Moduli spaces of Riemann surfaces and symmetric products: A combinatorial description of the Mumford–Miller–Morita classes

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Overview

Let $\mathfrak{M}^m_{g,n}$ be the moduli space of Riemann surfaces $F^m_{g,n}$ of genus g with n parametrised boundary curves and m punctures, cf. [FM12]. In the situation where $n \geq 1$, it turns out that $\mathfrak{M}^m_{g,n}$ is both a manifold and a classifying space for the mapping class group

$$\Gamma^m_{g,n} = \operatorname{Diff}^+(F^m_{g,n}, \partial F^m_{g,n}) / \operatorname{Diff}^+_0(F^m_{g,n}, \partial F^m_{g,n}).$$

Thus, cohomology classes of $\mathfrak{M}^m_{g,n}$ yield characteristic classes for orientable surface bundles. For technical reasons, we consider a closed surface F with m marked points $P_1, \dots, P_m \in F$ and replace each boundary curve by a point $Q \in F$ and a tangential direction $0 \neq X \in T_Q F$ and obtain an isomorphic moduli space. Thus, a modulus in $\mathfrak{M}^m_{g,n}$ is a conformal class

$$[\mathcal{F}] \coloneqq [F; (Q_1, X_1), \dots, (Q_n, X_n); P_1, \dots, P_m].$$

We consider the space $\mathfrak{H}^m_{g,n}$ of all conformal classes $[\mathcal{F}, u]$ where $u \colon F \longrightarrow \mathbb{R} \cup \{\infty\}$ is a map which is harmonic outside of the points Q_j and P_i. Then $\mathfrak{H}^m_{g,n} \longrightarrow \mathfrak{M}^m_{g,n}, [\mathcal{F}, u] \longmapsto [\mathcal{F}]$ is an affine bundle, in particular $\mathfrak{H}^m_{g,n} \simeq \mathfrak{M}^m_{g,n}$. We restrict ourselves to the case $n = 1$. Consider all critical points of u and cut the surface along the flow lines of ∇u leaving these points. The image of each holomorphic completion $w = u + iv$ lies in a slitted complex plane. This translates the bundle $\mathfrak{H}^m_{g,1}$ into a relative bi-semisimplicial complex (P, P') where the cells are combinatoric glueing data of a slitted complex plane, called *slit configurations*, see [Böd90] and [ABE08]. Due to [Mül96], there is an orientation system $\mathcal{O}$, which gives us a notion of Poincaré–Lefschetz duality for the relative manifold (P, P'). Thus, as decribed in [Böd90], we get an isomorphism

$$\mathrm{PL}_{\mathfrak{P}} \colon H^k(\Gamma^m_{g,1}) \cong H^k(\mathfrak{M}^m_{g,1}) \longrightarrow H_{3h-k}(P, P'; \mathcal{O}).$$

Using this general framework, this thesis pursues three different approaches to describe classes in the cohomology of the mapping class group:

In the first place, it turns out that the relative simplicial complex (P, P') has several interesting subcomplexes (W_s, W'_s), called *Weierstraß complexes*, which are of dimension $3h + 2 - 2s$ and contained within each other as $W_s \subseteq W_{s-1}$. We construct a simplicial fundamental class $[w_s] \in H_{3h+2-2s}(P, P'; \mathcal{O})$ for each subcomplex.

Secondly, for each genus g, there are universal cohomology classes $\kappa_{s-1} \in H^{2s-2}(\Gamma_{g,1})$, called *Mumford–Miller–Morita classes*. They are stable with respect to the canonical inclusion of mapping class groups $\Gamma_{g,1} \hookrightarrow \Gamma_{g+1,1}$, and Madsen and Weiss showed in [MW07] that the rational cohomology ring $H^*(\Gamma_\infty; \mathbb{Q})$ of the stable mapping class group $\Gamma_\infty \coloneqq \varinjlim \Gamma_{g,1}$ is a polynomial algebra in the Mumford classes, which means we have

$$H^*(\Gamma_\infty; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \dots].$$

The third approach studies the surface bundle $F \longrightarrow \mathfrak{F} \xrightarrow{\pi} \mathfrak{H}$ and on this the vertical tangent bundle $\mathbb{C} \longrightarrow L \coloneqq T^\perp \mathfrak{F} \longrightarrow \mathfrak{F}$. We consider the h-fold symmetric fibre product $\mathbb{C}P^h \longrightarrow \mathrm{SP}^h L \longrightarrow \mathfrak{F}$ and construct a section $\omega \colon \mathfrak{F} \longrightarrow \mathrm{SP}^h L$ which "scans" for critical points of u in a geodesic disk around the given point. We show in Construction 3.3.8 that $\mathrm{SP}^h L$ is homologically trivial, i. e. we get a Leray–Hirsch isomorphism

$$H^*(\mathrm{SP}^h L) \cong H^*(\mathfrak{F}) \otimes \mathbb{Z}[\chi]/\chi^{h+1}.$$

All three approaches finally yield the same classes in the cohomology resp. homology of $\Gamma_{g,1}$. This result is summarised in the main theorem of this thesis:

Theorem 5.2.11. *Let $m = 0$ and $s \geq 2$. Then*

$$\mathrm{PL}_{\mathfrak{P}}(\kappa_{s-1}) = \mathrm{PL}_{\mathfrak{P}}(\pi^! \omega^* \chi^s) = [w_s].$$

Thus, we found a simplicial description of the Mumford classes by representing subcomplexes of a triangulation of the moduli space. Similar ideas, using a completely different framework, can be found in [Igu04].

Organisation of the thesis

The thesis has five parts:

(I) In the first chapter, we introduce the common notation for working with the previously mentioned bi-semisimplicial complex (P, P'), and we formulate the general constructions to get the identification between $\mathfrak{M}^m_{g,1}$ and $|P| \setminus |P'|$.

(II) In the second chapter, we start with a few general remarks concerning explicit calculations in the complex P and continue in CONSTRUCTION 2.2.2 with the notion of Weierstraß cells with respect to a certain partition $\nu \vdash h$. Finally, we study the mentioned subcomplexes (W_s, W'_s) of (P, P') and their topological properties and construct a fundamental class $[w_s]$ for each of them.

(III) The third chapter contains some general remarks and constructions for treating the cohomology of real and complex vector bundles and their symmetric fibre products.

(IV) We start the fourth chapter by studying the surface bundle $\mathfrak{F} \longrightarrow \mathfrak{P}$ homologically and give a brief survey of the Mumford classes. After this, we perform some geometric constructions to get the geodesic disks. We finish this chapter by constructing the section $\omega \colon B \longrightarrow \mathrm{SP}^h L$ and clarifying the relation between $\pi^! \omega^* \chi^s$ and κ_{s-1}.

(V) In the fifth chapter, we compare the constructions: We "complete" the surface bundle $\mathfrak{F}$ and describe the above mentioned version of Poincaré–Lefschetz duality on the completion. Finally, we show the main theorem using a transversality argument.

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1 Introduction

1.1 Moduli spaces and Teichmüller theory

We define the mapping class group $\Gamma_{g,1}^m$ of a Riemann surface of genus g with m punctures and n boundary curves and the corresponding moduli space $\mathfrak{M}_{g,n}^m$, following [FM12].

1.1.1 Definition. A *surface with structure* is a tuple $\mathcal{F} \coloneqq (F, \mathcal{Q}, \mathcal{X}, \mathcal{P})$ where:

(I) F is a connected and closed Riemann surface of genus g.
(II) $\mathcal{P} = \{P_1, \dots, P_m\} \subseteq F$ is an finite unordered set.
(III) $\mathcal{Q} = \{Q_1 < \dots < Q_n\} \subseteq F \setminus \mathcal{P}$ is a finite ordered set.
(IV) $\mathcal{X} = \{X_{Q_1} < \dots < X_{Q_n}\} \subseteq TF$ is a finite ordered set with $0 \neq X_{Q_i} \in T_{Q_i}F$.

We say that $\mathcal{F}$ is *of type* (g, m, n) and call the P *sinks* and the (Q, X) *dipoles* for reasons which we will see soon in Construction 1.3.2.

1.1.2 Definition. A *morphism* $f\colon (F, \mathcal{Q}, \mathcal{X}, \mathcal{P}) \longrightarrow (F', \mathcal{Q}', \mathcal{X}', \mathcal{P}')$ between two surfaces of type (g, m, n) is a map $f\colon F \longrightarrow F'$ such that:

(I) f is an orientation-preserving diffeomorphism of real manifolds.
(II) f induces a (bijective) map $\mathcal{P} \longrightarrow \mathcal{P}'$ by restriction.
(III) f induces a monotone map $\mathcal{Q} \longrightarrow \mathcal{Q}'$ and fulfills $\mathrm{d}f(X_Q) = X_{f(Q)}$.

The surfaces with structure of type (g, m, n) together with the morphisms form a category which is a connected groupoid.

1.1.3 Construction (Mapping class group)**.** Let $\mathcal{F}$ be a surface of type (g, m, n). The endomorphisms of $\mathcal{F}$ form a group $\mathrm{Diff}^+(\mathcal{F})$ which, endowed with the $\mathcal{C}^\infty$-Whitney topology, becomes a topological group. The path component of the identity is denoted by $\mathrm{Diff}_0^+(\mathcal{F})$ and contains all morphisms isotopic to the identity. The quotient

$$\Gamma(\mathcal{F}) \coloneqq \mathrm{Diff}^+(\mathcal{F}) / \mathrm{Diff}_0^+(\mathcal{F}) = \pi_0 \mathrm{Diff}^+(\mathcal{F})$$

is a discrete group and called *mapping class group*. Since the groupoid is connected, the isomorphism type of $\Gamma(\mathcal{F})$ does not depend on the choice of a surface $\mathcal{F}$ and we may just write $\Gamma_{g,n}^m$ for the mapping class group of an arbitrary surface of type (g, m, n).

1.1.4 Remark. Instead of looking at a closed surface with structure $(F, \mathcal{Q}, \mathcal{X}, \mathcal{P})$, we consider a surface $F_{g,n}^m$ with m removed points $P_1, \dots, P_m$ and parametrised boundary curves around the $Q_1, \dots, Q_n$. We alternatively consider endomorphisms of such surfaces to be those which keep the boundary pointwise fixed and obtain an isomorphic mapping class group. Furthermore, we consider the remaining surface

$$F_{g,n}^m \subseteq F^\star \coloneqq F \setminus (\mathcal{P} \cup \mathcal{Q}) \subseteq F$$

where all dipoles and sinks are removed. We have $F^\star \simeq F_{g,1}^m \simeq \bigvee^{2g+m+n-1} \mathbb{S}^1$, and thus, the Euler characteristic of $F_{g,n}^m$ is given by $\chi(F_{g,1}^m) = 2 - 2g - m - n$. We restrict ourselves to the case $n = 1$, so $\chi(F_{g,1}^m) = 1 - 2g - m$. For $\chi(F_{g,1}^m) \leq 0$ we have $g \geq 1$ or $m \geq 1$.

1.1.5 Construction (Teichmüller space)**.** Fix a (g,m,n)-surface $\mathcal{F}$. Let $q\colon \mathcal{F} \longrightarrow \mathcal{F}'$ be a morphism into another (g,m,n)-surface. For each $R \subseteq F$ conformally equivalent to a rectangle $[0,a] \times [0,b]$, the image $q(R) \subseteq F'$ is conformally equivalent to another rectangle $[0,a'] \times [0,b']$. Call q *quasiconformal* if there is a $1 \leq K < \infty$ such that

$$\frac{1}{K} \cdot \frac{a}{b} \leq \frac{a'}{b'} \leq K \cdot \frac{a}{b}$$

holds for all such rectangles. The infimum of all such K is called the *dilatation* $K(q)$. Let $\widetilde{\mathfrak{T}}(\mathcal{F})$ be the set of all quasiconformal homeomorphisms. We call q_1 and q_2 *equivalent* if $q_2 \circ q_1^{-1}$ is isotopic to a conformal map. The quotient is called *Teichmüller space* $\mathfrak{T}(\mathcal{F})$. For x_1 and x_2 in the Teichmüller space, we define their *Teichmüller distance*

$$d(x_1, x_2) \coloneqq \frac{1}{2} \cdot \log \sup \Big(K(q_2 \circ q_1^{-1});\; q_i \in x_i \Big).$$

Then d is a metric and $\mathfrak{T}(\mathcal{F})$ has the structure of a $(6g-6+2m+4n)$-dimensional contractible manifold (see [FM12, Thm. 11.17 & Thm. 10.6]).

1.1.6 Construction (Moduli space)**.** We have a right action of $\mathrm{Diff}^+(\mathcal{F})$ on $\mathfrak{T}(\mathcal{F})$ by isometries, namely precomposing $[q] \cdot f \coloneqq [q \circ f]$. The kernel of $\mathrm{Diff}^+(\mathcal{F}) \longrightarrow \mathrm{Aut}(\mathfrak{T}(\mathcal{F}))$ is $\mathrm{Diff}^+_0(\mathcal{F})$, which yields an action of $\Gamma(\mathcal{F})$ on $\mathfrak{T}(\mathcal{F})$. We define the *moduli space* of $\mathcal{F}$ by

$$\mathfrak{M}(\mathcal{F}) \coloneqq \mathfrak{T}(\mathcal{F})/\Gamma(\mathcal{F}).$$

The homeomorphism type of $\mathfrak{M}^m_{g,n}$ does not depend on the choice of $\mathcal{F}$, and we write $\mathfrak{M}^m_{g,n}$ for the moduli space of (g,m,n)-surfaces. The above description yields

$$\mathfrak{M}^m_{g,n} \cong \{(g,m,n)\text{-surfaces}\}\Big/\text{conformal equivalence},$$

i.e. the *moduli*, the elements of $\mathfrak{M}^m_{g,n}$, are equivalence classes $[\mathcal{F}]$ of (g,m,n)-surfaces identified with each other if there is a conformal isomorphism $f\colon \mathcal{F} \longrightarrow \mathcal{F}'$.

1.1.7 Example.

(I) The mapping class group $\Gamma^m_{0,1}$ is the braid group $\mathrm{Br}(m)$, and $\mathfrak{M}^m_{0,1}$ is homotopy equivalent to the unordered configuration space $\mathrm{Conf}_m(\mathbb{R}^2)$.

(II) Each conformal class of $F^0_{1,0}$ is represented by $\mathbb{C}/\Lambda_\tau$ where $\Lambda_\tau \coloneqq \mathbb{Z} + \tau\mathbb{Z} \subseteq \mathbb{C}$. Here $(1,\tau)$ has to be a positively oriented $\mathbb{R}$-basis, so $\tau \in \mathbb{H}_2$. If $\tau' = M \cdot \tau$ under a Möbius transformation $M \in \mathrm{SL}_2(\mathbb{Z})$, Λ_τ and $\Lambda_{\tau'}$ yield equivalent conformal structures, thus,

$$\mathfrak{M}^0_{1,0} \cong \mathbb{H}_2/\,\mathrm{SL}_2(\mathbb{Z}).$$

1.1.8 Construction. In the case where $n \geq 1$, the operation of $\Gamma^m_{g,n}$ on the contractible Teichmüller space $\mathfrak{T}^m_{g,n}$ is free, see [FM12]. This has two crucial consequences:

(I) Since the of $\Gamma^m_{g,n}$ on $\mathfrak{T}^m_{g,n}$ is proper, it is now properly discontinuous and as a quotient, $\mathfrak{M}^m_{g,n} = \mathfrak{T}^m_{g,n}/\Gamma^m_{g,n}$ is a manifold of dimension $6g-6+2m+4n$.

(II) $\mathfrak{M}^m_{g,n}$ is a classifying space $B\Gamma^m_{g,n}$. Since $\mathrm{Diff}^+ \longrightarrow \Gamma^m_{g,n}$ is a homotopy equivalence, we get $\mathfrak{M}^m_{g,n} \simeq B\Gamma^m_{g,n} \simeq B\,\mathrm{Diff}^+$ and $\mathfrak{M}^m_{g,n}$ classifies orientable surface bundles.

In particular, since we found a finite-dimensional classifying space for $\Gamma^m_{g,n}$, the mapping class group is torsion-free for $n \geq 1$.

1.2 Slit configurations

We describe the combinatorial and homological structure of the (semi-)simplicial complex of parallel slit configurations, following [Böd90] and [BH14]. Some special notations for simplicial structures and the symmetric group are explained in the APPENDICES B and C.

1.2.1 Definition. Recall that $\overline{\mathfrak{S}}_p = \mathrm{Sym}\{0,\dots,p\}$ is the extended symmetric group. A tuple $\Sigma = (\sigma_q : \dots : \sigma_0) \in \overline{\mathfrak{S}}_p^{q+1}$ is called (p,q)-*cell* (for reasons we will see very soon). We define the *norm* and the *number of punctures* of Σ by (cf. DEFINITION C.3)

$$N(\Sigma) \coloneqq \sum_{j=1}^{q} N(\sigma_j\sigma_{j-1}^{-1}) \qquad \text{and} \qquad m(\Sigma) \coloneqq \mathrm{ncyc}(\sigma_q) - 1.$$

We sometimes switch the notation of the entries in order to simplify statements:

(I) We call $(\sigma_q : \dots : \sigma_0)$ the *homogeneous notation*,
(II) We call $(\sigma_q\sigma_{q-1}^{-1} \mid \dots \mid \sigma_1\sigma_0^{-1})$ the *inhomogeneous notation.*

For a cell $(\tau_q \mid \dots \mid \tau_1)$ in inhomogeneous notation, we have

$$N(\Sigma) = N(\tau_1) + \dots + N(\tau_q).$$

1.2.2 Construction. Let $\mathfrak{X}_{p,q}$ be the set of all (p,q)-cells. We define the following *vertical face operators* d' and the *horizontal face operators* d'' by

$$\begin{aligned} d'_j &: \mathfrak{X}_{p,q} \longrightarrow \mathfrak{X}_{p,q-1}, (\sigma_q : \dots : \sigma_0) \longmapsto (\sigma_q : \dots : \widehat{\sigma}_j : \dots : \sigma_0),\\ d''_i &: \mathfrak{X}_{p,q} \longrightarrow \mathfrak{X}_{p-1,q}, (\sigma_q : \dots : \sigma_0) \longmapsto (D_i(\sigma_q) : \dots : D_i(\sigma_0)),\end{aligned}$$

where D_i denotes the skipping of the i^{th} symbol, see APPENDIX C. This turns $(\mathfrak{X}, d', d'')$ into a bi-semisimplicial complex. We will translate the horizontal faces into inhomogeneous notation in CHAPTER 2.1. For the vertical face operator, we see

$$d'_j(\tau_q \mid \dots \mid \tau_1) = \begin{cases} (\tau_q \mid \dots \mid \tau_2) & \text{for } j = 0,\\ (\tau_q \mid \dots \mid \tau_j\tau_{j-1} \mid \dots \mid \tau_1) & \text{for } 0 < j < q,\\ (\tau_{q-1} \mid \dots \mid \tau_1) & \text{for } j = q.\end{cases}$$

1.2.3 Remark. We may soon consider the geometric realisation of $(\mathfrak{X}_{p,q})$, but instead of taking the usual barycentric coordinates for our simplices, we may alternatively consider $\{(a_q,\dots,a_1) \in [-\infty,\infty]^q;\ a_q \ge \dots \ge a_1\}$: Let $a_{q+1} \coloneqq +\infty$ and $a_0 \coloneqq -\infty$ and consider

$$\Psi\colon \overline{\mathbb{R}} \longrightarrow [0,1], x \longmapsto \frac{\pi + 2\cdot\arctan(x)}{2\pi}.$$

Then, via $(a_q,\dots,a_1) \longmapsto (\Phi(a_{q+1}) - \Phi(a_q),\dots,\Phi(a_1) - \Phi(a_0))$, this coordinates can be translated into the barycentric ones whence the realisations coincide.

1.2.4 Definition. We call a cell $\Sigma = (\sigma_q : \dots : \sigma_0) = (\tau_q \mid \dots \mid \tau_1)$ *inner* if:

$$\left.\begin{aligned} &\text{(I)}\ \ \sigma_0 = \langle 0,\dots,p\rangle,\\ &\text{(II)}\ \ \sigma_j \neq \sigma_{j-1} \text{ for all } 1 \le j \le q,\\ &\text{(III)}\ \ \textstyle\bigcap_{j=0}^{q} \mathrm{inert}(\sigma_j) = \{p\}.\end{aligned}\right\} \text{ OR EQUIVALENTLY } \left\{\begin{aligned} &\text{(I)}\ \ \tau_j \in \mathfrak{S}_p = \mathrm{Sym}\{1,\dots,p\},\\ &\text{(II)}\ \ \tau_j \neq \mathrm{id} \text{ for all } 1 \le j \le q,\\ &\text{(III)}\ \ \mathrm{fix}(\tau_q,\dots,\tau_1) = \emptyset.\end{aligned}\right.$$

Obviously, since σ_0 is fixed, both notations can be translated into each other.

1.2.5 Construction. For a given tuple (g, m), we define $h := 2g + m$ and

$$P_{p,q} := \left\{ \Sigma \in \mathfrak{X}_{p,q};\ \begin{array}{rl} \text{(I)} & p \le 2h \text{ and } q \le h, \\ \text{(II)} & N(\Sigma) \le h \text{ and } m(\Sigma) \le m, \\ \text{(III)} & \text{If } N(\Sigma) = h, \text{ then } \operatorname{fix}(\tau_q, \dots, \tau_1) = \emptyset \end{array} \right\},$$

$$P'_{p,q} := \{\Sigma \in P_{p,q};\ N(\Sigma) < h \text{ or } m(\Sigma) < m \text{ or } \Sigma \text{ not inner}\}.$$

One can easily see that the boundary operators d' and d'' restricted on these subsets are well-defined, and we get bi-semisimplicial subcomplexes $(P, d'|_P, d''|_P)$ and $(P', d'|_{P'}, d''|_{P'})$. We call P the *slit complex* and P' the *degenerate subcomplex*. Finally, we define

$$\mathfrak{P}^m_{g,1} := |P| \setminus |P'|.$$

In other words, points in $\mathfrak{P}^m_{g,1}$ are represented by classes $[\Sigma; a, b]$ where Σ is an inner cell with $N(\Sigma) = h$ and $m(\Sigma) = m$. We call such an element *slit configuration*.

1.2.6 Remark. For each cell $\Sigma \in P$, the boundaries $d'_0\Sigma$ and $d'_q\Sigma$ are degenerate since

$$N(d'_0\Sigma) = N(\tau_q) + \dots + N(\tau_2) = N(\Sigma) - N(\tau_1) < N(\Sigma)$$

and analogously for $d'_q\Sigma$. Similarly $d''_0\Sigma$ and $d''_p\Sigma$ are degenerate. Therefore, each cell has at least four degenerate boundaries in $|P|$.

1.2.7 Remark. Let $[\Sigma; a, b]$ be a slit configuration. We set $a_{q+1} := b_0 := -\infty$ as well as $a_0 := b_{p+1} := +\infty$ and divide $[-\infty, +\infty]^2$ into rectangles $R_{j,i} := [a_{j+1}, a_j] \times [b_i, b_{i+1}]$. Now, we number the top and the bottom edges of each rectangle such that

$$\operatorname{top}(R_{j,i}) = \operatorname{bottom}(R_{j,\sigma_j(i)})$$

holds for all $0 \le i < p$. For a more extended formulation, see for example [BH14, p. 29]. Doing this for $\Sigma := (\langle 2, 4\rangle \mid \langle 1, 3\rangle)$, we obtain the following glueing prescription

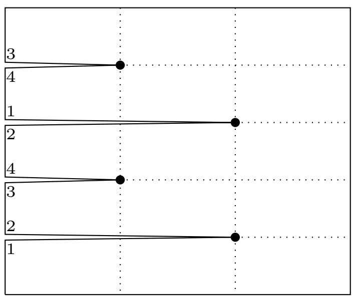


FIGURE 1.1: The slit picture for the cell $(\langle 2, 4\rangle \mid \langle 1, 3\rangle)$.

1.2.8 Definition. Let $[\Sigma; a, b] \in \mathfrak{P}$ be an inner point of an inner cell $\Sigma := (\tau_q \mid \dots \mid \tau_1)$. A *critical point* S of Σ is a non-trivial cycle of τ_j, and its *multiplicity* $\operatorname{mult}_S(\Sigma)$ is defined to be the norm of the cycle (the cycle length decreased by one). We get

$$\sum_S \operatorname{mult}_S(\Sigma) = \sum_{j=1}^{q} \sum_{\zeta \text{ cycle of } \tau_j} N(\zeta) = \sum_{j=1}^{q} N(\tau_j) = h.$$

It turns out that this description corresponds to a classical geometric multiplicity of a critical point, see DEFINITION 1.3.3 and CONSTRUCTION 1.3.8.

1.2.9 Reminder. A *relative n-manifold* is a topological pair (X, X') with:

(I) X is a Hausdorff space.
(II) $X' \subseteq X$ is closed (potentially empty).
(III) $X \setminus X'$ is a manifold of dimension n without boundary.

The classical example is the case where $(X, X') = (M, \partial M)$ is a manifold with boundary. We call the relative manifold (X, X') *compact* if X is compact, and *connected* if the complement $X \setminus X'$ is connected. In this situation, we obtain the following version of *Poincaré–Lefschetz duality* as presented in [Spa94, ch. 6.2]:

1.2.10 Theorem (Poincaré–Lefschetz)**.** *If (X, X') is a compact and connected relative n-manifold, there is a local system $\mathcal{O}$* (cf. Definition B.10) *and a fundamental class, i. e. a generator $[X, X'] \in H_n(X, X'; \mathcal{O})$, such that we have an isomorphism*

$$\begin{aligned} H^k(X \setminus X') &\longrightarrow H_{n-k}(X, X'; \mathcal{O}), \\ \vartheta &\longmapsto \vartheta \frown [X, X']. \end{aligned}$$

1.2.11 Proposition (Bödigheimer)**.** $(|P|, |P'|)$ *is a compact connected relative $3h$-manifold. There is a local system $\mathcal{O}$ and a fundamental class $[P, P'] \in H_{3h}(P, P'; \mathcal{O})$ such that*

$$\begin{aligned} \mathrm{PL}\colon H^k(\mathfrak{P}^m_{g,1}) &\longrightarrow H_{3h-k}(P, P'; \mathcal{O}), \\ \vartheta &\longmapsto \vartheta \frown [P, P']. \end{aligned}$$

is an isomorphism for each $0 \le k \le 3h$.

1.2.12 Construction. The orientation system is constructed in [Mül96] by considering an $m!$-fold covering $\widehat{\mathfrak{P}} \longrightarrow \mathfrak{P}$ with orientable total space where the extra information is given by numbering the punctures. For each cell Σ from $\mathfrak{P}$, there is a canonical numbering of the punctures and thus, a cell $\widehat{\Sigma}$ in $\widehat{\mathfrak{P}}$. We now have to track how the ordering of the punctures is changed if we reach the boundary of Σ. This geometric idea gives rise to an explicit algebraic description of the orientation system (cf. [Meh11, ch. 3.7] and [BH14, ch. 2.6]): For $\Sigma = (\sigma_q : \dots : \sigma_0) \in P_{p,q}$, we decompose σ_q into cycles

$$\sigma_q = \zeta_0 \cdots \zeta_m \quad \text{with} \quad \zeta_k = \langle \alpha_{k,1}, \dots, \alpha_{k,r_k} \rangle,$$

where we assume that $\beta_k \coloneqq \alpha_{k,1} \le \alpha_{k,l}$ and $\beta_k < \beta_{k+1}$ holds, i. e. the cycles are in standard form and indexed after the smallest start value. Consider the *standard puncture numbering*

$$\nu_\Sigma\colon [p] \longrightarrow [m], i \longmapsto k \text{ for } i \in \zeta_k.$$

We want to compare $\nu_\Sigma \circ \mathbf{d}_i$ and $\nu_{d''_i\Sigma}$. There are three cases:

(I) If $i \in \operatorname{fix}(\sigma_q)$, then $m(d''_i\Sigma) < m(\Sigma)$, and the cell is degenerate and not of interest.
(II) If $i \neq \beta_k$, then $\nu_\Sigma \circ \mathbf{d}_i = \nu_{d''_i\Sigma}$, we set $\varepsilon_i(\Sigma) = 1$.
(III) If $i = \beta_k$, i leaves another $\beta'_k = \alpha_{k,l}$ as the smallest element of ζ_k. If we swap β'_k with all β_l such that $\beta_k < \beta_l < \beta'_k$, we correct the numerbering of the punctures. Hence, we set $\varepsilon_i(\Sigma) = (-1)^{l-k}$ where l is the maximum with $\beta_k < \beta_l < \beta'_k$.

Then $\mathcal{O} \coloneqq (\varepsilon_i(\Sigma))_{i,\Sigma}$ is our local system and the modified differentials are

$$\partial'\Sigma = \sum_{j=0}^{q} (-1)^j \cdot d'_j\Sigma \quad \text{and} \quad \partial''\Sigma = \sum_{i=0}^{p} (-1)^i \cdot \varepsilon_i(\Sigma) \cdot d''_i\Sigma.$$

1.3 Hilbert uniformisation

We sketch the concept of Hilbert uniformisation presented in [Böd90] and use it to relate the homology of $\mathfrak{M}^m_{g,1}$ with the simplicial homology of the slit complex (P, P').

1.3.1 Definition. Let F be a Riemann surface and $u\colon F \longrightarrow \mathbb{R} \cup \{\infty\} \subseteq \overline{\mathbb{C}}$ be smooth.

(I) (Q, X) is called *dipole* if there is a $B_0 > 0$, a chart z around Q with $z(Q) = 0$ and $\mathrm{d}z(X) = \partial_x$, and a holomorphic function $f\colon U \longrightarrow \mathbb{C}$ around 0 such that

$$u(z) = \mathrm{Re}\left(\frac{1}{z} - B_0 \cdot \log(z) + f(z)\right).$$

(II) P_i is called *logarithmic sink* if there is $B_i > 0$ and z around P_i with $z(P_i) = 0$ and a holomorphic function $f_i\colon U \longrightarrow \mathbb{C}$ around 0 such that

$$u(z) = \mathrm{Re}\left(B_i \cdot \log(z) + f_i(z)\right)$$

1.3.2 Construction (Potential functions)**.** Let $\mathcal{F} \coloneqq (F, Q, X, \mathcal{P})$ be a $(g, m, 1)$-surface. A smooth map $u\colon F \longrightarrow \mathbb{R} \cup \{\infty\}$ is called *potential on* $\mathcal{F}$ if:

(I) (Q, X) is a dipole of u (with a coefficient called $B_0 > 0$).
(II) $P_1, \ldots, P_m$ are logarithmic sinks of u (with coefficients called $B_i > 0$).
(III) $u\colon F^\star \longrightarrow \mathbb{R}$ is harmonic, i. e. $\Delta u = 0$ for the Laplace operator Δ.

By the Residue theorem, we get $B_0 = B_1 + \cdots + B_m$ and by the Riemann–Roch theorem, $B_1, \ldots, B_m$ determine the map u uniquely up to an additive constant $C \in \mathbb{R}$.

1.3.3 Definition. A point $S \in F^\star = F \setminus \{Q, P_1, \ldots, P_m\}$ is called *critical* if $\mathrm{d}u(S) = 0$. After a choice of a Riemannian metric on the tangential bundle $TF^\star$, we get a vector field $\nabla u : F^\star \longrightarrow TF^\star$ and define the *multiplicity* of the critical point

$$\mathrm{mult}_S(u) \coloneqq -\,\mathrm{ind}(\nabla u, S)$$

by the negative index of ∇u at S. Since the index of the dipole is 2 and the index of each sink is 1, we get by the Poincaré–Hopf index theorem

$$\sum_{S \in F^\star} \mathrm{mult}_S(u) = 2 + m - \chi(F) = 2g + m = h.$$

1.3.4 Construction. Let u be a potential on $\mathcal{F}$. An integral curve $\gamma\colon \mathbb{R} \longrightarrow F^\star$ of $-\nabla u$ is called *critical* if there is a $1 \le i \le k$ such that γ starts in S_i, which means

$$\lim_{t \longrightarrow -\infty} \gamma(t) = S_i.$$

We define the *critical graph* $\mathcal{K} \subseteq F$ whose vertices are given by the dipole Q, the sinks $P_1, \ldots, P_m$ and the critical points $S_1, \ldots, S_k$ and the edges are given by the critical curves.

1.3.5 Construction. The gradient flow of ∇u yields a contraction of $F \setminus \mathcal{K}$ to a small area in front of the dipole. Therefore, $F \setminus \mathcal{K}$ is contractible, and since u is harmonic on $F \setminus \mathcal{K}$, there is a completion $v\colon F \setminus \mathcal{K} \longrightarrow \mathbb{R}$ such that we get a holomorphic map

$$w \coloneqq u + i \cdot v\colon F \setminus \mathcal{K} \longrightarrow \mathbb{C}$$

The completion v is unique up to an additive constant $D \in \mathbb{R}$, and $\mathrm{Im}(w)$ is the complex plane $\mathbb{C}$ with some left half rays removed. If we number the flow lines along which we cut the surface, we get a slit picture $\mathcal{H}(\mathcal{F}, u, D) \in |P| \setminus |P'|$.

1.3.6 Example. In the case of one sink P and one critical point S of multiplicity 1, we get the following picture as presented in [BH14, Fig. 2.6]:

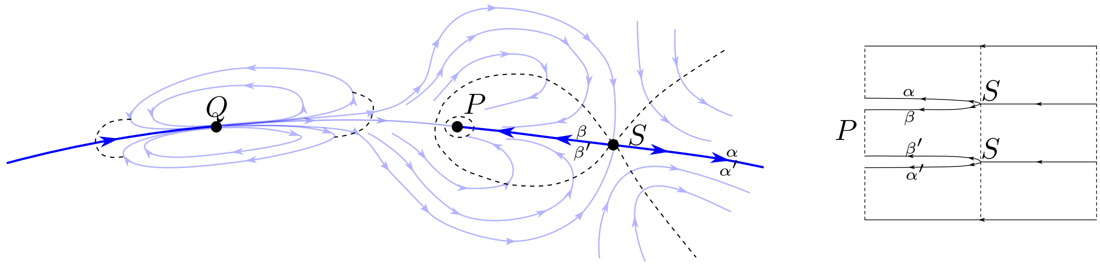


FIGURE 1.2: The gradient flow of a potential function and the slit picture.

1.3.7 Construction. For a given tuple (g, m), we consider on the set of tuples

$$\widetilde{\mathfrak{H}}^m_{g,1} := \{(\mathcal{F}, u, D);\ \mathcal{F} \text{ is an } (g, m, 1)\text{-surface, } u \text{ is a potential, } D \in \mathbb{R}\}$$

the equivalence relation where $(\mathcal{F}, u, D) \sim (\mathcal{F}', u', D')$ if there is a complex isomorphism $f\colon \mathcal{F} \longrightarrow \mathcal{F}'$ with $u = u' \circ f$ and if moreover $D' = D$. Define the quotient $\mathfrak{H}^m_{g,1} := \widetilde{\mathfrak{H}}^m_{g,1}/\sim$. The projection $(\mathcal{F}, u, D) \longmapsto \mathcal{F}$ factors through $\mathfrak{H} \longrightarrow \mathfrak{M}$, which we call *potential bundle.* The elements $\mathcal{N} = [\mathcal{F}, u, D]$ are called *potential classes.* For the fibres we have a bijection

$$\begin{aligned} \mathbb{A}_m := (\mathbb{R}_{>0})^m \times \mathbb{R}^2 &\longrightarrow \mathfrak{H}_{[\mathcal{F}]}, \\ (B_1, \ldots, B_m, C, D) &\longmapsto [\mathcal{F}, u_{(B_1,\ldots,B_m,C)}, D]. \end{aligned}$$

The above process assigning to each such tuple $(\mathcal{F}, u, D)$ a slit picture, factors through what Bödigheimer calls *Hilbert uniformisation* $\mathcal{H}\colon \mathfrak{H}^m_{g,1} \longrightarrow \mathfrak{P}^m_{g,1}$.

1.3.8 Construction. The Hilbert uniformisation $\mathcal{H}$ has an inverse: Let $[\Sigma; a, b]$ be a slit configuration. We define the rectangles $R_{j,i} \subseteq [-\infty, \infty]^2$ as above and consider

$$\widetilde{F} := \coprod_{j=0}^{q} \coprod_{i=0}^{p} R_{j,i}.$$

Now we construct the surface by identification (here we only sketch the geometric idea, we will give an explicit cellular decomposition in CONSTRUCTION 5.1.1).

(I) For $0 \leq i \leq q-1$, the right edge of $R_{j+1,i}$ with the left of $R_{j,i}$.

(II) For $0 \leq i \leq p-1$, the top edge of $R_{j,i}$ with the bottom edge of $R_{j,\sigma_j(i)}$.

(III) All top edges from $R_{j,p}$, bottom edges from $R_{j,0}$ and right edges from $R_{0,i}$ are contracted to Q. If $k := \nu_\Sigma(i) = 0$, contract the left edge from $R_{q,i}$ to Q, else to P_k.

We obtain a closed surface F with marked points $Q, P_1, \ldots, P_m$ which is triangulated into rectangles, triangles and digons. It carries a canonical structure, see [Böd90]:

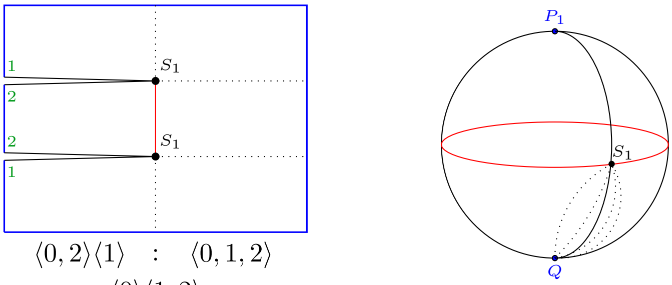
(I) A complex atlas is given by the rectangles which are glued together.

(II) The real coordinate of each point in the slit picture yields the potential u.

(III) The y-coordinate of the first slit yields an additive constant.

The critical points of $[\Sigma, a, b]$ correspond to those of u and their notions of multiplicity coincide. By the Poincaré–Hopf index theorem, the surface is of genus g. This glueing construction finally yields a map $\mathcal{G}\colon \mathfrak{P}^m_{g,1} \longrightarrow \mathfrak{H}^m_{g,1}$, which is inverse to $\mathcal{H}$. We endow $\mathfrak{H}^m_{g,1}$ with the induced topology from $\mathfrak{P}^m_{g,1}$.

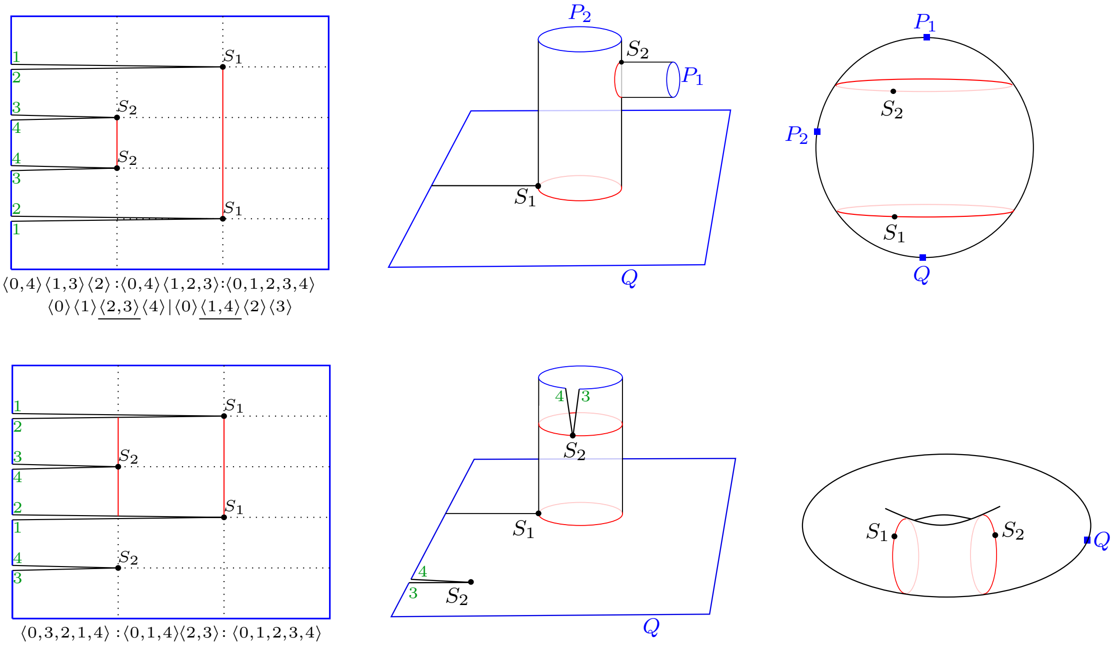
1.3.9 Example. An easy case is the following slit picture, from which we can read off the slit configuration and see that we have $m(\Sigma) = 1$ and $N(\Sigma) = 1$, i. e. $g(\Sigma) = 0$.



$$\begin{aligned}\Sigma &= \quad \langle 0,2\rangle\langle 1\rangle \quad : \quad \langle 0,1,2\rangle \\ &= \qquad \langle 0\rangle\langle 1,2\rangle\end{aligned}$$

FIGURE 1.3: The glueing construction for the case $m = 1$ and $g = 0$.

1.3.10 Example. Consider the following two slit configurations with two distinct critical points with multiplicity 1 (the underlined 2-cycles in τ_j): In the first situation, we have two sinks and genus 0, and in the second situation, we have no sink and genus 1.



$\langle 0,4\rangle\langle 1,3\rangle\langle 2\rangle : \langle 0,4\rangle\langle 1,2,3\rangle : \langle 0,1,2,3,4\rangle$
$\langle 0\rangle\langle 1\rangle\underline{\langle 2,3\rangle}\langle 4\rangle \mid \langle 0\rangle\underline{\langle 1,4\rangle}\langle 2\rangle\langle 3\rangle$

$\langle 0,3,2,1,4\rangle : \langle 0,1,4\rangle\langle 2,3\rangle : \langle 0,1,2,3,4\rangle$
$\langle 0\rangle\underline{\langle 1,3\rangle}\langle 2\rangle\langle 4\rangle \mid \langle 0\rangle\langle 1\rangle\underline{\langle 2,4\rangle}\langle 3\rangle$

FIGURE 1.4: The two possibilities for configurations with two critical points of multiplicity 1.

1.3.11 Theorem (Bödigheimer, [Böd90])**.** *The map $\mathfrak{H}^m_{g,1} \longrightarrow \mathfrak{M}^m_{g,1}$ is a fibre bundle with contractible fibre $\mathbb{A}_m$, hence a homotopy equivalence. In particular, we get for $0 \le k \le 3h$*

$$H^k(\mathfrak{M}^m_{g,1}) \cong H^k(\mathfrak{H}^m_{g,1}) \cong H_{3h-k}(P, P'; \mathcal{O}).$$

2 Weierstraß constructions

2.1 Calculations in the slit complex

We want to understand the horizontal boundaries $d_i''\Sigma$ and the situations in which these are shared by two cells given in inhomogeneous notation.

2.1.1 Definition. Recall that a cell $\Sigma \coloneqq (\tau_q \mid \cdots \mid \tau_1)$ is called *inner* if all τ_j are non-trivial and have no common fixed point. We furthermore define:

(I) Σ is called *vertically separated* if the τ_j have pairwise disjoint support. In terms of the potential, this means that there are no two critical points on the same v-level. Note that in this case, $\tau_q \cdots \tau_1 = \tau_{\pi(q)} \cdots \tau_{\pi(1)}$ for each $\pi \in \mathfrak{S}_q$.

(II) Σ is called *horizontally separated* if all τ_j are cycles. In terms of the potential, this means that there are no two critical points on the same u-level.

(III) Σ is called a *separated* if it is both horizontally and vertically separated.

Let $(\tau_q \mid \cdots \mid \tau_1)$ be a vertically separated inner (p,q)-cell. Then for each $1 \le i \le p$, there is a unique $1 \le j(i) \le q$ such that $i \in \operatorname{supp}(\tau_{j(i)})$.

2.1.2 Reminder. Let Σ be an inner cell, so Σ is non-degenerate in a given (P, P') with fixed m. The boundary $d_i''\Sigma$ is called *degenerate* if one of the following situations occurs:

(I) $N(d_i''\Sigma) < N(\Sigma)$.
(II) $m(d_i''\Sigma) < m(\Sigma)$.
(III) $d_i''\Sigma$ is not inner.

2.1.3 Reminder. As in the APPENDIX B, we denote the cosimplicial maps by

$$\mathbf{d}_i \colon [p-1] \longrightarrow [p] \qquad \text{and} \qquad \mathbf{s}_i \colon [p] \longrightarrow [p-1].$$

2.1.4 Proposition. *Let $\Sigma \coloneqq (\tau_q \mid \cdots \mid \tau_1)$ be a separated inner (p,q)-cell and let $0 \le i \le p$.*

(I) *We have $d_i''\Sigma = (\tau_q'' \mid \cdots \mid \tau_1'')$ where for each $1 \le j \le q$, we have*

$$\tau_j'' = \mathbf{s}_i \circ \langle i, \sigma_j(i)\rangle \circ \tau_j \circ \langle i, \sigma_{j-1}(i)\rangle \circ \mathbf{d}_i.$$

In particular, if $j \neq j(i)$, we obtain $\tau_j'' = \mathbf{s}_i \circ \tau_j \circ \mathbf{d}_i$.

(II) *If $i = 0$ or $i = p$ or $j(i) = j(i+1)$, then $d_i''\Sigma$ is degenerate.*

(III) *In all other cases, $d_i''\Sigma$ is non-degenerate. Let $j \coloneqq j(i) \neq j(i+1)$ and define*

$$\alpha \coloneqq \sigma_{j-1}(i) = (\tau_{j-1} \cdots \tau_1)(i+1).$$

Obviously, $\alpha \notin \operatorname{supp}(\tau_j)$. Consider the modified cycle $\overline{\tau}_j \coloneqq \langle i, \alpha\rangle \circ \tau_j \circ \langle i, \alpha\rangle$ where the symbol i is replaced by α. Then the above formula becomes

$$\tau_j'' = \mathbf{s}_i \circ \overline{\tau}_j \circ \mathbf{d}_i.$$

Proof. For the first statement, see [BH14, Prop. 2.3.10]. The third statement follows from $\sigma_{j-1}(i) = \sigma_j(i) = \alpha$, so we are left with showing the second: Since Σ is an inner cell, 1 is part of a non-trivial cycle of some τ_j. Then $\tau_j'' = \mathbf{s}_0 \circ \tau_j \circ \mathbf{d}_0$ has a non-trivial cycle containing 0. Moreover, p is in the support of some τ_j. Since $\sigma_j(p) = \sigma_{j-1}(p) = 0$, we get

$$\tau_j'' = \mathbf{s}_i \circ \langle p, 0\rangle \circ \tau_j \circ \langle i, 0\rangle \circ \mathbf{d}_i.$$

Thus, τ_j'' has a non-trivial cycle containing 0. Now consider the case where i and $i+1$ lie in the same cycle of some τ_j, so we may assume that τ_j is of the form $\langle i, \alpha_1, \dots, \alpha_r, i+1, \beta_1, \dots, \beta_s\rangle$. Then $\sigma_{j-1}(i) = i+1$ and $\sigma_j(i) = \tau_j\sigma_{j-1}(i) = \beta_1$ and we can calculate τ_j'' by

$$\begin{aligned}\tau_j'' &= \mathbf{s}_i \circ \langle i, \beta_1\rangle\langle i, \alpha_1, \dots, \alpha_r, i+1, \beta_1, \dots, \beta_s\rangle\langle i, i+1\rangle \circ \mathbf{d}_i \\ &= \langle i, \mathbf{s}_i\alpha_1, \dots, \mathbf{s}_i\alpha_r\rangle\langle \mathbf{s}_i\beta_1, \dots, \mathbf{s}_i\beta_s\rangle\end{aligned}$$

Thus, we get $N(\tau_j'') = r+s-1 < r+s+1 = N(\tau_j)$ and $d_i''\Sigma$ is degenerate. □

2.1.5 Definition. Let $\Sigma = (\tau_q \mid \cdots \mid \tau_1)$ be a (p,q)-cell. There are two systematic ways to get other (p,q)-cells, as described in [Sch03] and [ABE08]:

(I) For $\varrho \in \mathfrak{S}_p$, we have the *entry conjugation*

$$\varrho \cdot \Sigma := \left(\varrho\tau_q\varrho^{-1} \mid \cdots \mid \varrho\tau_1\varrho^{-1}\right)$$

(II) For $\pi \in \mathfrak{S}_q$, we have the *entry permutation*

$$\Sigma_\pi := \left(\tau_{\pi(q)} \mid \cdots \mid \tau_{\pi(1)}\right)$$

Both actions commute with each other, which means $\varrho \cdot (\Sigma_\pi) = (\varrho \cdot \Sigma)_\pi =: \varrho \cdot \Sigma_\pi$. Moreover, both actions fix the norm, but may change the puncture number. However, in the special case where Σ is vertically separated, we know that $\tau_q \cdots \tau_1 = \tau_{\pi(q)} \cdots \tau_{\pi(1)}$ and hence,

$$\begin{aligned} m(\Sigma_\pi) &= \operatorname{ncyc}(\tau_{\pi(q)} \cdots \tau_{\pi(1)}\sigma_0) - 1 \\ &= m(\Sigma). \end{aligned}$$

2.1.6 Corollary. *Let Σ be a separated inner cell and let $1 \le i \le p$ such that $d_i''\Sigma$ is non-degenerate. Now distinguish between the following cases as in* FIGURE 2.1*:*

CASE 1: *If $j(i) > j(i+1)$, let*

$$\alpha := \tau_{j(i+1)}(i+1).$$

There are two possible cases:

(I) *If $\alpha > i+1$, we get the coinciding boundaries $d_i''\Sigma = d_{\alpha-1}''(\varrho \cdot \Sigma)$ where*

$$\varrho := \langle i, \alpha, \alpha-1, \dots, i+1\rangle.$$

(II) *If $\alpha < i$, we get the coinciding boundaries $d_i''\Sigma = d_\alpha''(\varrho \cdot \Sigma)$ where*

$$\varrho := \langle \alpha+1, \alpha+2, \dots, i-1, i\rangle.$$

CASE 2: *If $j(i) < j(i+1)$, let*

$$\beta := \tau_{j(i)}^{-1}(i).$$

There are two possible cases:

(I) *If $\beta < i$, we get the coinciding boundaries $d_i''\Sigma = d_\beta''(\varrho \cdot \Sigma)$ where*

$$\varrho := \langle \beta, \beta+1, \dots, i, i+1\rangle.$$

(II) *If $\beta > i+1$, we get the coinciding boundaries $d_i''\Sigma = d_{\beta-1}''(\varrho \cdot \Sigma)$ where*

$$\varrho := \langle i+1, \beta-1, \beta-2, \dots, i+2\rangle.$$

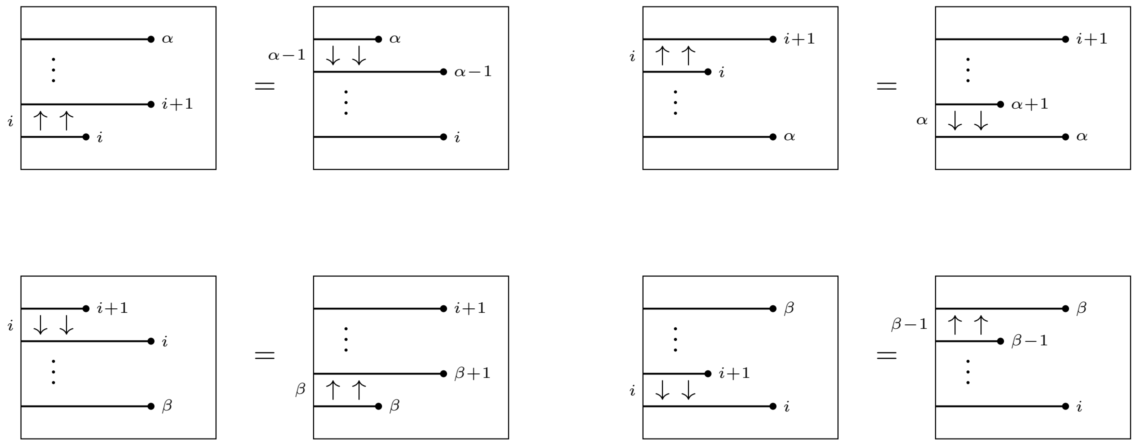


Figure 2.1: The four possibilities for coinciding horizontal boundaries.

2.1.7 Definition. Let $\mathcal{W}$ be a set of (p,q)-cells in (P,P'). A *pairing* is an involution $\Phi\colon \mathcal{W}\times\{1,\dots,p\}\longrightarrow \mathcal{W}\times\{1,\dots,p\}$, which means $\Phi^2=\mathrm{id}$, without fixed points.

2.1.8 Proposition. *For each boundary $d''_i\Sigma$ of a separated inner (p,q)-cell there is exactly one other $(\overline{\Sigma},\overline{i})\neq(\Sigma,i)$ where $\overline{\Sigma}$ is a separated inner (p,q)-cell and $1\leq\overline{i}\leq p$ such that*

$$d''_i\Sigma = d''_{\overline{i}}\overline{\Sigma}$$

and this other pair is given by the above cases. It might occur that $\overline{\Sigma}=\Sigma$, but not $\overline{i}=i$. This gives rise to a pairing $(\Sigma,i)\longmapsto(\overline{\Sigma},\overline{i})$ on the set of all separated inner (p,q)-cells.

Proof. The existence of another cell $\overline{\Sigma}$ sharing a different boundary with Σ is clear from the last corollary because in all cases, the face index differs from i. For the uniqueness, note that there is exactly one symbol $1\leq\alpha\leq p-1$ occuring twice in the inhomogeneous notation of $d''_i\Sigma$, e. g. in τ''_k and τ''_l with $k>l$. There are two possibilities:

(I) $i=\alpha$ and $d''_i\Sigma$ is a boundary of the above case 2.
(II) $i=(\tau''_l)^{-1}(\alpha)$ and $d''_i\Sigma$ is a boundary of the above case 1.

In both cases, the original cell is reconstructed by $\Sigma=(\tau_q\mid\cdots\mid\tau_1)$ where

$$\tau_j=\begin{cases}\mathbf{d}_i\circ\tau''_j\circ\mathbf{s}_i & \text{for } j\neq k,\\ \langle i,\mathbf{d}_i\alpha\rangle\circ\mathbf{d}_i\circ\tau''_k\circ\mathbf{s}_i\circ\langle i,\mathbf{d}_i\alpha\rangle & \text{for } j=k.\end{cases}$$

It is readily verified that this reconstruction really yields the original cell Σ. Clearly, the other cell $(\overline{\Sigma},\overline{i})$ has to be given by the corresponding other possibility. □

2.1.9 Remark. The situation for vertical boundaries of separated cells is easier: If Σ and $\overline{\Sigma}$ share a vertical boundary, i. e. if there are $1\leq j,\overline{j}\leq q$ with

$$d'_{\overline{j}}\overline{\Sigma}=d'_j\Sigma,$$

then $\overline{j}=j$ and $\overline{\Sigma}=\Sigma_{\langle j,j-1\rangle}$, which means the entries τ_j and τ_{j-1} are switched. We get a pairing with the explicit description $(\Sigma,j)\longmapsto(\Sigma_{\langle j,j-1\rangle},j)$.

2.2 Weierstraß cells

Based on ideas of [Böd18], we introduce the notion of *Weierstraß*[1] *cells* for a partition of h. In particular, we are interested in Weierstraß cells with a prescribed puncture number.

2.2.1 Definition. A non-increasing tuple $\nu := (h_k \geq \cdots \geq h_1)$ of positive integers is called *partition of* h, write $\nu \vdash h$, if $h_q + \cdots + h_1 = h$. We define for a partition $\nu \vdash h$:

(I) The *length* $|\nu| := k$ of the partition.
(II) For each $1 \leq i \leq h$ the *incidence number* $c(\nu, i) := \#\{1 \leq j \leq q;\ h_j = i\}$.
(III) We call the partition *primitive* if $c(\nu, i) \leq 1$ for each i.
(IV) We call the partition *homogeneous* if all h_j are equal.

Obviously, $c(\nu, 1) + \cdots + c(\nu, h) = |\nu|$ and the only primitive homogeneous partition is (h). Furthermore, for $\nu \vdash h$ and $\nu' \vdash h'$, there is a canonical sum $\nu \oplus \nu' \vdash h + h'$.

2.2.2 Construction. Let $\nu := (h_k \geq \cdots \geq h_1) \vdash h$. An inner (p, q)-cell $\Sigma := (\tau_q \mid \cdots \mid \tau_1)$ is called *Weierstraß cell of type* ν if it has k critical points with multiplicities $h_k, \ldots, h_1$.

(I) If Σ is a horizontally separated, then $q = k$ and each τ_j is (up to index permutation) a cycle of length $h_j + 1$, in particular, $N(\tau_j) = h_j$.
(II) If Σ is vertically separated, then $p = h + k$.
(III) If Σ is separated, then $\dim(\Sigma) = (h + q, q)$. We call these *Weierstraß top cells*. Let $\mathcal{W}^*_\nu$ be the set of all Weierstraß top cells of type ν (neglecting m).

2.2.3 Proposition. *Let $\nu := (h_1, \ldots, h_q) \vdash h$. We define $k_j := j + h_1 + \ldots + h_{j-1}$. We have a canonical inner Weierstraß top cell $\Sigma = (\tau_q \mid \cdots \mid \tau_1)$ of type ν given by*

$$\tau_j := \langle k_j, \ldots, k_j + h_j \rangle \in \mathfrak{S}_p.$$

Let $Z := Z(\tau_q \cdots \tau_1) \subseteq \mathfrak{S}_p$ be the centraliser of $\tau_q \cdots \tau_1$. Choose a system $\varrho_1, \ldots, \varrho_r \in \mathfrak{S}_p$ of representatives of $\mathfrak{S}_p / Z$. We obtain a bijection (depending on the choices of $\varrho_1, \ldots, \varrho_r$)

$$\begin{aligned} \Phi \colon \mathfrak{S}_q \times \mathfrak{S}_p / Z &\longrightarrow \mathcal{W}^*_\nu, \\ (\pi, \varrho_l) &\longmapsto \varrho_l \cdot \Sigma_\pi. \end{aligned}$$

Proof. It is clear that Φ is well-defined since all cells $\varrho_i \cdot \Sigma_\pi$ are inner Weierstraß top cells of type ν. We show surjectivity and injectivity separately:

(I) Let $\overline{\Sigma} = (\overline{\tau}_q \mid \cdots \mid \overline{\tau}_1)$ be an inner Weierstraß top cell. Clearly, there is a $\pi_1 \in \mathfrak{S}_q$ and a $\varrho \in \mathfrak{S}_p$ such that $\overline{\Sigma} = \varrho \cdot \Sigma_{\pi_1}$. Furthermore, there is a $1 \leq i \leq r$ and a $\alpha \in Z$ such that $\varrho = \varrho_l \alpha$. Since conjugation with α can only permute cycles of the same length, there is a $\pi_2 \in \mathfrak{S}_q$ with $\alpha \cdot \Sigma = \Sigma_{\pi_2}$. Now consider $\pi := \pi_2 \pi_1$. Then

$$\overline{\Sigma} = \varrho \cdot \Sigma_{\pi_1} = (\varrho_l \alpha) \cdot \Sigma_{\pi_1} = \varrho_l \cdot \Sigma_\pi.$$

(II) Let $\Phi(\pi', \varrho_{l'}) = \Phi(\pi, \varrho_l)$. Then $\varrho_{l'}$ and ϱ_l differ only by an element of Z since

$$\varrho_{l'} \tau_q \cdots \tau_1 \varrho_{l'}^{-1} = \prod_{j=1}^{q} \varrho_{l'} \tau_{\pi'(j)} \varrho_{l'}^{-1} = \prod_{j=1}^{q} \varrho_l \tau_{\pi(j)} \varrho_l^{-1} = \varrho_l \tau_q \cdots \tau_1 \varrho_l^{-1}.$$

□

[1] The idea to name these constructions after Karl Weierstrass steems from the conjecture that the arising complexes might have something to do with the Weierstraß points of Riemann surfaces.

2.2.4 Corollary. *We get a formula for the number of Weierstraß top cells*

$$\#\mathcal{W}_\nu^* = \frac{p!\cdot q!}{\prod_{k=1}^h (k+1)^{c(\nu,k)}\cdot c(\nu,k)!} = \frac{(h+q)!}{\prod_{k=1}^h (k+1)^{c(\nu,k)}}\cdot\binom{q}{c(\nu,1),\ldots,c(\nu,h)}.$$

2.2.5 Example. Consider $\nu := (1,1)$ and the cell $\Sigma := (\langle 3,4\rangle \,|\, \langle 1,2\rangle)$. The centraliser of $\langle 3,4\rangle\langle 1,2\rangle$ can be calculated explicitly and we get

$$Z = \{\,\mathrm{id}, \langle 1,2\rangle, \langle 3,4\rangle, \langle 1,2\rangle\langle 3,4\rangle, \langle 1,3\rangle\langle 2,4\rangle, \langle 1,4\rangle\langle 2,3\rangle\,\}.$$

There are different possibilities for a representative system of $\mathfrak{S}_4/Z$, e. g. $\{\mathrm{id}, \langle 1,4\rangle, \langle 1,3\rangle\}$ or $\{\mathrm{id}, \langle 2,3\rangle, \langle 1,3\rangle\}$. We get two different bijections Φ:

	id	$\langle 1,2\rangle$
id	$(\langle 3,4\rangle \mid \langle 1,2\rangle)$	$(\langle 1,2\rangle \mid \langle 3,4\rangle)$
$\langle 1,4\rangle$	$(\langle 1,3\rangle \mid \langle 2,4\rangle)$	$(\langle 2,4\rangle \mid \langle 1,3\rangle)$
$\langle 1,3\rangle$	$(\langle 2,3\rangle \mid \langle 1,4\rangle)$	$(\langle 1,4\rangle \mid \langle 2,3\rangle)$

	id	$\langle 1,2\rangle$
id	$(\langle 3,4\rangle \mid \langle 1,2\rangle)$	$(\langle 1,2\rangle \mid \langle 3,4\rangle)$
$\langle 2,3\rangle$	$(\langle 2,4\rangle \mid \langle 1,3\rangle)$	$(\langle 1,3\rangle \mid \langle 2,4\rangle)$
$\langle 1,3\rangle$	$(\langle 2,3\rangle \mid \langle 1,4\rangle)$	$(\langle 1,4\rangle \mid \langle 2,3\rangle)$

We see that a different choice of representatives may yield a permutation inside a row. It is clear that the common centraliser $Z(\tau_q)\cap\cdots\cap Z(\tau_1)$ also fixes the order of each row, so the relevant choice of representatives is made inside $Z(\tau_q\cdots\tau_1)/Z(\tau_q)\cap\cdots\cap Z(\tau_1)$. Thus, if ν is primitive, then the bijection does not depend on the choice.

2.2.6 Construction. There is a second way to index all Weierstraß top cells: Choose a system of representatives $\pi_1,\ldots,\pi_s$ of $\mathfrak{S}_q/(\mathfrak{S}_{c(\nu,1)}\times\cdots\times\mathfrak{S}_{c(\nu,h)})$. Then we get a bijection

$$\begin{aligned}\mathfrak{S}_q\Big/\prod_{i=1}^h \mathfrak{S}_{c(\nu,i)} \times \mathfrak{S}_p\Big/\bigcap_{j=1}^q Z(\tau_j) &\longrightarrow \mathcal{W}_\nu^*,\\ (\pi_l,[\varrho]) &\longmapsto \varrho\cdot\Sigma_{\pi_l}.\end{aligned}$$

Note that we do not have to choose representatives in the second factor, whence we just denote equivalence classes. In this description, we also see our special cases:

(i) If ν is primitive, then $c(\nu,k)=1$ for all k and the left side becomes $\mathfrak{S}_p/\bigcap Z(\tau_j)\times\mathfrak{S}_q$.

(ii) If ν is homogeneous, then there is exactly one k such that $c(\nu,k)=q$ and the left side becomes $\mathfrak{S}_p/Z(\tau_q,\ldots,\tau_1)$.

2.2.7 Problem. For a given (ν,m), we want to consider Weierstraß top cells inside $P^m_{g,1}$. Obviously, the norm $N(\Sigma)=h$ is prescribed by the partition, the problem is to find the Weierstraß cells with the correct puncture number $m(\Sigma)$. We see:

(i) Our "canonical" cell from above has a very specific $m = \#\{1\le j\le q;\ h_j \text{ is odd}\}$.

(ii) Under both bijections above, the puncture number does not depend on the index permutation π, but only on the conjugation permutation ϱ_l.

(iii) Since the choice of representatives only change the order of the $\tau_q,\ldots,\tau_1$, the puncture number does not depend on the representatives, only on the cosets of $\mathfrak{S}_p/Z$.

2.2.8 Definition. Let $\nu\vdash h$ and $m\le h$ with $h-m$ even. A cell Σ is called *of type* (ν,m), write $\Sigma : (\nu,m)$, if Σ is a Weierstraß cell of type ν having the correct puncture number. Let $\mathcal{W}^m_\nu\subseteq\mathcal{W}^*_\nu$ be the set of all Weierstraß top cells of type (ν,m).

2.2.9 Proposition. *For each $\nu \vdash h$ and $m \leq h$ having the same pairity as h, there is a non-degenerate Weierstraß cell Σ with the correct puncture number, i. e. $\mathcal{W}^m_\nu \neq \emptyset$.*

2.2.10 Lemma. *For a given $h \geq 0$ and $1 \leq k \leq h$ consider $p := h+1$ and the $(1,p)$-cell $\Sigma := (\tau^{h,k})$ with $\tau^{h,k} = \langle 1, \cdots, k, p, \ldots, k+1\rangle$. Then*

$$m(\Sigma) = \xi_{h,k} := \begin{cases} h-k+1 & \text{for } k \text{ odd,} \\ h-k & \text{for } k \text{ even.} \end{cases}$$

2.2.11 Lemma. *Let $\Sigma = (\tau_q \mid \cdots \mid \tau_1)$ be a (p,q)-cell, $\Sigma' = (\tau'_{q'} \mid \cdots \mid \tau'_1)$ be a (p',q')-cell.*

(I) *Define the Whitney sum $\Sigma' \times \Sigma := (\overline{\tau_{q'}}' \mid \cdots \mid \overline{\tau_1}' \mid \tau_q \mid \cdots \mid \tau_1) \in P_{p+p',q+q'}$ where*

$$\overline{\tau}'_j(i) := \begin{cases} i & \text{for } i \leq p, \\ \tau'_j(i-p)+p & \text{for } i \geq p+1. \end{cases}$$

The diagonal product is additive with respect to the puncture number:

$$m(\Sigma' \times \Sigma) = m(\Sigma') + m(\Sigma).$$

(II) *Define the twisted sum $\Sigma' \rtimes \Sigma$ arising from $\Sigma' \times \Sigma$ by conjugating all permutations with $\langle p, p+1\rangle$, i. e. switching the top slit of Σ and the bottom slit of Σ'. In the special case where $\Sigma' = (\langle 1, \ldots, p'\rangle)$ and $\Sigma = (\langle 1, \ldots, p\rangle)$ with p and p' even, we get*

$$m(\Sigma' \rtimes \Sigma) = 0.$$

For $\Sigma' := \Sigma := (\langle 1, 2\rangle)$, we can visualise $\Sigma' \times \Sigma$ and $\Sigma' \rtimes \Sigma$ as follows:

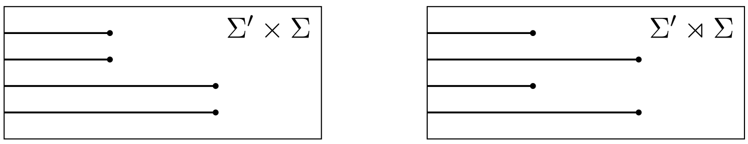


FIGURE 2.2: The diagonal cell $\Sigma' \times \Sigma$ and the twisted cell $\Sigma' \rtimes \Sigma$.

Proof of Proposition 2.2.9. We forget the order $h_q \geq \cdots \geq h_1$ and assume that $h_1, \ldots, h_k$ are odd and $h_{k+1}, \ldots, h_q$ are even. Now we explicitly construct a top cell recursively:

(I) Let $m_1 := m$ and $j_0 := k$.

(II) For $j = 1, \ldots, q$ do the following:

- Let $\overline{h}_j := \min(h_j, m_j)$.
- The difficult case is $m_j = 0$ and h_j odd. Then, we choose $\overline{\tau}_j := \langle 1, \ldots, h_j + 1\rangle$. If this occurs for the first time, $k - j$ is even and there is an even number of remaining odds $h_j, \ldots, h_k$; set $j_0 := j$. Finally let $m_{j+1} := m_j = 0$.
- Else, there is a $1 \leq k \leq h_j$ with $\xi_{h_j,k} = \overline{h}_j$. Set $\overline{\tau}_j := \tau^{h_j,k}$ and $m_{j+1} := m_j - \overline{h}_j$.

We finally build the desired cell by

$$\Sigma := \overline{\tau}_q \times \cdots \times \overline{\tau}_{k+1} \times (\overline{\tau}_k \rtimes \overline{\tau}_{k-1}) \times \cdots \times (\overline{\tau}_{j_0+1} \rtimes \overline{\tau}_{j_0}) \times \overline{\tau}_{j_0-1} \times \cdots \times \overline{\tau}_1.$$

Since $m \leq h$, we get $m_{q+1} = 0$ and finally conclude

$$m(\Sigma) = \sum_{j=1}^{j_0-1} \overline{h}_j + \sum_{j=k+1}^{q} \overline{h}_j = m - m_{q+1} = m. \qquad \square$$

2.2.12 Example. Consider the partition $\nu := (3,3,1,2) \vdash 9$ and let $m = 1$. We want to construct a non-degenerate Weierstraß cell in $P^m_{g,1}$. First, we re-order the tuple such that the odd elements are on the right side, $\nu := (2,3,3,1)$.

(I) $\overline{h}_1 = \min(1,1) = 1$. Since $\xi_{1,1} = 1$, we let $\overline{\tau}_1 = \tau^{1,1} = \langle 1,2\rangle$ and $m_2 := m - \overline{h}_1 = 0$.
(II) $m_2 = 0$ and h_2 odd for the first time, so $j_0 := 2$ and $\overline{\tau}_2 := \langle 1,\dots,4\rangle$. Set $m_3 := 0$.
(III) Again, we have $m_3 = 0$ and h_3 odd, so we set $\overline{\tau}_3 := \langle 1,\dots,4\rangle$ and $m_4 := 0$.
(IV) Finally, $m_4 = 0$ and h_4 even and we find that $\xi_{2,2} = 0$. Therefore, set $\overline{\tau}_4 := \langle 1,2,3\rangle$.

Hence, we get $\Sigma = \overline{\tau}_4 \times (\overline{\tau}_3 \rtimes \overline{\tau}_2) \times \tau_1 = (\langle 11,12,13\rangle \mid \langle 6,8,9,10\rangle \mid \langle 3,4,5,7\rangle \mid \langle 1,2\rangle)$.

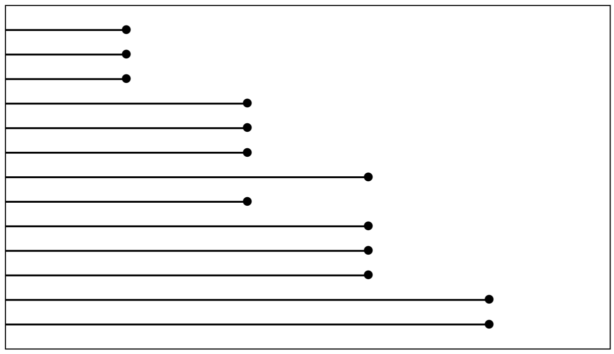


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FIGURE 2.3: A Weierstraß cell for the partition $(3,3,1,2)$ with $m = 1$.

2.3 Weierstraß complexes and classes

Given a partition $\nu \vdash h$ and a puncture number m, we are interested in subcomplexes (W_ν, W'_ν) of (P, P') and study their geometric realisation $\mathfrak{W}^m_\nu := |W^m_\nu| \setminus |W^m_\nu{}'|$ topologically. Finally, we construct simplicial fundamental classes $[w_\nu] \in H_{h+2q}(P, P'; \mathcal{O})$.

2.3.1 Construction (Weierstraß complex)**.** Let $\nu \vdash h$ and m a puncture number. We define the *Weierstraß complex* $W^m_\nu \subseteq P^m_{g,1}$ of type (ν, m) generated by all Weierstraß cells in $P^m_{g,1}$, i.e. by adding all boundary cells. We get a subcomplex $W^m_\nu{}' := W^m_\nu \cap P'$ of all degenerate cells in the Weierstraß complex. Define the geometric realisation

$$\mathfrak{W}^m_\nu := |W^m_\nu| \setminus |W^m_\nu{}'| \subseteq \mathfrak{P}^m_{g,1}.$$

2.3.2 Definition. Let $\nu \vdash h$. A partition $\mu \vdash h$ is called *subpartition* of ν, write $\mu \preceq \nu$, if the entries of μ arise from those of ν by adding up at least two entries. $\mu \preceq \nu$ is called *direct subpartition* if it arises from adding up exactly two entries.

2.3.3 Example. For $(1,1,1,1,1) \vdash 5$, we have the following lattice of subpartitions:

$$\begin{tikzcd}
& (1,1,1,1,1) \arrow[d, dash] & \\
& (2,1,1,1) \arrow[dl, dash] \arrow[dr, dash] & \\
(3,1,1) \arrow[d, dash] \arrow[drr, dash] & & (2,2,1) \arrow[d, dash] \arrow[dll, dash] \\
(4,1) \arrow[dr, dash] & & (3,2) \arrow[dl, dash] \\
& (5) &
\end{tikzcd}$$

2.3.4 Proposition. *If $\mu \preceq \nu$, then $W_\mu \subseteq W_\nu$. More precisely, the cells of W_ν^m are exactly the Weierstraß cells of type (μ, m) where $\mu \preceq \nu$. Thus, $\mathfrak{W}_\nu^m$ consists of all slit configurations where the multiplicities of the critical points form a subpartition of ν.*

2.3.5 Lemma. (I) *For each $\tau \in \mathfrak{S}_p$ there is a unique $\nu(\tau) = (h_r \geq \cdots \geq h_1) \vdash N(\tau)$ such that τ is decomposed into cycles of lengths $h_r + 1, \ldots, h_1 + 1$.*

(II) *Let $\tau, \tau' \in \mathfrak{S}_p$ and let $h := N(\tau)$ and $h' := N(\tau')$. Furthermore, we assume that $N(\tau \cdot \tau') = N(\tau) + N(\tau') = h + h'$. Then $\nu(\tau \cdot \tau')$ is a subpartition of $\nu(\tau) \oplus \nu(\tau')$.*

Proof of the Lemma. The first statement is clear and for the second statement, we can write $\tau = \zeta_1 \cdots \zeta_r$ for the decomposition into non-trivial cycles, i. e. $\zeta_i = \langle a_{i,1}, \ldots, a_{i,s_i} \rangle$. Analogously, we decompose $\tau' = \zeta'_1 \cdots \zeta'_l$ with $\zeta'_i = \langle a'_{i,1}, \ldots, a'_{i,t_i} \rangle$ and obtain

$$\tau = \prod_{i=1}^{r} \prod_{j=1}^{s_i - 1} \langle a_{i,j}, a_{i,j+1} \rangle \quad \text{and} \quad \tau' = \prod_{i=1}^{l} \prod_{j=1}^{t_i - 1} \langle a'_{i,j}, a'_{i,j+1} \rangle.$$

Then $N(\tau) = \sum_{i=1}^{r} (s_i - 1)$ as well as $N(\tau') = \sum_{i=1}^{l} (t_i - 1)$. Since (τ, τ') is a geodesic pair, there is no transposition contained in both decompositions above. Therefore, either the decompositions are disjoint or some cycles are glued together at one symbol. □

Proof of Proposition 2.3.4. It is enough to show "$W_\mu \subseteq W_\nu$" for direct subpartitions $\mu \preceq \nu$, so let $\mu := (h_q, \ldots, h_1) \vdash h$ be a partition and $\nu := (h_q, \ldots, \ldots, h_{J+1}, A, B, h_{J-1}, \ldots, h_1)$ with $A + B = h_J$. Let $\Sigma := (\tau_q \mid \cdots \mid \tau_1)$ be a Weierstraß top cell of type μ, in particular, $\tau_q, \ldots, \tau_1$ have pairwise disjoint support. After permutation of indices, we may assume that $\tau_j = \langle \alpha_{j,1}, \ldots, \alpha_{j,h_j+1} \rangle$ is a cycle of length $h_j + 1$. Now let $i := \alpha_{J,A+1}$ and define the cell $\overline{\Sigma} := (\overline{\tau}_q \mid \cdots \mid \overline{\tau}_{J+1} \mid \overline{\tau}_J^{(2)} \mid \overline{\tau}_J^{(1)} \mid \overline{\tau}_{J-1} \mid \cdots \mid \overline{\tau}_1)$ by:

- For $j \neq J$ let $\overline{\tau}_j := \mathbf{d}_i \tau_j := \langle \mathbf{d}_i \alpha_{j,1}, \ldots, \mathbf{d}_i \alpha_{j,h_j+1} \rangle$.
- Define $\overline{\tau}_J^{(1)} := \langle \mathbf{d}_i \alpha_{J,1}, \ldots, \mathbf{d}_i \alpha_{J,A+1} \rangle$
- Define $\overline{\tau}_J^{(2)} := \langle i, \mathbf{d}_i \alpha_{J,A+2}, \ldots, \mathbf{d}_i \alpha_{J,h_j+1} \rangle$.

It is clear that this new cell $\overline{\Sigma}$ is a Weierstraß cell of type ν. Moreover, by Lemma 2.1.4, $d''_i \overline{\Sigma} = (\tau_q \mid \cdots \mid \tau_{J+1} \mid \tau''_2 \mid \tau''_1 \mid \tau_{J-1} \mid \cdots \mid \tau_1)$ where $\tau''_1 = \langle \alpha_{J,1}, \ldots, \alpha_{J,A+1} \rangle$ and

$$\tau''_2 = \mathbf{s}_i \circ \langle i, \mathbf{d}_i \alpha_{J,1} \rangle \circ \overline{\tau}_J^{(2)} \circ \mathbf{d}_i = \langle \alpha_{J,1}, \alpha_{J,A+2}, \ldots, \alpha_{J,h_j+1} \rangle.$$

In particular, $\tau''_2 \tau''_1 = \tau_J$ and $d'_J d''_i \overline{\Sigma} = \Sigma$. It remains to show that $m(\overline{\Sigma}) = m(\Sigma)$. Since the vertical differential d'_J preserves m, the only possibility is that d''_i decreases the puncture number by deleting a trivial cycle from $\overline{\sigma}_q$. However, since the norm is preserved, the puncture number can only decrease by an even number which is not possible. To see that all cells in $(W_\nu^m, W_\nu^{m\prime})$ are of the prescribed from, we have to show that if $\Sigma = (\tau_q \mid \cdots \mid \tau_1)$ contains a τ_j with a cycle of length $h_j + 1$, then $d'_k \Sigma$ and $d''_k \Sigma$ also do if they are non-degenerate. We show this in both cases separately:

(I) $d''_k \Sigma$ arises from Σ by eliminating one element from a cycle or connecting two cycles. Since the norm is preserved, it is not possible to eliminate an element from a non-trivial cycle. Thus, all cycles are preserved.

(II) We know that $d'_0 \Sigma$ and $d'_p \Sigma$ are degenerate, so we only have to consider $0 < j < q$ and $d'_j \Sigma = (\tau_q \mid \cdots \mid \tau_j \tau_{j-1} \mid \cdots \mid \tau_1)$. Since the norm is preserved, we have that $N(\tau_j \tau_{j-1}) = N(\tau_j) + N(\tau_{j-1})$. Now apply Lemma 2.3.5. □

2.3.6 Remark. We already know that the non-degenerate Weierstraß top cells are of bidimension $(h+q,q)$, so $\dim(W_\nu^m, W_\nu^{m\prime}) = h+2q$. For a direct subpartition $\mu \preceq \nu$, the relative inclusion $(W_\mu^m, W_\mu^{m\prime}) \subseteq (W_\nu^m, W_\nu^{m\prime})$ is of codimension 2 since

$$\dim(W_\mu, W'_\mu) = h + 2\cdot(q-1) = \dim(W_\nu, W'_\nu) - 2.$$

2.3.7 Construction. We have Weierstraß complexes assigned to special partitions:

(I) For $1 \leq s \leq h$, we consider the special partition $\nu_s \coloneqq (s,1,\dots,1) \vdash h$. We sometimes say that the Weierstraß top cells are of type (s,h,m). Denote the corresponding Weierstraß complex by $(W_s^m, W_s^{m\prime})$. It has the relative dimension $3h+2-2s$.

(II) We have a sequence $\nu_s \preceq \nu_{s-1} \preceq \cdots \preceq \nu_1$, yielding a filtration

$$(P,P') = (W_1^m, W_1^{m\prime}) \supseteq \cdots \supseteq (W_h^m, W_h^{m\prime})$$

with relative codimension 2. This gives rise to a filtration

$$\mathfrak{P}_{g,1}^m = \mathfrak{W}_1^m \supseteq \cdots \supseteq \mathfrak{W}_h^m.$$

(III) The non-degenerate cells in $(W_s^m, W_s^{m\prime})$ are exactly those containing at least one critical point of at least multiplicity s.

2.3.8 Definition. Let $\nu \coloneqq (h_q \geq \cdots \geq h_1) \vdash h$. Let $X^h + \mathbb{C}[X]_{h-1} \cong \mathbb{C}^h$ be the space of all normed complex polynomials of degree exactly h.

(I) Define the polynomial $p_\nu \in X^h + \mathbb{C}[X]_{h-1}$ assigned to ν by

$$p_\nu \coloneqq \prod_{j=1}^{q} (X-j)^{h_j}.$$

(II) Define the variety $U_\nu \subseteq X^h + \mathbb{C}[X]_{h-1}$ assigned to ν by

$$U_\nu \coloneqq \left\{ \prod_{j=1}^{q} (X-z_j)^{h_j};\ z_j \in \mathbb{C} \right\} \subseteq X^h + \mathbb{C}[X]_{h-1}.$$

2.3.9 Remark. We see that $\deg(p_\nu) = h$ and $p_\nu \in U_\nu$, more general $p_\mu \in U_\nu$ for $\mu \preceq \nu$. Furthermore, we have a surjective parametrisation

$$\Phi\colon \mathbb{C}^q \longrightarrow U_\nu, (z_1,\dots,z_q) \longmapsto \prod_{j=1}^{q} (X-z_j)^{h_j},$$

which is a local isomorphism around $(z_1,\dots,z_q)$ with $z_i \neq z_j$ for $i \neq j$ by the implicit function theorem since $\mathrm{D}\Phi|_{(z_1,\dots,z_q)}$ is invertible.

2.3.10 Lemma. *Let $\mathcal{N} \coloneqq [\Sigma; a, b] \in \mathfrak{W}_\nu^m$. There is a subpartition $\mu \preceq \nu$ such that the distribution of critical points of $\mathcal{N}$ is of type μ. Then around $\mathcal{N}$, the realisation $\mathfrak{W}_\nu^m$ is locally homeomorphic to $\mathbb{R}^h \times U_\nu$ where $\mathcal{N}$ corresponds to $(0, p_\mu)$.*

Proof. Consider the slit picture and let μ be of the form $(h'_s \geq \cdots \geq h'_1)$. The possibilities of splitting and joining the critical points of $\mathcal{N}$ is exactly described by the behaviour of polynomials in U_ν lying in the neighbourhood of p_μ and their roots of zero. The missing coordinates are at each critical point the h'_j degrees of freedom to place the remaining copies of the critical point vertically. These sum up to h real degrees of freedom. □

2.3.11 Example. Let $\mathcal{N} := [\Sigma; a, b]$ be in the interior of a Weierstraß top cell. Then around $\mathcal{N}$, the geometric realisation is locally homeomorphic to $\mathbb{R}^h \times U_\nu$ around $(0, p_\nu)$. Since $p_\nu = \Phi(1, \ldots, q)$ and $\Phi \colon \mathbb{C}^q \longrightarrow U_\nu$ is a local isomorphism around $(1, \ldots, q)$ the realisation is, as expected, around $\mathcal{N}$ locally homeomorphic to $\mathbb{R}^h \times \mathbb{C}^q = \mathbb{R}^{h+2q}$.

2.3.12 Proposition. *If ν is a homogeneous partition, then $\mathfrak{W}^m_\nu$ is a manifold.*

Proof. Since ν is homogeneous, there is a divisor $d \mid h$ of h with $\nu = (d, \ldots, d)$ and $q := \frac{h}{d}$. Note that $U_{1,\ldots,1}$ is just the space of all normed complex polynomials of degree q and hence homeomorphic to $\mathbb{C}^q$. Furthermore, we have an isomorphism

$$U_{1,\ldots,1} \longrightarrow U_\nu, p \longmapsto p^d$$

Therefore, $U_\nu \cong \mathbb{C}^q$ and each neighbourhood is euclidean as desired. $\square$

2.3.13 Example. In general $\mathfrak{W}^m_\nu$ is not necessarily a manifold: Consider $\nu = (2, 1, 1)$ and $m = 0$ and the cell $\Sigma := (\langle 1, 2, 3\rangle \mid \langle 4, 5, 6\rangle)$. Then Σ is a Weierstraß cell of type $(2, 1 + 1)$ and around an interior point $[\Sigma; a, b]$, the complex $\mathfrak{W}^0_{2,1,1}$ is homeomorphic to

$$\mathbb{R}^4 \times U_{2,1,1} \subseteq \mathbb{R}^4 \times \Big\{(X - z_1) \cdot (X - z_2) \cdot (X - z_3)^2 \in \mathbb{C}[X]\Big\}$$

and in the second factor, we are in the neighbourhood of $p_{2,2} := (X - 1)^2 \cdot (X - 2)^2$. It is now readily verified that $(X - 1)^2 \cdot (X - 2)^2$ is a singularity: The explicit parametrisation

$$\Phi \colon \mathbb{C}^3 \longrightarrow U_{2,1,1}, (z_1, z_2, z_3) \longmapsto (X - z_1) \cdot (X - z_2) \cdot (X - z_3)^2$$

has the property $\Phi(1, 1, 2) = \Phi(2, 2, 1) = p_{2,2}$, but $\mathrm{Im}(\mathrm{D}\Phi|_{(1,1,2)}) \neq \mathrm{Im}(\mathrm{D}\Phi|_{(2,2,1)})$.

2.3.14 Construction. The direct boundaries of a Weierstraß top cell are those Weierstraß cells where exactly two critical points share either a u- or a v-line. We may assign to each $\nu \vdash h$ and puncture number m a coloured multigraph in the following way:

(I) The vertices are given by all Weierstraß top cells $\mathcal{W}^m_\nu$.
(II) Draw an edge between Σ and $\overline{\Sigma}$ for each shared non-degenerate horizontal boundary.
(III) Draw a red edge between Σ and $\overline{\Sigma}$ for each shared non-degenerate vertical boundary.

2.3.15 Example. The multigraph assigned to $\nu = (2, 1)$ and $m = 3$ is of the form

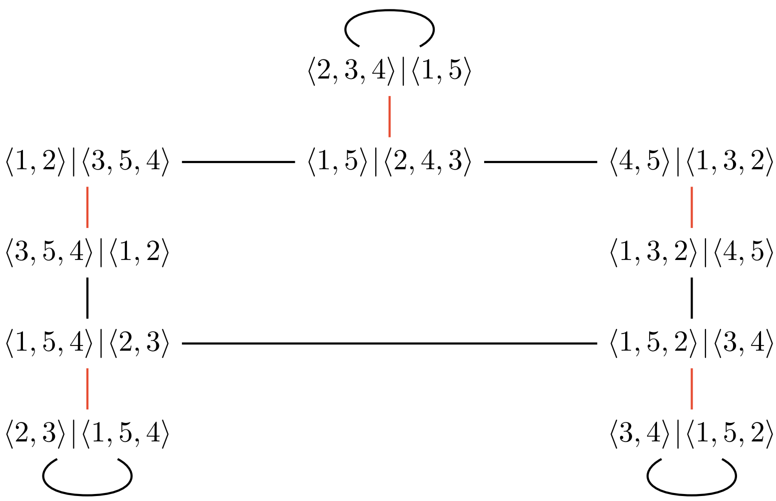


FIGURE 2.4: The multigraph assigned to $\nu = (2, 1)$ and $m = 3$.

2.3.16 Corollary. *Let ν be homogeneous. Then $\mathfrak{W}^m_\nu$ is connected if and only if the above multigraph assigned to the pair (ν, m) is connected.*

Proof. Obviously, if the multigraph is connected, the complex is connected, since all top cells are directly connected via boundaries of codimension 1. Conversely, since $\mathfrak{W}^m_\nu$ is a manifold, its top cells can not be connected only via boundaries of codimension ≥ 2. □

2.3.17 Example. Not all Weierstraß complexes $\mathfrak{W}^m_\nu$ are connected:

(I) Consider the partition $\nu = (h)$ and m arbitrary. Each Weierstraß top cell $\Sigma : (\nu, m)$ has only degenerate boundaries (since $j(i) = j(i+1)$ for each $1 \leq i \leq p$) and therefore, the (open) Weierstraß top cells are exactly the path components of $\mathfrak{W}^m_\nu$. For example, in the case $\nu = (4)$ and $m = 0$, we have eight Weierstraß top cells

(II) Consider $\nu \coloneqq (2, 2)$ and $m = 0$, then Corollary 2.3.16 applies. It is easily verified that there are 24 Weierstraß top cells and the assigned multigraph splits into two components, one containing 22 cells and one component of the form

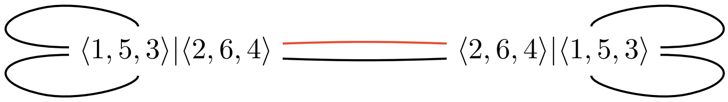


Figure 2.5: One component in the multigraph assigned to $\nu = (2, 2)$ and $m = 0$.

In [Bia18], Bianchi presented an algorithm to find a path in the multigraph from each Weierstraß top cell to a base cell for the case $c(\nu, 1) \geq 1$ (i. e. one critical point of multiplicity 1 is allowed) and $m = 0$. Hence $\mathfrak{W}^0_\nu$ is connected and even more, the assigned multigraph is connected. In particular $\mathfrak{W}^0_s$ is connected for $s \leq h - 1$.

2.3.18 Preparation. Consider our multigraph assigned to (ν, m) and give the edges signs:

(I) The edge representing a vertical pairing (Σ, j) with $(\overline{\Sigma}, \bar{j})$ gets the sign -1.

(II) The edge representing a horizontal pairing (Σ, i) with $(\overline{\Sigma}, \bar{i})$ gets the sign

$$(-1)^{i+\bar{i}+1} \cdot \varepsilon_i(\Sigma) \cdot \varepsilon_{\bar{i}}(\overline{\Sigma}),$$

where $(\varepsilon_i)_i$ are the coefficients of the orientation system from Construction 1.2.12. Due to [Mül96], the orientation system corrects all signs in such a way that the signed multigraphs are consistent, i. e. every cycle in the graph has the sign product $+1$.

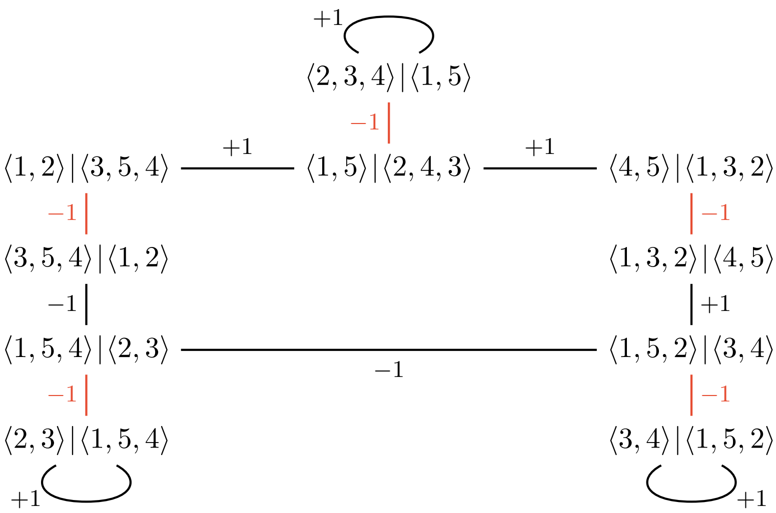


Figure 2.6: The multigraph from Example 2.3.15 with signs.

2.3.19 Construction. For each pair of $\nu \vdash h$ and $m \geq 0$, there is a system of signs $\delta\colon \{\Sigma : (\nu, m)\} \longrightarrow \{\pm 1\}$ such that the following fundamental chain is a cycle

$$w_\nu \coloneqq w_\nu^m \coloneqq \sum_{\Sigma:(\nu,m)} \delta(\Sigma)\cdot\Sigma \in C_{h+q,q}(P,P';\mathcal{O}).$$

The system δ is unique (up to one global sign) if the corresponding multigraph is connected (e. g. if $\mathfrak{W}_\nu^m$ is a connected manifold, see COROLLARY 2.3.16). We obtain classes

$$[w_\nu] \coloneqq [w_\nu^m] \in H_{h+2q}(P,P';\mathcal{O})$$

which are called *Weierstraß classes* and can be described as fundamental classes for the subcomplex (W_ν, W'_ν), as explicitly done in REMARK 5.2.6.

Proof. (I) Since $\partial' w_\nu \in C_{h+q,q-1}(P,P';\mathcal{O})$ and $\partial'' w_\nu \in C_{h+q-1,q}(P,P';\mathcal{O})$, we need to choose δ such that $\partial' w_\nu = 0$ and $\partial'' w_\nu = 0$. Recall from PROPOSITION 2.1.8 and REMARK 2.1.9 that sharing non-degenerate horizontal boundaries resp. sharing non-degenerate vertical boundaries yield two pairings among separated inner $(h+q,q)$-cells in the sense of DEFINITION 2.1.7. It is clear that if two such cells Σ and $\overline{\Sigma}$ are paired (horizontally or vertically) and one is a Weierstraß cell of type ν, then also the other one is a Weierstraß cell of type ν. Therefore, the edges on the assigned multigraph cover all boundaries of Weierstraß top cells and each boundary cell occurs exactly twice in the sum $\partial' w_\nu$ resp. $\partial'' w_\nu$.

(II) We choose in each component of the multigraph one cell Σ_0, set $\delta(\Sigma_0) = 1$ and for each other cell Σ in this component, choose a path $e_1 \cdots e_r$ from Σ_0 to Σ and set $\delta(\Sigma) = \mathrm{sg}(e_1)\cdots\mathrm{sg}(e_r)$. The sign consistency guarantees that δ is well-defined. By construction, the two coinciding summands in $\partial' w_\nu$ and $\partial'' w_\nu$ get a different sign and therefore cancel each other out.

(III) Note that the only freedom lies in the choice of the base cell in each component of the graph. Hence, if the graph is connected, two choices yield the same sign system up to one global sign. □

2.3.20 Remark. The explicit description of the fundamental class $[P,P'] \in H_{3h}(P,P';\mathcal{O})$ of the whole slit complex (P,P') as described in [ABE08] coincides with the above construction of the Weierstraß class $[w_1]$ (i. e. for the partition $\nu = (1,\dots,1)$). Hence, if we consider the induced Poincaré–Lefschetz duality isomorphism

$$\begin{aligned} \mathrm{PL}_{\mathfrak{P}}\colon H^\bullet(\mathfrak{P}_{g,1}^m) &\longrightarrow H_{3h-\bullet}(P,P';\mathcal{O}), \\ \vartheta &\longmapsto \vartheta \frown [P,P']. \end{aligned}$$

we get $\mathrm{PL}^{-1}[w_1] = \pm 1 \in H^0(\Gamma)$ (only one choice of sign is involved, since $\mathfrak{W}_1^m = \mathfrak{P}_{g,1}^m$ is connected). We are interested in the Poincaré–Lefschetz duals of other Weierstraß classes.

3 Symmetric fibre products

3.1 Vector and sphere bundles

We repeat some basic facts about real vector bundles and their orientability as in [Bre93] and, following [Hat03], continue with some complex clutching constructions for $\mathbb{S}^1$-bundles and their correspondence to Mayer–Vietoris sequences. We use the methods for working with fibre bundles presented in Appendix A.

3.1.1 Reminder. Let $(M, \partial M) \longrightarrow E \longrightarrow B$ be a fibre bundle where M is a compact and orientable m-dimensional manifold. We call the bundle *orientable* if $\pi_1(B)$ acts trivially on $H_m(M, \partial M)$, i. e. if the fundamental class $[M]$ is fixed by all transition functions.

3.1.2 Construction. Let $\mathbb{R}^k \longrightarrow E \longrightarrow B$ be a vector bundle. Applying the one-point compactification $(-)^+$ fibre-wise on $E \longrightarrow B$, we get an $\mathbb{S}^k$-bundle, the *Thom bundle*

$$(\mathbb{S}^k, \infty) \longrightarrow (E^+, B_\infty) \longrightarrow B,$$

where $s_\infty \colon B \longrightarrow E^+$ is the ∞-section and $B_\infty \coloneqq s_\infty(B) \subseteq E^+$ is its image. We have an obvious bundle inclusion $\jmath \colon E \hookrightarrow (E^+, B_\infty)$ over B.

3.1.3 Reminder. Let $\mathbb{R}^k \longrightarrow E \longrightarrow B$ be a real vector bundle. A *Riemannian metric* is a section $\mathbf{g} \colon B \longrightarrow E^* \otimes_B E^*$ such that $\mathbf{g}(x) \in E^*_x \otimes_{\mathbb{R}} E^*_x$ is a scalar product for all $x \in B$. Every real vector bundle over a paracompact space B admits a Riemannian metric.

(i) The set $R \subseteq \Gamma(E^* \otimes_{\mathbb{R}} E^*)$ of all Riemannian metrics is convex, in particular contractible, since every convex combination of scalar products is a scalar product.

(ii) For a given metric, we use the Gram–Schmidt process to find local trivialisations of $E \longrightarrow B$ such that the transition functions lie in $\mathrm{O}(n)$. This means that the transition functions are norm- and angle-preserving.

3.1.4 Construction. Let $\mathbb{R}^k \longrightarrow E \longrightarrow B$ be a real vector bundle, endowed with a Riemannian metric. Then we obtain subbundles of E

(i) We get a disk bundle $(\mathbb{D}^k, \mathbb{S}^{k-1}) \longrightarrow (\mathbb{D}E, \mathbb{S}E) \longrightarrow B$, the *induced disk bundle.* Going to the quotient bundle, we get an $\mathbb{S}^k$-bundle $(\mathbb{S}^k, \infty) \longrightarrow (\mathbb{D}/\mathbb{S})E \longrightarrow B$.

(ii) We have a natural isomorphism $\mathbb{D}/\mathbb{S} \longrightarrow (-)^+$ by rescaling the radius $[0,1] \xrightarrow{\cong} [0, \infty]$. Hence, we obtain a bundle isomorphism $(\mathbb{D}/\mathbb{S})E \longrightarrow E^+$.

3.1.5 Reminder. We call a real vector bundle $\mathbb{R}^k \longrightarrow E \longrightarrow B$ *orientable*, if one of the following equivalent conditions is satisfied:

(i) $(\mathbb{S}^k, \infty) \longrightarrow (E^+, B_\infty) \longrightarrow B$ is orientable.

(ii) After a choice of a Riemannian metric, $(\mathbb{D}^k, \mathbb{S}^{k-1}) \longrightarrow (\mathbb{D}E, \mathbb{S}E) \longrightarrow B$ is orientable.

(iii) The structure group of $E \longrightarrow B$ is given by the orientation preserving linear automorphisms $\mathbb{R}^k \longrightarrow \mathbb{R}^k$ (i. e. those with positive determinant).

3.1.6 Reminder. Let $\mathbb{R}^k \longrightarrow E \longrightarrow B$ be an orientable real vector bundle.

(I) A *Thom class* of $E \longrightarrow B$ is a class $\tau \in H^k(E^+, B_\infty)$ such that for all fibre inclusions $\imath_x \colon (E_x^+, \infty) \longrightarrow (E^+, B_\infty)$, the pullback $\imath_x^* \tau$ is a generator. Such a Thom class always exists for orientable vector bundles and is determined up to sign.

(II) We define the *Euler class* $e(E) \coloneqq s_0^* \jmath^* \tau \in H^k(B)$ where $s_0 \colon B \longrightarrow E$ is the zero section and $\jmath \colon E \longrightarrow (E^+, B_\infty)$ is the fibre-wise inclusion. The Euler class is natural, i. e. for a map $f \colon A \longrightarrow B$, we have $e(f^*E) = f^*e(E)$.

3.1.7 Remark. (I) If $\mathbb{C}^k \longrightarrow E \longrightarrow B$ is a complex vector bundle, it is orientable since the transitions are $\mathbb{C}$-linear and in particular orientation preserving as $\mathbb{R}$-linear maps.

(II) Let $\mathbb{R}^2 \longrightarrow E \longrightarrow B$ be an oriented vector bundle with a metric on it. We choose the trivialisations such that the transition functions lie in $\mathrm{SO}(2)$ and obtain a complex structure by defining multiplication by i to be a rotation counterclockwise by $\frac{\pi}{2}$. This turns $E \longrightarrow B$ into a complex line bundle.

3.1.8 Example. A *smooth surface bundle* is a bundle $M \longrightarrow E \xrightarrow{\pi} B$ whose fibre is a smooth surface and where the structure group is given by $\mathrm{Diff}(M)$. Consider its associated principal bundle $\mathrm{Diff}(M) \longrightarrow P \longrightarrow B$ and the vector bundle

$$T^\perp E \coloneqq P \times_{\mathrm{Diff}(M)} TM \longrightarrow P \times_{\mathrm{Diff}(M)} M = E$$

which we call *vertical tangent bundle*. If $\pi \colon E \longrightarrow B$ itself is a smooth map between manifolds, then it is a submersion and $T^\perp E \cong \mathrm{Ker}(T\pi) \subseteq TE$. If the bundle is orientable, the structure group is given by $\mathrm{Diff}^+(M)$ and vertical tangent bundle $T^\perp E \longrightarrow E$ is an orientable 2-dimensional real vector bundle and therefore also a complex line bundle.

3.1.9 Reminder. As a special case of Example A.8, we have a 1:1-correspondence between complex line bundles and principal $\mathbb{S}^1$-bundles, explicitly given as follows:

(I) Let $\mathbb{C} \longrightarrow L \longrightarrow B$ be a complex line bundle. We choose a Riemannian metric on it and can achieve by Gram–Schmidt that the structure group is given by $\mathrm{U}(1) = \mathbb{S}^1$. Hence, the restriction to the subfibre $\mathbb{S}^1 \longrightarrow \mathbb{S}L \longrightarrow B$ is a principal $\mathbb{S}^1$-bundle.

(II) Let $\mathbb{S}^1 \longrightarrow S \longrightarrow B$ be a principal $\mathbb{S}^1$-bundle. Since $\mathbb{S}^1 = \mathrm{U}(1)$, we can form the balanced product $\mathbb{L}S \coloneqq S \times_{\mathrm{U}(1)} \mathbb{C}$ and obtain a complex line bundle $\mathbb{C} \longrightarrow \mathbb{L}S \longrightarrow B$.

Thus, we can define the Euler class of a principal $\mathbb{S}^1$-bundle by $e(S) \coloneqq e(\mathbb{L}S) \in H^2(B)$. Moreover, we define the following two operations for two principal $\mathbb{S}^1$-bundles $S, S' \longrightarrow B$ where their associated complex line bundles are given by $L, L' \longrightarrow B$:

(I) Consider their *tensor product* $\mathbb{C} \longrightarrow L \otimes_B L' \longrightarrow B$ and restrict it again to

$$\mathbb{S}^1 \longrightarrow S \otimes_B S' \coloneqq \mathbb{S}(L \otimes_B L') \longrightarrow B.$$

(II) Consider their *Whitney sum* $\mathbb{C} \longrightarrow L \oplus_B L' \longrightarrow B$ and restrict it again to

$$\mathbb{S}^3 \longrightarrow S \oplus_B S' \coloneqq \mathbb{S}(L \oplus_B L') \longrightarrow B.$$

For a paracompact space B, we can identify $H^2(B; \mathbb{Z}) \cong \mathrm{Prin}_{\mathbb{S}^1}(B)$ via the Euler class. This behaves well with respect to the above operations, see [Hat03]:

(I) $e(S \otimes_B S') = e(S) + e(S')$, so sums of classes are tensor products of vector bundles.
(II) $e(S \oplus_B S') = e(S) \smile e(S')$, so cup products are Whitney sums of the bundles.

3.1.10 Construction. Call $(B; U, V)$ *excisive triad* if $U, V \subseteq B$ are open and $B = U \cup V$. Let $\pi\colon S \longrightarrow B$ be a principal $\mathbb{S}^1$-bundle. Then S is a right $\mathbb{S}^1$-space and we get a bijection

$$\begin{aligned} \{\text{maps } U \cap V \longrightarrow \mathbb{S}^1\} \quad &\longleftrightarrow \quad \{\text{bundle automorphisms } S\big|_{U\cap V} \longrightarrow S\big|_{U\cap V}\}, \\ \gamma \quad &\longmapsto \quad [\Phi_\gamma\colon \widehat{x} \longmapsto \widehat{x} \cdot \gamma(\varrho(x))]\,. \end{aligned}$$

For a section $s\colon U \cap V \longrightarrow S\big|_{U\cap V}$, the inverse of this map can be explicitly given by using the comparison map (see A.5) $C\colon S \times_B S \longrightarrow \mathbb{S}^1$:

$$\gamma(x) = \frac{\Phi_\gamma(s(x))}{s(x)} = C\big(s(x), \Phi_\gamma(s(x))\big) \in \mathbb{S}^1.$$

Each map $\gamma\colon U \cap V \longrightarrow \mathbb{S}^1$ gives us an $\mathbb{S}^1$-bundle $S_\gamma \longrightarrow B$ as the pushout

$$\begin{tikzcd} S\big|_{U\cap V} \arrow[r, "\Phi_\gamma"] \arrow[d, hook] & S\big|_{U\cap V} \arrow[r, hook] & S\big|_V \arrow[d] \\ S\big|_U \arrow[rr] & & S_\gamma. \end{tikzcd}$$

In this situation, we apparently have the following properties:

(I) For $\gamma\colon U \cap V \longrightarrow \mathbb{S}^1$, we get $S_\gamma\big|_U \cong S\big|_U$ and $S_\gamma\big|_V \cong S\big|_V$.

(II) If $\gamma \simeq \gamma'$, then Φ_γ and $\Phi_{\gamma'}$ are isotopic and thus, $S_\gamma \cong S_{\gamma'}$.

(III) Let $S, S' \longrightarrow B$ be two $\mathbb{S}^1$-bundles and assume that there are bundle isomorphisms $\Phi_U\colon S\big|_U \longrightarrow S'\big|_U$ and $\Phi_V\colon S\big|_V \longrightarrow S'\big|_V$. Then $S' \cong S_\gamma$ where

$$\Phi_\gamma = \left(\Phi_U\big|_{U\cap V}\right)^{-1} \circ \left(\Phi_V\big|_{U\cap V}\right) : S\big|_{U\cap V} \longrightarrow S\big|_{U\cap V}.$$

3.1.11 Lemma. *Let $\underline{\mathbb{S}^1}_B$ be the trivial $\mathbb{S}^1$-bundle over B and $\mathbb{S}_\gamma := (\underline{\mathbb{S}^1}_B)_\gamma$. Then for each principal $\mathbb{S}^1$-bundle $S \longrightarrow B$, we get $S_\gamma \cong S \otimes_B \mathbb{S}_\gamma$, which means $e(S_\gamma) = e(S) + e(\mathbb{S}_\gamma)$.*

Proof. For each $x \in U$, let $1_U^x \in \underline{\mathbb{S}^1}_U$ be the complex 1 in each fibre and analogously 1_V^x. Then $1_U^x \sim 1_V^x \cdot \gamma(x)$ in $\mathbb{S}_\gamma$. Thus, we can define the isomorphism $\Psi\colon S_\gamma \longrightarrow S \otimes_B \mathbb{S}_\gamma$ by

$$\begin{tikzcd} S\big|_{U\cap V} \arrow[r, "\Phi_\gamma"] \arrow[d, hook] & S\big|_{U\cap V} \arrow[r, hook] & S\big|_V \arrow[d] \arrow[ddr, bend left, "\mathrm{id}\otimes 1_V"] & \\ S\big|_U \arrow[rr] \arrow[drrr, bend right, "\mathrm{id}\otimes 1_U"'] & & S_\gamma \arrow[dr, dashed, "\exists!\Psi"] & \\ & & & S \otimes_B \mathbb{S}_\gamma \end{tikzcd}$$

□

3.1.12 Remark. Recall that $\mathrm{Prin}_{\mathbb{S}^1}(B)$ is the set of all isomorphism classes of principal $\mathbb{S}^1$-bundles over B and that $\mathrm{Prin}_{\mathbb{S}^1}(B) \cong H^2(B)$. Now let $(B; U, V)$ be an excisive triad. Then Lemma 3.1.11 translates a part of the Mayer–Vietoris sequence:

$$\begin{tikzcd} H^1(U\cap V) \arrow[r, "\delta^*"] \arrow[d, equal] & H^2(B) \arrow[r] \arrow[d, equal] & H^2(U) \oplus H^2(V) \arrow[d, equal] \\ {[U\cap V, \mathbb{S}^1]} \arrow[r] & \mathrm{Prin}_{\mathbb{S}^1}(B) \arrow[r] & \mathrm{Prin}_{\mathbb{S}^1}(U) \times \mathrm{Prin}_{\mathbb{S}^1}(V) \\ {[\gamma]} \arrow[r, mapsto] & {[\mathbb{S}_\gamma]} & \\ & {[S]} \arrow[r, mapsto] & \big([S|_U], [S|_V]\big). \end{tikzcd}$$

3.2 Symmetric products and polynomials

Following [Hat01, ch. 4.K], we introduce the notion of symmetric products of based topological spaces and discuss their correspondence to polynomials via their roots of zero for complex vector spaces. This gives rise to the notions of degree and multiplicity.

3.2.1 Construction. Let (X, ∞) be a based topological space and $h \geq 0$. Then $\mathfrak{S}_h$ acts on X^h by permuting the components and we define the *h-fold symmetric product* by

$$\mathrm{SP}^h X \coloneqq X^h/\mathfrak{S}_h.$$

Obviously, $\mathrm{SP}^1 X = X$. In contrast to configuration spaces, the action of $\mathfrak{S}_h$ is not free here. The elements of $\mathrm{SP}^h X$ are formal sums, denoted in two equivalent ways:

(I) $\sum_{x\in X} \nu_x \cdot x$ with $\nu_x \in \mathbb{Z}_{\geq 0}$ and $\sum_{x\in X} \nu_x = h$,
(II) $\sum_{x\neq\infty} \nu_x \cdot x$ with $\nu_x \in \mathbb{Z}_{\geq 0}$ and $\sum_{x\in X} \nu_x \leq h$.

In the second notation, we write $\emptyset$ for the empty sum. We have natural inclusions

$$\mathrm{SP}^h X \hookrightarrow \mathrm{SP}^{h+1} X, \sum_{x\neq\infty} \nu_x \cdot x \longmapsto \sum_{x\neq\infty} \nu_x \cdot x.$$

3.2.2 Definition (Degree and multiplicity)**.** Let $\Theta \coloneqq \sum_{x\neq\infty} \nu_x \cdot x \in \mathrm{SP}^h X$. We define the *degree of* Θ and for $\infty \neq x \in X$ the *multiplicity of* Θ *at* x by

$$|\Theta| \coloneqq \sum_{x\neq\infty} \nu_x \leq h \qquad \text{and} \qquad \mathrm{mult}_\Theta(x) \coloneqq \nu_x \leq h.$$

Intuitively speaking, the multiplicity is the number how often a point is seen and the degree is the number of points (counting multiplicity) distinct from ∞.

3.2.3 Construction. We have a graded addition $\mathrm{SP}^k X \times \mathrm{SP}^l X \longrightarrow \mathrm{SP}^{k+l} X$ by

$$\sum_{x\neq\infty} \nu_x \cdot x + \sum_{x\neq\infty} \nu'_x \cdot x \coloneqq \sum_{x\neq\infty} (\nu_x + \nu'_x) \cdot x.$$

This motivates the definition of the *infinite symmetric product*

$$\mathrm{SP}^\infty X \coloneqq \varinjlim_h \mathrm{SP}^h X.$$

Its elements are formal sums $\Theta \coloneqq \sum_{x\in X} \nu_x \cdot x$ with finite degree. The above addition turns $\mathrm{SP}^\infty X$ into a graded abelian topological monoid. In fact, it is the free abelian topological monoid over the space X, which means for every based map $f\colon (X, \infty) \longrightarrow (M, 1)$ into an abelian topological monoid, there is a unique homomorphism $\mathrm{SP}^\infty X \longrightarrow M$ extending f.

3.2.4 Construction (Functoriality)**.** Let $1 \leq h \leq \infty$ and let $f\colon X \longrightarrow Y$ be a map between based spaces. We obtain a morphism

$$f_*\colon \mathrm{SP}^h X \longrightarrow \mathrm{SP}^h Y, \sum_{x\in X} \nu_x \cdot x \longmapsto \sum_{x\in X} \nu_x \cdot f(x).$$

Thus, SP^h becomes a functor $\mathbf{Top_0} \longrightarrow \mathbf{Top_0}$. The functor preserves homotopy, which means if $f \simeq g$, then also $f_* \simeq g_*$. We can extend the functors on the category $\mathbf{Top}^2$ of topological pairs by setting $\mathrm{SP}^h(X, A) \coloneqq \mathrm{SP}^h(X/A)$ with basepoint $[A]$.

3.2.5 Motivation. Let $\mathbb{C}[X]_h$ be the space of all polynomials of degree at most h. Then $\mathbb{C}^\times$ acts on $\mathbb{C}[X]_h$ by scaling the coefficients and we identify $\mathbb{C}[X]_h/\mathbb{C}^\times \cong \mathbb{C}P^h$. Furthermore, every $[p] \in \mathbb{C}[X]_h/\mathbb{C}^\times$ has uniquely determined roots of zero $z_1, \dots, z_r \in \mathbb{C}$ with multiplicities $\nu_1, \dots, \nu_r \in \mathbb{Z}_{\geq 0}$, such that $\nu_1 + \dots + \nu_r = \deg(p) \leq h$. This describes an element in the h-fold symmetric product of the compactification $(\mathbb{C}^+, \infty)$. We get

$$\Phi\colon \mathbb{C}P^h \longrightarrow \mathrm{SP}^h\mathbb{C}^+, [a_h : \dots : a_0] \longmapsto \sum_{k=1}^{r} \nu_k \cdot z_k.$$

It is easy to see that Φ is an isomorphism, e. g. [Hat01, Expl. 4.K4]. Furthermore:

(I) By the fundamental theorem of algebra, the degree of Θ is the degree of the corresponding polynomial, e. g. $|\Theta| = k$ means $a_{k+1}, \dots, a_h = 0$ and $a_k \neq 0$.

(II) For $z \in \mathbb{C}$, the multiplicity of $\Theta \coloneqq \Phi([a_h : \dots : a_0])$ at z is exactly the multiplicity of the polynomial at z. Hence, $\mathrm{mult}_\Theta(0) = s$ means $a_0, \dots, a_{s-1} = 0$ and $a_s \neq 0$.

3.2.6 Construction. The above ideas can be generalised to complex vector spaces:

(I) Let V be a finite-dimensional $\mathbb{C}$-vector space. We form the space of *polynomials in* V as the symmetric algebra over V^*, so $\mathrm{Pol}\, V \coloneqq \mathrm{Sym}\, V^*$. We have an evaluation

$$(\mathrm{Pol}\, V) \otimes V \longrightarrow \mathbb{C}, \left(p \coloneqq \sum_{i\geq 0} \bigotimes_{j=1}^{i} u_{ij}, v\right) \longmapsto p(v) \coloneqq \sum_{i\geq 0} \prod_{j=1}^{k} u_{ij}(v).$$

Thus, we get a notion of degree and of roots of zero. The vector space of polynomials of degree at most h is given by the first $h+1$ graduation components

$$\mathrm{Pol}^h V \coloneqq \bigoplus_{i=0}^{h} \mathrm{Sym}^i V^*.$$

(II) For notational reasons let $\mathrm{Pol}^\infty V \coloneqq \mathrm{Pol}\, V$ and $1 \leq h \leq \infty$. Then Pol^h becomes a covariant functor: Let $\varphi\colon V \longrightarrow W$ be a linear isomorphism. Then we get

$$\varphi_* \coloneqq \mathrm{Sym}(\varphi^{-1})^*\colon \mathrm{Pol}^h V \longrightarrow \mathrm{Pol}^h W, \bigotimes_{j=1}^{i} u_{ij} \longmapsto \bigotimes_{j=1}^{i} (u_{ij} \circ \varphi^{-1}).$$

(III) Consider the category $\mathbf{C}$ of 1-dimensional complex vector spaces together with linear isomorphisms and let $\mathbb{P}$ be the projectivation. We have two functors $\mathbf{C} \longrightarrow \mathbf{Top}_0$,

$$\mathbb{P} \circ \mathrm{Pol}^h \qquad \text{and} \qquad \mathrm{SP}^h \circ (-)^+.$$

For $V \in \mathbf{C}$ and $[p] \in \mathbb{P}\,\mathrm{Pol}^h V$, we have roots of zero $v_1, \dots, v_r \in V$ with multiplicities $\nu_1, \dots, \nu_r \geq 0$ with $\nu_1 + \ldots + \nu_r = \deg(p) \leq h$. We get a natural isomorphism

$$\vartheta_V^h \colon \mathbb{P}\,\mathrm{Pol}^h V \longrightarrow \mathrm{SP}^h V^+, [p] \longmapsto \sum_{i=1}^{r} \nu_i \cdot v_i.$$

Proof. It is clear that ϑ_V^h is a homeomorphism. The naturality law follows since

$$(\varphi_* p)(\varphi(v)) = \sum_{i=0}^{h} \prod_{j=1}^{i} (u_{ij} \circ \varphi^{-1})(\varphi(v)) = \sum_{i=1}^{h} \prod_{j=1}^{j} u_{ij}(v) = p(v).$$

□

3.3 Homology of symmetric fibre products

We study the homology of symmetric fibre products of compactified complex line bundles, which are $\mathbb{C}P^h$-bundles. It turns out that they are homologically trivial.

3.3.1 Construction. Let $(F, \infty) \longrightarrow (E, B_\infty) \longrightarrow B$ be a bundle with based fibre. We consider the *h-fold symmetric fibre product* $\varrho\colon \mathrm{SP}^h E \longrightarrow B$.

(I) We have natural inclusions $\mathrm{SP}^h \hookrightarrow \mathrm{SP}^{h+1}$ which give rise to bundle morphisms

$$\mathrm{SP}^1 E \hookrightarrow \cdots \hookrightarrow \mathrm{SP}^\infty E.$$

(II) We have a fibre-wise multiplication induced by the monoid structure on $\mathrm{SP}^\infty E$

$$\begin{aligned} \mu\colon \mathrm{SP}^k E \times_B \mathrm{SP}^l E &\longrightarrow \mathrm{SP}^{k+l} E,\\ (x, y) &\longmapsto x \oplus y. \end{aligned}$$

Note that in order to avoid confusion with addition in homology, we denote the sum in the symmetric product sometimes by "$\oplus$". If $s\colon B \longrightarrow \mathrm{SP}^k E$ and $t\colon B \longrightarrow \mathrm{SP}^l E$ are two sections, we can add them fibre-wise and get $s \oplus t \coloneqq \mu \circ (s \times_B t)$.

3.3.2 Construction. Let $\mathbb{C} \longrightarrow L \longrightarrow B$ be a complex line bundle. We can use the two functors from the previous section to get a $\mathbb{C}P^h$-bundle:

(I) Applying $\mathscr{F}^h \coloneqq \mathbb{P} \circ \mathrm{Pol}^h$, we get a $\mathbb{C}P^h$-bundle $\mathbb{C}P^h \longrightarrow \mathbb{P}\,\mathrm{Pol}^h L \longrightarrow B$.
(II) Applying $\mathscr{G}^h \coloneqq \mathrm{SP}^h \circ (-)^+$, we get a $\mathbb{C}P^h$-bundle $\mathbb{C}P^h \longrightarrow \mathrm{SP}^h L^+ \longrightarrow B$.

The natural isomorphism ϑ^h from the previous section yields a bundle isomorphism between them. Furthermore, let $w^h\colon \mathscr{F}^h \longrightarrow \mathscr{F}^{h+1}$ and $u^h\colon \mathscr{G}^h \longrightarrow \mathscr{G}^{h+1}$ be the natural inclusions. They commute with the ϑ^h and we get the following diagram of bundle morphisms:

$$\begin{tikzcd} \mathbb{P}\,\mathrm{Pol}^h L \arrow[r, "\eta^h_*", "\cong"'] \arrow[d, "w^h_*"'] & \mathrm{SP}^h L^+ \arrow[d, "u^h_*"] \\ \mathbb{P}\,\mathrm{Pol}^{h+1} L \arrow[r, "\eta^{h+1}_*", "\cong"'] & \mathrm{SP}^{h+1} L^+ \end{tikzcd}$$

Summing up, after putting a Riemannian metric on $L \longrightarrow B$, we have three isomorphic $\mathbb{C}P^h$-bundles, which we often denote only by $\mathrm{SP}^h L$, given by

$$\mathbb{P}\,\mathrm{Pol}^h L \cong \mathrm{SP}^h L^+ \cong \mathrm{SP}^h(\mathbb{D}/\mathbb{S})L.$$

3.3.3 Remark. Let $(t_{ij}\colon U_{ij} \longrightarrow \mathrm{Homeo}(\mathbb{C}))$ be the transition maps of the complex line bundle $\mathbb{C} \longrightarrow L \longrightarrow B$. Then the transitions of $\mathbb{C}P^h \longrightarrow E \coloneqq \mathrm{SP}^h L \longrightarrow B$ are given by

$$\sum_{z\in\mathbb{C}} \nu_z \cdot z \longmapsto \sum_{z\in\mathbb{C}} \nu_z \cdot t_{ij}(x)(z)$$

Since the transitions $t_{ij}(x)$ are all linear, they fix 0 and thus, the transitions of $\mathrm{SP}^h L$ fix the multiplicity of 0. For $0 \le s \le h$, we get subbundles $\mathbb{C}P^{h-s} \longrightarrow E^s \longrightarrow B$ defined as

$$E^s \coloneqq \{(x, \Theta) \in E;\ \mathrm{mult}_\Theta(0) \ge s\} \subseteq E.$$

Thus, we get a filtration $E = E^0 \supseteq \cdots \supseteq E^s \cong B$.

3.3.4 Proposition. *The bundle $\varrho\colon \mathrm{SP}^\infty L \longrightarrow B$ is fibre-homotopically trivial.*

Proof. Step 1: Since $L \longrightarrow B$ is a complex line bundle, it is orientable and $\pi_1(B)$ acts trivially on $H^2(L_x^+) = H^2(\mathbb{S}^2)$. Now $[\gamma] \in \pi_1(B, x)$ yields $\gamma_\#^h\colon H_2(\mathrm{SP}^h\mathbb{S}^2) \longrightarrow H_2(\mathrm{SP}^h\mathbb{S}^2)$ and for the bundle inclusion $u\colon L^+ = \mathrm{SP}^1 L \hookrightarrow \mathrm{SP}^\infty L$, we get a diagram

$$\begin{tikzcd}
H_2(\mathbb{S}^2) \arrow[r, "\gamma_\#^1=\mathrm{id}"] \arrow[d, "H_2(u_x)"'] & H_2(\mathbb{S}^2) \arrow[d, "H_2(u_x)"] \\
H_2(\mathrm{SP}^\infty\mathbb{S}^2) \arrow[r, "\gamma_\#^\infty"'] & H_2(\mathrm{SP}^\infty\mathbb{S}^2)
\end{tikzcd}$$

Since $u_x\colon \mathbb{S}^2 = \mathbb{C}P^1 \longrightarrow \mathbb{C}P^\infty = \mathrm{SP}^\infty\mathbb{S}^2$ is the usual inclusion, $H_2(u_x)$ is an isomorphism and $\gamma_\#^\infty = \mathrm{id}$. By the Hurewicz isomorphism, $\pi_1(B)$ acts trivially on $\pi_2(\mathbb{C}P^\infty)$.

Step 2: We are in the special situation that the fibre $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$-space. Since $\pi_1(B)$ acts trivially on $\pi_2(K(\mathbb{Z},2))$, there is a map $f\colon B \longrightarrow K(\mathbb{Z},3)$ such that we get $\mathrm{SP}^h L \simeq f^*\mathscr{P}K(\mathbb{Z},3)$ for the path space fibration

$$\Omega K(\mathbb{Z},3) \longrightarrow \mathscr{P}K(\mathbb{Z},3) \xrightarrow{\sigma} K(\mathbb{Z},3),$$

see [MP12, Lem. 3.4.2]. Now consider the zero section $s\colon B \longrightarrow L$. We get the diagram

$$\begin{tikzcd}
L \arrow[r, hook] \arrow[d] & L^+ = \mathrm{SP}^1 L \arrow[r, hook] \arrow[d] & \mathrm{SP}^\infty L \arrow[r, "u"] \arrow[d, "\varrho"'] & \mathscr{P}K(\mathbb{Z},3) \arrow[d, "\sigma"] \\
B \arrow[u, bend left, "s"] \arrow[r, equal] & B \arrow[r, equal] & B \arrow[r, "f"'] & K(\mathbb{Z},3)
\end{tikzcd}$$

Then $\varrho \circ s = \mathrm{id}_B$ and $f = f \circ \varrho \circ s = \sigma \circ u \circ s$ and f is null-homotopic since it lifts over the path space $\mathscr{P}K(\mathbb{Z},3)$. Hence, $\mathrm{SP}^\infty L \simeq f^*\mathscr{P}K(\mathbb{Z},3)$ is homotopically trivial. $\square$

3.3.5 Reminder (Leray–Hirsch)**.** Let $F \longrightarrow E \xrightarrow{\varrho} B$ be a fibre bundle and $\imath_x\colon F \longrightarrow E$ a fibre inclusion and we have the following properties:

(i) $H^i(F)$ is free and finitely generated for each $i \geq 0$.

(ii) For each $i \geq 0$ there are classes $\beta_{i,1}, \ldots, \beta_{i,r_i} \in H^i(E)$ such that for $\alpha_{i,j} \coloneqq \imath_x^*\beta_{i,j}$ the family $(\alpha_{i,1}, \ldots, \alpha_{i,r_i})$ is a $\mathbb{Z}$-basis of $H^i(F)$.

Then we have an isomorphism in cohomology

$$\begin{aligned}
\mathrm{LH}_E\colon H^*(B) \otimes_{\mathbb{Z}} H^*(F) &\longrightarrow H^*(E), \\
\gamma \otimes \alpha_{i,j} &\longmapsto \varrho^*\gamma \smile \beta_{i,j}
\end{aligned}$$

This isomorphism is natural: Let $(u, f) : (F \longrightarrow E \longrightarrow B) \longrightarrow (F' \longrightarrow E' \longrightarrow B')$ be a morphism of bundles satisfying the above properties. For $x \in B$, consider $u_x : E_x \longrightarrow E'_{f(x)}$. After identifying $F \cong E_x$ via $\imath_x$ and $F' \cong E'_{f(x)}$ via $\imath'_{f(x)}$, we get a commuting square

$$\begin{tikzcd}
H^*(B') \otimes_{\mathbb{Z}} H^*(F') \arrow[r, "\mathrm{LH}_{E'}"] \arrow[d, "f^*\otimes u_x^*"'] & H^*(E') \arrow[d, "u^*"] \\
H^*(B) \otimes_{\mathbb{Z}} H^*(F) \arrow[r, "\mathrm{LH}_E"'] & H^*(E).
\end{tikzcd}$$

3.3.6 Reminder (Tautological line bundle)**.** Let V be an $(h+1)$-dimensional complex vector space. Then we have a *tautological line bundle*

$$\lambda V \coloneqq \big\{(v, A) \in V \times \mathbb{P}V;\ v \in A\big\} \longrightarrow \mathbb{P}V \cong \mathbb{C}P^h.$$

This bundle is a complex line bundle and its Euler class $\xi \in H^2(\mathbb{P}V) \cong \mathbb{Z}$ is a generator.

3.3.7 Construction. Let $\mathbb{C}^{h+1} \longrightarrow E \longrightarrow B$ be a complex vector bundle. The *fibre-wise tautological line bundle* $\mathbb{C} \longrightarrow \lambda E \longrightarrow \mathbb{P}E$, defined as

$$\lambda E \coloneqq \big\{(v, A) \in E \times_B \mathbb{P}E;\ v \in A\big\},$$

is a complex line bundle. If we choose a fibre inclusion $\imath_x \colon \mathbb{P}V \longrightarrow \mathbb{P}E$, we get $\lambda V = \imath_x^* \lambda E$, which means we have a pullback of bundles

$$\begin{tikzcd} \lambda V \arrow[r] \arrow[d] & \lambda E \arrow[d] \\ \mathbb{P}V \arrow[r, "\imath_x"'] & \mathbb{P}E. \end{tikzcd}$$

Let $\chi(E) \in H^2(\mathbb{P}E)$ be the Euler class of $\lambda E \longrightarrow \mathbb{P}E$ and identify the cohomology ring of the fibre $H^*(\mathbb{P}V) = H^*(\mathbb{C}P^h) = \mathbb{Z}[\xi]/\xi^{h+1}$. Then up to sign, $\imath_x^* \chi(E) = \xi$.

3.3.8 Construction. By identifying $\mathbb{P}\operatorname{Pol}^h L \cong \operatorname{SP}^h L$, we obtain the *universal classes*

$$\chi_h(L) \coloneqq \chi(\operatorname{Pol}^h L) \in H^2(\operatorname{SP}^h L).$$

A system of classes as occuring in Leray–Hirsch is given by powers $\chi_h(L)^s \in H^{2s}(\operatorname{SP}^h L)$. This yields the desired natural isomorphism in cohomology

$$\begin{aligned} \mathrm{LH}\colon H^*(B) \otimes_{\mathbb{Z}} H^*(\mathbb{C}P^h) &\longrightarrow H^*(\operatorname{SP}^h L), \\ \vartheta \otimes \xi^s &\longmapsto \varrho^* \vartheta \smile \chi_h(L)^s. \end{aligned}$$

3.3.9 Corollary. (I) *The inclusion $u\colon \operatorname{SP}^h L \hookrightarrow \operatorname{SP}^{h+1} L$ induces an isomorphism in cohomology in degree up to $2h$ and $u^* \chi_{h+1}(L) = \chi_h(L)$. Thus, we just write $\chi(L)$.*

(II) *Let $f\colon A \longrightarrow B$ be a morphism and $u\colon f^* \operatorname{SP}^h L \longrightarrow \operatorname{SP}^h L$ the morphism induced from the pullback. Then $u^* \chi(L) = \chi(f^* L)$.*

Proof. Both statements are immediate consequences of the naturality of the Leray–Hirsch isomorphism. The first statement uses the naturality in the following form

$$\begin{tikzcd} H^*(\mathbb{C}P^{h+1}) \otimes_{\mathbb{Z}} H^*(B) \arrow[r, "\mathrm{LH}", "\cong"'] \arrow[d, "\mathrm{pr} \otimes \mathrm{id}"'] & H^*(\operatorname{SP}^{h+1} L) \arrow[d, "u^*"] \\ H^*(\mathbb{C}P^h) \otimes_{\mathbb{Z}} H^*(B) \arrow[r, "\cong", "\mathrm{LH}"'] & H^*(\operatorname{SP}^h L). \end{tikzcd}$$

For the second statement, we note that $f^* \operatorname{SP}^h L \cong \operatorname{SP}^h f^* L$ and use the naturality

$$\begin{tikzcd} H^*(\mathbb{C}P^h) \otimes_{\mathbb{Z}} H^*(B) \arrow[r, "\mathrm{LH}", "\cong"'] \arrow[d, "\mathrm{id} \otimes f^*"'] & H^*(\operatorname{SP}^h L) \arrow[d, "u^*"] \\ H^*(\mathbb{C}P^h) \otimes_{\mathbb{Z}} H^*(A) \arrow[r, "\cong", "\mathrm{LH}"'] & H^*(f^* \operatorname{SP}^h L) = H^*(\operatorname{SP}^h f^* L) \end{tikzcd}$$

□

3.3.10 Proposition. *Consider the infinity section* $s_\infty \colon B \longrightarrow L^+$ *and the Thom class* $\tau \in H^2(L^+, B_\infty)$ *as well as the inclusion* $\jmath \colon L^+ \longrightarrow (L^+, B_\infty)$. *Then*

$$\jmath^*\tau = \pm\chi(L) \in H^2(L^+).$$

Proof. The Thom class is (up to sign) determined by the property that the restriction $\imath_x^*\jmath^*\tau \in H^2(\mathbb{S}^2, \infty)$ is a generator. This is satisfied by χ, see Construction 3.3.7. □

3.3.11 Corollary. *We can pull back* $\chi(L)$ *via the two canonical sections* $s_0, s_\infty \colon B \longrightarrow L^+$*:*

(i) *For the zero section* $s_0 \colon B \longrightarrow L^+ \subseteq \mathrm{SP}^h L$, *we get* $s_0^*\chi(L) = e(L)$.
(ii) *For the infinity section* $s_\infty \colon B \longrightarrow L^+ \subseteq \mathrm{SP}^h L$, *we get* $s_\infty^*\chi(L) = 0$.

3.3.12 Proposition. *Let* $s \colon B \longrightarrow \mathrm{SP}^k L$ *and* $t \colon B \longrightarrow \mathrm{SP}^l L$ *be sections and denote the addition in the symmetric product by "$\oplus$". Then for* $\chi \coloneqq \chi(L)$, *we get*

$$(s \oplus t)^*\chi = s^*\chi + t^*\chi.$$

Proof. Consider the multiplication in $\mathbb{C}P^\infty$, which we denote $\varepsilon \colon \mathbb{C}P^k \times \mathbb{C}P^l \longrightarrow \mathbb{C}P^{k+l}$. Furthermore, denote the generators by $\xi_k \in H^2(\mathbb{C}P^k)$. Then we know that

$$\varepsilon^*\xi_{k+l} = (\xi_k \times 1) + (1 \times \xi_l) \in H^2(\mathbb{C}P^k \times \mathbb{C}P^l).$$

Now back to our sections s and t. Since the bundles are homologically trivial, there are $\varphi \colon H^*(\mathbb{C}P^k) \longrightarrow H^*(B)$ and $\psi \colon H^*(\mathbb{C}P^l) \longrightarrow H^*(B)$ such that

$$\begin{tikzcd} H^*(B) \otimes H^*(\mathbb{C}P^k) \arrow[r, "\mathrm{LH}_k"] \arrow[d, "\mathrm{id} \smile \varphi"'] & H^*(\mathrm{SP}^k L) \arrow[d, "s^*"] \\ H^*(B) \arrow[r, equal] & H^*(B) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H^*(B) \otimes H^*(\mathbb{C}P^l) \arrow[r, "\mathrm{LH}_l"] \arrow[d, "\mathrm{id} \smile \psi"'] & H^*(\mathrm{SP}^l L) \arrow[d, "t^*"] \\ H^*(B) \arrow[r, equal] & H^*(B) \end{tikzcd}$$

We know that $s \oplus t = \mu \circ (s \times_B t)$ and we furthermore have the diagram

$$\begin{tikzcd} H^*(B) \otimes H^*(\mathbb{C}P^{k+l}) \arrow[r, "\mathrm{LH}_{k+l}"] \arrow[d, "\mathrm{id} \otimes \varepsilon^*"'] & H^*(\mathrm{SP}^{k+l} L) \arrow[d, "\mu^*"] \\ H^*(B) \otimes H^*(\mathbb{C}P^k) \otimes H^*(\mathbb{C}P^l) \arrow[r, "\mathrm{LH}_k \times_B \mathrm{LH}_l"] \arrow[d, "\mathrm{id} \smile \varphi \smile \psi"'] & H^*(\mathrm{SP}^k L \times_B \mathrm{SP}^l L) \arrow[d, "(s \times_B t)^*"] \\ H^*(B) \arrow[r, equal] & H^*(B). \end{tikzcd}$$

Thus, we can finally calculate the class by

$$\begin{aligned} (s \oplus t)^*\chi_{k+l} &= (s \times_B t)^*\mu^*\chi_{k+l} \\ &= (\varphi \smile \psi)(\varepsilon^*\xi_{k+l}) \\ &= (\varphi \smile \psi)\left((\xi_k \times 1) + (1 \times \xi_l)\right) \\ &= \varphi(\xi_k) + \psi(\xi_l) \\ &= s^*\chi_k + s^*\chi_l. \end{aligned}$$

□

4 Scanning critical points

4.1 The universal surface bundle

We study the universal surface bundle $F \longrightarrow \mathfrak{F}^m_{g,1} \longrightarrow \mathfrak{P}^m_{g,1}$ and define some subvarieties. Moreover, we give a short survey on the Mumford–Miller–Morita classes $\kappa_{s-1} \in H^{2s-2}(\Gamma_{g,1})$ following [Til12] and [Hat14]. Finally, we discuss the cohomology of $\mathfrak{F}^m_{g,1}$.

4.1.1 Construction. Recall that $\mathrm{Diff}^+ = \mathrm{Diff}^+(\mathcal{F})$ is the group of orientation preserving diffeomorphisms on the closed surface F fixing the dipole and permuting the sinks, and $\mathfrak{P}^m_{g,1} \cong \mathfrak{H}^m_{g,1}$ is a classifying space for Diff^+. We define the *universal surface bundle*

$$\mathfrak{F}^m_{g,1} \coloneqq E\,\mathrm{Diff}^+ \times_{\mathrm{Diff}^+} F \xrightarrow{\pi} B\,\mathrm{Diff}^+ \simeq \mathfrak{P}^m_{g,1}.$$

4.1.2 Remark. There are two facts we already know about $\mathfrak{F}$:

(I) The bundle is a smooth surface bundle in the sense of Example 3.1.8, so we can consider its vertical tangent bundle $T^\perp\mathfrak{F}$. Since it is orientable, it carries the structure of a complex line bundle. We often refer to it as in Chapter 3:

$$\begin{array}{ccccc} \mathbb{C} & \longrightarrow & T^\perp\mathfrak{F} & \longrightarrow & \mathfrak{F} \\ \shortparallel & & \shortparallel & & \shortparallel \\ \mathbb{C} & \longrightarrow & L & \longrightarrow & B. \end{array}$$

(II) The structure group of the surface bundle is given by $\mathrm{Diff}^+(\mathcal{F})$ and

$$\varphi \cdot (u, \zeta) = (u \circ \varphi^{-1}, \varphi(\zeta)).$$

Hence, subsets of $\mathfrak{F}$ defined by properties of u are fixed by the transpositions and therefore yield well-defined subbundles of $\mathfrak{F}$.

4.1.3 Construction. For each $0 \le s \le h$, we have two interesting subspaces

$$\mathfrak{U}_s \coloneqq \{(\mathcal{N}, \zeta);\ \mathrm{mult}_u(\zeta) \le s\} \qquad \text{and} \qquad \mathfrak{V}_s \coloneqq \{(\mathcal{N}, \zeta);\ \mathrm{mult}_u(\zeta) \ge s\}$$

Of course, these subspaces are well-defined since the condition is fixed by the transitions. However, they are in general not subbundles. We can conclude some properties for them:

(I) The images of $\mathfrak{V}_s$ under π are exactly the geometric realisations of the Weierstraß complexes (W_s, W'_s); this means we get $\pi(\mathfrak{V}_s) = \mathfrak{W}_s = |W_s| \setminus |W'_s|$. Therefore, we call these subspaces *lifted Weierstraß complexes*.

(II) The subspaces are complementary to each other, $\mathfrak{U}_s = \mathfrak{F} \setminus \mathfrak{V}_{s+1}$, and both filter the surface bundle in the following way:

$$\mathfrak{F} = \mathfrak{U}_h \supseteq \cdots \supseteq \mathfrak{U}_0 \qquad \text{and} \qquad \mathfrak{F} = \mathfrak{V}_0 \supseteq \cdots \supseteq \mathfrak{V}_h$$

(III) We saw in Remark 2.3.6 that $\mathfrak{W}_s$ is a subvariety of dimension $3h+2-2s$. Moreover, the fibres of $\mathfrak{V}_s \longrightarrow \mathfrak{W}_s$ are discrete for $s \ge 1$. Therefore, $\mathfrak{V}_s \subseteq \mathfrak{F}$ is a subvariety of codimension $2s$ for $s \ge 1$; in particular, all $\mathfrak{V}_s \subseteq \mathfrak{F}$ are closed.

4.1.4 Construction. Consider the Serre spectral sequence $(E_r^{p,q})$ of $\pi\colon \mathfrak{F}_{g,1} \longrightarrow \mathfrak{P}_{g,1}$. Since the bundle is orientable, the top and the bottom row of the second page equal the cohomology with integral coefficients. We get the corresponding edge maps

$$E_\infty^{k,2} \longrightarrow E_2^{k,2} = H^k(\Gamma_{g,1}).$$

There is a filtration $0 = F^0H \subseteq F^1H \subseteq F^2H = H^{k+2}(\mathfrak{F})$ of the cohomology of $\mathfrak{F}$ such that $F^2H/F^1H \cong E_\infty^{k,2}$. This yields the *fibre transfer morphism* in cohomology

$$\pi^!\colon H^k(\mathfrak{F}) \longrightarrow E_\infty^{k-2,2} \longrightarrow H^{k-2}(\Gamma_{g,1}).$$

Let $e \coloneqq e(T^\perp\mathfrak{F}) \in H^2(\mathfrak{F})$ be the Euler class of the vertical tangent bundle $T^\perp\mathfrak{F} \longrightarrow \mathfrak{F}$. Then we define the $(s-1)^{th}$ *Mumford–Miller–Morita class*[1] κ_{s-1}, short *Mumford class*,

$$\kappa_{s-1} \coloneqq (-1)^s \cdot \pi^! e^s \in H^{2s-2}(\Gamma_{g,1}).$$

4.1.5 Survey. (I) The above construction is possible for each smooth orientable surface bundle $F \longrightarrow E \longrightarrow B$ of genus g. Because of the naturality of the Euler class, these classes are characteristic classes of orientable surface bundles.

(II) The classes were traditionally defined in [Mum83] for the surface bundle over $\mathfrak{M}_g$, i. e. for surfaces without a fixed dipole. However, the first part of Harer's stability theorem, see [Har84], improved by Ivanov, Boldsen and Randal-Williams, see [Wah13], yields that the canonical map $\Gamma_{g,1} \longrightarrow \Gamma_g$ induces isomorphisms

$$H_k(\Gamma_{g,1}) \xrightarrow{\cong} H_k(\Gamma_g) \quad \text{for} \quad k \leq \frac{2g}{3}.$$

and thus, by the universal coefficient theorem, also in cohomology for $k \leq \frac{2g}{3} - 1$. Under this isomorphism, the two notions for the Mumford classes coincide.

(III) So far, each genus has its own Mumford classes as characteristic classes. However, the second statement of Harer's stability theorem, see [Har84] and [Wah13], yields that the canonical inclusion $\imath\colon \Gamma_{g,1} \hookrightarrow \Gamma_{g+1,1}$ (by extending the diffeomorphisms trivially on the complement) induces isomorphisms in homology

$$H_k(\Gamma_{g,1}) \xrightarrow{\cong} H_k(\Gamma_{g+1,1}) \quad \text{for} \quad k \leq \frac{2g-2}{3},$$

and thus, by the universal coefficient theorem, also in cohomology. The Mumford classes are stable with respect to this inclusion, i. e. $\imath^*\kappa_s^{(g+1)} = \kappa_s^{(g)}$.

(IV) Using the above inclusions, we obtain the *stable mapping class group* by

$$\Gamma_\infty \coloneqq \varinjlim_g \Gamma_{g,1}.$$

By Harer's stability theorem, the stable Mumford classes are classes in $H^*(\Gamma_\infty)$. The Madsen–Weiss theorem, see [MW07], states that the rational cohomology ring $H^*(\Gamma_\infty;\mathbb{Q})$ is a polynomial algebra generated by the Mumford classes,

$$H^*(\Gamma_\infty;\mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \dots].$$

[1] In the literature, various versions of aligning the names can be found. We use the chronological order in which the relevant papers appear: Mumford [Mum83], Miller [Mil86] and Morita [Mor87].

4.1.6 Construction. There is another interesting subspace of $\mathfrak{F}$: Since the structure group fixes the set $\{Q, P_1, \ldots, P_m\}$ in each fibre, we can consider the subbundle where the fibre is restricted to $F^\star := F \setminus \{Q, P_1, \ldots, P_m\}$. We call this subbundle the *open surface bundle*

$$F^\star \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{P}.$$

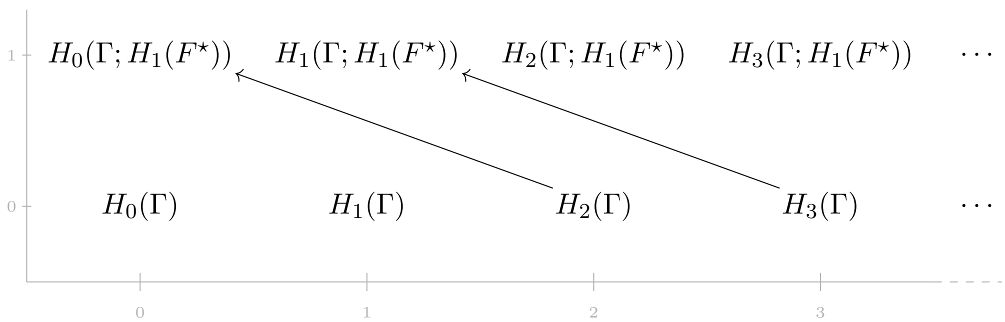
4.1.7 Proposition. *Consider the action of* Γ *on the first homology group* $H_1(F^\star)$*. Then*

$$H_k(\mathfrak{E}) \cong H_k(\Gamma) \oplus H_{k-1}(\Gamma; H_1(F^\star)).$$

Proof. We choose a Riemannian metric on $T^\perp\mathfrak{E} \longrightarrow \mathfrak{E}$ and in each fibre a geodesic neighbourhood around Q. We get a continuous map $\varepsilon : \mathfrak{P} \longrightarrow \mathbb{R}_{>0}$ such that $\varepsilon(\mathcal{N})$ is smaller than the injectivity radius for each $[\mathcal{N}] \in \mathfrak{P}$. Thus, we obtain a section

$$s \colon \mathfrak{P} \longrightarrow \mathfrak{E},\ [\mathcal{N}] \longmapsto \exp_Q(\varepsilon \cdot X)$$

and conclude that $H_k(\mathfrak{E}) \cong H_k(\Gamma) \oplus \mathrm{Ker}(\pi_*)$. The fibre $F^\star$ is homotopy equivalent to a bouquet and therefore, $H_k(F^\star) = 0$ for $k \geq 2$. Since the action of $\pi_1(\mathfrak{P})$ on $H_0(F^\star)$ is trivial, the E^2-page of the corresponding Serre spectral sequence is of the form



Because of the section, the bottom row survives and the differentials of the E^2-page are 0. Thus, the E^2-page is already the E^∞-page and we get a short exact sequence

$$0 \longrightarrow H_{k-1}(\Gamma; H_1(F^\star)) \longrightarrow H_k(\mathfrak{E}) \longrightarrow H_k(\Gamma) \longrightarrow 0.$$

With the above section, the sequence splits, yielding the desired identity. $\square$

4.1.8 Remark. To the bare result from the spectral sequence, we can add the following:

(i) The cohomological version works completely analogously and we get

$$H^k(\mathfrak{E}) \cong H^k(\Gamma) \oplus H^{k-1}(\Gamma; H^1(F^\star)).$$

(ii) In the case $m = 0$, the action of $\Gamma_{g,1}$ on $H^1(F^\star) = \mathbb{Z}^{2g}$ corresponds to the well-known symplectic representation of the mapping class group, so $H^{k-1}(\Gamma_{g,1}; H^1(F^\star))$ is isomorphic to the group cohomology $H^{k-1}(\Gamma_{g,1}; \mathbb{Z}^{2g})$ with symplectic coefficients.

(iii) We know that $H^1(F^\star) \cong \mathbb{Z}^{2g+m}$ is a $\Gamma^m_{g,1}$-module without invariants since to each generating curve a, we can add a curve b intersecting a via a Dehn twist in $\Gamma^m_{g,1}$. Therefore, $H^0(\Gamma; H_1(F^\star)) = 0$ and we conclude the identity

$$H^1(\mathfrak{E}) \cong H^1(\Gamma) \cong \mathrm{Hom}_\mathbb{Z}(\Gamma, \mathbb{Z}).$$

4.1.9 Example. The above result helps us to understand the cohomological relation between $\mathfrak{F}$ and $\mathfrak{E}$ in small degrees, i. e. we see what the dipole "changes" homologically:

(I) The inclusion $\mathfrak{F} \setminus \mathfrak{E} \hookrightarrow \mathfrak{F}$ is a smooth embedding of a subbundle of codimension 2. Therefore, the Thom isomorphism yields

$$H^{k-2}(\mathfrak{F} \setminus \mathfrak{E}) \cong H^k(\mathfrak{F}, \mathfrak{E}).$$

In particular, $H^2(\mathfrak{F}, \mathfrak{E}) \cong H^0(\mathfrak{F} \setminus \mathfrak{E})$. The remaining bundle $\mathfrak{F} \setminus \mathfrak{E} \longrightarrow \mathfrak{P}$ is the sum of two coverings of $\mathfrak{P}$: One 1-sheeted covering for Q and for $m \neq 0$ one m-sheeted covering for $P_1, \ldots, P_m$. Hence, we get for the 2$^{\text{nd}}$ relative cohomology

$$H^2(\mathfrak{F}, \mathfrak{E}) \cong H^0(\mathfrak{F} \setminus \mathfrak{E}) \cong \begin{cases} \mathbb{Z} & \text{for } m = 0, \\ \mathbb{Z}^2 & \text{for } m \geq 1. \end{cases}$$

(II) Again by the Thom isomorphism, we get that $H^1(\mathfrak{F}, \mathfrak{E}) \cong H^{-1}(\mathfrak{F} \setminus \mathfrak{E}) \cong 0$. Therefore, $H^1(\mathfrak{F}) \longrightarrow H^1(\mathfrak{E})$ is injective, since the long exact sequence of $(\mathfrak{F}, \mathfrak{E})$ becomes

$$0 = H^1(\mathfrak{F}, \mathfrak{E}) \longrightarrow H^1(\mathfrak{F}) \longrightarrow H^1(\mathfrak{E}).$$

(III) Powell showed that $\Gamma_{g,1}$ is perfect[2] for $g \geq 3$, [Pow78]. Hence, there is no non-trivial morphism $\Gamma_{g,1} \longrightarrow \mathbb{Z}$ and $H^1(\mathfrak{F}) \hookrightarrow H^1(\mathfrak{E}) = 0$. Our long exact sequence becomes

$$0 = H^1(\mathfrak{E}) \longrightarrow H^2(\mathfrak{F}, \mathfrak{E}) \cong \mathbb{Z} \longrightarrow H^2(\mathfrak{F}) \longrightarrow H^2(\mathfrak{E}).$$

In particular, the kernel of $H^2(\mathfrak{F}) \longrightarrow H^2(\mathfrak{E})$ is given by $\mathbb{Z}$.

4.1.10 Motivation. We want to consider for each point $[\mathcal{N}, \zeta] \in \mathfrak{F}$ in the surface bundle a small closed disk $G(\zeta) \subseteq F$ around ζ which forms a disk bundle $\mathfrak{G} \subseteq \mathfrak{F} \times_{\mathfrak{P}} \mathfrak{F}$ such that "scanning for critical points of u in $G(\zeta)$" becomes a section

$$\mathfrak{F} \longrightarrow \mathrm{SP}^h \mathfrak{G}, [\mathcal{N}, \zeta] \longmapsto \sum_{i=1}^{k} \nu_i \cdot S_i \in \mathrm{SP}^h(F^\star, F^\star \setminus G(\zeta)).$$

The canonical approach would be to choose a Riemannian metric on $L := T^\perp \mathfrak{F} \longrightarrow \mathfrak{F}$ and consider geodesic disks coming from this metric. We furthermore get an exponential map

$$\exp\colon (\mathbb{D}L, \mathbb{S}L) \longrightarrow (\mathfrak{G}, \partial \mathfrak{G})$$

which is a bundle isomorphism. However, in order to get a better control over the form of the disks in terms of our slit coordinates, we go a slightly different way and give a construction of small disks in terms of the harmonic function u, provisionally only for points in $\mathfrak{E}$. It turns out that they are polygons and the number of vertices depends on the number of seen critical points counting multiplicities. Since the concrete form of the disks is not relevant for our homological studies, this can be seen as a little digression. We continue with studying the above scanning section homologically in Chapter 4.3.

[2] The author and his advisor do not like the rather characterless and vacuous word "perfect" and endorse the usage of the phrase "cainian" instead, in order to emphasise the contrast to abelianity.

4.2 Geodesic disks via u

For each class $[\mathcal{F}, u, D] \in \mathfrak{P}^m_{g,1}$, we give a combinatorial description of a metric on $F^\star$ and for $\zeta \in F^\star$, there is a maximal radius $\varrho(\zeta) > 0$ such that $B(\zeta; r)$ is a disk for all $r < \varrho(\zeta)$. This yields a disk bundle $\mathfrak{G}^m_{g,1} \longrightarrow \mathfrak{E}^m_{g,1}$. The crucial geometric idea is due to [Böd18].

4.2.1 Construction. Let $[\mathcal{F}, u, D] \in \mathfrak{P}^m_{g,1}$ be a potential class. A path $\gamma\colon [0,1] \longrightarrow F^\star$ is called *u-path* if it follows piecewise the gradient flow of u or its contour lines, i. e. piecewise, we have $\mathrm{d}u(\gamma'(t)) = 0$ or $\mathrm{d}u(i \cdot \gamma'(t)) = 0$. Then, γ is piecewise smooth and we define

$$\Delta_u(\gamma) := \int_0^1 \left| \mathrm{d}u|_{\gamma(t)}(\gamma'(t)) \right| \mathrm{d}t \quad \text{and} \quad \Delta_v(\gamma) := \int_0^1 \left| \mathrm{d}u|_{\gamma(t)}(i \cdot \gamma'(t)) \right| \mathrm{d}t,$$

Now define the *length* of γ by $L(\gamma) := \max(\Delta_u(\gamma), \Delta_v(\gamma))$ and let

$$\widetilde{d}(\zeta, \eta) := \inf \Big(L(\gamma);\, \gamma \text{ is a } u\text{-path from } \zeta \text{ to } \eta \Big).$$

4.2.2 Remark. We can consider this distance function to define disks around a point $\zeta \in F^\star$. This turns out not to be what we want: Consider the simple case $g = 0$ and $m = 1$, so we have one critical point with multiplicity 1. If we choose $\zeta \in F^\star$ as in the picture below, the closed disk with radius 2 is of the following form:

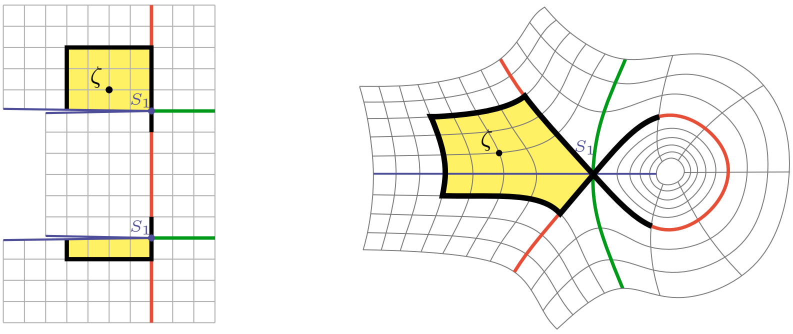


FIGURE 4.1: The disks induced by $\widetilde{d}$ do not behave well.

The boundary with respect to the metric (coloured black) is not the topological boundary of the actual disk (coloured yellow). Therefore, we are going to modify the notions of disks.

4.2.3 Construction. Let $\zeta \in F^\star$ and $r > 0$. We define $B^k(\zeta; r)$ recursively:

(I) Let $s_0 := r$ and $B^0(\zeta; r) := \{\zeta\}$.

(II) In the $(k+1)^{\text{th}}$ step, we extend the disk $B^k(\zeta; r)$ to $B^{k+1}(\zeta; r)$ until the next critical point is seen or the radius is exhausted. More formally, define the set of remaining critial points by $C_k := \mathrm{Cr}[\mathcal{F}, u, D] \setminus \overline{B^k(\zeta; r)}$ and the temporary radius

$$s := \min\Big(\widetilde{d}(B^k(\zeta; r), C_r), s_k\Big) \le s_k.$$

Now let $s_{k+1} := s_k - s \ge 0$ and extend the disk to the next filtration component

$$B^{k+1}(\zeta; r) := B^k(\zeta; r) \cup \{\eta \in F^\star;\, \widetilde{d}(B^k(\zeta; r), \eta) < s\}.$$

Since $\mathrm{Cr}[\mathcal{F}, u, D]$ is a finite set, $s \neq 0$ until $s_k = 0$ and in each step the set C_k of possible critical points in the neighbourhood of ζ becomes smaller. Thus, $B^0 \subseteq B^1 \subseteq \cdots$ becomes stationary after $s_K = 0$ and we define the *k-disk* by $B(\zeta; r) := B^K(\zeta; r)$.

4.2.4 Example. We want to construct $B(\zeta;3)$ in the example above. In the first step, we obtain a critical point with distance 2 to our ζ, so $B^1(\zeta;3)$ looks as follows:

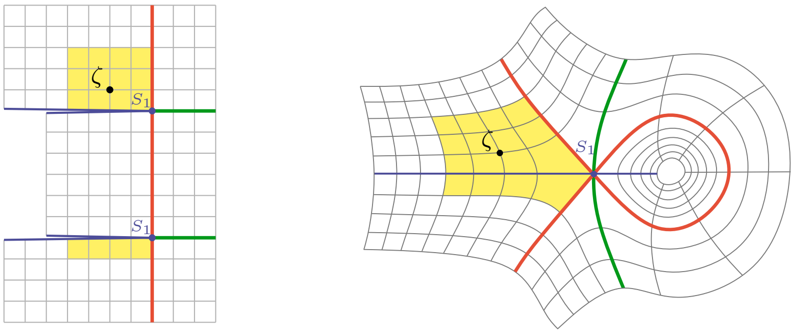


FIGURE 4.2: The first component B^1 of the disk $B(\zeta;3)$.

In the second (and last) step, we see no other critical point and add a strip of width $3-2=1$ to $B^1(\zeta;3)$. Therefore, we get the final disk $B(\zeta;3)=B^2(\zeta;3)$ by

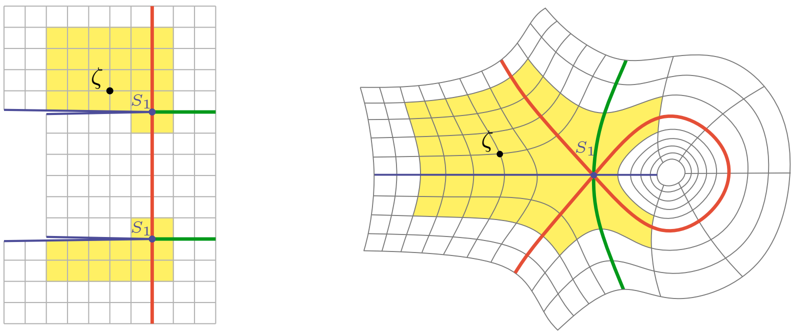


FIGURE 4.3: The final disk $B(\zeta;3)=B^2(\zeta;3)$ as described above.

4.2.5 Lemma. *The disks define, a posteriori, a metric which induces the disks,*

$$d(\zeta,\eta) := \inf\Big(r;\ \eta\in B(\zeta;r)\Big).$$

Then $\widetilde{d}(\zeta,\eta)\le d(\zeta,\eta)$ *and the topology of* $F^\star$ *is induced by the metric. Furthermore,*

$$\overline{B(\zeta;r)} = \{\eta\in F^\star;\ d(\zeta,\eta)\le r\} \quad \text{and} \quad \partial B(\zeta;r) = \{\eta\in F^\star;\ d(\zeta,\eta)=r\}.$$

Proof. The proof is just an easy and straight-forward checking of many little assertions, e. g. that d is symmetric: We have to show that $\eta\in B(\zeta;r)$ implies $\zeta\in B(\eta;r)$ for all $\zeta,\eta\in F^\star$ and $r>0$. By induction, we may assume that $B(\eta;r)$ contains exactly one critical point and let $s:=d(\eta,S)<r$. Then $\widetilde{d}(B(\eta;s),\eta)<r-s$ and $\zeta\in B(\eta;r)$. □

4.2.6 Lemma. *Let* $\zeta\in F^\star$ *and* $r>0$*. The boundary components of a disk* $B(\zeta;r)$ *can be parametrised by a u-path and are in particular piecewise smooth.*

Proof. Since $(B^k(\zeta;r))_{k\ge0}$ becomes stationary, it is enough to show the statement for each $\partial B^k(\zeta;r)$. We do this by induction over k. Here, the case "$k=0$" is clear and for "$k\longrightarrow k+1$" we see that the boundary $\partial B^{k+1}(\zeta;r)$ is given by curves piecewise parallel (in the slit picture) to the boundary $\partial B^k(\zeta;r)$, and therefore also a u-path. □

4.2.7 Remark. Denote by $\mathbb{B}^2 := \mathbb{D}^2 \setminus \mathbb{S}^1$ the open 2-dimensional disk. If the considered radius $r > 0$ becomes "too big", it can occur that $B(\zeta; r)$ is no longer homeomorphic to $\mathbb{B}^2$, e. g. for $r = 4$ in the following case:

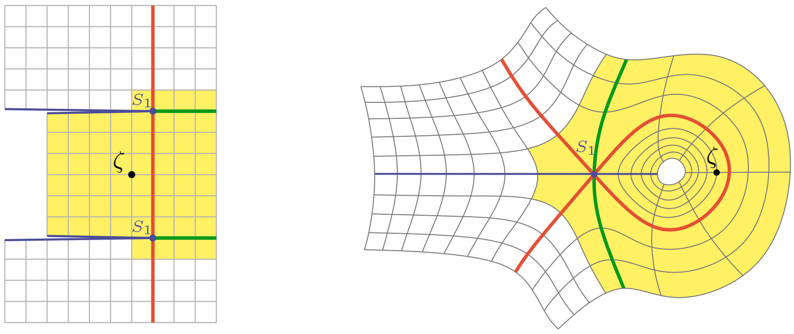


FIGURE 4.4: A disk with large radius which is no longer homeomorphic to $\mathbb{D}^2$.

4.2.8 Lemma. *There are minimal and maximal radii in the above sense:*

(I) *There is a $r > 0$ such that for each $\zeta \in F^\star$, we have $B(\zeta; r) \cong \mathbb{B}^2$.*
(II) *For each $\zeta \in F^\star$, there is a $R > 0$ such that $B(\zeta; R) \ncong \mathbb{B}^2$.*

Proof. (I) Consider the slit coordinates (a, b) and we may assume $b_i < b_{i+1}$ for each $0 \le i \le p$. Now let $r := \min(b_{i+1} - b_i) \cdot 2^{-1}$. Then $B(\zeta; r)$ is a square.

(II) Since $h = 1 - \chi(F^\star) \ge 1$, there is a critical point $S \in F^\star$. Now for $\zeta \in F^\star$ let $R := d(\zeta, S) + b_p - b_1 + 1$. By the triangular inequality, $B(S; b_p - b_1 + 1) \subseteq B(\zeta; R)$ and this disk contains a loop based in S which is not contractible in $F^\star$. $\square$

4.2.9 Lemma. *Let $\zeta \in F^\star$ and $s > 0$. Then the following statements are equivalent:*

(I) $B(\zeta; s) \ncong \mathbb{B}^2$
(II) $B(\zeta; s)$ *contains a loop which is not contractible in $F^\star$.*

Proof. We only have to show "(I) $\Rightarrow$ (II)": Define the radius

$$s' := \inf(r > 0;\ B(\zeta; r) \ncong \mathbb{B}^2).$$

Then $s' \le s$ and we conclude $B(\zeta; s') \subseteq B(\zeta; s)$. Moreover, there is $\eta \in \partial B(\zeta; s')$ which is both a top and bottom point. We want to consider a loop based in η as follows:

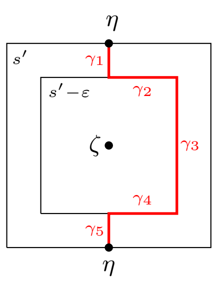


FIGURE 4.5: The loop γ.

For $0 < \varepsilon \ll s'$ we have $\overline{B(\zeta; s' - \varepsilon)} \cong \mathbb{D}^2$. Now consider the loop $\gamma = \gamma_1 \star \cdots \star \gamma_5$ from η to η along one side of the boundary of $B(\zeta; s' - \varepsilon)$ as shown in FIGURE 4.5. This path γ cannot be null-homotopic in $F^\star$ since $u\colon F^\star \longrightarrow \mathbb{R}$ is harmonic, but

$$\int_\gamma u = \underbrace{\left(\int_{\gamma_1} + \int_{\gamma_5}\right) u}_{\ll \int_{\gamma_3}} + \underbrace{\int_{\gamma_3} u}_{\neq 0} \neq 0.$$

$\square$

4.2.10 Corollary. *Let $\zeta, \eta \in F^\star$ and $r, s > 0$ with $B(\eta, s) \subseteq B(\zeta, r)$. Then we have*

$$B(\zeta; r) \cong \mathbb{B}^2 \implies B(\eta; s) \cong \mathbb{B}^2$$

.

4.2.11 Construction (Geodesic disck)**.** For $[\mathcal{N},\zeta] \in \mathfrak{E}$, we define the *injectivity radius*

$$\varrho(\zeta) \coloneqq \varrho[\mathcal{N},\zeta] \coloneqq \frac{\sup(r;\ B(\zeta;r) \cong \mathbb{B}^2)}{2}$$

and the *geodesic disk* by $G(\zeta) \coloneqq \overline{B(\zeta;\varrho(\zeta))}$. We obtain a disk bundle, the *geodesic bundle* $(\mathbb{D}^2,\mathbb{S}^1) \longrightarrow (\mathfrak{G},\partial\mathfrak{G}) \longrightarrow \mathfrak{E}$, where we define the space $\mathfrak{G}$ by

$$\mathfrak{G} \coloneqq \{[\mathcal{N},\zeta,\eta] \in \mathfrak{E} \times_{\mathfrak{P}} \mathfrak{E};\ \eta \in G(\zeta)\} \subseteq \mathfrak{E} \times_{\mathfrak{P}} \mathfrak{E}.$$

Proof. (I) By Lemma 4.2.8, there are minimal and maximal radii with $B(\zeta;r) \cong \mathbb{B}^2$ and $\varrho(\zeta) > 0$. Since $G(\zeta) \subseteq B(\zeta, 2\cdot\varrho(\zeta)) \cong \mathbb{B}^2$, we get $G(\zeta) \cong \mathbb{D}^2$ and $\partial G(\zeta) \cong \mathbb{S}^1$.

(II) The subbundle $\mathfrak{G} \subseteq \mathfrak{E} \times_{\mathfrak{P}} \mathfrak{E}$ is well-defined since the transitions preserve values of u, as we mentioned in Remark 4.1.2.

(III) We see that the fibre varies continuously: For $\zeta,\eta \in F^\star$, consider $r \coloneqq 2\cdot\varrho(\zeta) - d(\zeta,\eta)$. By the triangular inequality, we get $B(\eta;r) \subseteq B(\zeta;2\cdot\varrho(\zeta))$. By Corollary 4.2.10, $B(\eta;r) \cong \mathbb{D}^2$ and thus $r < 2\cdot\varrho(\eta)$. We conclude

$$\varrho(\zeta) - \varrho(\eta) < \frac{d(\zeta,\eta)}{2}.$$

For symmetry reasons, $|\varrho(\zeta) - \varrho(\eta)| \leq \frac{d(\zeta,\eta)}{2}$ and $\varrho\colon \mathfrak{E} \longrightarrow \mathbb{R}_{>0}$ is continuous. □

4.2.12 Construction. We call $e \in \partial G(\zeta)$ *vertex* of the disk if the u-path describing it changes the direction (from $\mathrm{d}u(\gamma') = 0$ to $\mathrm{d}u(i\cdot\gamma') = 0$ or vice versa). Let $\operatorname{vert}(\zeta)$ be the number of vertices. If $G(\zeta)$ contains ν critical points (counting multiplicity), then

$$\operatorname{vert}(\zeta) = 4\cdot(\nu+1).$$

Proof. By induction over the number k of critical points *without* multiplicity. Here, the case "$k=0$" is clear, because $G(\zeta)$ has obviously four vertices. For "$k-1 \longrightarrow k$" consider k critical points $S_1,\dots,S_k$ with $\operatorname{mult}(S_l) = \nu_l$, and we assume our formula to be true for the case where $G(\zeta)$ contains $S_1,\dots,S_{k-1}$. Now if we add the critical point S_k, there are exactly ν_k+1 copies of S_k in the slit picture. Since $G(\zeta)$ is homeomorphic to $\mathbb{D}^2$, on the part of the boundary around each of these copies, we have four vertices. The disk around one copy of S_k is not new since it comes from the old disk. Therefore,

$$\operatorname{vert}(\zeta) = 4\cdot(\nu_k+1) - 4 + 4\cdot\left(\sum_{l=1}^{k-1}\nu_l + 1\right) = 4\cdot\left(\sum_{l=1}^{k}\nu_l+1\right).$$
□

4.2.13 Example. Consider $g=0$ and $m=2$ and the slit configuration $(\langle 2,3\rangle \mid \langle 1,4\rangle)$. Then the following maximal disk ($r = 2-\varepsilon$) has 12 vertices:

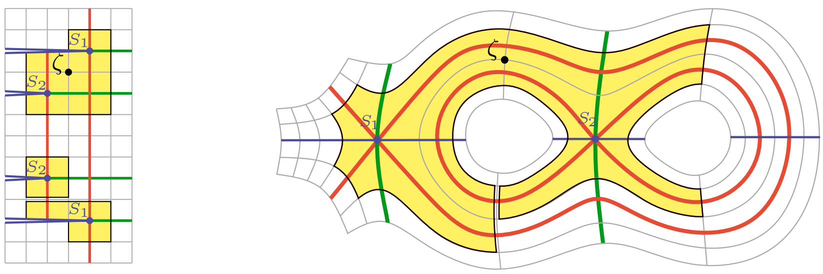


Figure 4.6: A disk containing two critical points of multiplicity 1 has 12 vertices.

4.2.14 Construction. We can give the construction of an explicit contraction

$$H\colon \mathfrak{G} \times [0,1] \longrightarrow \mathfrak{G}, ([\mathcal{N},\zeta,\eta],t) \longmapsto H_\zeta(\eta,t) = \gamma_{\zeta,\eta}(t)$$

with the following useful properties:

(I) The contraction ends in ζ, i. e. $H_\zeta(\zeta,1) = \zeta$,
(II) The contraction has constant speed, i. e. $H_\zeta(\partial G(\zeta), 1-t) = \partial B(\zeta; t\cdot\varrho(\zeta))$,
(III) $\gamma_{\zeta,\eta}\colon [0,1] \longrightarrow G(\zeta)$ is piecewise smooth.

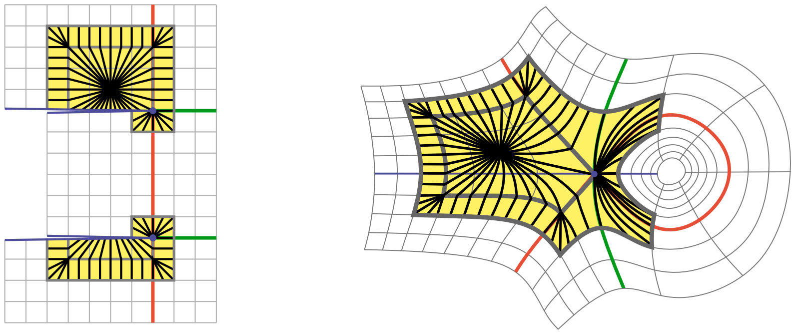


FIGURE 4.7: The paths of the explicit disk contraction constructed below.

STEP 1. Let $r > 0$ and $\zeta \in F^\star$. The construction of the disk $B(\zeta;r)$ yields a sequence $\zeta = B^0 \subseteq \cdots \subseteq B^K = B(\zeta;r)$ such that $\overline{B^{k+1}} \setminus \overline{B^k}$ does not contain any critical point. Since ∂B^k is compact, for each $\eta \in \overline{B^{k+1}} \setminus \overline{B^k}$, there is a nearest $\vartheta \in \partial B^k$ satisfying

$$d(\zeta,\eta) = d(\zeta,\vartheta) + d(\vartheta,\eta).$$

If we add that, if possible, $u(\vartheta) = u(\eta)$ or $v(\vartheta) = u(\vartheta)$, then $\vartheta[\eta] = \vartheta_\eta$ becomes unique and we get a continuous function $\overline{B^{k+1}} \setminus \overline{B^k} \longrightarrow \partial B^k, \eta \longmapsto \vartheta_\eta$.

STEP 2. We may assume that $u(\eta) \le u(\vartheta_\eta)$ and $v(\eta) \le u(\vartheta_\eta)$. Consider the rectangle

$$R := \Big\{u(\eta) \le u \le u(\vartheta_\eta)\Big\} \cap \Big\{v(\eta) \le v \le v(\vartheta_\eta)\Big\} \subseteq G(\zeta).$$

Then $\varphi := (u,v)$ is a homeomorphism $R \longrightarrow [u(\eta),u(\vartheta_\eta)] \times [v(\eta),v(\vartheta_\eta)]$. We define

$$\alpha_\eta\colon [0,1] \longrightarrow R, t \longmapsto \varphi^{-1}\Big(t\cdot\varphi(\vartheta_\eta) + (1-t)\cdot\varphi(\eta)\Big)$$

and parametrise α_η by arc-length with respect to the metric on $T^\perp\mathfrak{E}$ and rescale it again linearly to $[0,1]$. Then we get $\|\alpha'_\eta\| \equiv d(\eta,\vartheta_\eta)$.

STEP 3. Now let $\eta \in G(\zeta)$ be arbitrary. There is a $k \ge 0$ with $\eta \in \overline{B^{k+1}} \setminus \overline{B^k}$. Now we recursively define $\vartheta_1 := \eta$ and $\vartheta_{l+1} := \vartheta[\vartheta_l]$ and furthermore, let $t_l := d(\eta,\vartheta_{l+1})\cdot d^{-1}(\eta,\zeta)$. Then $\vartheta_{k+1} = \zeta$ and $0 = t_0 < \cdots < t_k = 1$. Define $\gamma_{\zeta,\eta} := \alpha_{\vartheta_1} \star \cdots \star \alpha_{\vartheta_k}$; each α_{ϑ_l} linearly rescaled to the intervall $[t_{l-1},t_l]$. Then $\gamma_{\zeta,\eta}$ is parametrised by distance, which means $\|\gamma_{\zeta,\eta}\| \equiv d(\eta,\zeta)$. This gives rise to the desired global fibre-wise disk contraction

$$H_\zeta\colon \mathfrak{G} \times [0,1] \longrightarrow \mathfrak{G}, ([\mathcal{N},\zeta,\eta],t) \longmapsto \gamma_{\zeta,\eta}(t).$$

4.2.15 Construction. We choose a Riemannian metric on $L|_{\mathfrak{E}} := T^{\perp}\mathfrak{E} \longrightarrow \mathfrak{E}$ and consider the vertical tangential disk bundle $(\mathbb{D}^2, \mathbb{S}^1) \longrightarrow (\mathbb{D}L|_{\mathfrak{E}}, \mathbb{S}L|_{\mathfrak{E}}) \longrightarrow \mathfrak{E}$. There is a bundle morphism $\Phi\colon (\mathfrak{G}, \partial\mathfrak{G}) \longrightarrow (\mathbb{D}L|_{\mathfrak{E}}, \mathbb{S}L|_{\mathfrak{E}})$ such that for each $\eta \in G(\zeta)$, we have

$$\begin{aligned} \eta \in \partial G(\zeta) &\iff \Phi(\zeta, \eta) \in \mathbb{S}^1, \\ \eta = \zeta &\iff \Phi(\zeta, \eta) = 0. \end{aligned}$$

We construct the morphism Φ explicitly in separate steps, starting with a preliminary map $\widetilde{\Phi}$ which has nearly the correct property, but have some direction "jumps".

STEP 1. We consider the derivative of the contraction paths

$$\widetilde{\Phi}\colon \mathfrak{G} \longrightarrow \mathbb{D}L|_{\mathfrak{E}}, \ [\mathcal{N}, \zeta, \eta] \longmapsto -\frac{\gamma'_{\zeta,\eta}(1)}{\varrho(\zeta)},$$

which gives us the negative velocity of the contraction path when arriving ζ. This morphism has the nice property that $\|\widetilde{\Phi}(\zeta, \eta)\| = d(\zeta, \eta)$ (in particular, $\|\widetilde{\Phi}\|$ is continuous), but is not continuous near critical points. The area "behind" a critical point always yields the same direction until finally the critical point is arrived; then suddenly everything is visible:

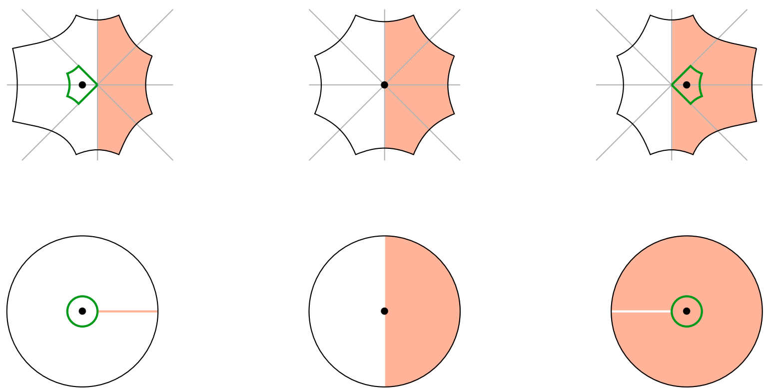


FIGURE 4.8: A range of directions is not seen until we arrive the critical point.

The basic aim of the following construction is to artificially correct our visible field behind the critical points in order to achieve a different schematic picture:

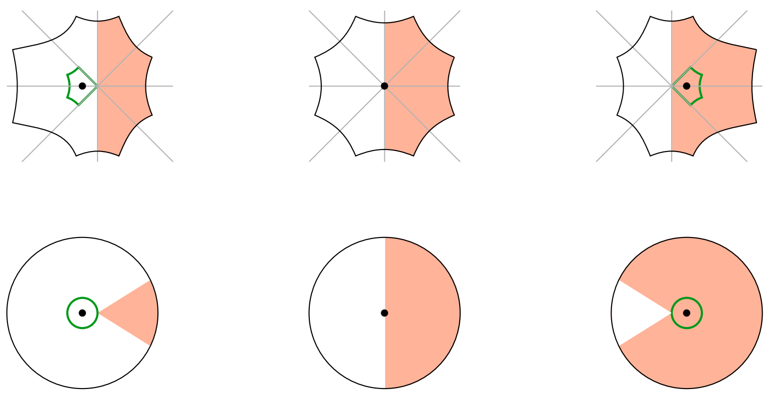


FIGURE 4.9: The corrected visible field behind the critical point extends continuously.

Step 2. For $[\mathcal{N}, S] \in \mathfrak{E}$ there is a maximal disk $D_{\mathcal{N}}(S) \subseteq F^\star$ and a $r_{\mathcal{N}}(S) > 0$ such that

$$\exp\colon T_S F^\star \supseteq B(0; r_{\mathcal{N}}(\zeta)) \longrightarrow D_{\mathcal{N}}(\zeta)$$

is an isomorphism. Note that r and D depend continuously on $(\mathcal{N}, S)$. For each $\zeta \in D(S)$ there is a unique Riemannian geodesic $\beta_{S,\zeta}\colon [0,1] \longrightarrow F^\star$ from S to ζ. Since the Riemannian metric induces a Levi–Civita connection on $T^\perp\mathfrak{E}$, this construction gives us a parallel transport between the corresponding tangent spaces of S and of ζ by

$$P[S,\zeta]\colon T_S^\perp \mathfrak{E} \longrightarrow T_\zeta^\perp \mathfrak{E}.$$

Step 3. In each fibre $\mathfrak{E}_{\mathcal{N}} \cong F^\star$, let $S_1, \dots, S_{k(\mathcal{N})}$ be the critical points without multiplicity. There are maps $\varphi_1, \dots, \varphi_{k(\mathcal{N})}\colon \mathfrak{E}_{\mathcal{N}} \longrightarrow [0,1]$ with the following properties (recall that $\mathfrak{U} := \mathfrak{U}_0 \subseteq \mathfrak{E}$ is the subspace of all points $[\mathcal{N}, \zeta]$ where ζ is not critical):

- $\overline{\operatorname{supp}(\varphi_i)} \subseteq D(S_i)$
- The map $\varphi\colon \mathfrak{E} \longrightarrow [0,1], [\mathcal{N},\zeta] \longmapsto \sum_{i=1}^{k(\mathcal{N})} \varphi_i(\zeta)$ is continuous and $\varphi\big|_{\mathfrak{U}} \equiv 1$.

An explicit way to construct them is the following: For $\zeta \in D(S)$ let $d_{\mathrm{Riem}}(S,\zeta)$ be the length of the Riemannian geodesic with respect to the Riemannian metric. For each $1 \le i \le k(\mathcal{N})$ define the preliminary functions $\widetilde{\varphi}_i$ by

$$\widetilde{\varphi}_i(\zeta) := \begin{cases} 1 - \frac{2 \cdot d_{\mathrm{Riem}}(S_i,\eta)}{r_{\mathcal{N}}(S_i)} & \text{for } 2 \cdot d_{\mathrm{Riem}}(S_i,\eta) < r_{\mathcal{N}}(S_i), \\ 0 & \text{else.} \end{cases}$$

Since D depends continuously on $[\mathcal{N}, S]$, the set $\mathfrak{D} := \bigcup_{\mathcal{N}} \bigcup_{i=1}^{k(\mathcal{N})} D_{\mathcal{N}}(S_i) \subseteq \mathfrak{E}$ is an open neighbourhood of $\mathfrak{V}_1$. Therefore, $(\mathfrak{U}, \mathfrak{D})$ is an open covering and there is a partition $(\varepsilon_{\mathfrak{U}}, \varepsilon_{\mathfrak{D}})$ of unity subordinated to $(\mathfrak{U}, \mathfrak{D})$. Finally, define $\varphi_i\colon \mathfrak{E}_{\mathcal{N}} \longrightarrow [0,1]$ by

$$\varphi_i := \frac{\widetilde{\varphi}_i \cdot \varepsilon_{\mathfrak{D}}}{\widetilde{\varphi}_1 + \cdots + \widetilde{\varphi}_{k(\mathcal{N})}}.$$

Step 4. For each $\mathcal{N} \in \mathfrak{P}$ define the scaling functions $\widetilde{a}_1, \dots, \widetilde{a}_{k(\mathcal{N})}\colon \mathfrak{E}_{\mathcal{N}} \longrightarrow [0,1]$ by

$$\widetilde{a}_i(\zeta,\eta) := \begin{cases} \max\left(0, \frac{d(\zeta,\eta) - d(\zeta,S_i)}{\varrho(\zeta) - d(\zeta,S_i)}\right) & \text{for } S_i \in G(\zeta)^\circ, \\ 0 & \text{else,} \end{cases}$$

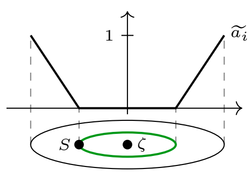


Figure 4.10

which detects everything behind the critical point S_i. Now set the actual factor to be $a_i(\zeta,\eta) := \varphi_i(\zeta) \cdot \widetilde{a}_i(\zeta,\eta)$ and the morphism $\Psi\colon \mathfrak{G} \longrightarrow L\big|_{\mathfrak{E}}$ to be

$$\Psi(\mathcal{N},\zeta,\eta) := \left(1 - \sum_{i=1}^{k(\mathcal{N})} a_i(\zeta,\eta)\right) \cdot \widetilde{\Phi}(\zeta,\eta) + \sum_{i=1}^{k(\mathcal{N})} a_i(\zeta,\eta) \cdot P[S_i,\zeta]\left(\widetilde{\Phi}(S_i,\zeta)\right).$$

Then Ψ is continuous and since only η behind the S_i are affected, the summands cannot cancel each other. Hence, we can rescale Ψ by $\widetilde{\Phi}$ and finally obtain the desired Φ.

4.3 The scanning section

As suggested in Motivation 4.1.10, we want to scan for critical points lying inside $G(\zeta)$. This finally gives a section $\omega\colon \mathfrak{F} \longrightarrow \mathrm{SP}^h T^\perp \mathfrak{F}$ and we are interested in $\omega^*\chi \in H^2(\mathfrak{F})$.

4.3.1 Construction. For the disk bundle $(\mathbb{D}^2, \mathbb{S}^1) \longrightarrow (\mathfrak{G}, \partial\mathfrak{G}) \longrightarrow \mathfrak{E}$ we have transitions $t_{ij}\colon U_{ij} \longrightarrow \mathrm{Homeo}(\mathbb{D}^2, \mathbb{S}^1)$. For each $\zeta \in U_{ij}$, we get homeomorphisms $s_{ij}(\zeta)$ defined by

$$\begin{array}{ccc} \mathrm{SP}^h(F, F \setminus G(\zeta)) & \xrightarrow{\;s_{ij}(\zeta)\;} & \mathrm{SP}^h(F, F \setminus G(\zeta)) \\ \| & & \| \\ \mathrm{SP}^h(G(\zeta)/\partial G(\zeta)) & \xrightarrow[\mathrm{SP}^h(t_{ij}(\zeta))]{\cong} & \mathrm{SP}^h(G(\zeta)/\partial G(\zeta)) \end{array}$$

Since SP^h is a continuous functor, this yields new transitions $s_{ij}\colon U_{ij} \longrightarrow \mathrm{Homeo}(\mathbb{C}P^k)$ and we obtain a new fibre bundle $\mathrm{SP}^h\mathfrak{G} \longrightarrow \mathfrak{E}$ with fibre $\mathrm{SP}^h(F, F \setminus G(\zeta)) \cong \mathbb{C}P^h$.

4.3.2 Remark. The morphism $\Phi\colon \mathfrak{G} \longrightarrow \mathbb{D}L|_{\mathfrak{E}}$ from Construction 4.2.15 yields

$$\mathrm{SP}^h\Phi\colon \mathrm{SP}^h\mathfrak{G} \longrightarrow \mathrm{SP}^h(\mathbb{D}/\mathbb{S})L|_{\mathfrak{E}} \cong \mathrm{SP}^h L|_{\mathfrak{E}}^+,$$

which inherits the following two crucial properties from Φ:

(i) Since $\Phi(\zeta, \eta) = 0$ if and only if $\eta = \zeta$, the map $\mathrm{SP}^h\Phi$ preserves the multiplicity of the center of the disk, which means we get

$$\mathrm{mult}_{\mathrm{SP}^h\Phi(\Theta)}(0) = \mathrm{ord}_\Theta(\zeta).$$

(ii) Since $\Phi(\zeta, \eta) \in \mathbb{S}^1$ if and only if $\eta \in \partial G(\zeta)$, the map $\mathrm{SP}^h\Phi$ preserves the degree of the considered configuration in $\mathrm{SP}^h\mathfrak{G}$, which means we get

$$|\mathrm{SP}^h\Phi(\Theta)| = |\Theta|.$$

4.3.3 Construction (The scanning section)**.** We have the interesting section

$$\mathfrak{E} \longrightarrow \mathrm{SP}^h\mathfrak{G},\ [\mathcal{N}, \zeta] \longmapsto \sum_{i=1}^k \nu_i \cdot S_i \in \mathrm{SP}^h(F, F \setminus G(\zeta))$$

where $S_1, \ldots, S_k$ are the critical points of the potential u, with multiplicities $\nu_1, \ldots, \nu_k$. Postcomposing with $\mathrm{SP}^h\Phi$ gives us $\widetilde{\omega}\colon \mathfrak{E} \longrightarrow \mathrm{SP}^h L|_{\mathfrak{E}}$, which can be extended on $\mathfrak{F}$ by $\omega(Q) \coloneqq \omega(P_i) = \emptyset$. This finally gives us the *scanning section*

$$\omega\colon \mathfrak{F} \longrightarrow \mathrm{SP}^h L.$$

Proof. We show that for each $[\mathcal{N}] \in \mathfrak{P}$, there is a neighbourhood $U \subseteq F$ around Q resp. P_i such that $\omega(\zeta) = \emptyset$ for each $\zeta \in U$: For each critical point S_i, there is a radius $r_i > 0$ such that $B(S_i; r_i)$ contains a non-trivial loop in S_i. Consider $r \coloneqq \max(r_1, \ldots, r_k)$. Since $\{S_1, \ldots, S_k\}$ is finite, there is a $\zeta_0 \in F^\star$ and a $R > 0$ such that $S_1, \ldots, S_k \subseteq B(\zeta_0; R)$ and $d(S_i, \partial B(\zeta_0; R)) \geq r$. Let $\zeta \in F^\star \setminus \overline{B(\zeta_0; R)}$. If there is a $S_i \in G(\zeta)$, then

$$2 \cdot \varrho(\zeta) \geq 2 \cdot d(S_i, \zeta) > r + d(S_i, \zeta)$$

and $B(S_i; r_i) \subseteq B(S_i; r) \subseteq B(\zeta; 2 \cdot \varrho(\zeta))$. Then $B(\zeta; 2 \cdot \varrho(\zeta))$ would contain a non-contractible loop, which is not possible. Hence $G(\zeta)$ contains no critical point. $\square$

4.3.4 Remark. The initial idea for the scanning section $\omega : \mathfrak{F} \longrightarrow \mathrm{SP}^h L$ comes from Carl-Friedrich Bödigheimer and has the following geometric interpretation:

(I) Since $G(\zeta)$ is contractible, there is a new holomorphic completion $\widetilde{v}$ of u in $G(\zeta)$ and $p_\zeta := u + i \cdot \widetilde{v}\colon G(\zeta) \longrightarrow \mathbb{C}$ is holomorphic. In fact it is a branched covering of the square $[-\varrho(\zeta), \varrho(\zeta)]^2 \subseteq \mathbb{C}$ with the critical points as branch points.

(II) By an adequate choice of a chart for $G(\zeta)$, we may consider p_ζ as a complex polynomial with $p_\zeta(0) = 0$ whose critical points (i. e. roots of zero of the derivation) are exactly the critical points of u. By identifying $\mathrm{SP}^h T_\zeta F \cong \mathbb{P}\,\mathrm{Pol}^h\, T_\zeta F$, we have $\omega(\zeta) = p'_\zeta$.

The interpolating polynomial has contour lines corresponding to our metric grid. In our Example 4.2.4, after identifying $T_\zeta F \cong \mathbb{C}$ via the chart, we want p_ζ with one a critical point $S \in \mathbb{C}$ of multiplicity 1 such that $p_\zeta(S) = 2 - i$. A possible polynomial is given by

$$\widetilde{p}_\zeta(z) := \int_0^z (w-1)\,\mathrm{d}w = z^2 - 2z \qquad \text{and} \qquad p_\zeta(z) := \frac{2-i}{\widetilde{p}(1)} \cdot \widetilde{p}_\zeta(z).$$

The contour lines of the real and the imaginary parts of p_ζ are shown in Figure 4.11.

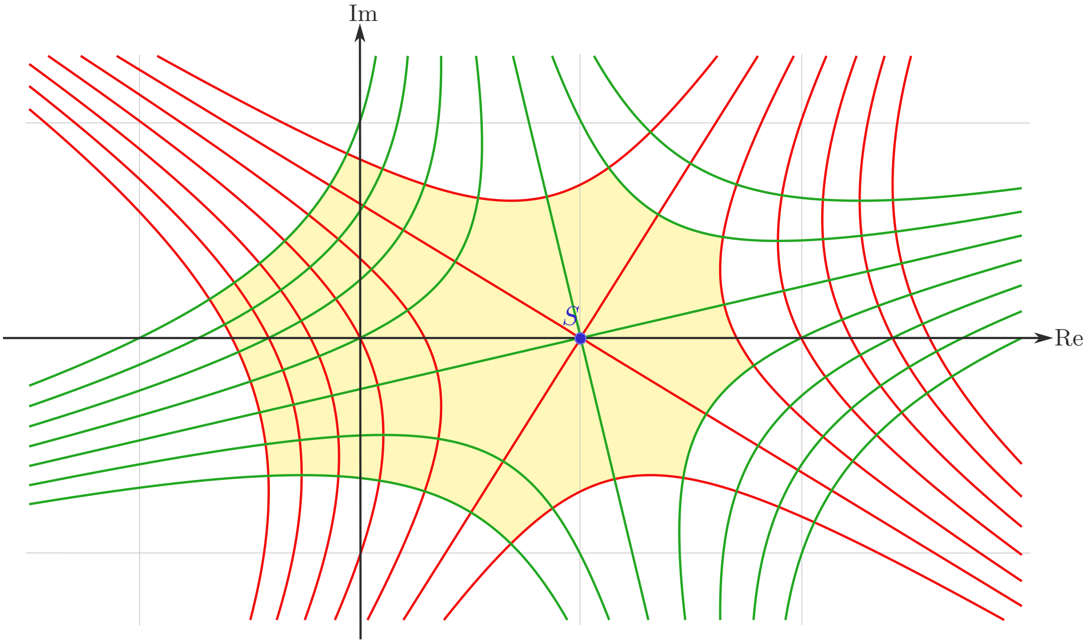


Figure 4.11: The contour lines of the polynomial interpolating our above conditions.

4.3.5 Preparation. Let $\mathbb{S}^1 \longrightarrow S \xrightarrow{\pi} B$ be an $\mathbb{S}^1$-bundle. Consider the $\mathbb{R}P^1$-bundle

$$\mathbb{S}^1/\mathbb{Z}^\times \longrightarrow S/\mathbb{Z}^\times \xrightarrow{\varrho} B$$

If ϱ has a global section $v\colon B \longrightarrow S/\mathbb{Z}^\times$, then the Euler class $e(S) \in H^2(B)$ has order 2: Consider the fibre-wise cone $(\mathbb{D}^2, \mathbb{S}^1) \longrightarrow (D, S) \longrightarrow B$ and the two-fold zero section $s := s_0 \oplus s_0\colon B \longrightarrow \mathrm{SP}^2(D, S)$. We have a homotopy between s and the empty section.

$$\begin{aligned} H\colon B \times [0,1] &\longrightarrow \mathrm{SP}^2(D,S), \\ (x,t) &\longmapsto t \cdot \Big(v(x) \oplus (-v(x))\Big). \end{aligned}$$

Therefore, $0 = s^*\chi = s_0^*\chi + s_0^*\chi = 2 \cdot e$. The crucial point here is the fact that v is only well-defined up to sign, but the section always uses the combination of both possibilities.

4.3.6 Lemma. *Let M be a manifold and $N \hookrightarrow M$ a subvariety of codimension k. Then the inclusion $\jmath\colon M \setminus N \hookrightarrow M$ induces an isomorphism in H^i for $i \leq k-2$.*

Proof. (I) Let $i \leq k-1$, $|\mathfrak{X}|$ an i-dimensional polyhedron and let $f\colon |\mathfrak{X}| \longrightarrow M$ be a morphism. By induction on the skeleta of $\mathfrak{X}$, there is a homotopy $f \simeq f'$ such that $\mathrm{Im}(f') \subseteq M \setminus N$. By the homotopy invariance of the homology groups we get that $\jmath^*\colon H_i(M \setminus N) \longrightarrow H_i(M)$ is surjective for $n \leq k-1$.

(II) Let $n \leq k-2$ and $\sigma, \sigma'\colon \Delta^n \longrightarrow M$ be homologous, i. e. there is a chain $c \in S_{i+1}(M)$ with $\partial c = \sigma - \sigma'$. As before, there is a chain $\widetilde{c} \in S_{i+1}(M \setminus N)$ homologous to c, which means $\partial \widetilde{c} = \sigma - \sigma'$. We conclude that $\jmath_*\colon H_i(M \setminus N) \longrightarrow H_i(M)$ is also injective.

(III) Let $i \leq k-2$. We get by the universal coefficient theorem the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(M), \mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{Ext}(\jmath_*, \mathrm{id}_{\mathbb{Z}})"'] & H^i(M) \arrow[r] \arrow[d, "\jmath^*"'] & \mathrm{Hom}_{\mathbb{Z}}(H_i(M), \mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{Hom}(\jmath_*, \mathrm{id}_{\mathbb{Z}})"'] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(M \setminus N), \mathbb{Z}) \arrow[r] & H^i(M \setminus N) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(H_i(M), \mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

where the both rows are exact. Now the statement follows from the five lemma. □

4.3.7 Proposition. *As usual, let $L \coloneqq T^\perp \mathfrak{F}$ and $\chi \coloneqq \chi(L) \in H^2(L)$. Recall that, after a choice of a Riemannian metric, we have a section $\nabla u\colon \mathfrak{F} \longrightarrow L^+$. Then*

$$\omega^* \chi = -\nabla u^* \chi.$$

Proof. Consider $\mathfrak{U} \coloneqq \mathfrak{U}_1$ and the restriction $\mathrm{SP}^h L|_{\mathfrak{U}}$. In $\mathfrak{U}$, it is not possible that two critical points meet in ζ. By a rescaling of the injectivity radius ϱ_ζ by some $\varepsilon\colon \mathfrak{F} \longrightarrow (0,1)$ we may assume that ω sees no critical point of higher multiplicity. Hence we can homotopically replace ∇u and ω such that there are small charts φ_i around each $[\mathcal{N}, S_i] \in \mathfrak{U}$ with:

- In the chart ∂_x corresponds to the outgoing, ∂_y to the incoming gradient line in S_i.
- Outside the charts, nothing is seen, neither from ∇u nor from ω.
- Inside the charts, both ∇u and ω yield exactly one point in $\mathrm{SP}^h L_\zeta$.

There is no canonical choice of the direction ∂_x in ζ, it is only well-defined up to sign. In this chart, $\varphi_i(S_i) = 0$ and u is in halfed normal form, this means $u = \mathrm{Re}\left(\frac{z^2}{2}\right) = \frac{x^2-y^2}{2}$. Hence, we have as local coordinates for the sections ω and ∇u

$$\begin{aligned}
\omega(x,y) &= \Phi(\varphi_i(S_i) - \varphi_i(\zeta)) = -x\partial_x - y\partial_y, \\
\nabla u(x,y) &= x\partial_x - y\partial_y.
\end{aligned}$$

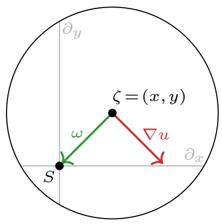


FIGURE 4.12: ∇u and ω.

Now consider the section $s \coloneqq \nabla u \oplus \omega\colon \mathfrak{U} \longrightarrow \mathrm{SP}^2 L|_{\mathfrak{U}}$. There is a homotopy from s to the empty section by moving $\omega(\zeta)$ and $\nabla u(\zeta)$ away from each other along ∂_x. Note that, as in PREPARATION 4.3.5, we use the direction ∂_x only up to sign. By PROPOSITION 3.3.12,

$$\nabla u^* \chi|_{\mathfrak{U}} + \omega^* \chi|_{\mathfrak{U}} = (\nabla u \oplus \omega)^* \chi|_{\mathfrak{U}} = s^* \chi|_{\mathfrak{U}} = 0.$$

Since $\mathfrak{F} \setminus \mathfrak{U} = \mathfrak{V}_2 \hookrightarrow \mathfrak{F}$ is a subvariety of codimension 4, the restriction $H^2(\mathfrak{F}) \longrightarrow H^2(\mathfrak{U})$ is injective by LEMMA 4.3.6 and we get the desired result $\omega^* \chi = -\nabla u^* \chi$. □

4.3.8 Remark. We want to compare $\nabla u^*\chi$ and $e := e(L)$, the Euler class of L. We restrict ourselves to the case where $m = 0$, so have no punctures. We already know:

(I) Let $s_0 \colon \mathfrak{F} \longrightarrow L$ be the zero section. Then $s_0^*\chi = e$ by COROLLARY 3.3.11.

(II) Let $U := \mathfrak{E} \subseteq \mathfrak{F}$. We know that $\nabla u|_U \simeq s_0|_U$ since ∇u does not attend the empty configuration here. Therefore, we get for the restrictions

$$\nabla u^*\chi|_U = s_0^*\chi|_U = e|_U.$$

(III) $K := \mathfrak{F} \setminus U$ is a subbundle of $\mathfrak{F}$ with the single point Q as the fibre and thus homeomorphic to $\mathfrak{P}$. We find a tubular neighbourhood $V \subseteq \mathfrak{F}$ of K which is geodesic around Q in each fibre. Then $(V \longrightarrow K) \cong (\mathbb{D}L|_K \longrightarrow \mathfrak{F}|_K)$ and since the dipole direction is a non-vanishing section, $V \longrightarrow K$ is trivial. Thus, $V \cong \mathfrak{P} \times \mathbb{D}^2$ and

$$U \cap V \simeq \partial V \cong \mathfrak{P} \times \mathbb{S}^1 \implies H^1(U \cap V) \cong H^1(\Gamma) \oplus \mathbb{Z}\langle[\mathbb{S}^1]\rangle.$$

(IV) Since we are near the dipole, $\nabla u|_V$ is homotopic to the infinity section, but on the other hand, $L|_V$ is trivial since we have a global section $V \longrightarrow L|_V, [\mathcal{N}, \zeta] \longmapsto P_\zeta(X)$ by the parallel transport of the dipole direction. Therefore, we get

$$\nabla u^*\chi|_V = 0 = s_0^*\chi|_V = e|_V.$$

Obviously, $\mathfrak{F} = U \cup V$. If we denote the inclusions by $\eta_U \colon U \hookrightarrow \mathfrak{F}$ and $\eta_V \colon V \hookrightarrow \mathfrak{F}$, the Mayer–Vietoris sequence of the excisive triad $(\mathfrak{F}; U, V)$ is of the form

$$H^1(\Gamma) \oplus \mathbb{Z}\langle[\mathbb{S}^1]\rangle \xrightarrow{\delta^*} H^2(\mathfrak{F}) \xrightarrow{\eta_U^* \oplus \eta_V^*} H^2(U) \oplus H^2(V)$$

We call the class $D := \delta^*[\mathbb{S}^1]$ *dipole class*, since it is induced by a loop around the dipole. Since $(\eta_U^* \oplus \eta_V^*)(\nabla u^* - s_0^*)\chi = 0$, there is a $\vartheta \in H^1(\Gamma) \oplus \mathbb{Z}\langle[\mathbb{S}^1]\rangle$ with $\nabla u^*\chi = e + \delta^*\vartheta$.

4.3.9 Proposition. *For $m = 0$ and using the above notation, we get $\nabla u^*\chi = e \pm 2 \cdot D$. In particular, we get $\omega^*\chi = -e + 2 \cdot D$.*

Proof. Recall that $\chi(L) \in H^2(L^+)$ is the Euler class $e(E)$ of the tautological line bundle $E := \lambda L^+ \longrightarrow L^+$. After translating the above Mayer–Vietoris situation into isomorphism classes of bundles as done in REMARK 3.1.12, we have to compare the two $\mathbb{S}^1$-bundles given by the pullbacks $S := \mathbb{S}L = s_0^*\mathbb{S}E$ and $S' := \nabla u^*\mathbb{S}E$ over $\mathfrak{F}$.

(I) Since $\nabla u|_U \simeq s_0|_U$, we know that their pullbacks are isomorphic, whence $S|_U \cong S'|_U$. We denote the isomorphism by $\Phi_U \colon S|_U \longrightarrow S'|_U$.

(II) We know that $S|_V$ is trivial, since the parallel transport $P_\zeta X$ of the dipole direction is a global section. Furthermore, $S'|_V$ is trivial since $\nabla u|_V \simeq s_\infty|_V$. Therefore, $S|_V \cong S'|_V$. We denote the isomorphism by $\Phi_V \colon S|_V \longrightarrow S'|_V$.

On the intersection $U \cap V$, let $\Phi := (\Phi_U|_{U\cap V})^{-1} \circ (\Phi_V|_{U\cap V}) \colon S|_{U\cap V} \longrightarrow S|_{U\cap V}$ as in CONSTRUCTION 3.1.10. We have two interesting sections of $S|_{U\cap V}$, namely the parallel transport of the (normed) dipole direction and the normed gradient of u,

$$s(\zeta) := P_\zeta X \qquad \text{and} \qquad t(\zeta) := \frac{\nabla u(\zeta)}{\|\nabla u(\zeta)\|}.$$

Note that the above trivialisation of $S|_V$ was induced by s, whereas the above trivialisation of $S'|_V$ restricts over $U \cap V$ to the trivialisation induced by the section $\Phi_U \circ t$. Since Φ_V is given by the composition of these two trivialisations, it translates these sections into each other, which means $\Phi_V \circ s = \Phi_U \circ t$. Thus, $\Phi \circ s = t$ and $\Phi = \Phi_\gamma$ where

$$\gamma\colon U \cap V \longrightarrow \mathbb{S}^1, \zeta \longmapsto \frac{\Phi(s(\zeta))}{s(\zeta)} = \frac{t(\zeta)}{s(\zeta)}.$$

Recall that $U \cap V \simeq \partial V \cong \mathfrak{P} \times \mathbb{S}^1$. Apparently the map γ just compares the indices $\operatorname{ind}(s, Q) = 0$ and $\operatorname{ind}(t, Q) = 2$ of the dipole and thus factors, up to homotopy, through

$$\begin{tikzcd}
\mathfrak{P} \times \mathbb{S}^1 \arrow[r, dash, "\simeq"] \arrow[d, "\mathrm{pr}"'] & U \cap V \arrow[r, "\gamma"] & \mathbb{S}^1 \\
\mathbb{S}^1 \arrow[urr, bend right, "z \longmapsto z^2"'] & &
\end{tikzcd}$$

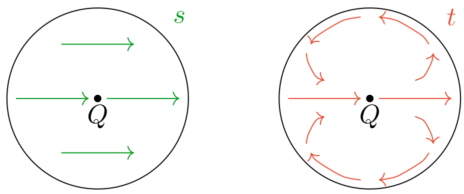


Figure 4.13: The two vector fields s and t.

Therefore, under the correspondence $[U \cap V, \mathbb{S}^1] \cong H^1(U \cap V) \cong H^1(\Gamma) \oplus \mathbb{Z}\langle[\mathbb{S}^1]\rangle$, the above γ is identified with $2 \cdot [\mathbb{S}^1]$. Since $S' = S \otimes \mathbb{S}_\gamma$, we finally get as in Remark 3.1.12

$$\nabla u^* \chi = e(S') = e(S \otimes \mathbb{S}_\gamma) = e(S) + e(\mathbb{S}_\gamma) = e + 2 \cdot D. \qquad \square$$

4.3.10 Lemma. *Let $(X; U, V)$ be an excisive triad, $\alpha \in H^i(U \cap V)$ and $\beta \in H^j(X)$ with $\beta|_V = 0$. Then $\delta^* \alpha \smile \beta = 0$ where δ^* is the Mayer–Vietoris connecting homomorphism.*

Proof. Denote the following homomorphisms:

- Let $\imath\colon U \cap V \longrightarrow U$ and $\jmath\colon U \longrightarrow X$ be the inclusions.
- Let $\eta\colon U \longrightarrow (U, U \cap V)$ and $\vartheta\colon X \longrightarrow (X, V)$ be the inclusions.
- Let $\varphi\colon (U, U \cap V) \longrightarrow (X, V)$ the excisive inclusion, i. e. φ^* is an isomorphism.
- Let $\varepsilon^*\colon H^{i+j}(U \cap V) \longrightarrow H^{i+j+1}(U, U \cap V)$ be the connecting homomorphism.

There is a $\gamma \in H^j(X, V)$ with $\beta = \vartheta^* \gamma$. Moreover, see [May99, ch. 14.5], we have a diagram

$$\begin{tikzcd}
H^\bullet(U \cap V) \arrow[r, "\delta^*"] \arrow[d, "\varepsilon^*"'] & H^{\bullet+1}(X) \\
H^{\bullet+1}(U, U \cap V) \arrow[r, "(\varphi^*)^{-1}"'] & H^{\bullet+1}(X, V) \arrow[u, "\vartheta^*"']
\end{tikzcd}$$

Let $\gamma' := \varphi^* \gamma \in H^j(U, U \cap V)$ and $\beta' := \eta^* \gamma' \in H^j(U)$. We know that $\jmath^* \vartheta^* = \eta^* \varphi^*$ and thus, $\beta' = \jmath * \beta$. According to [Bre93, ch. VI.4], the connecting homomorphism ε^* satisfies

$$\varepsilon^* \alpha \smile \gamma' = \varepsilon^*(\alpha \smile \imath^* \eta^* \gamma') = 0,$$

where the last identity follows from $\imath^* \eta^* = 0$. On the other hand, we get

$$\vartheta^* (\varphi^*)^{-1} \left(\varepsilon^* \alpha \smile \gamma'\right) = \delta^* \alpha \smile \left(\vartheta^* (\varphi^*)^{-1} \gamma'\right) = \delta^* \alpha \smile \beta.$$

$\square$

4.3.11 Corollary. *Let $m = 0$, i. e. we have no punctures. Then we get*

$$\omega^*\chi^s = \begin{cases} (-1)^s \cdot e^s & \textit{for } s \geq 2, \\ -e + 2 \cdot D & \textit{for } s = 1. \end{cases}$$

In particular, we obtain for the fibre transfer

$$\pi^!\omega^*\chi^s = \begin{cases} \kappa_{s-1} & \textit{for } s \geq 2, \\ \kappa_0 + 2 & \textit{for } s = 1. \end{cases}$$

Proof. Consider the above cover $\mathfrak{F} = U \cup V$ and let $D := \delta^*[\mathbb{S}^1] \in H^2(\mathfrak{F})$. Then

$$\omega^*\chi^s = (-e + 2 \cdot D)^s = (-1)^s \cdot e^s + \sum_{k=1}^{s} \binom{s}{k} \cdot (-1)^{s-k} \cdot 2^k \cdot \left(e^{s-k} \smile D^k\right).$$

We see $D\big|_V = 0$ since the Mayer–Vietoris sequence is exact, and $e\big|_V = 0$ since $T^\perp\mathfrak{F}\big|_V$ is trivial. By Lemma 4.3.10, $D^2 = 0$ and $e \smile D = 0$. This yields the first statement. Moreover, applying the fibre transfer yields

$$\pi^!\omega^*\chi^s = \begin{cases} \kappa_{s-1} & \text{for } s \geq 2, \\ \kappa_0 + 2 \cdot \pi^! D & \text{for } s = 1. \end{cases}$$

We finally see $\pi\big|^!_{U \cap V}(1 \times [\mathbb{S}^1]) = 1 \in H^0(\Gamma)$ for the transfer $\pi\big|^!_{U\cap V} : H^1(U \cap V) \longrightarrow H^0(\Gamma)$ of the $\mathbb{S}^1$-subbundle $U \cap V \simeq \mathbb{S}^1 \times \mathfrak{P} \longrightarrow \mathfrak{P}$ and conclude that

$$\pi^! D = \pi^! \delta^* \left(1 \times [\mathbb{S}^1]\right) = \pi\big|^!_{U\cap V} \left(1 \times [\mathbb{S}^1]\right) = 1.$$

□

4.3.12 Remark. We identify $H^0(\Gamma) \cong \mathbb{Z}$ and know that $\kappa_0 = -\chi(F_g) = 2g - 2 \in H^0(\Gamma)$. Thus, we know that $\pi^!\omega^*\chi = 2g$. We want to use the geometric description of our section ω in order to understand the other classes $\pi^!\omega^*\chi^s$ and thus, the Mumford classes κ_{s-1}.

5 Weierstraß complexes in cohomology

5.1 Duality on the surface bundle

We complete the surface bundle $\mathfrak{F} \longrightarrow \mathfrak{P}$ to a pair $(\overline{\mathfrak{F}}, \overline{\mathfrak{F}}')$ and extend the structure of (P, P') on it in order to get a notion of Poincaré–Lefschetz duality for $\mathfrak{F}$.

5.1.1 Construction. We know that every $\Sigma \in P$ yields a prescription how to glue a 2-dimensional complex. An explicit bi-cellular decomposition is constructed as follows:

(i) The $(0,0)$-skeleton is given by the following points:

- Let $m = m(\Sigma)$. Consider points $Q, P_1, \dots, P_m$.
- For each $1 \le j \le q$ and $1 \le i \le p-1$ consider two points $(j,i)^*$ and $(j,i)_*$ which are the "top right" and the "bottom right" corner of the rectangle $R_{j,i}$ resp. the "top left" and the "bottom left" corner of the rectangle $R_{j-1,i}$. Consider also $(j,p)_*$ and $(j,0)^*$. Now we write $[j,i]_*$ resp. $[i,j]^*$ by identifying

$$(j, \sigma_{j-1}(i))_* \sim (j,i)^* \sim (j, \sigma_j(i))_*.$$

(ii) The $(0,1)$-skeleton is given by the following edges:

- For $1 \le j \le q$ and $1 \le j \le p-1$, we have vertical edges $v_{j,i}$ from $[j,i]_*$ to $[j,i]^*$ which are considered as the "right" edge of $R_{j,i}$ resp. the "left" edge of $R_{j-1,i}$.
- Define vertical edges $v_{j,p}$ from $[j,p]_*$ to Q and $v_{j,0}$ from Q to $[j,0]^*$.

(iii) The $(1,0)$-skeleton is given by the following edges:

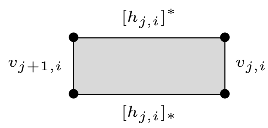
- Consider edges $(h_{j,i})^*$ from $[j+1,i]^*$ to $[j,i]^*$ and $(h_{j,i})_*$ from $[j+1,i]_*$ to $[j,i]_*$ which are the "top" and "bottom" edge of $R_{j,i}$. Write $[h_{j,i}]^*$ resp. $[h_{j,i}]_*$ by

$$(h_{j,i})^* \sim (h_{j,\sigma_j(i)})_*$$

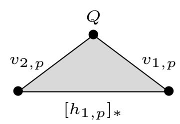
- We have certain horizontal edges outside our grid:
  - From $[1,i]_*$ to Q (which is the same as from $[1,i-1]^*$ to Q).
  - From Q to $[q,i]_*$ if $\nu_\Sigma(i) = 0$.
  - From P_k to $[q,i]_*$ if $\nu_\Sigma(i) = k$.

(iv) The $(1,1)$-skeleton is given by the following rectangles (resp. triangles and digons):

- For $1 \le j \le q-1$ and $1 \le i \le p-1$, glue $R_{j,i}$ canonically along



- Else, contract at least one of the faces of $R_{j,i}$ to Q or P_k, e. g. for $R_{1,p}$:



5.1.2 Remark. The above construction turns the glued surface $F(\Sigma)$ into a "generalised" bi-semisimplicial complex $\mathfrak{Z} = \mathfrak{Z}[\Sigma]$ where some of the boundaries are lower-dimensional. Our boundary operators are of the following form:

$$\begin{tikzcd} & \mathfrak{Z}_{1,1} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] \arrow[dd, dashed, Rrightarrow, "\text{or}"', "\text{or}"] & \\ \mathfrak{Z}_{1,0} \arrow[dr, Rightarrow] & & \mathfrak{Z}_{0,1} \arrow[dl, Rightarrow] \\ & \mathfrak{Z}_{0,0} & \end{tikzcd}$$

We will call the face operators e' and e'' (in contrast to d' and d'' in the complex P) and denote the general cells in $\mathfrak{Z}$ by Π. The unusual point is that we may lose more than one dimension when taking the face of a 2-cell. However, the usual formulae for simplicial homology work if we use the boundary operators (analogously for η'')

$$\eta'\Pi = \sum_{|e'_k\Pi|=|\Pi|-1} (-1)^k \cdot e'_k\Pi$$

This is immediately clear if we consider the complex as a finite cell complex and calculate their cellular chain using the factors prescribed by the mapping degree of the glueing maps. Note that for each two (p,q)-cells Σ and Σ' in P, there is a canonical identification between the (r,s)-cells of $\mathfrak{Z}[\Sigma]$ and $\mathfrak{Z}[\Sigma']$ (e. g. there are always $(p+1)\cdot(q+1)$ rectangles to glue, indexed canonically.) Only the glueing depends heavily on the entries of Σ resp. Σ'. Hence, we may write $\mathfrak{Z}^{p,q}_{r,s}$ for the (r,s)-cells and note that an explicit (p,q)-cell Σ has to be given in order to determine the faces e' and e'' of a given cell $\Sigma \times R_{kl}$.

5.1.3 Construction. We can complete the surface bundle $\mathfrak{F}$ to a $(3h+2)$-dimensional 4-semisimplicial complex $\overline{\overline{\mathfrak{F}}}$ in the following way:

(I) The set $\overline{\overline{\mathfrak{F}}}_{p,q,r,s}$ of (p,q,r,s)-cells is given by $P_{p,q} \times \mathfrak{Z}^{p,q}_{r,s}$.

(II) The last two boundary operators are canonically given the face operators e' and e'' using the glueing structure given by the cell Σ:

$$e'_k(\Sigma \times \Pi) := \Sigma \times e'_k\Pi \qquad \text{and} \qquad e''_l(\Sigma \times \Pi) := \Sigma \times e''_l\Pi.$$

(III) The first two boundary operators are a bit more complicated, we have to say what $d'_j\Pi \in \mathfrak{Z}^{p,q-1}$ is and then let $d'_j(\Sigma \times \Pi) := d'_j\Sigma \times d'_jR_{k,l}$ (analogously for d''_i). It is geometrically clear what to do; the explicit term for $(1,1)$-cells $R_{k,l}$ is given by

$$d'_jR_{k,l} := \begin{cases} R_{\mathbf{s}_j(k),l} & \text{for } j \neq k, \\ v_{k,l} & \text{for } j = k. \end{cases}$$

5.1.4 Remark. The complex $\overline{\overline{\mathfrak{F}}}$ has several properties similar to the completion P of $\mathfrak{P}$:

(I) We have a simplicial map $\overline{\pi}\colon \overline{\overline{\mathfrak{F}}} \longrightarrow P$, given by the projection onto the first two factors. This gives rise to a map between the geometric realisations $\overline{\pi}\colon \overline{\overline{\mathfrak{F}}} \longrightarrow |P|$.

(II) Consider the subspace $\overline{\overline{\mathfrak{F}}}' := \overline{\pi}^{-1}(|P'|) \subseteq \overline{\overline{\mathfrak{F}}}$. Then $\overline{\overline{\mathfrak{F}}} \setminus \overline{\overline{\mathfrak{F}}}' \cong \mathfrak{F}$ and $\pi = \overline{\pi}|_{\mathfrak{F}}$. Thus, $(\overline{\overline{\mathfrak{F}}}, \overline{\overline{\mathfrak{F}}}')$ is a relative manifold of dimension $3h+2$.

(III) The pair $(\overline{\overline{\mathfrak{F}}}, \overline{\overline{\mathfrak{F}}}')$ is compact and connected in the sense of Reminder 1.2.9.

5.1.5 Construction. We can extend the semisimplicial orientation system $\mathcal{O}$ from P to $\overline{\mathfrak{F}}$ by ignoring the second two factors, i. e. $\varepsilon_i(\Sigma \times \Pi) \coloneqq \varepsilon_i(\Sigma)$. We get a fundamental cycle by

$$u_{\overline{\mathfrak{F}},\overline{\mathfrak{F}}'} \coloneqq \sum_{\Sigma:(1,h,m)} \delta(\Sigma) \cdot \sum_{k,l} (\Sigma \otimes R_{k,l}) \in C_{2h,h,1,1}(\overline{\mathfrak{F}}, \overline{\mathfrak{F}}'; \mathcal{O}).$$

This gives a fundamental class $[\overline{\mathfrak{F}}, \overline{\mathfrak{F}}'] \in H_{3h+2}(\overline{\mathfrak{F}}, \overline{\mathfrak{F}}')$ and a Poincaré–Lefschetz isomorphism

$$\begin{aligned} \mathrm{PL}_{\mathfrak{F}} \colon H^\bullet(\mathfrak{F}) &\longrightarrow H_{3h+2-\bullet}(\overline{\mathfrak{F}}, \overline{\mathfrak{F}}'; \mathcal{O}), \\ \vartheta &\longmapsto \vartheta \frown [\overline{\mathfrak{F}}, \overline{\mathfrak{F}}']. \end{aligned}$$

Proof. We have to show that $u = u_{\overline{\mathfrak{F}},\overline{\mathfrak{F}}'}$ is a cycle, i. e. that $\partial' u, \partial'' u, \eta' u, \eta'' u = 0$. The last two differentials are easy since here we fix the configuration cell Σ and we know that $\sum_{k,l} R_{k,l} = 0$ is a cycle in $C_{1,1}(\mathfrak{Z}[\Sigma])$. We show that $\partial' u = 0$ as $\partial'' u$ follows analogously:

$$\partial' u = \sum_j (-1)^j \sum_{\Sigma:(1,h,m)} \delta(\Sigma) \cdot \sum_l \sum_{k \neq j} (d'_j \Sigma \otimes d'_j R_{k,l}) =: \sum_l u_l.$$

Now we use the definition of $d'_j R_{k,l}$ to get

$$\sum_{k \neq j} d'_j R_{k,l} = \sum_{k<j} R_{k,l} + \sum_{k>j} R_{k-1,l} = \sum_{k<q} R_{k,l} =: R_l.$$

The crucial point is that this term no longer depends on j. Therefore, we get

$$u_l = \sum_j (-1)^j \sum_{\Sigma:(1,h,m)} \delta(\Sigma) \cdot (d'_j \Sigma \otimes R_l) = \underbrace{\partial' u_{P,P'}}_{=0} \otimes R_l. \qquad \square$$

5.1.6 Construction (Lifted Weierstraß constructions)**.** Let (s, h, m) be a triple.

(I) Let Σ be a Weierstraß cell of type (s, h, m). Since all critical points are corners of our rectangles $R_{k,l}$, we can look at cells $\Sigma \otimes \zeta \in \overline{\mathfrak{F}}$ where $\mathrm{mult}(\zeta) = s$.

(II) Let $V_s \subseteq \overline{\mathfrak{F}}$ be the subcomplex spanned by all those cells and let $V'_s \coloneqq V_s \cap \overline{\mathfrak{F}}$. Then $(V_s, V'_s) \subseteq (\overline{\mathfrak{F}}, \overline{\mathfrak{F}}')$ is a relative subcomplex of dimension $(2h+1-s, h+1-s, 0, 0)$.

(III) Consider the simplicial cycle

$$v_s \coloneqq \sum_{\Sigma:(s,m)} \delta(\Sigma) \cdot \sum_{\mathrm{mult}(\zeta) \geq s} (\Sigma \otimes \zeta) \in C_{2h+1-s,h+1-s,0,0}(\overline{\mathfrak{F}}, \overline{\mathfrak{F}}'; \mathcal{O}).$$

This yields to homology classes $[v_s] \in H_{3h+2-2s}(\overline{\mathfrak{F}}, \overline{\mathfrak{F}}'; \mathcal{O})$ and we obtain

$$\overline{\pi}_*[v_s] = \begin{cases} h \cdot [w_s] & \text{for } s = 1, \\ [w_s] & \text{for } s \geq 2. \end{cases}$$

Proof. It is clear that v_s is a cycle: $\partial' v_s = 0$ and $\partial' v_s = 0$ because of our sign system and $\eta' v_s = 0$ and $\eta' v_s = 0$ for dimensional reasons. Moreover, we consider Weierstraß top cells for the partition $(s, 1, \ldots, 1)$, so over Σ, there are h critical points of multiplicity s for $s = 1$ and only one critical point of multiplicity s for $s \geq 2$. Thus, the statement follows from

$$\overline{\pi}_*[v_s] = \sum_{\Sigma:(s,h,m)} \delta(\Sigma) \cdot \sum_{\mathrm{mult}(\zeta)=s} \pi(\Sigma \otimes \zeta) = \sum_{\Sigma:(s,h,m)} \delta(\Sigma) \cdot \#\{\mathrm{mult}(\zeta) = s\} \cdot \Sigma. \qquad \square$$

5.2 Weierstraß and Mumford classes

We want to understand the Poincaré–Lefschetz duals $\mathrm{PL}_{\mathfrak{P}}^{-1}[w_s]$ of the Weierstraß classes. It turns out that they equal the $\pi^! \omega^* \chi^s \in H^{2s-2}(\Gamma)$ studied in Chapter 4.3.

5.2.1 Reminder. For a complex line bundle $\mathbb{C} \longrightarrow L \longrightarrow B$, the transitions of $E \coloneqq \mathrm{SP}^h L$ preserve the multiplicity of 0, Remark 3.3.3. We get subbundles $\mathbb{C}P^{h-s} \longrightarrow E^s \longrightarrow B$ by

$$E^s \coloneqq \{(x, \Theta) \in E;\, \mathrm{mult}_\Theta(0) \geq s\} \subseteq E.$$

We obtain a filtration $E = E^0 \supseteq \cdots \supseteq E^s \cong B$.

5.2.2 Motivation. As usual, let $L \coloneqq T^\perp \mathfrak{F} \longrightarrow \mathfrak{F}$. The crucial idea of this chapter is the combination of two simple observations:

(I) For our scanning section, we have $\omega^{-1}(E^s) = \mathfrak{V}_s$.
(II) For each $x \in \mathfrak{F}$, the class $i_x^*[E^s] \in H_{2h-2s}(E_x)$ is dual to $\xi^s \in H^{2s}(E_x)$.

We want to deduce a homological statement from this fact.

5.2.3 Remark. Let M be an m-dimensional complex and let $N \hookrightarrow M$ be a closed and oriented n-dimensional submanifold and subcomplex of M. For a simplex Σ, write $\Sigma : N$ if $\dim(\Sigma) = n$ and $\Sigma \subseteq N$. There is a sign system δ such that

$$\sum_{\Sigma : N} \delta(\Sigma) \cdot \Sigma \in C_n(M)$$

is a cycle and then the fundamental class $[N]_M \in H_n(M)$ is represented by it up to sign. If N is connected, then δ is uniquely determined up to one global sign.

5.2.4 Reminder (Transversality)**.** Let M be a closed and oriented m-dimensional manifold together with a fundamental class $[M] \in H_m(M)$ and Poincaré duality isomorphisms

$$\begin{aligned} \mathrm{PD}_M \colon H^k(M) &\longrightarrow H_{m-k}(M), \\ \vartheta &\longmapsto \vartheta \frown [M]. \end{aligned}$$

Due to [Bre93], we introduce the concept of the intersection product:

(I) Let $a \in H_i(M)$ and $b \in H_j(M)$ be two homology classes. Then we can define their *intersection product*, dual to the cup product, by

$$a \bullet b \coloneqq (\mathrm{PD}_M^{-1} a \smile \mathrm{PD}_M^{-1} b) \frown [M] \in H_{i+j-m}(M).$$

(II) Let $N \hookrightarrow M$ a closed and oriented submanifold. We define the *Thom class* of the embedding by $\tau_N^M \coloneqq \mathrm{PD}_M [N]_M \in H^{m-n}(M)$. We obtain

$$[N]_M \bullet [K]_M = \left(\tau_N^M \smile \tau_K^M\right) \frown [M].$$

(III) Let M be smooth and two submanifolds $N, K \hookrightarrow M$ smoothly embedded. N and K *intersect transversally*, write $N \pitchfork K$, if $T_pM = T_pN + T_pK$ for each $p \in N \cap K$. In this case, $N \cap K$ is again a submanifold, now of dimension $\dim(N) + \dim(K) - \dim(M)$, and we have $\tau_{K \cap N}^N = (\imath_N^M)^* \tau_K^M$. In particular, we get for the intersection product

$$[N \cap K]_M = [N]_M \bullet [K]_M.$$

5.2.5 Lemma. *Let $\mathbb{C} \longrightarrow L \longrightarrow B$ be a line bundle over a closed, oriented m-manifold B. Then $\varrho\colon E \coloneqq \mathrm{SP}^h L \longrightarrow B$ is a closed and oriented $(m+2h)$-manifold.*

(i) *$E^s \hookrightarrow \mathrm{SP}^h L$ is a $(m+2h-2s)$-dimensional submanifold and $\mathrm{PD}_E(\chi^s) = [E^s]_E$.*

(ii) *Let $N \subseteq E$ be a smooth n-dimensional submanifold and $\omega\colon B \longrightarrow E$ be a section such that $\omega(B) \pitchfork N$. Then $\omega^{-1}(N)$ is an $(n-2h)$-dimensional submanifold and*

$$\left(\mathrm{PD}_B \circ \omega^* \circ \mathrm{PD}_E^{-1}\right)[N]_E = [\omega^{-1}(N)]_B.$$

Proof. Statement "(i)": Consider the infinity section $s_\infty\colon B \longrightarrow E$ and $B_\infty \coloneqq s_\infty(B)$. Since $s_\infty^*\chi^s = 0$ for $s \geq 1$, we get $\mathrm{PD}_E^{-1}[B_\infty] = \chi^h \in H^{2h}(E)$. We show "$\mathrm{PD}_E(\chi) = [E^1]$": Let $\imath_x\colon E_x \hookrightarrow E$ be a fibre inclusion. Then obviously, $E_x \pitchfork E^1$ and we get

$$\xi = \mathrm{PD}_{E_x}^{-1}[E_x^1] = \tau_{E_x \cap E^1}^{E_x} = \imath_x^* \tau_{E^1}^E.$$

After applying Leray–Hirsch, $\imath_x^*\colon H^2(B) \oplus H^2(\mathbb{C}P^h) \longrightarrow H^2(\mathbb{C}P^h)$ is just the projection. Thus, there is a $\alpha \in H^2(B)$ with $\tau_{E_1}^E = \chi + \varrho^*\alpha$. Since E^1 and B_∞ are disjoint, we get

$$\mathrm{LH}(\alpha \otimes \xi^h) = \varrho^*\alpha \smile \chi^h = \tau_{E_1}^E \smile \chi^h = \mathrm{PD}_E^{-1}[B_\infty \cap E^1] = 0$$

Hence, $\alpha = 0$ and $\mathrm{PD}_E^{-1}[E^1] = \tau_{E_1}^E = \chi$. Now consider E^1 and E^s. Obviously, $E^s \subseteq E^1$, in particular, they do not intersect transversally. However we can alternatively consider the image $\widetilde{E}^s \coloneqq D(E^s)$ of the fibre-wise s^{th} derivative, defined as

$$\begin{tikzcd}
(\widehat{x}, [a_s : \cdots : a_h]) \arrow[r, mapsto] & \left(\widehat{x}, \left[s! \cdot a_s : \cdots : \frac{h!}{(h-s)!} \cdot a_h : 0 : \cdots : 0\right]\right) \\
S \times \mathbb{C}P^{h-s} \arrow[r] \arrow[d] & S \times \mathbb{C}P^h \arrow[d] \\
E^s \arrow[r, "D"'] & E
\end{tikzcd}$$

where $S \coloneqq \mathbb{S}L$ is the principal $\mathbb{S}^1$-bundle associated to L and we identify $E \cong S \times_{\mathbb{S}^1} \mathbb{C}P^h$. We have $[E^s] = [\widetilde{E}^s]$ and $\widetilde{E}^s$ intersects E^1 transversally and moreover, $\widetilde{E}^s \cap E^1 = D(E^{s+1})$. Thus, we get $[E^s] \bullet [E^1] = [E^{s+1}]$ and the main statement follows by induction as

$$\mathrm{PD}_E(\chi^{s+1}) = \mathrm{PD}_E(\chi^s \smile \chi) = \mathrm{PD}_E(\chi^s) \bullet \mathrm{PD}_E(\chi) = [E^s] \bullet [E^1] = [E^{s+1}].$$

Statement "(ii)": Consider the inclusion $\imath\colon \omega(B) \hookrightarrow E$. Then $\omega\colon B \longrightarrow E$ lifts over $\omega = \imath \circ \eta$ where $\eta\colon B \longrightarrow \omega(B)$ is a homeomorphism (in particular, $\deg(\eta) = 1$) and

$$\begin{tikzcd}
& H^{m+2h-n}(E) \arrow[dl, bend right, "\imath^*"'] \arrow[dr, bend left, "\omega^*"] & \\
H^{m+2h-n}(\omega(B)) \arrow[rr, "\eta^*", "\cong"'] \arrow[d, "\mathrm{PD}_{\omega(B)}"'] & & H^{m+2h-n}(B) \arrow[d, "\mathrm{PD}_B"] \\
H_{n-2h}(\omega(B)) \arrow[rr, "\eta_*", "\cong"'] & & H_{n-2h}(B).
\end{tikzcd}$$

Finally, we see for the left vertical path in the diagram

$$\left(\mathrm{PD}_{\omega(B)} \circ \imath^* \circ \mathrm{PD}_E^{-1}\right)[N] = \left(\mathrm{PD}_{\omega(B)} \circ \imath^*\right) \tau_N^E = \mathrm{PD}_{\omega(B)}\, \tau_{N \cap \omega(B)}^{\omega(B)} = [N \cap \omega(B)]_{\omega(B)}.$$

Thus, $\left(\mathrm{PD}_B \circ \omega^* \circ \mathrm{PD}_E^{-1}\right)[N] = \eta_*[N \cap \omega(B)]_{\omega(B)} = [\omega^{-1}(N)]_B$. $\square$

5.2.6 Remark. Instead of B being a closed manifold, B is the interior of a compact relative manifold (X, X'). We have to change the following subtleties, cf. [Spa94]:

(I) A subspace $(S, S') \subseteq (X, X')$ is called *relative n-dimensional submanifold* if $S \cap B \subseteq B$ is an n-dimensional submanifold. Then we have:

- (S, S') is a relative manifold itself and (after installing an orientation system) we get an embedded relative fundamental class $[S, S'] \in H_n(X, X'; \mathcal{O})$.
- Let X be triangulated, $X' \subseteq X$ be a subcomplex of codimension at least 1 and $(S, S') \subseteq (X, X')$ be a relative submanifold such that $S \subseteq X$ is a connected subcomplex. Then there is a (up to one global sign) unique sign system δ such that the following $u \in C_n(X, X'; \mathcal{O})$ is a cycle and $[u] = [S, S']$:

$$u := \sum_{\Sigma : S} \delta(\Sigma) \cdot \Sigma \in C_n(X, X'; \mathcal{O})$$

- Let B carry a smooth structure and let $(S, S'), (T, T') \subseteq (X, X')$ be smoothly embedded. We say S and T *intersect transversally* if $S \cap B$ and $T \cap B$ intersect transversally. Then $(S \cap T, S' \cap T')$ is again a relative submanifold and the above formulae for transversality hold.

(II) The bundle $\varrho\colon E = \mathrm{SP}^h L \longrightarrow B$ can be extended over X to the *contracted extension* (which is no longer a fibre bundle): Consider $Y := E \sqcup A$ with the topology where

$$W \subseteq Y \text{ open } :\Longleftrightarrow \begin{array}{l} \text{there is } U \subseteq X \text{ open and } V \subseteq E \text{ open} \\ \text{with } U \cap B = \varrho(V) \text{ and } W = V \sqcup (U \cap X'). \end{array}$$

We get a subspace $Y' := A \subseteq Y$ and a map $\overline{\varrho}\colon (Y, Y') \longrightarrow (X, X')$ with:

- $E \subseteq Y$ open, $\overline{\varrho}\big|_E = \varrho$ and for $x \in X'$ the fibre $\overline{\varrho}^{-1}(x)$ has one point.
- Y is compact and (Y, Y') is a relative manifold with interior E.
- Every section $\omega\colon B \longrightarrow E$ can be uniquely extended to $\overline{\omega}\colon X \longrightarrow Y$.

(III) Let $\mathcal{O}$ be an orientation system for (X, X'). Since $E \longrightarrow B$ is an orientable bundle, this extends to an orientation system for (Y, Y') and we get Poincaré–Lefschetz maps

$$\begin{aligned} \mathrm{PL}_E\colon H^\bullet(E) &\longrightarrow H_{m+2h-\bullet}(Y, Y'; \mathcal{O}), \\ \mathrm{PL}_B\colon H^\bullet(B) &\longrightarrow H_{m-\bullet}(X, X'; \mathcal{O}). \end{aligned}$$

Lemma 5.2.5 can now be generalised without any problems to the relative case:

5.2.7 Lemma. *Use the notion of (X, X') and (Y, Y') resp. B and E from above.*

(I) *We can complete $E^s \subseteq E$ trivially to $(T^s, T^{s\prime}) \subseteq (Y, Y')$ with $T^s \cap E = E^s$ and get classes $[T^s, T^{s\prime}] \in H_{m+2h-2s}(Y, Y'; \mathcal{O})$. Then*

$$\mathrm{PL}_E(\chi^s) = [T^s, T^{s\prime}].$$

(II) *Let $(T, T') \subseteq (Y, Y')$ be a smooth submanifold meeting $(\overline{\omega}(X), \overline{\omega}(X'))$ transversally. If $(S, S') \subseteq (X, X')$ is a submanifold with $S \setminus S^{-1} = \omega^{-1}(T \setminus T')$, then*

$$\left(\mathrm{PL}_B \circ \omega^* \circ \mathrm{PL}_E^{-1}\right)[T, T'] = [S, S'].$$

5.2.8 Proposition. *We have the dual correspondence* $\mathrm{PL}_{\mathfrak{F}}(\omega^*\chi^s) = [v_s]$.

Proof. Step 1. Consider our subcomplex $(V_s, V_s') \subseteq (\overline{\mathfrak{F}}, \overline{\mathfrak{F}}')$. Unfortunately, the interior $V_s \setminus V_s' = \mathfrak{V}_s \subseteq \mathfrak{F}$ is not a submanifold since it has branch points. However, all such branch points lie in V_{s+1} and $V_{s+1} \subseteq \overline{\mathfrak{F}}$ is a subcomplex of codimension $2s+2$. Hence, by Lemma 4.3.6, the inclusion $\mathfrak{U} := \mathfrak{U}_{s+1} \hookrightarrow \mathfrak{F}$ induces isomorphisms in cohomology up to degree $2s$ and we may equivalently show for the restrictions

$$\mathrm{PL}_{\mathfrak{F}}^{-1}[v_s]\Big|_{\mathfrak{U}} = \omega^*\chi^s|_{\mathfrak{U}}$$

Step 2. Consider the slightly altered situation where the base is $(X, X') := (\overline{\mathfrak{F}}, \overline{\mathfrak{F}}' \cup V_{s+1})$ and $B := X \setminus X' = \mathfrak{U}$. The bundle $\overline{\varrho}\colon (Y, Y') \longrightarrow (X, X')$ is the contracted extension of $\mathrm{SP}^h T^\perp B \longrightarrow B$ and $\overline{\omega}\colon X \longrightarrow Y$ is the unique extension of our scanning section on X. We have a submanifold given by $(S, S') = (V_s, V_s' \cup V_{s+1}) \subseteq (X, X')$. By Remark 5.2.3, its fundamental class $[S, S']$ is given by $\imath_*[v_s]$ where $\imath\colon (\overline{\mathfrak{F}}, \overline{\mathfrak{F}}') \longrightarrow (X, X')$ is the relative inclusion. Let $\jmath\colon B \longrightarrow \mathfrak{F}$ be the inclusion. Since $\mathrm{PL}_B \circ \jmath^* = \imath^* \circ \mathrm{PL}_{\mathfrak{F}}$, we get

$$\mathrm{PL}_{\mathfrak{F}}^{-1}[v_s]\Big|_B = \jmath^*\, \mathrm{PL}_{\mathfrak{F}}^{-1}[v_s] = \mathrm{PL}_B^{-1}\, \imath_*[v_s] = \mathrm{PL}_B^{-1}[S, S'].$$

Step 3. Consider the embedded relative submanifold $(\overline{\omega}(X), \overline{\omega}(X')) \subseteq (Y, Y')$. On the interior, we have $\omega(\mathfrak{U})$ which intersects E^s transversally: Let $p \in \omega(\mathfrak{U}) \cap E^s$. The space of all directions in $\mathfrak{U}$ where the critical point of multiplicity s is preserved has dimension $3h+2-2s$ since we cannot move the point on the surface, but we are in the interior of a Weierstraß simplex $\Delta^{2h+1-s} \times \Delta^{h+1-s}$. Thus, $\dim(T_p\omega(\mathfrak{U}) \cap T_pE^s) = 3h+2-2s$ and

$$\begin{aligned}\dim\left(T_p\omega(\mathfrak{U}) + T_pE^s\right) &= (3h+2) + (3h+2+2h-2s) - (3h+2-2s)\\ &= \dim(T_pE).\end{aligned}$$

Since we restricted everything on $\mathfrak{U}$, we know that $(\omega|_{\mathfrak{U}})^{-1}(E^s) = \mathfrak{V}_s \cup \mathfrak{U} = S \setminus S'$. Now, we finally combine both parts of Lemma 5.2.7 and get the identity

$$\omega^*\chi^s|_{\mathfrak{U}} \overset{(\mathrm{I})}{=} ((\omega|_{\mathfrak{U}})^* \circ \mathrm{PL}_E^{-1})[T^s, T^{s\prime}] \overset{(\mathrm{II})}{=} \mathrm{PL}_B^{-1}[S, S'] = \mathrm{PL}_{\mathfrak{F}}^{-1}[v_s]\Big|_{\mathfrak{U}}. \qquad \square$$

5.2.9 Corollary. *The intersection product of the lifted Weierstraß classes is given by*

$$[v_s] \bullet [v_t] = [v_{s+t}] \in H_{3h-2-2s-2t}(\overline{\mathfrak{F}}, \overline{\mathfrak{F}}'; \mathcal{O}).$$

5.2.10 Corollary. *We have in the relative complex* (P, P')

$$\mathrm{PL}_{\mathfrak{P}}(\pi^!\omega^*\chi^s) = \begin{cases} h \cdot [w_1] & \text{for } s = 1,\\ [w_s] & \text{for } s \geq 2.\end{cases}$$

Proof. We use the previous Proposition 5.2.8 and apply the diagram

$$\begin{tikzcd} H^{2s}(\mathfrak{F}) \arrow[r, "\mathrm{PL}_{\mathfrak{F}}"] \arrow[d, "\pi^!"'] & H_{3h+2-2s}(\overline{\mathfrak{F}}, \overline{\mathfrak{F}}'; \mathcal{O}) \arrow[d, "\overline{\pi}_*"] \\ H^{2s-2}(\mathfrak{P}) \arrow[r, "\mathrm{PL}_{\mathfrak{P}}"'] & H_{3h+2-2s}(P, P'; \mathcal{O}). \end{tikzcd}$$

$\square$

This finally leads immediately to the central theorem of this thesis as a combination of Corollary 4.3.11 and Corollary 5.2.10: Let $m = 0$. We have the easy case

$$\mathrm{PL}_{\mathfrak{P}}(\kappa_0 + 2) = \mathrm{PL}_{\mathfrak{P}}(\pi^!\omega^*\chi) = (2 - \chi(F_g)) \cdot [w_1] = 2g \cdot [w_1].$$

and for the other Mumford–Miller–Morita classes, we get the main result:

5.2.11 Theorem. *Let $m = 0$ and $s \geq 2$. Then*

$$\mathrm{PL}_{\mathfrak{P}}(\kappa_{s-1}) = \mathrm{PL}_{\mathfrak{P}}(\pi^!\omega^*\chi^s) = [w_s].$$

5.2.12 Outlook. There are several interesting questions arising from this duality:

(i) We can study the divisibility of the Mumford classes or, equivalently, the divisibility of the Weierstraß classes. There are some famous results for the divisibility of the Mumford classes in [Ebe08] and [GMT06].

(ii) We may study the intersection products $[w_\nu] \bullet [w_\mu]$ of Weierstraß classes. These correspond to cup products of the Mumford classes. One conjecture is that

$$[w_s] \bullet [w_t] = [w_{s,t}]$$

where (s, t) means the partition $(s, t, 1, \ldots, 1)$. It has the correct degree since

$$\dim[w_{s,t}] = 3h + 4 - 2s - 2t = \dim[w_s] + \dim[w_t] - 3h.$$

(iii) Some Weierstraß cycles w_ν may be boundaries and hence $[w_\nu] = 0$. If, as conjectured in (ii), cup products of Mumford classes correspond again to Weierstraß classes $[w_\nu]$, we may find algebraic relations which the Mumford classes fulfill for $g < \infty$.

Appendix

A General bundle constructions

Following [Hus66] and [Ste99], we present the basic tools for working with fibre bundles, principal G-bundles as well as real and complex vector bundles.

A.1 Reminder. Let $\mathbf{C}$ be a subcategory of $\mathbf{Top}^2$ and let B be a topological space. If $E \longrightarrow B$ is a fibre bundle with fibre F in $\mathbf{C}$, there is a cover $(U_i)_{i\in I}$ of B and trivialisations $\varphi_i\colon E|_{U_i} \longrightarrow U_i \times F$. For $i, j \in I$ consider $U_{ij} := U_i \cap U_j$ and the map

$$\varphi_{ij}\colon U_{ij} \times F \xrightarrow{\varphi_j^{-1}} E|_{U_{ij}} \xrightarrow{\varphi_i} U_{ij} \times F$$

Its adjoints $t_{ij}\colon U_{ij} \longrightarrow \mathrm{Homeo}(F)$ are called *transition functions*. If $\mathrm{Im}(t_{ij}) \subseteq \mathrm{Aut}_{\mathbf{C}}(F)$, we say, the bundle is *of type* $\mathbf{C}$. The transition functions fulfill the *transition identities*

(I) $t_{ii}(x) = \mathrm{id}_F$,

(II) $t_{jk}(x) \circ t_{ij}(x) = t_{ik}(x)$ for all $i, j, k \in I$ and $x \in U_{ij} \cap U_{jk}$.

Conversely, every system $(t_{ij}\colon U_{ij} \longrightarrow \mathrm{Homeo}(F))_{i,j}$ fulfilling the transition identities yields a fibre bundle by glueing the charts together along the adjoints φ_{ij}.

A.2 Construction (Subfibre)**.** Let $F \longrightarrow E \longrightarrow B$ be a fibre bundle with transitions $(t_{ij})_{i,j}$ and let $F' \subseteq F$ be a subset of the fibre which is fixed by all transitions, i. e. $t_{ij}(x)(F') = F'$. Then we obtain a subbundle, the *induced subfibre bundle*

$$F' \longrightarrow E' := \bigcup_{x\in B} F'_x \xrightarrow{\pi|_{E'}} B.$$

A.3 Construction. Let $\mathbf{C}$ and $\mathbf{D}$ be subcategories of $\mathbf{Top}^2$. $\mathscr{F}\colon \mathbf{C} \longrightarrow \mathbf{D}$ is called *continuous* if for each $X, Y \in \mathbf{C}$ the map $\mathscr{F}_{X,Y}\colon \mathbf{C}(X,Y) \longrightarrow \mathbf{D}(\mathscr{F}X, \mathscr{F}Y)$ is continuous in the compact–open topology. Now let $p\colon E \longrightarrow B$ be a fibre bundle of type $\mathbf{C}$ with transitions $(t_{ij})_{i,j}$ and let $\mathscr{F}\colon \mathbf{C} \longrightarrow \mathbf{D}$ be a continuous functor. Then the functions

$$s_{ij}\colon U_{ij} \longrightarrow \mathbf{D}(\mathscr{F}F, \mathscr{F}F), x \longmapsto \mathscr{F}_{F,F}(t_{ij}(x))$$

fulfill the transition identities and therefore define a new fibre bundle $\mathscr{F}_B E \longrightarrow B$ with fibre $\mathscr{F}F$, called the *fibre-wise application of* $\mathscr{F}$. Very often, we suppress the index B, do not confuse this with the application of $\mathscr{F}$ only on the total space E.

A.4 Remark. Let $\mathscr{F}, \mathscr{G}\colon \mathbf{C} \longrightarrow \mathbf{D}$ be two continuous functors. A natural transformation $\eta\colon \mathscr{F} \longrightarrow \mathscr{G}$ is called *continuous* if all $\eta_X\colon \mathscr{F}X \longrightarrow \mathscr{G}X$ are continuous. In this case, we get a bundle morphism $\eta_*\colon \mathscr{F}_B E \longrightarrow \mathscr{G}_B E$ given locally by

$$\begin{tikzcd}
\mathscr{F}_B E|_U \arrow[r, "\eta_*|_U"] \arrow[d, "\cong"'] & \mathscr{G}_B E|_U \arrow[d, "\cong"] \\
U \times \mathscr{F}F \arrow[r, "\mathrm{id}_U \times \eta_F"'] & U \times \mathscr{G}F
\end{tikzcd}$$

A.5 Construction (Principal G-bundles)**.** Let G be a topological group. For a continuous map $\pi\colon P \longrightarrow B$, the following statements are equivalent:

(I) π is a fibre bundle with fibre G and structure group G where the action of G on itself is given by right translation.

(II) P is a free right G-space, π is isomorphic to the projection $\mathrm{pr}\colon P \longrightarrow P/G$, we have a local sections and a continuous *comparison map* detecting the action of G:

$$C\colon P \times_B P \longrightarrow G, (\widehat{x}, \widehat{x} \cdot g) \longmapsto g.$$

In this case call $\pi\colon P \longrightarrow B$ *principal G-bundle*. Let $\mathrm{Prin}_G(B)$ be the set of all isomorphism classes of principal G-bundles over B. By the pullback, Prin_G becomes a contravariant homotopy functor, which is represented by the *classifying space BG*, i. e. there is a natural bijection between $\mathrm{Prin}_G(B)$ and $[B, BG]$ for all CW complexes B.

A.6 Construction (Balanced products)**.** Let G be a topological group. Let X be a right and Y a left G-space. Then we can form their *balanced product*

$$X \times_G Y \coloneqq \frac{X \times Y}{(x \cdot g, y) \sim (x, g \cdot y)}.$$

The projection $\mathrm{pr}_X\colon X \times Y \longrightarrow X$ is equivariant. Thus, we get $\pi\colon X \times_G Y \longrightarrow X/G$.

A.7 Construction. Let $P \longrightarrow B$ be a principal G-bundle and F a left G-space. Then

$$\pi\colon P \times_G F \longrightarrow B$$

is a fibre bundle with fibre F and structure group G, called the *associated fibre bundle*. Conversely, we call $P \longrightarrow B$ its *associated principal bundle*.

A.8 Example (Classification of vector bundles)**.**

(I) Let V be a real k-dimensional vector space. Then define the *space of frames*

$$\mathrm{Fr}(V) \coloneqq \{(v_1, \dots, v_k) \in V^k;\ (v_1, \dots, v_k) \text{ is a basis of } V\}.$$

Each linear isomorphism $\varphi\colon V \longrightarrow W$ yields a morphism $\varphi_*\colon \mathrm{Fr}(V) \longrightarrow \mathrm{Fr}(W)$ by

$$\varphi_*(v_1, \dots, v_k) \coloneqq (\varphi(v_1), \dots, \varphi(v_k))$$

This turns Fr into a continuous functor of finite-dimensional vector spaces.

(II) If $E \longrightarrow B$ is a k-dimensional vector bundle, then the fibre-wise application $\mathrm{Fr}(E)$ is a principal $\mathrm{GL}(k, \mathbb{R})$-bundle and $E \cong \mathrm{Fr}(E) \times_{\mathrm{GL}(k,\mathbb{R})} \mathbb{R}^k$, i. e. $\mathrm{Fr}(E)$ is the associated principal bundle. This yields a 1:1-correspondence between $\mathrm{Vect}_k^{\mathbb{R}}(B)$, the isomorphism classes k-dimensional vector bundles over B, and $\mathrm{Prin}_{\mathrm{GL}(k,\mathbb{R})}(B)$.

(III) We know that $B\mathrm{GL}(k, \mathbb{R}) \simeq B\mathrm{O}(k) \simeq \mathrm{Gr}_k(\mathbb{R}^\infty)$ where $\mathrm{Gr}_k(\mathbb{R}^\infty)$ denotes the real (k, ∞)-Graßmannian. Thus, we get a bijection between $\mathrm{Vect}_k^{\mathbb{R}}(B)$ and $[B, \mathrm{Gr}_k(\mathbb{R}^\infty)]$.

(IV) The same holds for complex vector bundles since $B\mathrm{GL}(k, \mathbb{C}) \simeq B\mathrm{U}(k) \simeq \mathrm{Gr}_k(\mathbb{C}^\infty)$, where $\mathrm{Gr}_k(\mathbb{C}^\infty)$ denotes the complex (k, ∞)-Graßmannian. Thus, we get a bijection between $\mathrm{Vect}_k^{\mathbb{C}}(B)$ and $[B, \mathrm{Gr}_k(\mathbb{C}^\infty)]$. In particular, we have for complex line bundles

$$\mathrm{Vect}_1^{\mathbb{C}}(B) \cong [B, \mathbb{C}P^\infty] \cong H^2(B; \mathbb{Z}).$$

B Simplicial and semisimplicial structures

B.1 Definition. Define the *simplicial category* $\mathbf{\Delta}$, whose objects are finite cardinals $[n] = \{0, \dots, n\}$ and whose morphisms are order-preserving maps $\alpha\colon [n] \longrightarrow [m]$.

B.2 Construction (Coface maps)**.** For $n \geq 1$ and $0 \leq i \leq n$, there is a unique injective morphism $\mathbf{d}_i\colon [n-1] \longrightarrow [n]$ in $\mathbf{\Delta}$ not attaining $i \in [n]$. For every injective morphism $\mathbf{d}\colon [n] \longrightarrow [m]$, there are unique indices $0 \leq i_{n+1}, \dots, i_m \leq j-1$ with

$$\mathbf{d} = \mathbf{d}_{i_m}^{(m)} \cdots \mathbf{d}_{i_{n+1}}^{(n+1)}.$$

We call the $\mathbf{d}_i$ *coface maps.* They generate the injective morphisms in $\mathbf{\Delta}$.

B.3 Construction (Codegenerace maps)**.** For $n \geq 0$ and $0 \leq i \leq n$, there is a unique surjective morphism $\mathbf{s}_i\colon [n+1] \longrightarrow [n]$ in $\mathbf{\Delta}$ repeating $i \in [n]$. For every surjective morphism $\mathbf{s}\colon [n] \longrightarrow [m]$, there are unique indices $0 \leq i_m, \dots, i_{n-1} \leq j$ with

$$\mathbf{s} = \mathbf{s}_{i_m}^{(m)} \cdots \mathbf{s}_{i_{n-1}}^{(n-1)}.$$

We call the $\mathbf{s}_i$ *codegenerace maps.* They generate the surjective morphisms in $\mathbf{\Delta}$.

B.4 Proposition (Cosimplicial identities)**.**

(I) *The codegenerace and coface maps fulfill the simplicial identities*

$$\begin{aligned} \mathbf{d}_j\mathbf{d}_i &= \mathbf{d}_i\mathbf{d}_{j-1} && \text{for } i < j, \\ \mathbf{s}_j\mathbf{s}_i &= \mathbf{s}_i\mathbf{s}_{j+1} && \text{for } i \leq j, \\ \mathbf{s}_j\mathbf{d}_i &= \begin{cases} \mathbf{d}_i\mathbf{s}_{j-1} & \text{for } i < j, \\ \mathrm{id} & \text{for } i = j, j+1, \\ \mathbf{d}_{i-1}\mathbf{s}_j & \text{for } i > j+1. \end{cases} \end{aligned}$$

(II) *Every morphism* $\alpha\colon [n] \longrightarrow [m]$ *can be written uniquely as* $\alpha = \mathbf{ds}$ *with* $\mathbf{d}$ *injective and* $\mathbf{s}$ *surjective; therefore, the coface and the codegenerace maps generate the morphisms.*

In particular $\mathbf{s}_i\mathbf{d}_i = \mathrm{id}$ *and* $\mathbf{d}_i\mathbf{s}_i(j) = j$ *for* $j \neq i$.

B.5 Definition. We define the *semisimplicial category* $\mathbf{\Delta}_+$ to be the subcategory containing all injective morphisms, i. e. all strictly order preserving maps. We see:

(I) We have the inclusion functor $\mathcal{I}\colon \mathbf{\Delta}_+ \longrightarrow \mathbf{\Delta}$.
(II) The morphisms in $\mathbf{\Delta}_+$ are exactly the compositions of coface maps.

The following constructions can be done either over the simplicial or the semisimplicial category. For our purposes, it is enough to restrict ourselves to the semisimplicial case.

B.6 Construction. A *semisimplicial set* is a contravariant functor

$$\mathfrak{X}\colon \mathbf{\Delta}_+^{\mathrm{op}} \longrightarrow \mathbf{Set}.$$

A *morphism* $f\colon \mathfrak{X} \longrightarrow \mathfrak{Y}$ of two semisimplicial sets is a natural transformation. Hence the category of semisimplicial sets is exactly the functor category $[\mathbf{\Delta}_+^{\mathrm{op}}, \mathbf{Set}]$. Note that defining a semisimplicial set $\mathfrak{X}$ is equivalent to defining a family $(\mathfrak{X}_n)_{n\geq 0}$ together with $d_i\colon \mathfrak{X}_n \longrightarrow \mathfrak{X}_{n+1}$ fulfilling $d_jd_i = d_id_{j-1}$ for $i < j$. We call the d_i *face maps.*

B.7 Construction. A *k-semisimplicial set* is a functor

$$\mathfrak{X}\colon \underbrace{\boldsymbol{\Delta}^{\mathrm{op}}_{(+)} \times \cdots \times \boldsymbol{\Delta}^{\mathrm{op}}_{(+)}}_{k} \longrightarrow \mathbf{Set}.$$

We restrict ourselves to the case $k = 2$ and call the 2-semisimplicial sets *bi-semisimplicial sets*. In other words, a bi-semisimplicial set is a family $(\mathfrak{X}_{p,q})_{p,q\geq 0}$ together with maps

$$d'_j\colon \mathfrak{X}_{p,q} \longrightarrow \mathfrak{X}_{p,q-1} \quad \text{and} \quad d''_i\colon \mathfrak{X}_{p,q} \longrightarrow \mathfrak{X}_{p-1,q}$$

such that $d'_j d''_i = d''_i d'_j$ and d' and d'' are face maps.

B.8 Remark. We have a covariant functor

$$\begin{aligned}(-)_*\colon \boldsymbol{\Delta}_+ &\longrightarrow \mathbf{Top},\\ [n] &\longmapsto \Delta_n,\\ \mathbf{d} &\longmapsto \left[\mathbf{d}_*\colon \sum_{k=0}^{n} a_k \cdot e_k \longmapsto \sum_{k=0}^{m} a_k \cdot e_{\mathbf{d}(k)}\right].\end{aligned}$$

B.9 Construction. Let $\mathfrak{X}\colon \boldsymbol{\Delta}^{\mathrm{op}}_{+} \times \boldsymbol{\Delta}^{\mathrm{op}}_{+} \longrightarrow \mathbf{Set}$ be a bi-semisimplicial set. We define the *geometric realisation* to be the quotient

$$|\mathfrak{X}| \coloneqq \coprod_{p\geq 0}\coprod_{q\geq 0} \mathfrak{X}_{p,q} \times \Delta_p \times \Delta_q \Big/ \begin{array}{l}(\Sigma; (\mathbf{d}_i)_* a, b) \sim (d''_i\Sigma; a, b),\\ (\Sigma; a, (\mathbf{d}_j)_* b) \sim (d'_j\Sigma; a, b).\end{array},$$

For a morphism $f\colon \mathfrak{X} \longrightarrow \mathfrak{Y}$ of bi-semisimplicial sets, we have a continuous map

$$|f|\colon |\mathfrak{Y}| \longrightarrow |\mathfrak{Y}|, [\Sigma; a, b] \longmapsto \left[f_{[p]\times[q]}(\Sigma); a, b\right].$$

This gives rise to functors $|\cdot|\colon [\boldsymbol{\Delta}^{\mathrm{op}}_{+} \times \boldsymbol{\Delta}^{\mathrm{op}}_{+}, \mathbf{Set}] \longrightarrow \mathbf{Top}$.

B.10 Definition (Local system)**.** Let $\mathfrak{X}$ be a bi-semisimplicial set and $\mathbf{C}$ be a category. By abuse of notation, denote by $\mathfrak{X} = \coprod_{p,q\geq 0} \mathfrak{X}_{p,q}$ also the set of all simplices. A *local* $\mathbf{C}$*-system* is a family $\mathcal{O} \coloneqq (E_\Sigma)_{\Sigma\in\mathfrak{X}}$ of objects from $\mathbf{C}$ together with isomorphisms

$$\varepsilon'_j(\Sigma)\colon E_\Sigma \longrightarrow E_{d'_j\Sigma} \quad \text{and} \quad \varepsilon''_i(\Sigma)\colon E_\Sigma \longrightarrow E_{d''_i\Sigma}$$

fulfilling the semisimplicial identities. For us, the following examples are important:

(I) The constant family $E_\Sigma = E$ and $\varepsilon''_j(\Sigma) = \varepsilon''_i(\Sigma) = \mathrm{id}_E$ is a local $\mathbf{C}$-system.

(II) An *orientation system* is a local $\mathbf{Ab}$-system with $E_\Sigma = \mathbb{Z}$. In this case, the morphisms $\varepsilon''_i(\Sigma)$ and $\varepsilon'_j(\Sigma)$ can be identified with signs ± 1.

B.11 Construction. Let $\mathfrak{X}$ be a bi-semisimplicial set and $\mathcal{O}$ an orientation system. The simplicial double chain complex $C_{\bullet,\bullet}(\mathfrak{X})$ can be modified by using the following differentials

$$\begin{aligned}\partial'\colon C_{p,q}(\mathfrak{X}) &\longrightarrow C_{p,q-1}(\mathfrak{X}),\\ \Sigma &\longmapsto \sum_{j=0}^{q}(-1)^j \cdot \varepsilon'_j(\Sigma) \cdot d'_j\Sigma,\end{aligned} \qquad \begin{aligned}\partial''\colon C_{p,q}(\mathfrak{X}) &\longrightarrow C_{p-1,q}(\mathfrak{X}),\\ \Sigma &\longmapsto \sum_{i=0}^{p}(-1)^i \cdot \varepsilon''_i(\Sigma) \cdot d''_i\Sigma.\end{aligned}$$

C Special notation for the symmetric group

C.1 Definition. We define the *standard symmetric group* $\mathfrak{S}_p \coloneqq \mathrm{Sym}\{1, \dots, p\}$ and the *extended symmetric group* $\overline{\mathfrak{S}}_p \coloneqq \mathrm{Sym}[p] = \mathrm{Sym}\{0, \dots, p\}$. Most of the time, we are dealing with $\overline{\mathfrak{S}}_p$. Furthermore, we use the convention $\sigma \cdot \tau \coloneqq \sigma \circ \tau$.

C.2 Definition. For a permutation $\sigma \in \overline{\mathfrak{S}}_p$, we define the *set of fixed points*, *set of inertia* and the *support* to be the following subsets of $[p]$:

$$\begin{aligned} \mathrm{fix}(\sigma) &\coloneqq \{k;\, \sigma(k) = k\}, \\ \mathrm{inert}(\sigma) &\coloneqq \{k;\, \sigma(k) = k + 1 \mod p + 1\} \\ \mathrm{supp}(\sigma) &\coloneqq \{k;\, \sigma(k) \neq k\} = \{0, \dots, p\} \setminus \mathrm{fix}(\sigma). \end{aligned}$$

For $0 \leq i \leq p$, define $\mathrm{cyc}(\sigma, i) \in \overline{\mathfrak{S}}_p$ as the (possibly trivial) cycle of σ containing i. Moreover, for a collection $\sigma_1, \dots, \sigma_r \in \overline{\mathfrak{S}}_p$ of permutatons, we define

$$\begin{aligned} \mathrm{fix}(\sigma_1, \dots, \sigma_r) &\coloneqq \mathrm{fix}(\sigma_1) \cap \dots \cap \mathrm{fix}(\sigma_r), \\ \mathrm{supp}(\sigma_1, \dots, \sigma_r) &\coloneqq \mathrm{supp}(\sigma_1) \cup \dots \cup \mathrm{supp}(\sigma_r). \end{aligned}$$

C.3 Definition (Cycle number and norm)**.** Let $p \geq 1$ and let $\sigma \in \overline{\mathfrak{S}}_p$.

(I) For $1 \leq k \leq p + 1$ let $T_k(\sigma)$ be the number of k-cycles. The *cycle number* (counting trivial cycles) is then defined by $\mathrm{ncyc}(\sigma) \coloneqq T_1 + \dots + T_{p+1}$ and we obtain

$$p + 1 = \sum_{k=1}^{p+1} k \cdot T_k.$$

(II) The *norm* $N(\sigma)$ is the minimal number of transpositions needed to generate σ. Then $\mathrm{ncyc}(\sigma) + N(\sigma) = p + 1$, more precisely

$$N(\sigma) = \sum_{k=1}^{p+1} (k - 1) \cdot T_k.$$

C.4 Example. For $\sigma \in \overline{\mathfrak{S}}_p$ consider its decomposition into non-trivial cycles

$$\sigma = \prod_{i=1}^{r} \langle \alpha_{i,1}, \dots, \alpha_{i,s_i} \rangle.$$

Then $\mathrm{supp}(\sigma) = \{a_{ij}\}_{i,j}$ and $\mathrm{sg}(\sigma) = (-1)^{N(\sigma)}$ and moreover,

$$N(\sigma) = \sum_{i=1}^{r} (s_i - 1) = -r + \sum_{i=1}^{r} s_i.$$

C.5 Remark. We have the *triangular inequality*

$$N(\sigma \cdot \tau) \leq N(\sigma) + N(\tau).$$

We call a pair (σ, τ) of permutations *geodesic pair* if $N(\sigma \cdot \tau) = N(\sigma) + N(\tau)$. Obviously, if $\mathrm{supp}(\sigma)$ and $\mathrm{supp}(\tau)$ are disjoint, then (σ, τ) is a geodesic pair.

C.6 Definition. For $0 \leq i \leq p$, define the *skipping*

$$D_i \colon \overline{\mathfrak{S}}_p \longrightarrow \overline{\mathfrak{S}}_{p-1}, \sigma \longmapsto \mathbf{s}_i \circ \langle i, \sigma(i) \rangle \circ \sigma \circ \mathbf{d}_i,$$

where $\mathbf{d}_i \colon [p-1] \longrightarrow [p]$ resp. $\mathbf{s}_i \colon [p] \longrightarrow [p-1]$ are the i^{th} coface resp. codegeneracy maps (see B.2 and B.3). In other words, we remove the element i from the cycle decomposition and apply the (now injective) morphism $\mathbf{s}_i$ on the remaining elements.

C.7 Remark. Obviously, D_i is surjective, but not a homomorphism. However, we have

$$\begin{aligned} D_i(\sigma^{-1}) \cdot D_i(\sigma) &= \mathbf{s}_i \circ \langle i, \sigma(i) \rangle \circ \sigma^{-1} \circ \mathbf{d}_i \circ \mathbf{s}_i \circ \langle i, \sigma(i) \rangle \circ \sigma \circ \mathbf{d}_i \\ &= \mathbf{s}_i \circ \sigma^{-1}\sigma \circ \mathbf{d}_i \\ &= \mathrm{id} \end{aligned}$$

and we obtain $D_i(\sigma)^{-1} = D_i(\sigma^{-1})$. Furthermore, we have $D_i D_j = D_{j-1} D_i$ for $i < j$ which turns $[p] \longmapsto \overline{\mathfrak{S}}_p$ into a semisimplicial set.

C.8 Remark. For a permutation $\sigma \in \overline{\mathfrak{S}}_p$ and $0 \leq i \leq p$, we have

$$N(D_i(\sigma)) = \begin{cases} N(\sigma) & \text{for } i \in \operatorname{fix}(\sigma), \\ N(\sigma) - 1 & \text{else,} \end{cases}$$

since in the first case, the cycle number and p is decreased by one and in the second case, no cycle is removed.

Symbol Index

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Bibliography

[ABE08] Jochen Abhau, Carl-Friedrich Bödigheimer, and Ralf Ehrenfried. "Homology of the mapping class group $\Gamma_{2,1}$ for surfaces of genus 2 with a boundary curve". In: *Geometry & Topology Monographs* 14 (2008), pp. 1–25.

[BH14] Felix Jonathan Boes and Anna Hermann. "Moduli Spaces of Riemann Surfaces – Homology Computations and Homology Operations". Master's thesis. Rheinische Friedrich-Wilhelms-Universität Bonn, 2014.

[Bia18] Andrea Bianchi. *Personal communication.* 2018.

[Böd18] Carl-Friedrich Bödigheimer. *Personal communication.* 2018.

[Böd90] Carl-Friedrich Bödigheimer. "On the topology of moduli spaces of Riemann surfaces. Part I: Hilbert Uniformization". In: *Mathematica Gottingensis* (1990).

[Bre93] Glen E. Bredon. *Topology and Geometry.* Springer, 1993.

[Ebe08] Johannes Ebert. "Divisibility of Miller–Morita–Mumford classes of spin surface bundles". In: *The Quarterly Journal of Mathematics* 59 (2008), pp. 207–212.

[FM12] Benson Farb and Dan Margalit. *A Primer on Mapping Class Groups.* Princeton: Princeton University Press, 2012.

[GMT06] Søren Galatius, Ib Madsen, and Ulrike Tillmann. "Divisibility of the stable Miller–Morita–Mumford classes". In: *Journal of the American Mathematical Society* 19 (2006), pp. 759–779.

[Har84] John L. Harer. "Stability of the homology of the mapping class group of orientable surfaces". In: *Annals of Mathematics* 121 (1984), pp. 215–249.

[Hat01] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2001.

[Hat03] Allen Hatcher. "Vector bundles and K-theory". Preprint. 2003. URL: https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf.

[Hat14] Allen Hatcher. "A short exposition of the Madsen–Weiss theorem". Preprint. 2014. URL: https://arxiv.org/abs/1103.5223.

[Hus66] Dale Husemoller. *Fibre bundles.* Springer, 1966.

[Igu04] Kiyoshi Igusa. "Combinatorial Miller–Morita–Mumford classes and Witten cycles". In: *Algebraic & Geometric Topology* 4 (2004), pp. 473–520.

[May99] J. Peter May. *A Concise Course in Algebraic Topology.* University of Chicago Press, 1999.

[Meh11] Stefan Mehner. "Homologieberechnungen von Modulräumen Riemannscher Flächen durch diskrete Morse-Theorie". Diplomarbeit. Rheinische Friedrich-Wilhelms-Universität Bonn, 2011.

[Mil69] R. James Milgram. "The Homology of Symmetric Products". In: *Transactions of the American Mathematical Society* 138 (1969), pp. 251–265.

[Mil86] Ed Y. Miller. "The homology of the mapping class group". In: *J. Differential Geometry* 24 (1986), pp. 1–14.

[Mor87] Shigeyuki Morita. “Characteristic classes of surface bundles”. In: *Inventiones mathematicae* 90 (1987), pp. 551–577.

[MP12] J. Peter May and Kate Ponto. *More Concise Algebraic Topology: Localization, Completion, and Model Categories*. University of Chicago Press, 2012.

[Mül96] Meinhard Müller. “Orientierbarkeit des Raumes der Parallelschlitzgebiete”. Diplomarbeit. Rheinische Friedrich-Wilhelms-Universität Bonn, 1996.

[Mum83] David Mumford. “Towards an enumerative geometry of the moduli space of curves”. In: *Arithmetic and Geometry* 2 (1983), pp. 271–328.

[MW07] Ib Madsen and Michael Weiss. “The stable moduli space of Riemann surfaces: Mumford’s conjecture”. In: *Annals of Mathematics* 165 (2007), pp. 843–941.

[Pow78] Jerome Powell. “Two theorems on the mapping class group of a surface”. In: *Proceedings of the American Mathematical Society* 68 (1978), pp. 374–350.

[Sch03] Sarah Schardt. “Die Hopf-Struktur auf der Konjugation-Shuffle-Algebra der symmetrischen Gruppen”. Diplomarbeit. Rheinische Friedrich-Wilhelms-Universität Bonn, 2003.

[Spa94] Edwin H. Spanier. *Algebraic Topology*. Springer, 1994.

[Ste99] Norman Steenrod. *The topology of fibre bundles*. Princeton University Press, 1999.

[Til12] Ulrike Tillmann. “Mumford’s Conjecture – A topological outlook”. In: *Handbook of Moduli* 3 (2012), pp. 399–429.

[Vis11] Balázs Visy. “Factorable Groups and their Homology”. PhD thesis. Rheinische Friedrich-Wilhelms-Universität Bonn, 2011.

[Wah13] Nathalie Wahl. “The Mumford conjecture, Madsen–Weiss and homological stability for mapping class groups of surfaces”. In: *IAS/Park City Mathematics Series* 20 (2013), pp. 109–138.