Homework 2 (optional)

(1) (Exercise 3.21, McDuff–Salamon) Prove Darboux's theorem "by hand" in the 2-dimensional case.

(McDuff–Salamon suggest a coordinate-free approach, using the fact that locally, a nonvanishing 1-form can be written as $f dg$. You could also try a coordinate-based approach. Start by writing the symplectic form ω locally as $h(x,y)dx \wedge dy$. If our Darboux coordinates are called p, q, set $p = x$. What should q be? Why do the resulting p, q form a local coordinate system?)

(2) (Example 3.8, McDuff–Salamon) In this exercise, we construct the *Kodaira–Thurston manifold*. This is a closed symplectic manifold that does not admit a Kähler structure.

Define a group $\Gamma \coloneqq \mathbb{Z}^2 \times \mathbb{Z}^2$, with the following nonabelian group operation:

$$(j', k') * (j, k) \coloneqq (j + j', A_{j'}k + k'), \qquad A_j \coloneqq \begin{pmatrix} 1 & j_2 \\ 0 & 1 \end{pmatrix},$$

where j_2 denotes the second entry of j. Γ acts on $\mathbb{R}^4$ like so:

$$\Gamma \to \mathrm{Diff}(\mathbb{R}^4),\ (j,k) \mapsto \rho_{jk}, \qquad \rho_{jk}(x,y) \coloneqq (x + j, A_j y + k).$$

(a) Prove that $M \coloneqq \mathbb{R}^4/\Gamma$ is a closed manifold.
(b) Prove that the symplectic form $dx_1 \wedge dx_2 + dy_1 \wedge dy_2$ on $\mathbb{R}^4$ descends to a symplectic form on M.
(c) Using standard facts about covering spaces, argue that $\pi_1(M) = \Gamma$. From this, compute $H_1(M;\mathbb{Z})$. Using the fact that odd Betti numbers of Kähler manifolds are even, conclude that M is a closed symplectic manifold that does not admit a Kähler structure.

(In fact, this fits in to a larger framework. Γ is a lattice in a nilpotent Lie group. This can be used to give quick proofs that Γ acts properly discontinuously and cocompactly, for instance.)

(3) (Exercise 3.18, McDuff–Salamon) Give examples of symplectic, isotropic, coisotropic, and Lagrangian submanifolds of the Kodaira–Thurston manifold.

(Start with a linear subspace of $\mathbb{R}^4$).

(4) If (V, ω) is a symplectic vector space, then a *compatible complex structure* is a linear map $J \colon V \to V$ such that (a) $J^2 = -\mathrm{Id}$, and (b) $g_J \coloneqq \omega(-, J-)$ defines a metric on V. ((b) means that $\omega(v, Jw) = \omega(w, Jv)$ for all v, w, and $\omega(v, Jv) > 0$ for all $v \neq 0$. The first of these conditions is equivalent to $\omega(v,w) = \omega(Jv, Jw)$ for all v, w.)

Prove the following fact, which we used in class in our proof of Weinstein's Lagrangian neighborhood theorem: Suppose that V and J are as above, and that $\Lambda \subset V$ is a Lagrangian subspace. Then $J\Lambda$ is equal to the orthogonal complement of Λ with respect to g_J.