SEMINAR PLAN: INTRODUCTION TO HOMOLOGICAL MIRROR SYMMETRY

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Last update: 10.09.2026.

1. Summary

The term "mirror symmetry" designates a class of related phenomena bridging symplectic and algebraic geometry. This is not a subject which can be encapsulated by a single conjecture – rather, there are certain heuristics which one picks up over time, according to which certain structures on one side are related to certain structures on the other side. In practice, the process of making these heuristics precise has led to a wealth of interesting mathematics.

In this seminar, we will focus on *categorical* aspects of mirror symmetry. There is a celebrated conjecture, known as the "homological mirror symmetry conjecture", which was first proposed by Kontsevich and has since be refined and extended in several directions by many other mathematicians. In its original form, this conjecture says if $M, M^\vee$ are "mirror pairs", then the Fukaya category of M is derived equivalent to the bounded derived category of coherent sheaves on $M^\vee$.

It is worth noting that at the time of formulating the conjecture, Kontsevich did not have a rigorous definition of the Fukaya category of a symplectic manifold. In fact, from today's perspective, we understand that there is no "the" Fukaya category – rather, there are several related constructions giving rise to Fukaya-like categories, which can be carried out under different assumptions on the ambient symplectic geometry.

The upshot is that an enormous amount of mathematics is required to even formulate homological mirror symmetry precisely in the simplest cases. It is easy to get lost or overwhelmed. To mitigate this risk, we will limit our attention the simplest examples. Namely, our goal is to understand homological mirror symmetry for the following pairs of spaces:

(1) $\mathbb{C}^* \leftrightarrow T^*S^1$
(2) $\mathbb{CP}^1 \leftrightarrow (T^*S^1, \partial_\infty T_0^*)$

Pre-requisites. This seminar covers a large amount of mathematics, drawn from several different subfields. It would be helpful for participants to have some prior background in at least one of the following three topics: (i) symplectic geometry, (ii) algebraic geometry (including schemes, coherent sheaves, etc), (iii) higher category theory. It is by no means expected that participants are familiar with all three topics.

2. Instructions and advice for speakers

Please be advised that the following instructions apply only to this particular seminar! Other instructors may have different expectations regarding both the preparation and content of seminar talks.

Preparatory meetings. Preliminary meeting: I am available to meet the speaker of week N on Wednesday on week $N-1$, right after the talk. This meeting is completely optional. If you are speaking in the first week and would like to meet, please contact me to fix a meeting time.

Timing. All talks should be "board talks" (i.e. please no slides). Talks should last 90 minutes, including a five minute break. Since there will almost surely be questions during the talk, I would suggest to plan to speak for roughly 80 minutes, while keeping some material "in reserve", in case there are fewer questions than usual. *It is extremely important to end on time.* If you go over time, I will ask you to stop after a few minutes, and this will likely affect your grade.

General advice. In general, the focus of the talks should be on explaining *definitions* and *statements* – definitely not proofs.[1] Please give examples and non-examples: for every new concept that is introduced, the audience members should ideally have in mind at least one key example.

Finally, please write legibly on the board and place statements in the appropriate environments (so e.g. if you write a new definition on the board, please leave a small space, write the word "Definition:" and then insert your definition). This is all common practice in mathematics lectures, but in my experience many beginning students forget to do this.

3. Plan of talks

Note: the references provided below are not intended to be exhaustive – you are of course welcome to follow other sources if you wish.

Talk 1: Abelian categories. Please review the definitions of: categories, morphisms/isomorphisms, functors, equivalences. Define additive and abelian categories. Give examples of additive categories which are not abelian. Discuss examples of abelian categories, including: modules over a ring, chain complexes, sheaves valued in abelian groups/modules.

Many interesting rings (and hence abelian categories) arise from *quivers*, possibly with relations. Please define the notion of a quiver and of its path algebra, and give some examples, including Dynkin quivers, the Kronecker quiver, maybe a cyclic quiver. If you wish, you may also introduce the notion of a *representation* of a quiver (in which case, please state and possibly prove

[1] I suspect that most talks will contain zero complete proofs.

that the category of representations of a quiver is equivalent to the category of modules over the path algebra.) Reference: [4, Sec. 1.1].

Talk 2: The abelian category of coherent sheaves. Review the notion of a coherent sheaf on a variety (we may assume all varieties in this seminar are over $\mathbb{C}$). Give a few examples and non-examples. State that the category of coherent sheaves is abelian. Remark that on an affine variety, the category of coherent sheaves is equivalent to the category of finitely-generated modules over the coordinate ring. (If you have time, if would be good to briefly recall how to deduce this from the equivalence $\Gamma : \mathrm{QCoh}(\mathrm{Spec}(A)) \to \mathrm{Mod}(A)$.)

Classify coherent sheaves on smooth curves (they are direct sums of vector bundles and torsion sheaves). State (and time allowing, prove) Grothendieck's theorem on the classification of vector bundles on $\mathbb{P}^1$. You will need to use black-box some facts from algebraic geometry – that's fine, but please make sure to state clearly what you are using. Reference: [4].

Talk 3: Triangulated categories. Define triangulated categories. Mention some examples: the derived category of an abelian category (to be discussed in the next talk), Fukaya categories (to be discussed later), and if you wish, you could also mention examples coming from homotopy theory. Illustrate by example the non-canonicity of cones. Define the notion of a summand. Define what it means for a subcollection of objects in a triangulated category to *generate* the category [2] Define the notion of a semi-orthogonal decomposition. References: [4, Sec. 1.2, 1.4].

Talk 4: The derived category of an abelian category. The goal of this talk is to explain the construction of the derived category of an abelian category. To this end, you should introduce the Verdier quotient. Explain how, under appropriate assumptions (enough injectives/projectives, upper/lower boundedness), the homs in the derived category can be computed using resolutions. Explain why an arbitrary functor between abelian categories does not tautologically induce a functor on derived categories, and discuss assumptions under which one can construct derived functors. Reference: [4, Sec. 2.1].

Talk 5: The bounded derived category of coherent sheaves. This talk should introduce the bounded derived category of coherent sheaves. There is quite a bit of flexibility about what material to include, but it is essential that you give a detailed proof that $D^b\,\mathrm{Coh}(\mathbb{P}^1) \simeq \langle \mathcal{O}, \mathcal{O}(1)\rangle$. I suggest to focus on explaining [4, Cor 3.15] (this requires some preparatory results – please use your judgement in deciding what to black-box). You can then combine [4, Cor 3.15] with the characterization of coherent sheaves on

[2] Let us adopt the convention to allow summands as well as cones, i.e. our definition of generation is what some authors call 'split generation'.

$\mathbb{P}^1$ from talk 2 to deduce the desired characterization of $D^b\operatorname{Coh}(\mathbb{P}^1)$. Time allowing, you could also state Beilinson's theorem on $D^b\operatorname{Coh}(\mathbb{P}^n)$.

Talk 6: Introduction dg and A_∞ categories. The goal of this talk is to introduce the notion of a higher category. There are several equivalent approaches which give rise to the same theory. The formalism which is most relevant for this seminar is the dg/A_∞-formalism – this is because Fukaya categories naturally carry an A_∞ structure which comes from the geometry of moduli spaces of holomorphic disks.

Please define both dg and A_∞ categories. Define the notion of a functor of dg/A_∞ categories. Define the 'homotopy category' $H^0(-)$ of a dg/A_∞ category. Define dg/A_∞ modules. Please also state that an A_∞ category is equivalent to a dg category ("homological perturbation"). As a general rule, is fine to only write formulas in the dg setting if the A_∞ analogs seem too messy for a talk.

Given a topological space X, one can construct a A_∞ category as follows: the objects are points of X. The set of morphisms between two points p, q is the complex of chains[3] over the path-space from p to q. The higher operation interpolate between different ways to concatenate paths. Please explain this example in detail, introducing the Stasheff associahedron along the way. Time allowing, you can also define the Moore path space, which naturally gives rise to a dg category by the same construction.

Talk 7: (pre)-triangulated dg and A_∞ categories. The goal of this talk is to explain the notion of an exact triangle and a summand in the context of dg/A_∞-categories. Here again, it is fine to only write formulas in the dg setting if the A_∞ analogs seem too messy for a talk.

Define the notion of a mapping cone and of a summand for dg/A_∞-categories. [5, (3e)] or [1, Sec. 3.1]. Explain in what sense the cone of a morphism is unique up to unique isomorphism. Define the notion of the pre-triangulated closure of a *dg*/A_∞-category $\mathcal{C}$ (following the appendix of [3], we shall denote this by $\operatorname{Perf}(\mathcal{C})$). State its universal property. Please remark that, in the word of 1-categories, the exact triangles are *data* – in contrast, the existence and universal property of the pretriangulated closure of a *dg*/A_∞-category implies that the underlying category already "knows" all mapping cones. Also state the existence and universal property of the pretriangulated idempotent completion of a *dg*/A_∞ category.

State that the Verdier quotient lifts to dg/A_∞ categories, so that the derived category of an abelian category carries a natural dg/A_∞ structure. Time allowing, you can give an indication of the construction in the dg setting (Drinfeld quotient), but this is not necessary. Conclude that the derived category of an abelian category carries a canonical dg structure.

[3] It is usually convenient to take *cubical* chains, for which Künneth works on the nose.

3.1. **Talk 8: Introduction to symplectic manifolds.** Define symplectic manifolds. Give examples. Define Lagrangian submanifolds. Give examples. Define symplectomorphisms. Define (time-dependent) Hamiltonians, Hamiltonian symplectomorphisms. Explain what is means for a symplectic manifold to be exact, and for a Lagrangian in an exact sympectic manifold to be exact. Give examples and non-examples.

3.2. **Talk 9: Lagrangian Floer theory.** The goal of this talk is to give a lightning tour of Lagrangian Floer theory. You should work throughout in the exact setting (exact Lagrangians in an exact symplectic manifold, with $\mathbb{Z}/2$-gradings and $\mathbb{Z}/2$-coefficients. Define almost-complex structures. Define J-holomorphic disk (more generally, J-holomorphic curves). Explain the definition of the Floer complex for two transversally intersecting Lagrangians: what are the generators and what does the differential count. Briefly sketch the proof that $d^2 = 0$. Explain how to compute the Lagrangian Floer (co)homology of the zero section in T^*S^1 with a transversally intersecting Hamiltonian displacement. (Assume for simplicity $\mathbb{Z}/2$-coefficents and $\mathbb{Z}/2$-gradings. Time allowing, comment on what is needed to lift coefficients to characteristic zero. Reference: [1, Sec. 1].

Talk 10: Higher operations in Lagrangian Floer theory. Start the talk by giving an informal and preliminary sketch of what a Fukaya category should be: the objects are (some) Lagrangian submanifolds in a given symplectic manifold, and the morphisms are the Lagrangian Floer cohomology groups. The main goal of this talk is to explain what why Fukaya categories should be A_∞ categories. The key point is to study compactifications of moduli spaces of 1-dimensional holomorphic disks. I suggest to explain carefully how to define the Floer product (i.e. μ^2), and to be a bit sketchier for the higher operations. I also suggest to work with $\mathbb{Z}/2$-coefficients throughout – i.e. ignore all signs. Finally, please comment briefly on the difficulties (transversality and compactness) which need to be overcome in order to turn the intuitive description of the Fukaya category into a rigorous construction. Reference: [1, Sec. 2.1, 2.2].

Talk 11: Exact symplectic manifolds. Define the notion of a 2-dimensional Liouville manifold. Draw some examples in (real) dimension 2 ($\mathbb{R}^2$ with the radial Liouville form, T^*S^1, the pair of pants, etc). Define what it means for a Lagrangian to be cylindrical at infinity. Briefly comment on the generation to higher dimensions, which requires the notion of a contact manifold. Define the *core* (also known as the *skeleton*) Key example: the zero section in the cotangent bundle. Define *cocore Lagrangians*. Keym example: the contangent fiber. Remark that a Liouville manifold is homotopy equivalent to its core. Illustrate these notions by drawing the core and cocores of $\mathbb{R}^2, T^*S^1$, the pair of pants ... Define the linking disks of a Legendrian, in dimension 2. Briefly comment on the existence of a higher dimensional generalisation of this notion. References: I suggest to start with [2, Sec. 2].

You can also consult [3] for a longer discussion of cores/cocores. The book of Cieliebak–Eliashberg is the definitive reference for Liouville geometry, but it is probably too detailed for the purpose of this talk.

Talk 12: The partially wrapped Fukaya category of a surface. Define the partially wrapped Fukaya category of a surface, preferably following the definition of [2]. The objects are all exact Lagrangians which are cylindrical at infinity (suitably decorated with orientation and grading data depending on one's choice of gradings/coefficients). Explain *wrapping*. Compute the full A_∞ algebra structure of endomorphism algebra of the the cotangent fiber of T^*S^1. (This is explained in [1, Sec. 4.2], using a somewhat different definition for the Fukaya category.

Finally, consider the wrapped Fukaya category of T^*S^1 with stops at $\partial_\infty T^*q$ (i.e. the boundary at infinity of the cotangent fiber over some $q \in S^1$). Let L_0 be the cotangent fiber over some point $0 \neq q$ and let L_1 be obtained by wrapping L_0 once past one of the stops. Compute $\operatorname{end}(L_0 \oplus L_1)$: it is isomorphic to the path algebra of the Kronecker quiver.

Talk 13: Generation for partially wrapped Fukaya categories. State the generation theorem of Ganatra–Pardon–Shende [3], namely that the wrapped Fukaya category of a Liouville manifold with stops (and appropriately isotropic core) is generated by the cocores and linking disks. You are welcome to state the theorem only for surfaces. Explain the intuition behind this theorem (under scaling, an arbitrary lagrangian converges to a union of cocores over its intersection points with the skeleton).

Deduce that the cotangent fiber generates the wrapped Fukaya category of T^*Q. Deduce the homological mirror symmetry equivalence $\operatorname{Perf}\mathcal{W}(T^*S^1) \simeq \operatorname{Perf}(k[x, x^{-1}]) = D^b \operatorname{Coh}(\mathbb{C}^*)$.

Finally, state the wrapping exact triangle [3] (you can state it and illustrate it only for surfaces). Deduce that the wrapped Fukaya category of T^*S^1 with stops at $\partial_\infty T^*_q$ is generated by L_0, L_1 from the previous talk (the linking disks are the cones on the two obvious degree zero homs $L_0 \to L_1$). Deduce the mirror symmetry equivalence $\operatorname{Perf}\mathcal{W}(T^*S^1, \partial_\infty T^*_q) \simeq D^b \operatorname{Coh}(\mathbb{CP}^1)$.

References

[1] Denis Auroux, *A beginner's introduction to Fukaya categories*, available at arXiv:1301.7056.

[2] Sheel Ganatra, John Pardon, and Vivek Shende, *Covariantly functorial wrapped Floer theory on Liouville sectors*, Publications Mathématiques de l'IHES **131** (2020), no. 1, 73–200.

[3] ———, *Sectorial descent for wrapped Fukaya categories*, Journal of the American Mathematical Society **37**, no. 2, 499–635.

[4] Daniel Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford University Press, 2006.

[5] Paul Seidel, *Fukaya categories and Picard–Lefschetz theory*, 2008.