f))$ by (1) and (2). For the other inclusion suppose that 0 is not in the essential range of f. Then $g := 1/f \in L^\infty(E)$ and in particular $fg = 1$. Therefore $\Phi(f)\Phi(g) = I$ and $\mathcal{R}(\Phi(f)) = H$. $E(\omega_0) = 0$, hence $|f| > 0$ E-a.e. and $\Phi(f)$ is one-to-one, as was shown in (2). It follows from the closed graph theorem ([4], Chapter 2, 2.14, 2.15) that the inverse T^{-1} lies in $B(H)$, hence $0 \notin \sigma(\Phi(f))$ and (3) is proved. □

We are now capable of proving the spectral theorem for unbounded, normal operators.

Theorem 4.3.6. *For every normal operator N in H exists an unique spectral measure E such that*

$$(Nx, y) = \int_{\sigma(N)} \lambda \, dE_{x,y}(\lambda) \tag{4.16}$$

for all $x \in \mathcal{D}(N), y \in H$ and $E(\sigma(N)) = I$.

Proof. We prove the existence of E in two steps:

Firstly, we will find self-adjoint projections $P_i \in B(H)$ with pairwise orthogonal ranges, such that:

(1) $x = \sum_{n=1}^{\infty} P_i x$ for all $x \in H$.

(2) $P_i N \subset NP_i \in B(H)$ and all NP_i are normal.

Secondly, the spectral theorem for bounded operators will be applied to every NP_i and the corresponding spectral measures $E^i := E_{NP_i}$ will be used to define E.

(1) Recall that by proposition (4.2.8) there are $B, C \in B(H)$, such that

$$C = NB$$

and $CB = BC$, as N is normal.

The operator B is positive and $\|B\| \leq 1$. Hence the spectral decomposition E_B of B is concentrated on $\sigma(B) \subset [0,1]$. Proposition (4.2.8) shows also, that B is one-to-one and therefore $E_B(\{0\}) = 0$ by proposition (4.3.5). Hence

$$E_B((0,1]) = I. \tag{4.17}$$

Consider the sequence $\{\frac{1}{i}\}_{i \in \mathbb{N}\setminus\{0\}}$ and let $\chi_i := \chi_{(\frac{1}{i+1},\frac{1}{i}]}$. Using the symbolic calculus, we define

$$P_i = \chi_i(B). \tag{4.18}$$

Every P_i is a self-adjoint projection, as $\chi_i(B) = E((\frac{1}{i+1}, \frac{1}{i}])$ and moreover

$$\sum_{i=1}^{\infty} P_i x = \sum_{i=1}^{\infty} E((\tfrac{1}{i+1}, \tfrac{1}{i}])x = E_B((0,1])x = x \tag{4.19}$$

by (4.17) and the fact that $\omega \mapsto E_B(\omega)x$ is countably additive. Hence (1) is proved.

Define

$$f_i(t) = \frac{\chi_i(t)}{t}$$

on $(0,1]$ and note that f_i is bounded. Hence, $P_i = Bf_i(B) = f_i(B)B$. By proposition (4.2.8) $C \subset BN$ and $C = NB$. Therefore the following equalities hold:

$$P_iN = \ f_i(B)BN \ \subset f_i(B)C. \tag{4.20}$$

$$NP_i = \ NBf_i(B) \ = Cf_i(B). \tag{4.21}$$

Since C commutes with B, C commutes with every $f_i(B)$ and therefore

$$P_iN \subset NP_i. \tag{4.22}$$

From the second equality it also follows, that $NP_i \in B(H)$. In particular, this implies $\mathcal{R}(P_i) \subset \mathcal{D}(N)$ and if $x \in \mathcal{R}(P_i)$, then $Nx = NP_ix = P_iNx$ holds by (4.22). $\mathcal{R}(P_i)$ is therefore an invariant subspace of $N$and this shows especially that

$$P_i(NP_i) = (NP_i)P_i. \tag{4.23}$$

To see that NP_i is normal, use proposition (4.1.9) to obtain $(NP_i)^* \subset (P_iN)^* = N^*P_i$. As $(NP_i)^* \in B(H)$, we obtain $(NP_i)^* = N^*P_i$ and by proposition (4.2.11)

$$\|(NP_i)^*x\| = \|N^*P_ix\| = \|NP_ix\|$$

holds. NP_i is therefore normal by proposition (4.2.11) and the first part is proved.

(2) Let E^i be the spectral decomposition of each NP_i. P_i commutes with NP_i by (4.23) and therefore with every projection $E^i(\omega)$. Hence

$$E^i(\omega)P_ix = P_iE^i(\omega)x \in \mathcal{R}(P_i)$$

for all Borel sets $\omega \in \mathbb{C}$ and $i =\in \mathbb{N} \setminus \{0\}$. Note that the ranges $\mathcal{R}(P_i)$ are pairwise orthogonal. Since $\|P\| \leq 1$ for every self-adjoint projection,

$$\sum_{i=1}^{\infty} \|E^i(\omega)P_ix\|^2 \leq \sum_{i=1}^{\infty} \|P_ix\|^2 = \|x\|^2 \tag{4.24}$$

by (4.19) and proposition (3.1.2). Then

$$E(\omega) = \sum_{i=1}^{\infty} E^i(\omega)P_i \in B(H) \tag{4.25}$$

for every Borel set ω by proposition (3.1.2) and equation (4.24).

In fact, E is a spectral measure. We show, that $E(\omega)$ it is a self-adjoint projection for every Borel set ω and that $\omega \to (E(\omega)x, y)$ is a complex measure. The other properties follow similarly. For $x \in H$

$$E(\omega)^2 x = \sum_{i=1}^{\infty} E^i(\omega) P_i \left(\sum_{j=1}^{\infty} E^j(\omega) P_j x \right) = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} E^i(\omega) P_i E^j(\omega) P_j x \right)$$
$$= \sum_{i=1}^{\infty} E^i(\omega) P_i x = E(\omega)x,$$

as $E^i(\omega)P_i = P_i E^i(\omega)$ and $P_i P_j = 0$ for $i \neq j$. Hence, E is a projection.

$E(\omega)$ is self-adjoint, as $\sum_{i=1}^{\infty} E^i(\omega)P_i x$ converges for every $x \in H$ and the inner product is continuous.

Suppose now, $(\omega_n)_{n\in\mathbb{N}}$ are pairwise disjoint and let $\tilde{\omega} = \bigcup_{n\in\mathbb{N}} \omega_n$. For $x, y \in H$:

$$(E(\tilde{\omega})x, y) = \sum_{i=1}^{\infty} (E^i(\tilde{\omega})P_i x, y) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} (E^i(\omega_n)P_i x, y).$$

The sum converges absolutely, as

$$\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} |(E^i(\omega_n)P_i x, y)| \leq \|y\| \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \|(E^i(\omega_n)P_i x\| = \|y\| \|E(\tilde{\omega})x\| < \infty$$

and we can change the order of summation. Therefore

$$(E(\tilde{\omega})x, y) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} (E^i(\omega_n)P_i x, y) = \sum_{n=1}^{\infty} (E(\tilde{\omega})x, y)$$

and E is a spectral measure.

Now by theorem (4.3.3) there exists a closed operator M in H, such that

$$(Mx, y) = \int \lambda \, dE_{x,y}(\lambda) \tag{4.26}$$

for $x \in \mathcal{D}(M) = \mathcal{D}_{id}$, $y \in H$.

We show $N \subset M$ and then $N = M$ follows from the maximality of N.

Consider $x \in H$ and let $x_i = P_i x$. Then for any Borel set $\omega \subset \mathbb{C}$:

$$(E(\omega)x, x) = \|E(\omega)x\|^2 = \sum_{i=1}^{\infty} \|E^i(\omega)P_i x\|^2 = \sum_{i=1}^{\infty} E^i_{x_i,x_i}(\omega). \tag{4.27}$$

Now pick $x \in \mathcal{D}(N)$. Since $NP_i x = P_i N x$, we obtain

$$\sum_{i=1}^{\infty} \int_{\mathbb{C}} |\lambda|^2 dE^i_{x_i,x_i}(\lambda) = \sum_{i=1}^{\infty} \|NP_i x_i\|^2 = \sum_{i=1}^{\infty} \|P_i N x_i\|^2 = \|Nx\|^2. \tag{4.28}$$

and now it follows from (4.27), that

$$\mathcal{D}(N) \subset \mathcal{D}(M).$$

Finally suppose $x \in \mathcal{R}(P_i)$. Since $x = P_i x$, we obtain $E(\omega)x = E^i(\omega)x$ and the measures $E_{x,y}$ and $E^i_{x,y}$ are therefore equal. This implies

$$(Nx, y) = (NP_i x, y) = \int_{\mathbb{C}} \lambda \, dE^i_{x,y} \int_{\mathbb{C}} \lambda \, dE_{x,y} = (Mx, y) \tag{4.29}$$

for every $y \in H$. Observe that for $x \in \mathcal{D}(N)$

$$P_i N x = N P_i x = M P_i x.$$

Therefore $\big(\sum_{i=1}^k P_i\big)Nx = M\big(\sum_{i=1}^k P_i\big)x$ for all $k \in \mathbb{N}$ and

$$\left\{\left(\sum_{i=1}^k P_i\right)x, \left(\sum_{i=1}^k P_i\right)Nx\right\} \in \mathcal{G}(M).$$

But $\mathcal{G}(M)$ is closed and $(\sum_{i=1}^\infty P_i)x = x$. Therefore $\mathcal{G}(N) \subset \mathcal{G}(M)$.

Proposition (4.3.5), shows that $\sigma(N)$ is the essential range of the identity with respect to E. Therefore $E(\sigma(E)) = I$ and (2) is proved.

For the uniqueness we use the fact that for every *positive* operator A in H, there exists an unique, positive operator $\sqrt{A}$ in H, such that $A = \sqrt{A} \circ \sqrt{A}$. (An operator A in H is called *positive*, if $(Ax, x) \geq 0$ for all $x \in \mathcal{D}(A)$.) As in the bounded case positivity is equivalent to A being self-adjoint and $\sigma(A) \subset [0, \infty)$. For the proof see ([4], Chapter 13, 13.31).

For every $x \in \mathcal{D}(N^*N) \subset \mathcal{D}(N)$ we obtain

$$(N^*Nx, x) = (Nx, N^{**}x) = (Nx, Nx) \geq 0,$$

as N is closed and therefore $N = N^{**}$. The operator N^*N is positive and therefore it makes sense to define

$$T = N(I + \sqrt{N^*N})^{-1}.$$

The inverse of $I + \sqrt{N^*N}$ exists, as $\sqrt{N^*N}$ is positive and by proposition (4.2.8).

Let E_N be a spectral measure satisfying $N = \int \lambda \, dE_N(\lambda)$. Then

$$N^*N = \int_{[0,\infty)} \lambda\overline{\lambda} \, dE_N(\lambda) = \int_{[0,\infty)} |\lambda| \, dE_N(\lambda) \int_{[0,\infty)} |\lambda| \, dE_N(\lambda) \tag{4.30}$$

by theorem (4.3.3), hence $\sqrt{N^*N} = \int_{[0,\infty)} |\lambda| \, dE_N(\lambda)$.

Consider $f(\lambda) = \frac{\lambda}{1+|\lambda|}$ and observe that theorem (4.3.3) implies

$$T = \int \frac{\lambda}{(1 + |\lambda|)} \, dE_N.$$

Note that equality holds, as $\lambda \mapsto (1 + |\lambda|)^{-1}$ is bounded.

The function f is also bounded, hence $\mathcal{D}(T) = \mathcal{D}_f = H$. Theorem (4.3.3) shows that T is closed and T is therefore continuous by the closed graph theorem ([4], Chapter 2, 2.14, 2.15). As f is also one-to-one, we can apply proposition (3.3.7) to define the spectral measure $E(\omega) = E_N(f(\omega))$ and obtain

$$T = \int (f \circ f^{-1})\, dE = \int \lambda\, dE(\lambda). \tag{4.31}$$

Theorem (4.3.3) shows that E is and so is E_N. □

4.4 The Laplacian on $\mathbb{R}$

The spectral theorem for unbounded, normal operators will be applied to the Laplacian on $\mathbb{R}$. The *Fourier transform* will be the main tool for this:

Definition 4.4.1. ***(Fourier transform)*** To begin with the *Fourier transform* is defined as the linear, continuous map

$$\tilde{\mathcal{F}} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R}), \qquad f \mapsto \left(t \to \int_{\mathbb{R}} f(x) e^{-2\pi i x t}\, dx\right). \tag{4.32}$$

However, the Parseval formula ([5], Chapter 9, 9.13) shows that for $\varphi \in C_c^\infty(\mathbb{R})$ $\|\varphi\|_{L^2} = \|\tilde{\mathcal{F}}(\varphi)\|_{L^2}$. Note that $\tilde{\mathcal{F}}(\varphi)$ does a priori not lie in $L^2(\mathbb{R})$.

As $C_c^\infty(\mathbb{R})$ lies dense in $L^2(\mathbb{R})$, the Fourier transform extends isometrically to a linear operator on $L^2(\mathbb{R})$

$$\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \tag{4.33}$$

which is surjective ([5], Chapter 9, 9.13). $\mathcal{F}$ is therefore an isometric bijection and ([4], Chapter 12, 12.13) shows that, $\mathcal{F}^* = \mathcal{F}^{-1}$.

In general an operator $T \in B(H)$ is called *unitary*, if $T^* = T^{-1}$ holds.

The Fourier transform has the important property, that for every $\varphi \in C_c^\infty(\mathbb{R})$

$$\mathcal{F}\left(\frac{\partial^n}{\partial x^n}\varphi\right) = (2\pi i x)^n \mathcal{F}(\varphi) \tag{4.34}$$

holds, see ([4], Chapter 7, 7.4).

Definition 4.4.2. Let T, S be operators in H. Then we say T is *unitarily equivalent* to S, if there exists an unitary $U \in B(H)$, such that $S = U^{-1}TU$. Especially $\mathcal{D}(S) = \{x \in H \mid Ux \in \mathcal{D}(T)\}$.

Remark 4.4.3. For unitarily equivalent operators S, T in H, the following hold:
(1) T self-adjoint implies S self-adjoint.
(2) T closed implies S closed.

Proposition 4.4.4. *Consider the Laplacian $\Delta = -\frac{\partial^2}{\partial x^2}$ on $C_c^\infty(\mathbb{R})$ and let $\mathcal{F}$ be the Fourier transform on $L^2(\mathbb{R})$. Then the following hold:*
(1) *Δ is unitarily equivalent to the multiplication operator $M_{4\pi^2 x^2}$ on $\mathcal{F}(\mathcal{D}(C_c^\infty(\mathbb{R})))$.*

(2) $\mathcal{F}^{-1}M_{4\pi^2x^2}\mathcal{F}(\varphi)$ *on* $\mathcal{F}^{-1}(\mathcal{D}(M_{x^2}))$ *is a self-adjoint extension of* Δ.
(3) *The extension in* (2) *is the only self-adjoint extension of* Δ *on* $C_c^\infty(\mathbb{R})$.

Proof. (1) Let $\varphi \in C_c^\infty(\mathbb{R})$. Equation (4.34) implies

$$\mathcal{F}(\Delta\varphi) = M_{4\pi^2x^2}\mathcal{F}(\varphi),$$

which shows in particular, that $M_{4\pi^2x^2}$ is well-defined on $\mathcal{F}(\mathcal{D}(C_c^\infty(\mathbb{R})))$. Applying the inverse of the Fourier transform to both sides proves (1), as

$$\Delta\varphi = \mathcal{F}^{-1}M_{4\pi^2x^2}\mathcal{F}(\varphi). \tag{4.35}$$

(2) In example (4.1.6) it was shown, that $M_{4\pi^2x^2}$ is self-adjoint on $D(M_{4\pi^2x^2})$. We can therefore use remark (4.4.3) and obtain: $\mathcal{F}^{-1}M_{4\pi^2x^2}\mathcal{F}$ on $\mathcal{F}^{-1}(\mathcal{D}(M_{x^2}))$ is a self-adjoint extension of Δ.

(3) We that the closure $\bar{\Delta}$ of the laplacian is self-adjoint. Then

$$\bar{\Delta} \subset \mathcal{F}^{-1}M_{4\pi^2x^2}\mathcal{F},$$

because self-adjoint operators are closed and $\bar{\Delta}$ is the smallest closed extension of the Laplacian. Since self-adjoint operators are maximally self-adjoint by proposition (4.1.8), equality holds . The same arguments works for any self-adjoint extension of Δ and uniqueness follows.

Note firstly that $\overline{\mathcal{G}(\Delta)}$ is actually the graph of an operator in H, as it is a subset of $\mathcal{G}(\mathcal{F}^{-1}M_{4\pi^2x^2}\mathcal{F})$.

To show that $\bar{\Delta}$ is self-adjoint we use the following criterion from ([3], Chapter VIII, Coro. of VIII.3): The closure $\bar{T}$ of a symmetric operator T in H is self-adjoint, if $\mathcal{R}(T+iI)$ and $\mathcal{R}(T-iI)$ are dense in H.

Suppose $\varphi \in L^2(\mathbb{R})$ is orthogonal to $\mathcal{R}(\Delta+i)$. Then $0 = (\varphi, (\Delta+iI)\psi)$ for all $\psi \in C_c^\infty(\mathbb{R})$ and applying the Fourier transform gives

$$0 = (F(\varphi), \mathcal{F}((\Delta+i)\psi)) = (\mathcal{F}(\varphi), (M_{4\pi^2}+i)\mathcal{F}(\psi)) = ((M_{4\pi^2}+i)\mathcal{F}(\varphi), \mathcal{F}(\psi)).$$

Since $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, so is $\mathcal{F}(C_c^\infty(\mathbb{R}))$, as the Fourier transform is a bijective isometry. Therefore

$$0 = (M_{4\pi^2}+iI)\mathcal{F}(\varphi)$$

and $\varphi = 0$, as $\mathcal{F}(\varphi) = 0$. Hence, $\mathcal{R}(\Delta+iI)$ is dense in H. The same arguments hold for $-i$ and we see that $\bar{\Delta}$ is self-adjoint. □

We will now determine the spectral decomposition of the multiplication operator M_{x^2}. Note first, that $M_{x^2} - \alpha I = M_{x^2-\alpha}$ has a bounded inverse $M_{1/(x^2-\alpha)}$ for every $\alpha \in \mathbb{C}\setminus[0,\infty)$. The spectrum of M_{x^2} is therefore a subset of $[0,\infty)$.

Proposition 4.4.5. *The spectral decomoposition* E*of the Multiplication operator* M_{x^2} *is defined on the Borel subsets of* $[0,\infty)$ *and for* $\omega \in \mathcal{B}([0,\infty))$

$$E(\omega) = M_{\chi_\omega(\cdot)^2}, \tag{4.36}$$

where χ_ω *is the characteristic function of* ω.

Proof. Showing that E is a spectral measure is straightforward. Fix $\omega \in \mathcal{B}([0,\infty))$ and $\varphi, \psi \in L^2(\mathbb{R})$. Then $E(\omega)$ is a self-adjoint projection, as $\chi_\omega \circ \chi_\omega = \chi_\omega$ and χ_ω is real-valued. Finite additivity and multiplicativity follow directly from the properties of characteristic functions. For $\varphi, \psi \in L^2(\mathbb{R})$, we obtain

$$E_{\varphi,\psi}(\omega) = (E(\omega)\varphi, \psi)_{L^2} = \int_{\mathbb{R}} \chi_\omega(x^2)\varphi(x)\psi(x)\ dx$$

and $E_{\varphi,\psi}$ is therefore a complex measure.

We will now proceed by showing that $E_{\varphi,\psi}$ is the sum of two measures induced by complex density functions. For $\omega \in \mathcal{B}([0,\infty))$, we obtain:

$$E_{\varphi,\psi}(\omega) = \int_{\mathbb{R}^+} \chi_\omega(x^2)\varphi(x)\psi(x)\ dx + \int_{\mathbb{R}^-} \chi_\omega(x^2)\varphi(x)\psi(x)\ dx$$

We apply change of variables to both parts of the sum. For the first one we use $\mathbb{R}^+ \to \mathbb{R}^+, x \mapsto \sqrt{x}$, for the second one $\mathbb{R}^- \to \mathbb{R}^-, x \mapsto -\sqrt{-x}$. Here $\sqrt{\cdot} : \mathbb{R}^+ \to \mathbb{R}^+$ sends a positive $x \in \mathbb{R}$ on its unique positive square root $\sqrt{x}$. We obtain:

$$E_{\varphi,\psi}(\omega) = \int_{\mathbb{R}^+} \frac{\chi_\omega(x)}{2\sqrt{x}}\varphi(\sqrt{x})\psi(\sqrt{x})\ dx + \int_{\mathbb{R}^-} \frac{\chi_\omega(-x)}{2\sqrt{-x}}\varphi(-\sqrt{-x})\psi(-\sqrt{-x})\ dx.$$

Another change of variables to the second part, namely $\mathbb{R}^+ \to \mathbb{R}^-, x \mapsto -x$, gives

$$E_{\varphi,\psi}(\omega) = \int_{\mathbb{R}^+} \frac{\chi_\omega(x)}{2\sqrt{x}}\varphi(\sqrt{x})\psi(\sqrt{x})\ dx + \int_{\mathbb{R}^+} -\frac{\chi_\omega(x)}{2\sqrt{x}}\varphi(-\sqrt{x})\psi(-\sqrt{x})\ dx. \tag{4.37}$$

This implies $E_{x,y} = \mu_+ + \mu_-$, where μ_+ and μ_- are the measures induced by the integrals in equation (4.37).

Now let $\varphi \in \mathcal{D}(M_{x^2})$ and $\psi \in L^2(\mathbb{R})$. Then

$$\int_{[0,\infty)} x\ dE_{\varphi,\psi}(x) = \int_{[0,\infty)} x\ d\mu_+ + \int_{[0,\infty)} x\ d\mu_-$$

and furthermore

$$\int_{[0,\infty)} x\ d\mu_+ = \int_{\mathbb{R}^+} x\frac{1}{2\sqrt{x}}\varphi(\sqrt{x})\psi(\sqrt{x})\ dx$$
$$\int_{[0,\infty)} x\ d\mu_- = \int_{\mathbb{R}^+} -x\frac{1}{2\sqrt{x}}\varphi(-\sqrt{x})\psi(-\sqrt{x})\ dx.$$

Reversing the change of variables made above gives:

$$\int_{[0,\infty)} x\ d\mu_+ = \int_{\mathbb{R}^+} x^2\varphi(x)\psi(x)\ dx \tag{4.38}$$
$$\int_{[0,\infty)} x\ d\mu_- = \int_{\mathbb{R}^-} x^2\varphi(x)\psi(x)\ dx. \tag{4.39}$$

From this and the uniqueness assertion of theorem (4.3.6) it follows that E is the spectral decomposition of M_{x^2}, as

$$(M_{x^2}\varphi, \psi)_{L^2} = \int_{[0,\infty)} x \; dE_{\varphi,\psi}(x).$$

□

Corollary 4.4.6. *The spectral decomposition $E_{\bar{\Delta}}$ of $\bar{\Delta}$ on $\mathcal{F}(\mathcal{D}(M_{x^2}))$ is defined on the Borel subsets of $[0,\infty)$ and given by:*

$$E_{\bar{\Delta}}(\omega) = \mathcal{F}^{-1}E(\omega)\mathcal{F},$$

for all Borel sets ω, where E is the spectral decomposition of M_{x^2}.

Proof. As $\mathcal{F}$ is unitary and E is a spectral measure, we see that $E_{\bar{\Delta}}$ is a spectral measure. $\bar{\Delta}$ is unitarily equivalent to M_{x^2} and therefore $\sigma(\bar{\Delta}) \subset [0,\infty)$.

It remains to show that

$$(\bar{\Delta}\varphi, \psi)_{L^2} = \int_{[0,\infty)} x \; dE_{\bar{\Delta};\varphi,\psi} \tag{4.40}$$

for all $\varphi \in \mathcal{F}^{-1}(\mathcal{D}(M_{x^2}))$ and $\psi \in L^2(\mathbb{R})$. Note, that for such φ and ψ

$$E_{\bar{\Delta};\varphi,\psi}(\omega) = (\mathcal{F}^{-1}E(\omega)\mathcal{F}(\varphi), \psi)_{L^2} = (E(\omega)\mathcal{F}(\varphi), \mathcal{F}(\psi))_{L^2} = E_{\mathcal{F}(\varphi),\mathcal{F}(\psi)}(\omega). \tag{4.41}$$

and therefore by proposition (4.4.5)

$$\begin{aligned} (\bar{\Delta}\varphi, \psi)_{L^2} &= (\mathcal{F}^{-1}M_{x^2}\mathcal{F}(\varphi), \psi)_{L^2} = (M_{x^2}\mathcal{F}(\varphi), \mathcal{F}(\psi))_L^2 \\ &= \int_{[0,\infty)} x \; dE_{\mathcal{F}(\varphi),\mathcal{F}(\psi)} = \int_{[0,\infty)} x \; dE_{\bar{\Delta};\varphi,\psi}. \end{aligned}$$

The uniqueness assertion of theorem (4.3.6) shows that $E_{\bar{\Delta}}$ is the spectral decomposition of Δ. □

Corollary 4.4.7. *For $\bar{\Delta}$ the spectrum $\sigma(\bar{\Delta}) = [0,\infty)$ and $\bar{\Delta}$ has no eigenvalues.*

Proof: The essential range of the identity on $[0,\infty)$ regarding the spectral measure $E_{\bar{\Delta}}$ is $[0,\infty)$: Fix $t \in [0,\infty)$ and consider an open ball $\mathcal{B}$ containing t. Then $\chi_{\mathcal{B}} \neq 0$ and therefore $\mathcal{F}^{-1}E(\mathcal{B})\mathcal{F} \neq 0$. Now proposition (4.3.5) shows that $\sigma(\bar{\Delta}) = [0,\infty)$ and also that no $t \geq 0$ is an eigenvalue, as $E(\{t\}) = 0$ for all $t \in [0,\infty)$.

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Bibliography

[1] E. Freitag, R. Busam *Funktionentheorie*, Springer Lehrbuch, Springer-Verlag, 1995

[2] S. Lang, *Real and Functional Analysis*, third ed., Graduate Texts in Mathematics, Springer-Verlag, 1993

[3] M. Reed, B. Simon, *Functional Analysis*, revised and enlarged ed., Methods of modern mathematical physics, Academic Press, Inc., 1980

[4] W. Rudin, *Functional Analysis*, second ed., McGraw-Hill Book Company, New York, 1991.

[5] W. Rudin, *Real and Complex Analysis*, third ed., McGraw-Hill Book Company, New York, 1987.

Zusammenfassung

Das Thema der vorliegende Bachelorarbeit ist der Spektralsatz für unbeschränkte Operatoren im Hilbertraum. Die wesentlichen Resultate sind der Spektralsatz für beschränkte, normale Operatoren, sowie der Spektralsatz für unbeschränkte normale Operatoren.

Im ersten Kapitel wird zunächst der Begriff der Banachalgebra eingeführt und einige Eigenschaften des Spektrums in Banachalgebren werden bewiesen. Desweiteren wird der Gelfandraum einer Banachalgebra definiert und sein Zusammenhang mit den maximalen Idealen der Banachalgebra erläutert.

Das zweite Kapitel behandelt die Gelfandtransformierte. Die Involution auf einer Banachalgebra wird eingeführt und der Satz von Gelfand-Naimark bewiesen.

Die Hauptaussage des dritten Kapitels ist der Spektralsatz für beschränkte, normale Operatoren auf einem Hilbertraum. Nachdem in Kapitel eins und zwei allgemeine Banachalgebren behandelt wurden, ist Kapitel drei auf die Banachalgebra $B(H)$ fokussiert. Die Hilbertraumadjungierte, sowie Spektralmaße werden eingeführt und die Integration von Funktionen bezüglich einem Spektralmaß wird definiert. Die zentrale Aussage des Spektralsatzes ist die Darstellung eines normalen, beschränkten Operators N in der Form:

$$N = \int_{\sigma(N)} \lambda \, dE,$$

Als Anwedung werden die Eigenwerte eines normalen Operators bezüglich des korrespondierendem Spektralmaß bestimmt und die Existenz von Quadratwurzeln von positiven Operatoren wird gezeigt.

Im vierten Kapitel wird der Spektralsatz für unbeschräkte Operatoren bewiesen. Zunächst werden unbeschränkte Operatoren *im* Hilbertraum definiert, sowie die Adjungierte und abgeschloßene Operatoren. Die in Kapitel drei bewiesenen Theoreme werden auf den unbeschränkten Fall übertragen und der Beweis des Spektralsatz für unbeschränkte, normale Operatoren folgt.

Als Anwendung wird die Spektralzerlegung des Multiplikationsoperatos M_{x^2} und des Laplaceoperators $\Delta = \frac{\partial^2}{\partial x^2}$ bestimmt.