Operads in Algebra and Topology

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Time and Place: Wednesdays, 14:15–15:45, SR 1.007

Seminar's webpage on BASIS
Seminar's official webpage

Summary Sometimes a topological space X is endowed with a nice, continuous product $X \times X \longrightarrow X$ (think for example of a Lie group). If this happens, then one can deduce many interesting properties about the topology of X: for example, the homology of X has a natural structure of graded ring, and the fundamental group $\pi_1(X)$ is abelian. However, many spaces X are naturally endowed with *several* natural operations, with two or more inputs and one output, and there is no canonical way to choose a single operation on X. An *operad* helps us keeping track of all operations that we want to consider. This notion naturally arose in the 1970s from the study of iterated loop spaces by Boardman, Vogt and May.

Similarly, in the context of linear algebra (over a commutative ring R), operads classify various types of multilinear operations that one can put on an R-module, and the relations that we want to impose among these operations (think of an R-algebra structure on an R-module, with associativity and commutativity as properties).

This seminar aims to provide an introduction to operad theory, with a focus on its applications to algebraic topology. We will in particular see how operads help us understanding the homology of spaces and sequences of spaces, and detect properties of their homotopy type.

One of our main goals will be the *recognition principle* of May [May72], stating that if X has an action of the operad $\mathcal{E}_d$ of little d-cubes (and assuming that X is *grouplike*), then X is homotopy equivalent to a d-fold loop space. We will also consider the following situation: we have a space X which naturally decomposes as a disjoint union of spaces X_n indexed by natural numbers n; an operad $\mathcal{O}$ acts on X, and for all n there are natural maps $X_n \longrightarrow X_{n+1}$ coming from this action. In this case the operad $\mathcal{O}$ can help us understand how the different spaces X_n are interrelated, and obtain some information of the *stable homology* of X, i.e. the *colimit* of $H_*(X_n)$ for n going to infinity. We will in particular discuss the *group completion theorem* for the homology of certain algebras $X = \coprod_{n\geq 0} X_n$ over the operad $\mathcal{E}_1$ [MS76, FM94].

Finally, we will consider the surface operad $\mathcal{M}$ introduced by Tillmann [Til00] and state the main theorem of [BBP+16]: if X has an action of an *operad with homological stability* (for example $\mathcal{M}$) and X is grouplike, then X is homotopy equivalent to an infinite loop space.

Seminar concept The talks are supposed to be 90 minutes long. Be aware that some time will be needed for questions. Please discuss your talk with one of us two weeks before you are going to give it. If you have questions during the preparation of the talk, feel free to contact us!

Prerequisites Good understanding of elementary homotopy theory, basic category theory, homological algebra and singular homology.

Organisational meeting Thursday, February 6$^{\text{th}}$ 2020, 11:00–12:00, SR 0.006

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(1) **H-spaces and d-fold loop spaces** .... Agata Sienicka, **22.04.2020**
[tD08, §4.4+5], [MSS07, §II.2.1+2], [Whi78, §III.2], [May72, §4.1, §5.1]

Recall the notion of an H-space X and give the examples of Lie groups. If X is an H-space, then $\pi_1(X)$ is abelian and acts trivially on higher homotopy groups. Distinguish between H- and strict unitality, associativity and commutativity of an H-space. A topological monoid is a strictly associative H-space with strict unit. A map of H-spaces is required to preserve the product only up to homotopy: explain what this means. Describe the Pontryagin product in the homology of an H-space, and say when it is associative or (graded) commutative.

For a pointed topological space $(Y,*)$, define the classical loop space $\Omega Y = \mathrm{map}((I,\partial I),(Y,*))$. It is always homotopy-associative, but normally not strictly associative; the unit (the constant loop at $*$) is not strict in general. Explain how the Moore loop space $\Omega^{\mathrm{Moore}}Y$ is a *strictly* associative replacement of ΩY (with *strict* unit).

Introduce the d-fold loop spaces $\Omega^d Y = \mathrm{map}((I^d,\partial I^d),(Y,*))$ for $d \geq 2$, and show that they are H-associative and H-commutative H-spaces. Unfortunately, one cannot always replace an Ω^d-space by a strictly commutative H-space: we will argue why during the seminar.

Define the spaces $\mathcal{E}_d(n)$ for $n \geq 0$, and prove that a point in $\mathcal{E}_d(n)$ gives a way to combine n elements of $\Omega^d Y$ and obtain a new element of $\Omega^d Y$. So there are not just single operations, but entire *spaces* of n-fold products on $\Omega^d Y$. In particular, the space of binary products is homotopy equivalent to S^{d-1}.

(2) **Symmetric monoidal categories** .... Janina Bernardy, **Fri, 24.04.2020**
[Bor94, §6.1+2+5,§7.1–2], [ML71, §VII.1+3,§XI.1+2], [May72, §10], [MSS07, §II.1.1]

To formalise the notion of operad and apply it also to other, non-topological contexts, it is useful to work in the general setting of monoidal categories.

Introduce the notion of (symmetric) monoidal category $(\mathscr{C},\otimes,1)$. Be extremely brief in discussing the coherence axioms for associators. Give examples: sets, abelian groups, (graded) R-modules, chain complexes, topological spaces (be brief on the necessity to work with compactly generated, weakly Hausdorff spaces); based spaces. These examples are all symmetric. A non-symmetric example is, for an arbitrary category C, the functor category $\mathsf{Fun}(\mathsf{C},\mathsf{C})$, with monoidal product given by composition of endofunctors.

Define (commutative) monoid objects in $\mathscr{C}$. If $\mathscr{C}$ is symmetric, then monoid objects in $\mathscr{C}$ form again a symmetric monoidal category $\mathsf{Mon}(\mathscr{C})$. In the previous list of examples we obtain: usual monoids; (graded) R-algebras, differential graded R-algebras; topological monoids.

Introduce (symmetric) lax and strong monoidal functor between symmetric monoidal categories. Examples: $\pi_0\colon \mathsf{Top} \longrightarrow \mathsf{Set}$ is strong monoidal; $H_*(-;R)\colon \mathsf{Top} \longrightarrow R\text{-}\mathsf{Mod}^{\mathbb{N}}$ is lax monoidal, and strong monoidal if R is a field. More tricky examples: $C_*\colon \mathsf{Top} \longrightarrow \mathsf{Ch}$ and $C^*\colon \mathsf{Top} \longrightarrow \mathsf{DGA}$ are lax monoidal, but not symmetric.

Observe that a lax monoidal functor $\mathscr{C} \longrightarrow \mathscr{D}$ maps monoid objects of $\mathscr{C}$ to monoid objects in $\mathscr{D}$. Recover the Pontryagin product in the homology of a topological monoid, or more generally of a homotopy associative H-space (which is the same as a monoid object in $\mathsf{ho}(\mathsf{Top})$).

(3) **Operads and little cubes** .... Jakub Löwit, **29.04.2020**
[MSS07, §II.1.2+4, §II.2.2], [May, §1], [May72, §1, §3.2+3, §4.1],

Give intuition for operads: $\mathcal{O}$ consists of a sequence of objects $\mathcal{O}(n)$; the object $\mathcal{O}(n)$ parametrises all possible ways to compose n inputs and obtain one output. Draw trees and give the formal definition of operad in a *symmetric* monoidal category $\mathscr{C}$, describing briefly structure maps and which coherences are required. Define morphisms of operads.

Give examples: $\mathcal{A}ss$ and $\mathcal{C}omm$ are operads in every *cocomplete* symmetric monoidal category. Describe them explicitly in the case of Top and R-Mod, for R a commutative ring. Introduce algebras over an operad and their morphisms. Algebras in $\mathscr{C}$ over $\mathcal{A}ss$ are exactly monoid objects in $\mathscr{C}$, whereas algebras over $\mathcal{C}omm$ are commutative monoid objects. Recover (commutative) topological monoids and (commutative) R-algebras as examples.

Now consider operads in Top. Show that for a topological operad $\mathcal{O}$ and an $\mathcal{O}$-algebra X, each choice of $m \in \mathcal{O}(2)$ endows the space X with a multiplication $X \times X \longrightarrow X$. When is this H-commutative, H-associative, H-unital or H-unique? Recall the definitions of the spaces $\mathcal{E}_d(n)$ from the first talk, and say how they assemble into an operad $\mathcal{E}_d$. Our aim is to use this operad $\mathcal{E}_d$ to detect Ω^d-spaces.[1] Show how operads and their algebras can be pushed forward along symmetric lax functors, e.g. π_0 and H_*^{sing}. What is $\pi_0(\mathcal{E}_d)$? Using this, we obtain the following: for an $\mathcal{E}_1$-algebra X, theset $\pi_0(X)$ is naturally a monoid and for an $\mathcal{E}_d$-algebra X with $d \geq 2$, the set $\pi_0(X)$ is naturally a *commutative* monoid.

Define *grouplike* $\mathcal{E}_d$-algebras. For example, $\Omega^d Y$ is a grouplike $\mathcal{E}_d$-algebra for all $d \geq 1$ (we are essentially saying that $\pi_d(Y)$ is a group and not just a monoid). Discuss the example of the unordered configuration space $\mathrm{Conf}(I^d) = \coprod_{n\geq 0} \mathrm{Conf}_n(I^d)$ as an $\mathcal{E}_d$-algebra and draw pictures. This algebra is not grouplike. State the recognition principle: if X is a grouplike $\mathcal{E}_d$-algebra, then X can be replaced, up to homotopy equivalence, by $\Omega^d B$ for some space B.

[1] These spaces live naturally in the category Top_* of *pointed* topological spaces; we will see in the next talk how to formally switch from the category Top to the category Top_*.

(4) **Monads** . . . . Nicholas Schwab, **06.05.2020**
[tD08, §4.4], [Bor94, §4.1+2], [ML71, §VI.1+2,§VII.9], [BBP+16, Def. 2.5/Ex. 2.6], [May, §2.1+4+9+10,§9.4]

Define monads T on a category C (we do not need any monoidal structure on C). Give an alternative description as monoids in the endomorphism category. Show how monads can arise from monoid objects and from adjunctions. Examples: an R-algebra A gives a monad $(A \otimes_R -)$ in the category R-Mod; the adjunction $\Sigma^d \colon \mathsf{Top}_* \longleftrightarrow \mathsf{Top}_* \colon \Omega^d$ gives a monad $\Omega^d\Sigma^d$ in Top_*.

Define algebras over a monad T, and morphisms of T-algebras. Note that T can be enhanced to a functor from C to the category $\mathsf{Alg}_T(\mathsf{C})$ of algebras over T: this functor is left adjoint to the forgetful functor $\mathsf{Alg}_T(\mathsf{C}) \longrightarrow \mathsf{C}$ and hence is called the *free algebra functor*. Examples: algebras over $(A \otimes_R -)$ are precisely A-modules; algebras over $\Omega\Sigma$ are more tricky. Maybe one can at least see that for all spaces B, $\Omega^d B$ is an algebra over the monad $\Omega^d\Sigma^d$ (a similar property holds for *any* adjunction).

Show how an operad $\mathcal{O}$ in a symmetric monoidal category $\mathscr{C}$ gives a monad $T_{\mathcal{O}}$ in $\mathscr{C}$, and that the notions of "algebra" over $\mathcal{O}$ and over $T_{\mathcal{O}}$ coincide. Describe the free $\mathcal{O}$-algebra $\mathcal{O}[X]$ over a $\mathscr{C}$-object X. Discuss explicitly what $\mathcal{E}_d[X]$ for a space X is. What is $\mathcal{E}_d[*]$? Argue how operads in Top give monads in Top_* by adding extra relations which identify 0-aries with the basepoint. Give the explicit example of the free algebra $\mathcal{E}_d[X,*]$ for a based space X and compare it to $\mathcal{E}_d[X]$. Is $\mathcal{E}_d[X,*]$ connected if X is?

Define morphisms of monads. Examples: a morphism of R-algebras $A \longrightarrow A'$ gives morphism of monads $(A \otimes_R -) \longrightarrow (A' \otimes_R -)$ defined on R-Mod. If $T \longrightarrow T'$ is a morphism of monads, each T'-algebra becomes a T-algebra. Define for a monad T in a category $\mathscr{C}$, what a right T-module functor S is. Example in R-Mod: if A is an associative R-algebra and S is a right A-module, then $S \otimes_R - \colon R\text{-}\mathsf{Mod} \longrightarrow R\text{-}\mathsf{Mod}$ is a right $(A \otimes_R -)$-module functor.

(5) **Simplicial objects and bar constructions** . . . . Andrea Lachmann, **13.05.2020**
[Bau95, pp. 8–10], [May72, §9], [Pen19, §4], [Ebe19, §1–2]

Recall the notion of simplicial object in a category C, and in the case of $\mathsf{C} = \mathsf{Top}$, define the geometric realisation of a simplicial space.

For a topological monoid M, define the (topologically enriched) nerve, which is a simplicial space NM; the geometric realisation is called BM and is a connected topological space, called the *bar construction*. The assignment $M \longmapsto BM$ gives a functor from the category $\mathsf{Mon}(\mathsf{Top})$ of topological monoids to the category Top. This functor is strong monoidal, i.e. there are natural homeomorphisms $B(M \times M') \cong BM \times BM'$ for all topological monoids M and M'.

More generally, if X is a right M-space and Y is a left M-space, then we can define the simplicial space $N(X, M, Y)$ and its geometric realisation $B(X, M, Y)$, which is just a topological space. Examples: We recover the space $BM = B(*, M, *)$ whereas $EM \coloneqq B(*, M, M)$ is contractible. For topological groups G, the canonical map $EG \longrightarrow BG$ is a universal principal G-bundle, and BG is a classifying space for principal G-bundles.[1]

We will need the following generalisation of the previous construction: For a category C, define the nerve $N\mathsf{C}$, which is a simplicial set; the geometric realisation of $N\mathsf{C}$ is called $B\mathsf{C}$ and is a topological space, in general disconnected. If C and D are two categories, argue that $B(\mathsf{C} \times \mathsf{D}) \cong B\mathsf{C} \times B\mathsf{D}$. If $\mathscr{C}$ is a strict monoidal category, then $B\mathscr{C}$ is a topological monoid.[2]

Introduce the bar construction $N(S,T,X)$ for a monad T in a category C, a T-algebra X and a right T-module functor S: it is a simplicial object in C, and if $\mathsf{C} = \mathsf{Top}$ then we have a geometric realisation $B(S,T,X)$. Give some concrete example.

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[1] In general, $EM \longrightarrow BM$ is only a continuous map with fibres homeomorphic to M.

[2] If $\mathscr{C}$ is only monoidal, then $B\mathscr{C}$ is a homotopy associative H-space.

(6) **Two applications of quasifibrations** . . . . Raphael Floris, **20.05.2020**
[Pen19, §5+7], [Hat01, §4K], [Hat14, Appendix D], [May72, §6–7]

Introduce the notion of quasifibration of topological spaces: a quasifibration does not have the strict homotopy lifting property with respect to all pairs of CW complexes, but satisfies a milder, yet useful condition expressed in terms of homotopy groups.

Recall the bar construction BM and the map $EM \longrightarrow BM$ from the last talk. Show that if M is grouplike, then the natural map $M \longrightarrow \Omega BM$ is a homotopy equivalence. Argue that $EM \longrightarrow BM$ is a quasifibration with fibre M, by filtering BM appropriately.

Note that for a based space X, there is always a canonical map $\mathcal{E}_d[X] \longrightarrow \Omega^d\Sigma^d X$. State the approximation principle: For a connected[1] space X, this map is a homotopy equivalence. Sketch the proof of the approximation theorem by arguing that there is a quasifibration $\mathcal{E}_d[X] \hookrightarrow P_d[X] \longrightarrow \mathcal{E}_{d-1}[\Sigma X]$, where the space $P_d[X]$ is contractible (we do not describe explicitly how this space looks like).

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[1] If X were not connected, this would not be a quasifibration (you do not need to show it, but it is important which step fails when an hypothesis is dropped); indeed it is easy to see that if X is not connected, then $\mathcal{E}_d[X]$ is not grouplike, and therefore ca not be homotopy equivalent to $\Omega^d\Sigma^d X$.

(7) **Two applications of the bar construction** . . . . Ben Steffan, Jonathan Pampel, **03.06.2020**
[MSS07, §II.2.2], [BBP+16, §3], [May72, §4, §13.5]

In this talk we sketch the proof of the recognition principle; we will first discuss the case of a connected $\mathcal{E}_d$-algebra X, proving that it is homotopy equivalent to a Ω^d-space. Show that Σ^d is a right $\mathcal{E}_d$-module functor, and also a right $\Omega^d\Sigma^d$-module functor. Recall the approximation principle and sketch the proof of the recognition principle for a connected $\mathcal{E}_d$-algebra.

There is a map $\mathcal{E}_1 \longrightarrow \mathcal{A}ss$ of topological operads which is a levelwise homotopy equivalence. Even more, $\mathfrak{S}_n$ acts freely both on $\mathcal{E}_1(n)$ and $\mathcal{A}ss(n)$, and the quotients are homotopy equivalent. We expect that each $\mathcal{E}_1$-algebra is homotopy equivalent to some $\mathcal{A}ss$-algebra in Top, i.e. a topological monoid. Describe briefly how the Moore trick works in this case. In general, we call operad $\mathcal{O}$ an $\mathcal{A}_\infty$-operad if there is levelwise $\mathfrak{S}_n$-equivariant homotopy equivalence $\mathcal{O} \longrightarrow \mathcal{A}ss$. If $\mathcal{O}$ is an $\mathcal{A}_\infty$-operad, we consider the bar construction $B(\mathcal{A}ss, \mathcal{O}, X)$ for any $\mathcal{O}$-algebra X: this is homotopy equivalent to X, and is a topological monoid.

Introduce the operad $\mathcal{E}_\infty$. For $d \geq 2$, including $d = \infty$, there is a map $\mathcal{E}_d \longrightarrow \mathcal{C}omm$, but only for $d = \infty$, it is levelwise a homotopy equivalence. Still, the action of $\mathfrak{S}_n$ is free on $\mathcal{E}_\infty(n)$, with quotient an Eilenberg-MacLane space $K(\mathfrak{S}_n, 1)$ (not contractible for $n \geq 2$), and it is non-free on $\mathcal{C}omm(n) = *$, with quotient a point. Hence we do not expect that an $\mathcal{E}_d$-algebra can be always strictified to a commutative topological monoid, not even for $d = \infty$.

(8) **Algebraic operads**. . . . . Daniel Mulcahy, Christian Kremer, 10.06.2020
[Sin06, §2], [MSS07, §II.1.5+9], [LV12, §5.1.1+3+4+8+9, §5.2.1+7+10+14, §5.4, §5.5.1+2+4], [May, §4, Def. 5.1+5]

The aim of this talk is to use operads to classify certain algebraic structures (e.g. commutative algebras, Lie algebras . . .) and to relate them to topological operads.

Firstly, we consider two alternative definitions of operads: given a symmetric monoidal category $\mathscr{C}$, introduce the category of symmetric sequences in $\mathscr{C}$ and on this, the composition product, yielding a monoidal category $\mathsf{SymSeq}(\mathscr{C})$. Note that this is not symmetric in general. Now operads are exactly monoid objects in $\mathsf{SymSeq}(\mathscr{C})$.

For the second alternative, define the tree monad on $\mathsf{SymSeq}(\mathscr{C})$. Notice that operads in $\mathscr{C}$ correspond to algebras over this monad. Define free operads over a symmetric sequence in $\mathscr{C}$ (here we need again $\mathscr{C}$ cocomplete) and explain the analogy to free monoids.

Introduce operadic ideals (and what it means to give *generators* of an ideal) and show that each operad (in Ab or more generally, in an abelian category) has a presentation.

Define the following notions in the context of (graded) R-modules: associative, commutative, graded commutative algebras; Lie d-algebras, Poisson d-algebras for $d \geq 0$. Give examples.

Define the $\mathsf{Ab}^{\mathbb{N}}$-operads $\mathcal{Lie}_d$ and $\mathcal{Pois}_d$ by presentation. What are algebras over them?

We already noticed that the functor $H_*\colon \mathsf{Top} \longrightarrow \mathsf{Ab}^{\mathbb{N}}$ is lax monoidal, so for each topological operad $\mathcal{O}$, its homology $H_*(\mathcal{O})$ is an operad in $\mathsf{Ab}^{\mathbb{N}}$. State without proof that $H_*(\mathcal{E}_d)$, as an operad in $\mathsf{Ab}^{\mathbb{N}}$, is isomorphic to $\mathcal{Pois}_d$. Give some intuition for this fact, e.g. describe homology classes in $H_*(\mathcal{E}_d)$ in terms of the the Poisson bracket and the Pontryagin product and draw them. How does this help us understanding the homology of $\mathcal{E}_d$-algebras?

(9) **Homological stability in many examples** . . CONSTANZE SCHWARZ, JONAH EPSTEIN, 17.06.2020 [RWW17, §5], [van80, §4.11], [Wah13], [FH01], [RW13]

Define homological stability for sequences $(X_n)_{n\geq 0}$ of spaces with maps $X_n \longrightarrow X_{n+1}$ and for sequences of groups $(G_n)_{n\geq 0}$. If needed, recall that the homology of a group G is $H_*(BG)$.

Introduce families of spaces and groups that exhibit (or do not exhibit) homological stability: firstly, introduce ordered and unordered configuration spaces of $\mathbb{R}^d$, for $d \geq 2$: for each $n \geq 0$ we have spaces $P\mathrm{Conf}_n(\mathbb{R}^d)$ and $\mathrm{Conf}_n(\mathbb{R}^d)$.[1] For $d = 2$, the space $\mathrm{Conf}_n(\mathbb{R}^2)$ resp. $P\mathrm{Conf}_n(\mathbb{R}^2)$ is a classifying space for the braid group β_n resp. the pure braid group $P\beta_n$. Show how loops in these spaces look like braids. For $d = \infty$, the space $P\mathrm{Conf}_n(\mathbb{R}^\infty)$ is contractible and $\mathrm{Conf}_n(\mathbb{R}^\infty)$ is a classifying space for the symmetric group $\mathfrak{S}_n$.

For symmetric groups we consider the canonical inclusions $\mathfrak{S}_n \hookrightarrow \mathfrak{S}_{n+1}$; for (pure) braid groups we have inclusions $\beta_n \hookrightarrow \beta_{n+1}$ (resp. $P\beta_n \hookrightarrow P\beta_{n+1}$) given by adding a strand on right; for configuration spaces in $\mathbb{R}^d$, we can adjoin a new point on right with respect to the first coordinate of $\mathbb{R}^d$, obtaining maps $\mathrm{Conf}_n(\mathbb{R}^d) \hookrightarrow \mathrm{Conf}_{n+1}(\mathbb{R}^d)$ and $P\mathrm{Conf}_n(\mathbb{R}^d) \hookrightarrow P\mathrm{Conf}_{n+1}(\mathbb{R}^d)$. Homological stability holds for symmetric groups, braid groups and unordered configuration spaces, but not for ordered configuration spaces and pure braid groups (it is enough to state these results, with explicit stability ranges). Another example in which homological stability does not hold is the sequence of groups $G_n = \mathbb{Z}^n$ with canonical inclusions $\mathbb{Z}^n \longrightarrow \mathbb{Z}^{n+1}$ on the first n coordinates.[2]

Let R be a PID,[3] and let $\mathrm{GL}_n(R)$ be the group of invertible matrices with coefficients in R; then $\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R)$ by adding a 1 in the lower right corner, and homological stability holds for the sequence of groups $(\mathrm{GL}_n(R))_{n\geq 0}$ (state this result with explicit stability ranges).

For an orientable surface $\Sigma_{g,n}$ of genus $g \geq 0$ with $n \geq 1$ boundary components, define the group $\Gamma_{g,n}$, called the *mapping class group*.[4] Give some of the following examples of mapping class groups, without proof: $\Gamma_{0,2} \cong \mathbb{Z}$ generated by the Dehn twist; $\Gamma_{0,3} \cong \mathbb{Z}^3$ generated by the three Dehn twists around the three boundary components; $\Gamma_{1,1} \cong \beta_3$. Note that in this case, we have two indices, g and n, parametrising the groups.

By gluing a $\Sigma_{1,2}$ along one boundary component, and extending diffeomorphisms by the identity on the new piece, we obtain natural maps $\Gamma_{g,n} \longrightarrow \Gamma_{g+1,n}$. Homological stability holds for mapping class groups, for $g \longrightarrow \infty$, and fixed n; (state this result with explicit stability ranges). Moreover the stable homology is the same for all values of n: say what this means by introducing maps $\Gamma_{g,n} \longrightarrow \Gamma_{g,n+1}$.

Interpret the stable homology of a sequence of spaces $(X_n)_{n\geq 0}$ as the homology of the mapping telescope $X_\infty = \mathrm{hocolim}_{n\longrightarrow\infty} X_n$, and the stable homology of a sequence of groups $(G_n)_{n\geq 0}$ as the homology of the colimit group $G_\infty = \mathrm{colim}_{n\longrightarrow\infty} G_n$. Often these stable homology groups are easier to compute than the unstable homology groups.

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[1] In the first talk we considered configuration spaces of the open cube I^d; however since I^d and $\mathbb{R}^d$ are *homeomorphic*, also the corresponding ordered/unordered configuration spaces are homeomorphic (why?).

[2] This shows that homological stability is very tricky, and that it is a priori difficult to guess which natural sequences of groups and spaces satisfy it!

[3] We consider only PIDs for simplicity, but there are theorems about more general rings.

[4] There are several nice and geometric models for a classifying space of $\Gamma_{g,n}$, but introducing them properly requires another seminar.

(10) **The surface operad** . . . . Malte Kornemann, 24.06.2020
[Til00], [BBP+16]

We consider a surface $\Sigma_{g,n+1}$ with $n+1$ parametrised boundary components. The extended mapping class group $\Gamma_{g,(n),1}$ is the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,n+1}$ that are allowed to permute the first n boundary components, respecting their parametrisation, and fix the last boundary component pointwise. There is a short exact sequence of groups $1 \longrightarrow \Gamma_{g,n+1} \longrightarrow \Gamma_{g,(n),1} \longrightarrow \mathfrak{S}_n \longrightarrow 1$.

Recall the wreath product $G \wr H = H^n \rtimes G$, for a group G endowed with a map $G \longrightarrow \mathfrak{S}_n$ and any group H. Define maps of groups $\Gamma_{g,(n),1} \wr \Gamma_{h,(k),1} \longrightarrow \Gamma_{g+kh,(kn),1}$ and show how these define a topological operad $\mathcal{M}$ with $\mathcal{M}(n) \simeq \coprod_{g\geq 0} B\Gamma_{g,n+1}$, called the *surface operad*.

Introduce the notion of an *operad with homological stability*. Note that $\mathcal{M}$ is an operad with homological stability. Argue that $\mathcal{E}_\infty$ and $\mathcal{Comm}$ are operads with homological stability, but $\mathcal{E}_d$ is not for finite d, and hence, also $\mathcal{Ass}$ is not. We will see in the last talk a very interesting result concerning operads with homological stability.

(11) **The group completion theorem** . . . . Branko Juran, 01.07.2020
[MS76], [FM94, Appendix], [Ebe19, §6.2]

The goal of this talk is to properly state and prove the group completion theorem for a topological monoid whose π_0 is isomorphic to $\mathbb{N}$ and generated by a homology-central element $m \in M$.

In order to do so, define the localisation of the homology $H_*(M)[\pi_0(M)^{-1}] = H_*(M)[m^{-1}]$, introduce the notion of homology fibration. Some spectral sequence argument is needed.

Deduce that $\Omega BM \simeq M_\infty^+ \times \mathbb{Z}$, where we use the Quillen plus construction and discuss the examples of $H_*(\beta_\infty) \cong H_*(\Omega_0^2 S^2)$ and $H_*(\mathrm{Conf}_\infty(I^d)) \cong H_*(\Omega_0^d S^d)$.

(12) **Ω^∞-spaces and operads with homological stability** . . . Urs Flock, Robin Louis, 08.07.2020
[BBP+16, §2–3], [Hat14]

A spectrum Y is a sequence $(Y_i)_{i\geq 0}$ of spaces, together with maps $Y_i \longrightarrow \Omega Y_{i+1}$. As example, introduce the suspension spectrum associated to a based space $(X, *)$. Another example is the spectrum $\mathrm{MTSO}(2)$, where $\mathrm{MTSO}(2)_n$ is the one-point compactification of the space of oriented, *affine* 2-planes in $\mathbb{R}^n$ (how is this a spectrum?).

Define the space $\Omega^\infty Y = \mathrm{hocolim}_{n\longrightarrow\infty} \Omega^n Y_n$. A space obtained in this way is called an infinite loop space (Ω^∞-space). If we know that the maps $Y_i \longrightarrow \Omega Y_{i+1}$ are cofibrations (which can always be achieved by a homotopy replacement), then instead of the homotopy colimit we can take the increasing union $\bigcup_{n\geq 0} \Omega^n Y_n$. The latter has an action of the operad $\mathcal{E}_\infty$ because for $n \geq d$ the inclusion $\Omega^n Y_n \longrightarrow \Omega^{n+1} Y_{n+1}$ is a map of $\mathcal{E}_d$-algebras. The recognition principle extends to this case: grouplike algebras over $\mathcal{E}_\infty$ are homotopy equivalent to Ω^∞-spaces.

State the main theorem of [BBP+16]: a grouplike algebra X over an operad $\mathcal{O}$ with homological stability (e.g. the surface operad $\mathcal{M}$) is homotopy equivalent to an Ω^∞-space. If an algebra X is not grouplike, the theorem ensures that its group completion ΩBX is an Ω^∞-space.[1]

Example 1: $X = \coprod_{n\geq 0} \mathrm{Conf}_n(I^\infty)$ is an algebra over $\mathcal{E}_\infty$ which is not grouplike.[2] However, we can consider its group completion ΩBX, and this is the infinite loop space loop space[3] $\Omega^\infty\Sigma^\infty S^0$. We obtain the Barrat-Priddy-Quillen theorem stating that $\mathrm{colim}_{n\longrightarrow\infty} H_*(\mathfrak{S}_n) \cong H_*(\Omega_0^\infty\Sigma^\infty S^0)$.

Example 2: $X = \coprod_{g\geq 0} B\Gamma_{g,1}$ is an algebra over $\mathcal{M}$ which is not grouplike. First, note that $\mathcal{M}$ contains a copy of $\mathcal{E}_2$, which contains $\mathcal{E}_1$. Hence X is a $\mathcal{E}_1$-algebra and we can form its *group completion* ΩBX. By the theorem, there is a spectrum Y with $\Omega BX \simeq \Omega^\infty Y$. The Madsen-Weiss theorem identifies *which* infinite loop space it is: we get $\Omega BX \simeq \mathbb{Z}\times\Omega^\infty\mathrm{MTSO}(2)$ from which we conclude that $\mathrm{colim}_{g\longrightarrow\infty} H_*(\Gamma_{g,1}) \cong H_*(\Omega^\infty\mathrm{MTSO}(2))$.

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[1] As we saw in the previous talk, usually ΩBX contains interesting information about the stable homology of X, especially if X is graded by $\mathbb{N}$, i.e. X splits as a disjoint union $\coprod_{n\geq 0} X_n$.

[2] Here I^∞ denotes the infinite dimensional cube: how is this defined?

[3] Note that the space $\Omega^\infty\Sigma^\infty S^0$ has the *stable homotopy groups* of spheres as homotopy groups, so it must be a very interesting space!

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