A brief introduction to the trace formula and its stabilization

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Contents

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1 Introduction

The trace formula is a fundamental tool in the theory of automorphic forms. In its early form it was developed by Selberg [Sel56], [Sel63] in the setting of modular forms on the upper half complex plane. Langlands [Lan63] realized its potential for the study of general automorphic forms. He employed it to the study base change for the group GL_2 and the transfer of automorphic forms between inner forms of GL_2 and of SL_2 [JL70], [LL79], [Lan80]. The formula was greatly generalized and refined by Arthur, and the result is now commonly referred to as the Arthur–Selberg trace formula, [Art05].

In its most basic shape, the trace formula asserts the identity of two distributions,

$$J_{\text{geom}}(f) = J_{\text{spec}}(f),$$

each accepting as an argument a smooth compactly supported complex valued "test" function f on the group $G(\mathbb{A})$ of adelic points of a connected reductive group G defined over a number field F.

When G is anisotropic, both distributions are rather explicit. J_{geom} consists of the integrals of f over the various conjugacy classes in $G(\mathbb{A})$ of elements of $G(F)$, while J_{spec} consists of the traces of f on the various irreducible representations of $G(\mathbb{A})$ occurring in $L^2(G(F)\backslash G(\mathbb{A}))$. The formula is obtained by developing two different expressions for the integration of a certain "kernel" function $k_f\colon (G(F)\backslash G(\mathbb{A})) \times (G(F)\backslash G(\mathbb{A})) \to \mathbb{C}$ over the diagonal copy of $G(F)\backslash G(\mathbb{A})$. The function k_f is the kernel of an integral operator that describes the action of the test function f by convolution on the Hilbert space $L^2(G(F)\backslash G(\mathbb{A}))$.

When G is isotropic the situation is considerably more complicated. While the dominant terms on each side of the trace formula are still of the same shape as before, various supplementary terms appear. On the geometric side, they correspond to conjugacy classes of elements of $G(F)$ that are not semisimple and elliptic. On the spectral side, they are related to the fact that the Hilbert space $L^2(G(F)\backslash G(\mathbb{A}))$ does not decompose as a Hilbert direct sum of irreducible representations.

The supplementary terms on both sides do not just complicate the picture by bringing undesired information into the formula. They also bring undesired behavior: they are not invariant under conjugation. Therefore, even after the trace formula has been derived, it needs to undergo an additional refinement step, in which the distributions J_{geom} and J_{spec} are replaced by conjugation-invariant distributions I_{geom} and I_{spec}, leading to the identity

$$I_{\text{geom}}(f) = I_{\text{spec}}(f),$$

called the *invariant* trace formula. The dominant terms in this formula are the same as those of the non-invariant trace formula, but the supplementary terms are different. The price that one pays for the pleasure of working with invariant distributions is that now the clean separation of geometric and spectral terms onto each side of the identity has been given up, because the process of making the distributions invariant involves combining geometric and spectral terms in order to cancel non-invariance. Nonetheless, the subscripts "geom" and "spec" have been kept in order to emphasize the nature of the dominant terms in each distribution.

For many applications of the trace formula, a further refinement is required, in which I_{geom} and I_{spec} are replaced by distributions S_{geom} and S_{spec} that have an even stronger invariance property, namely under stable conjugation (which is closely related to, but not exactly the same as, geometric conjugation, i.e. conjugation by points of G defined over the separable closure of the base field, or its adele ring). This leads to the identity

$$S_{\text{geom}}(f) = S_{\text{spec}}(f),$$

called the *stable* trace formula. The stabilization of the geometric side of the trace formula has been a long process, initiated by Langlands [Lan83] for the geometric terms corresponding to regular elliptic conjugacy classes, continued by Kottwitz [Kot86] for the singular elliptic conjugacy classes, and completed by Arthur for general conjugacy classes [Art02], [Art01], [Art03]. The dominant terms in the distribution S_{geom} are the integrals of f over the stable conjugacy classes of elliptic semi-simple elements of $G(\mathbb{A})$. The distribution S_{spec} is in general rather abstract. The explicit description of its dominant terms requires the validity of the local and global Arthur conjectures for the group G. Its dominant term was derived conditionally in [Kot84]. Unconditional derivations are available only in the few cases where (versions) of these conjectures are known, such as the case of classical groups [Art13].

These notes aim to present an overview of the Arthur–Selberg trace formula and some of its applications. In preparing them, we have benefitted from many excellent texts with a similar goal, such as [Whi10], [Gel96], [Art05], [GJ79]. We have attempted to minimize the overlap with these references and provide a different point of view on the subject.

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2 Precursors

In this subsection we will discuss two well-known elementary formulas that motivate the general Arthur–Selberg trace formula.

2.1 Poisson summation for locally compact abelian groups

Consider a function $f\colon \mathbb{R} \to \mathbb{C}$ that is of Schwartz type, i.e. smooth and all of its derivatives decay faster than polynomials. Then the Fourier transform

$$\widehat{f}(y) = \int_{\mathbb{R}} f(x)e^{-2\pi ixy}dx$$

is also of this kind. The Poisson summation formula states

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{k\in\mathbb{Z}} \widehat{f}(k).$$

This is a very useful and important formula. For example, it can be used to derive the functional equation of the Riemann zeta function.

This formula has a considerable generalization to the setting of locally compact abelian groups. A locally compact abelian group is an abelian topological group A for which there exists an open set $U \subset A$ containing 0 whose closure $\bar{U}$ is compact.

Example 2.1.1. The additive group $\mathbb{R}$ and the multiplicative group $\mathbb{R}^\times$ are locally compact. Any discrete group is locally compact.

On the other hand, the additive group $\mathbb{Q}$, with its relative topology inherited from $\mathbb{R}$, is not locally compact. Indeed, take an open set U in $\mathbb{Q}$ containing 0. It is the intersection with $\mathbb{Q}$ of an open set of $\mathbb{R}$ containing 0. The latter is a disjoint union of open intervals, so U itself is a disjoint union of the intersections with $\mathbb{Q}$ of open intervals. If U_0 is one such intersection containing 0, then $\bar{U}$ being compact would imply that $\bar{U}_0$ is. But $\bar{U}_0$ is the intersection of $\mathbb{Q}$ with a closed interval. So it is enough to show that such an intersection is not compact. For this, let $a_0 < a_1 < \cdots < a_n < \cdots$ be a sequence such that $a_0 = \inf(U_0)$, $a_i \notin \mathbb{Q}$ for $i > 0$, $a_\infty = \lim_{n\to\infty} a_n$ exists and satisfies $a_\infty \notin \mathbb{Q}$ and $a_\infty < \sup(U_0)$. Then the open intervals $(a_n, a_{n+1}) \cap \mathbb{Q}$ for $n \geq 0$ together with $(a_\infty, \sup(U_0)) \cap \mathbb{Q}$ form an infinite disjoint open cover.

Let A be a locally compact abelian group. It carries a regular measure on its Borel σ-algebra that is invariant under the group operation. This measure is unique up to scalar, and is called the Haar measure. This gives rise to the spaces $L^p(A)$ for any $1 \leq p \leq \infty$ in the usual way.

Furthermore we have the Pontryagin dual $A^* = \mathrm{Hom}_{\mathrm{cts}}(A, \mathbb{S}^1)$. It is again a locally compact abelian group, when equipped with the compact open topology, that is, the topology with subbase given by the sets $V(K,U) = \{f\colon A \to \mathbb{S}^1 \mid f(K) \subset U\}$ for $K \subset A$ compact and $U \subset \mathbb{S}^1$ open.

For $f \in L^1(A)$ we can form its Fourier transform $\widehat{f}\colon A^* \to \mathbb{C}$ by

$$\widehat{f}(\xi) = \int_A f(a)\xi(a)da.$$

The function $\widehat{f}$ is continuous, but need not be integrable. If it is, we can form its Fourier transform $\widehat{\widehat{f}} : A^{**} \to \mathbb{C}$. The Pontryagin duality theorem states that the natural map $A \to A^{**}$ is an isomorphism of topological groups. Therefore we can think of $\widehat{\widehat{f}}$ as a function on A.

Note that $\widehat{f}$ depends on the Haar measure da on A, unique up to constant, and $\widehat{\widehat{f}}$ further depends on the Haar measure da^* on A^*, again unique up to constant. For the following basic result see [HR70, Theorem 31.17].

Theorem 2.1.2 (Fourier inversion formula)**.** *Given a choice of Haar measure da on A there exists a (necessarily unique) Haar measure da^* on A^*, called the measure* dual *to da, such that*

$$\hat{\hat{f}}(a) = f(-a),$$

provided $f \in L^1(A)$ and $\widehat{f} \in L^1(A^*)$.

Let B be a closed subgroup of A and set $C = A/B$. Then $0 \to B \to A \to C \to 0$ is an exact sequence in the category of topological groups, and Pontryagin duality converts it into an exact sequence $0 \to C^* \to A^* \to B^* \to 0$. Fix Haar measures db and da and let $dc = da/db$. Note that C^* is the annihilator of B. For the following result, see [HR70, (31.46)(e)].

Theorem 2.1.3 (Poisson summation formula)**.** *If dc^* and da^* are dual to dc and da, then $da^*/dc^* = db^*$. For sufficiently nice functions $f\colon A \to \mathbb{C}$ one has*

$$\int_B f(b)db = \int_{C^*} \widehat{f}(c^*)dc^*.$$

The classical Poisson summation formula applies to the case where $A = \mathbb{R}$ and $B = \mathbb{Z}$. In that case, $A^* = A$ via the bicharacter

$$\mathbb{R} \times \mathbb{R} \to \mathbb{S}^1, \qquad (x, y) \mapsto e^{-2\pi i x y},$$

and under this bicharacter $\mathbb{Z}$ is its own annihilator, so $C^* = B^\perp = \mathbb{Z}$.

Example 2.1.4. Further examples of locally compact groups are the additive and multiplicative groups of the field $\mathbb{Q}_p$ of p-adic numbers, as well as the additive and multiplicative groups of the ring $\mathbb{A}$ of adeles. The Poisson summation formula applied to these groups plays a key role in the analytic study of L-functions as developed by Tate, cf. [CF86, Chapter XV].

2.2 The trace formula for finite groups

The Arthur–Selberg trace formula can be considered a generalization of the Poisson summation formula beyond the abelian case. To see what the general structure might look like, unencumbered by any analytic difficulties, we consider now the case of a finite group.

Let G be a finite group and Γ a subgroup. We consider the representation R of G on the space of functions

$$\mathbb{C}[\Gamma\backslash G] = \{\varphi\colon \Gamma\backslash G \to \mathbb{C}\}$$

under right translation, i.e. $(R(x)\phi)(g) = \phi(gx)$. This representation can be extended to a representation of the algebra of functions $\mathbb{C}[G] = \{f\colon G \to \mathbb{C}\}$, by the formula

$$R(f)\phi(g) = \sum_{x \in G} f(x)\phi(gx).$$

We want to understand the trace of the operator $R(f)$. There are two expressions for it, a spectral and a geometric one. To derive the spectral expression we remind ourselves that $\mathbb{C}[G]$ carries two actions of G, by left and right translations, respectively, and is thus a representation of $G \times G$. The natural inclusion

$$\mathbb{C}[\Gamma \backslash G] \subset \mathbb{C}[G],$$

realizes $\mathbb{C}[\Gamma \backslash G]$ as the submodule of $\mathbb{C}[G]$ with respect to the right action that is precisely the set of fixed points of $\Gamma \times \{1\} \subset \Gamma \times \Gamma$. Basic representation theory of finite groups [Ser77, §6.2, Proposition 10] implies that the $G \times G$-representation $\mathbb{C}[G]$ decomposes into irreducible factors as

$$\mathbb{C}[G] = \bigoplus_{\pi} \pi^{\vee} \boxtimes \pi,$$

where π runs over all irreducible representations of G. Therefore, the G-representation $\mathbb{C}[\Gamma \backslash G]$ decomposes into irreducible factors as

$$\mathbb{C}[\Gamma \backslash G] = \bigoplus_{\pi} \dim(\pi^{\Gamma}) \cdot \pi.$$

Consequently, the operator $R(f)$ takes the form

$$\sum_{\pi} \dim(\pi^{\Gamma}) \pi(f), \qquad \pi(f) = \sum_{g \in G} f(g) \pi(g),$$

leading to the desired spectral expression

$$\operatorname{tr} R(f) = \sum_{\pi} \dim(\pi^{\Gamma}) \operatorname{tr}(\pi(f)).$$

To derive the geometric expression for $\operatorname{tr} R(f)$ we observe that $\mathbb{C}[\Gamma \backslash G]$ is the induced representation of G from the trivial representation of Γ. The Frobenius character formula [Ser77, §3.3, Theorem 12] then implies, for $g \in G$

$$\operatorname{tr} R(g) = \sum_{\substack{x \in \Gamma \backslash G \\ xgx^{-1} \in \Gamma}} 1.$$

Thus, for $f \in \mathbb{C}[G]$ we obtain

$$\operatorname{tr} R(f) = \sum_{g \in G} \sum_{\substack{x \in \Gamma \backslash G \\ xgx^{-1} \in \Gamma}} f(g) = \frac{1}{|\Gamma|} \sum_{g \in G} \sum_{\substack{x \in G \\ xgx^{-1} \in \Gamma}} f(g).$$

This double sum is taken over the set

$$\{g \in G, x \in G, \gamma \in \Gamma \mid xgx^{-1} = \gamma\}.$$

Of course γ is determined by g and x. But in the same way, g is determined by x and γ, and rewriting the sum this way we obtain

$$\text{tr}\, R(f) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sum_{x \in G} f(x\gamma x^{-1}).$$

Writing G_γ for the centralizer of γ in G, it is clear that replacing x by xy with $y \in G_\gamma$ does not change the summand, and that replacing γ with a conjugate element does not change the inner sum. With these observations, we arrive at

$$\text{tr}\, R(f) = \frac{1}{|\Gamma|} \sum_{[\gamma] \in [\Gamma]} \frac{|\Gamma|}{|\Gamma_\gamma|} |G_\gamma| \sum_{x \in G/G_\gamma} f(x\gamma x^{-1}) = \sum_{[\gamma] \in [\Gamma]} \frac{|G_\gamma|}{|\Gamma_\gamma|} \sum_{x \in G/G_\gamma} f(x\gamma x^{-1}),$$

where $[\Gamma]$ denotes the set of conjugacy classes in Γ. This is the geometric expansion of tr $R(f)$. Together with the spectral expansion, we obtain:

Theorem 2.2.1 (Trace formula for a finite group)**.**

$$J_{spec}(f) := \sum_\pi dim(\pi^\Gamma) tr\, \pi(f) = \sum_{[\gamma] \in [\Gamma]} \frac{|G_\gamma|}{|\Gamma_\gamma|} \sum_{x \in G/G_\gamma} f(x\gamma x^{-1}) =: J_{geom}(f).$$

3 Trace formula for a compact quotient

As a first step towards generalizing the discussion §2 we will consider here a locally compact topological group G, not assumed abelian, and a closed discrete subgroup $\Gamma \subset G$ such that the quotient is compact.

This case includes both Theorem 2.2.1 and the classical Poisson summation formula, where $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$. It also covers the study of automorphic representations of anisotropic reductive $\mathbb{Q}$-groups $\mathbb{G}$, where the locally compact topological group $G = \mathbb{G}(\mathbb{A})$ is never compact (unless it is trivial), but the quotient $\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A})$ is compact. An example of such $\mathbb{G}$ is given by taking the group of elements of norm 1 in a division algebra.

3.1 Compact groups

The case that is most immediately analogous to Theorem 2.2.1 is that when G is itself compact. This forces Γ to be finite. This case can be handled by an argument analogous to the one given in §2.2. In place of $\mathbb{C}[G]$ one now considers the Hilbert space $L^2(G)$ and the right regular representation R of G on $L^2(G)$ given by the same formula as in the case of finite groups. Since it is infinite-dimensional, one needs to invoke functional analysis in its study. Nonetheless, the theorem of Peter–Weyl states the identity

$$L^2(G) = \widehat{\bigoplus}_\pi \pi^\vee \boxtimes \pi,$$

of representations of $G \times G$, where π runs over the (now infinite) set of irreducible representations of G and the hat symbol denotes a Hilbert direct sum (more precisely, the algebraic direct sum $\bigoplus_\pi \pi^\vee \boxtimes \pi$ embeds into $L^2(G)$, with $\lambda \boxtimes v \in \pi^\vee \boxtimes \pi$ being sent to the matrix coefficient $g \mapsto \lambda(\pi(g)v)$, and the image is dense). The closed subspace $L^2(\Gamma\backslash G)$ is again the subspace of invariants under $\Gamma \times \{1\}$ and therefore

$$L^2(\Gamma\backslash G) = \widehat{\bigoplus_\pi} \dim(\pi^\Gamma) \cdot \pi.$$

As in the case of a finite group we extend the right regular representation of G on $L^2(G)$ to an action of the algebra of functions on G. Taking the topology of G into account, we consider only the continuous functions $\mathcal{C}(G)$. The algebra structure is given by convolution, i.e. $f_1 * f_2(g) = \int_G f_1(gx^{-1})f_2(x)dx$.

Any Banach space representation π of G can be extended to a representation of the algebra $\mathcal{C}(G)$ by

$$\pi(f)v = \int_G f(x)\pi(x)v dx,$$

where we are using the Haar measure normalized so that $\mathrm{vol}(G; dx) = 1$. In the case of the right regular representation of G on $L^2(G)$ this takes the form $R(f)\varphi = \varphi * f^-$, where $f^-(g) = f(g^{-1})$ and this convention guarantees that we obtain a usual left action, rather than a right action. More precisely, one has the formulas

$$\begin{aligned} (f_1 * f_2) * f_3 &= f_1 * (f_2 * f_3) \\ (f_1 * f_2)^- &= f_2^- * f_1^-. \end{aligned}$$

We now obtain

$$\mathrm{tr}\, R(f) = \sum_\pi \dim(\pi^\Gamma) \mathrm{tr}\, \pi(f).$$

Remark 3.1.1. Any irreducible representation π of G is finite-dimensional, so $\mathrm{tr}\, \pi(f)$ is well-defined. On the other hand, the sum is now infinite, and it is a fair question under what conditions it converges. This is equivalent to asking the conditions under which the operator $R(f)$ is of trace class, which is a non-trivial question since $L^2(\Gamma\backslash G)$ is generally of infinite dimension.

It is clear that some condition is necessary: if we take $G = \mathbb{S}^1$ and $\Gamma = \{1\}$, then the set $\{\pi\}$ is the set of characters $\{e^{2\pi i n x} \mid n \in \mathbb{Z}\}$ and $\mathrm{tr}\, \pi(f) = \widehat{f}(n)$ for $\pi = e^{-2\pi i n x}$. We are thus asking under what conditions on f the Fourier coefficients of f are summable. It is well known that there exist continuous functions f on $\mathbb{S}^1$ whose Fourier coefficients are not summable. On the other hand, if f is continuously differentiable, then the Fourier coefficients are summable.

We will address this question in Remark 3.2.8.

On the other hand, the geometric expansion becomes a bit different. First, neither G_γ nor G/G_γ is finite in general. The term $|G_\gamma| \cdot |\Gamma_\gamma|^{-1}$ becomes replaced with $\mathrm{vol}(G_\gamma/\Gamma_\gamma; dx_\gamma/d\gamma)$, where dx_γ is an arbitrary choice of Haar measure on the centralizer G_γ of γ, and $d\gamma$ is the counting measure on Γ. The term

$\sum_{x\in G/G_\gamma} f(x\gamma x^{-1})$ becomes replaced by $\int_{G/G_\gamma} f(x\gamma x^{-1})dx/dx_\gamma$. We see that the product of both terms is independent of the choice of Haar measure dx_γ. The analog of Theorem 2.2.1 is now as follows.

Theorem 3.1.2 (Trace formula for a compact group)**.** *For a sufficiently nice function $f\colon G\to\mathbb{C}$ the following identity holds*

$$\sum_{\pi} dim(\pi^\Gamma) tr\,\pi(f) = \sum_{[\gamma]\in[\Gamma]} vol(G_\gamma/\Gamma_\gamma; dx_\gamma/d\gamma)\int_{G/G_\gamma} f(x\gamma x^{-1})dx/dx_\gamma.$$

We will not give the details of the proof of this formula. They are roghly analogous to that in the setting of a finite group, and we will derive a more general formula in the next subsection.

3.2 Compact quotient

Theorem 3.1.2 is not yet powerful enough to be of much use. While it does cover the case of Fourier series of periodic functions, it is not able to handle aperiodic functions on $\mathbb{R}$. It is also not able to handle automorphic representations of anisotropic $\mathbb{Q}$-groups $\mathbb{G}$, because the group $\mathbb{G}(\mathbb{A})$ is almost never compact.

However, Theorem 3.1.2 admits a further generalization, which does cover all these cases, and has an almost identical structure. In this set-up, the topological group G is no longer assumed compact, and instead Γ is assumed to be *cocompact*, i.e. it is assumed that the quotient $\Gamma\backslash G$ is compact. Note that, unlike in §3.1, now Γ is generally infinite.

Thus, we let G be a locally compact group, but without assuming that it is abelian. We recall from [HR79, §15] that there exists, and is unique up to scalar, a right Haar measure μ_r, as well as a left Haar measure μ_l. Both are unique up to a positive scalar multiple and one has $\mathbb{R}_{>0}\cdot d\mu_l(g) = \mathbb{R}_{>0}\cdot d\mu_r(g^{-1})$, but in general $\mathbb{R}_{>0}\cdot d\mu_l \neq \mathbb{R}_{>0}\cdot d\mu_r$. The failure of this identity is measured by the *modulus* function $\Delta\colon G\to\mathbb{R}_{>0}$, which is a continuous homomorphism that can be described by either of the following equivalent equations

$$\Delta(g) = \mu_r(g^{-1}X)/\mu_r(X), \quad \Delta(g) = \mu_l(Xg)/\mu_l(X), \quad \Delta(g) = \frac{d\mu_l(g)}{d\mu_r(g)}$$

where in the first two equalities X is an arbitrary compact subset of G, and in the third equality both Haar measures are normalized so that $\Delta(1) = 1$, see [HR79, (15.11)-(15.15)]. The group is called *unimodular*, if a right Haar measure is automatically also a left Haar measure, equivalently Δ is the trivial homomorphism. This is the case with abelian groups (trivial), compact groups (since $\mathbb{R}_{>0}$ has no nontrivial compact subgroups), but also many other locally compact groups, such as semi-simple Lie groups (since these admit no non-trivial continuous homomorphisms to $\mathbb{R}_{>0}$).

We assume that G is unimodular and let Γ be a closed discrete subgroup. Mimicking the discussion in the case of a compact group G, we consider the

Hilbert space $L^2(G)$, its closed subspace $L^2(\Gamma\backslash G)$, and the convolution algebra $\mathcal{C}_c(G)$ consisting of those continuous functions on G that have compact support. The convolution product is given by the same formula, except that now there is no canonical choice of a Haar measure, so we need to make an arbitrary choice[1]. As in the case of a compact group G, the Hilbert space $L^2(G)$ is a representation of $G \times G$, its closed subspace $L^2(\Gamma\backslash G)$ is the subspace of invariants under $\Gamma \times \{1\}$, and both are representations of $\mathcal{C}_c(G)$ with $R_f(\phi) = \phi * f^-$.

There is however one fundamental difference. The Hilbert space $L^2(G)$ does not generally decompose as a Hilbert direct sum of irreducible representations of $G \times G$.

Example 3.2.1. The simplest example is the case $G = \mathbb{R}$ as an additive group. Since G is commutative, a $G \times G$-stable subspace is the same as a G-stable subspace. We claim that the vector space $L^2(G)$ has plenty of proper non-zero closed G-invariant subspaces, but none of them is irreducible.

Indeed, it is known that there is a bijection between the set of closed G-invariant subspaces of $L^2(G)$ and the set of measurable subsets of $\mathbb{R}$ taken up to equivalence, where two measurable subsets $A, B \subset \mathbb{R}$ are considered equivalent (we write $A \equiv B$) if $\mu(A - B) = 0 = \mu(B - A)$, see [Rud64, Theorem 9.17]. This bijection assigns to a measurable subset A the subspace $M_A \subset L^2(G)$ consisting of those $f \in L^2(G)$ such that $\widehat{f}|_A = 0$ (almost everywhere). It is immediate that $M_B \subseteq M_A$ if and only if $\mu(A - B) = 0$ (we write $A \leq B$), $\{0\} = M_{\mathbb{R}}$, and $L^2(G) = M_{\varnothing}$. Let us further write $A \lneq B$ for $A \leq B \wedge A \not\equiv B$, which is equivalent to $M_B \subsetneq M_A$.

If $\{0\} \subsetneq M_A \subsetneq L^2(G)$ then $A \subset \mathbb{R}$ is a measurable set such that $\varnothing \lneq A \lneq \mathbb{R}$, equivalently $\mu(A) \neq 0$ and $\mu(\mathbb{R} - A) \neq 0$. We will produce a measurable set B such that $A \lneq B \lneq \mathbb{R}$, equivalently $\mu(B - A) \neq 0$, $\mu(A - B) = 0$, $\mu(\mathbb{R} - B) \neq 0$, which would imply $\{0\} \subsetneq M_B \subsetneq M_A$. For this it would be enough to find two disjoint open intervals $I_0, I_1 \subset \mathbb{R}$ such that $\mu(A \cap I_0) < \mu(I_0)$ and $\mu(A \cap I_1) < \mu(I_1)$ and set $B = A \cup I_0$, because then $\mu(B - A) = \mu(I_0 - A) = \mu(I_0) - \mu(A \cap I_0) > 0$, $\mu(\mathbb{R} - B) = \mu((\mathbb{R} - I_0) - A) \geq \mu(I_1 - A) > 0$, and $\mu(A - B) = \mu(\varnothing) = 0$. To find I_0, I_1, we first choose $x < y \in \mathbb{R}$ such that $\mu(A \cap (x, y)) < y - x$, which exists since $\mu(\mathbb{R} - A) > 0$. Then we do a binary search: set $z = (x + y)/2$ and check if $\mu(A \cap (x, z)) < z - x$ and $\mu(A \cap (z, y)) < y - z$. If both hold we set $I_0 = (x, z)$ and $I_1 = (z, y)$ and are done. If only the first holds replace y by z, and if only the second holds replace x by z. This binary search must terminate in finite time due to the assumption $\mu(A \cap (x, y)) \neq y - x$.

We have thus shown that the right regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$ has no non-zero irreducible closed subspaces. This precludes the decomposition of $L^2(\mathbb{R})$ into a Hilbert direct sum of irreducible closed subspaces in the most radical terms. One can still decompose $L^2(\mathbb{R})$ into irreducible representations, but the decomposition is no longer discrete, i.e. a Hilbert direct sum. Instead,

[1] This is the reason why some authors prefer to work with the convolution algebra of compactly supported measures, which is a more canonical object.

it is purely continuous, i.e. a Hilbert direct integral

$$L^2(\mathbb{R}) = \int_{\mathbb{R}}^{\oplus} \mathbb{C} \cdot e^{2\pi ixy} dy.$$

This is just a formalization of the Fourier inversion formula

$$f(x) = \int_{\mathbb{R}} \widehat{f}(y) e^{2\pi ixy} dy, \qquad \widehat{f}(y) = \int_{\mathbb{R}} \widehat{f}(x) e^{-2\pi ixy} dx.$$

Other interesting features to note are that the decomposition of $L^2(\mathbb{R})$ does not involve all irreducible representations (i.e. characters) of $\mathbb{R}$, but only the unitary ones, and that moreover none of these irreducible unitary representations are actually contained in the space $L^2(\mathbb{R})$, because none of the functions $x \mapsto e^{2\pi ixy}$ are square-integrable on $\mathbb{R}$.

In general, the decomposition of the representation $L^2(G)$ can have both discrete and continuous pieces. We will have to deal with the continuous part later. What helps in the current situation is the following result, which states that the closed subspace $L^2(\Gamma\backslash G)$ does decompose discretely.

Theorem 3.2.2 (Spectral decomposition of $L^2(\Gamma\backslash G)$)**.**

$$L^2(\Gamma\backslash G) = \widehat{\bigoplus_{\pi}} m(\pi) \cdot \pi,$$

where the sum runs over the set of unitary representations of G *and* $m(\pi)$ *is a natural number.*

The proof will require some functional analysis. We briefly review here what we need, both for the proof and for the discussion of the geometric expansion that will follow.

Let H be a separable Hilbert space and $F\colon H \to H$ a bounded operator.

1. If the image under F of the unit ball in H is relatively compact, then F is called a *compact* operator.

2. For any orthonormal basis B of H, the quantity $\sum_{b\in B} \|F(b)\|^2$ is independent of B. If it is finite, then F is called a *Hilbert–Schmidt* operator. Its *Hilbert–Schmidt norm* $\|F\|_2$ is defined to be the square root of that number.

3. Any Hilbert–Schmidt operator is compact.

4. Let $|F| = \sqrt{F^* \circ F}$ denote the positive semi-definite Hermitian square root of F. The sum $\sum_{b\in B}\langle |F|b, b\rangle \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ does not depend on the choice of Hilbert basis of H. If it is finite, then F is called *of trace class*. In that case tr $X := \sum_{b\in B}\langle Fb, b\rangle$ is independent of the orthonormal basis B, and is called the *trace* of F.

5. Any trace class operator is Hilbert–Schmidt.

6. If F_1, F_2 are Hilbert–Schmidt operators, then $F_1 \circ F_2$ is of trace class.

7. In the case $H = L^2(X)$ for a locally compact Hausdorff space X and $k \in L^2(X \times X)$, the integral operator $F(\phi)(x) = \int_X k(x,y)\phi(y)dy$ is Hilbert–Schmidt and $\|F\|_2 = \|k\|_2$. Given two $k_1, k_2 \in L^2(X \times X)$, the composition $F = F_1 \circ F_2$ is still an integral operator with kernel $k(x,y) = \int_X k_1(x,z)k_2(z,y)dz$ and it follows from the Cauchy-Schwartz inequality that the restriction of k to the diagonal copy of X in $X \times X$ is a well-defined element of $L^1(X)$. Moreover, $\operatorname{tr} F = \int_X k(x,x)dx$, which is checked easily in the case $F_2 = F_1^*$, and extended to the general case via the polarization identity for the Hilbert-Schmidt inner product and the identity $(F_1, F_2)_{HS} = \operatorname{tr}(F_2^* \circ F_1)$.

8. If F is compact and self-adjoint, then H is the Hilbert direct sum of eigenspaces

$$H = \widehat{\bigoplus}_{r \in \mathbb{R}} H_r, \qquad H_r = \{v \in H \mid F(v) = rv\}.$$

   For $r \neq 0$ the eigenspace is finite-dimensional. The set of eigenvalues is discrete and the only possible accumulation point is 0.

The key to proving Theorem 3.2.2 is the following result whose proof will be given further below; it is a consequence of Lemma 3.2.6.

Lemma 3.2.3. *Let $f \in \mathcal{C}_c(G)$. The operator $R(f)$ on $L^2(\Gamma\backslash G)$ is Hilbert–Schmidt.*

Proof of Theorem 3.2.2. By Zorn's lemma we can choose a maximal set S of orthogonal closed irreducible submodules of $L^2(\Gamma\backslash G)$. We then claim that the closure of $\bigoplus_{\pi \in S} \pi$ cannot be a proper submodule of $L^2(\Gamma\backslash G)$, and hence must equal to it. For this, it is enough to prove that any non-zero closed invariant subspace of $L^2(\Gamma\backslash G)$ contains a non-zero irreducible subspace (this is exactly what we showed fails in Example 3.2.1). Indeed, when applied to the orthocomplement of $\bigoplus_{\pi \in S} \pi$ this would contradict the maximality of S.

To see the claim, let H be a non-zero closed invariant subspace. Then we have the action of $\mathcal{C}_c(G)$ on it by the operators $R(f)$. Each $R(f)$ is Hilbert–Schmidt by Lemma 3.2.3, hence compact. We can choose a sequence $f_n \in \mathcal{C}_c(G)$ approaching Dirac δ such that $f_n(g) = f_n(g^{-1})$. Then $R(f_n)$ is a sequence of self-adjoint operators approaching the identity operator, thus one of them is non-zero. The spectral theorem applied to this operator implies the existence of a non-zero eigenvalue λ, and that $H(\lambda)$ is finite-dimensional.

Among all closed invariant subspaces M of H choose one for which the dimension of $M(\lambda)$ is minimal but non-zero. We claim that we can arrange M to be a cyclic module. To see this, let $0 \neq v \in M(\lambda)$ and let $E \subset M$ be the closed invariant subspace of M generated by v. Then $M = E \oplus E^\perp$ is an invariant decomposition, and $M(\lambda) = E(\lambda) \oplus E^\perp(\lambda)$. Since $v \in E(\lambda)$, the minimality of the dimension of $M(\lambda)$ shows $E^\perp(\lambda) = 0$. Thus E is also a closed invariant subspace of H for which $\dim E(\lambda)$ is minimal non-zero. We replace M by E, achieving that M is a cyclic module.

We claim that now M is irreducible. To see this, let $E_1 \subset M$ be a closed invariant subspace. Then $M = E_1 \oplus E_1^\perp$, and $M(\lambda) = E_1(\lambda) \oplus E_1^\perp(\lambda)$. By minimality of dimension, either $E_1(\lambda)$ or $E_1^\perp(\lambda)$ is zero. But $v \in M(\lambda)$ must then belong to the non-zero subspace, say $E_1(\lambda)$. But M being cyclic, this implies $M = E_1$.

We have now completed the proof of the existence of a non-zero irreducible subspace in any non-zero closed invariant subspace H, which implies that $L^2(\Gamma\backslash G)$ decomposes as a Hilbert direct sum of irreducible closed invariant subspaces. The fact that each isomorphism class of such subspaces only contributes finitely many copies follows from a similar argument: If H denotes an isotypic component in $L^2(\Gamma\backslash G)$, the finite-dimensionality of $H(\lambda)$ discussed above implies that H is a sum of finitely many irreducible closed subspaces. □

We can now derive the spectral side of the trace formula.

Corollary 3.2.4. *Let $f_1, f_2 \in \mathcal{C}_c(G)$ and let $f = f_1 * f_2$. Then the operator $R(f)$ is of trace class and*

$$tr R(f) = \sum_{\pi} m_\Gamma(\pi) tr\, \pi(f).$$

Proof. By Lemma 3.2.3, the operators $R(f_1)$ and $R(f_2)$ are Hilbert–Schmidt, hence $R(f) = R(f_1) \circ R(f_2)$ is of trace class. The second statement follows from Theorem 3.2.2. □

Remark 3.2.5. Unlike the case of a compact group, where $m_\Gamma(\pi) = \dim(\pi^\Gamma)$, the natural numbers $m_\Gamma(\pi)$ that appear in the above identity do not have such a simple interpretation. In fact, it is often the computation of these numbers that is the goal of applying the trace formula.

We now turn to the derivation of the geometric side of the trace formula, and the proof of Lemma 3.2.3. For $f \in \mathcal{C}_c(G)$ we have

$$R(f)\varphi(g) = \int_G f(x)\varphi(gx)dx.$$

This is an operator on the Hilbert space $L^2(\Gamma\backslash G)$ and we can rewrite it as an integral operator as follows.

$$\begin{aligned} R(f)\varphi(y) &= \int_G f(x)\varphi(yx)dx \\ &= \int_G f(y^{-1}x)\varphi(x)dx \\ &= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(y^{-1}\gamma x)\varphi(\gamma x)dx \\ &= \int_{\Gamma\backslash G} \left(\sum_{\gamma\in\Gamma} f(y^{-1}\gamma x) \right) \varphi(x)dx. \end{aligned}$$

In other words, $R(f)$ is the integral operator defined by the kernel

$$k_f(y,x) = \sum_{\gamma \in \Gamma} f(y^{-1}\gamma x). \tag{3.2.1}$$

Lemma 3.2.6. *The function k_f is continuous, and hence square-integrable.*

Proof. Let U and V be open neighborhoods of y and x whose closures $\bar{U}$ and $\bar{V}$ are compact. Let K be the support of f, by assumption compact. Then $\bar{U} \cdot K \cdot \bar{V}^{-1}$ is also compact, hence intersects Γ in a finite set. It follows that the restriction of k_f to $U \times V$ is the sum of $f(y^{-1}\gamma x)$ for only finitely many γ, and hence continuous. Since $\Gamma \backslash G$ is assumed compact, continuity implies square-integrability. □

We have now completed the proof of Lemma 3.2.3. Taking again $f = f_1 * f_2$ with $f_1, f_2 \in \mathcal{C}_c(G)$ we see that $R(f)$ is of trace class. The geometric side of the trace formula is now derived as follows.

$$\begin{aligned}
\mathrm{tr} R(f) &= \int_{\Gamma \backslash G} k_f(x,x)dx \\
&= \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} f(x^{-1}\gamma x)dx \\
&= \int_{\Gamma \backslash G} \sum_{[\gamma] \in [\Gamma]} \sum_{\delta \in \Gamma_\gamma \backslash \Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)dx \\
&= \sum_{[\gamma] \in [\Gamma]} \int_{\Gamma_\gamma \backslash G} f(x^{-1}\gamma x)dx \\
&= \sum_{[\gamma] \in [\Gamma]} \mathrm{vol}(G_\gamma / \Gamma_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)dx.
\end{aligned}$$

Equating the geometric and spectral sides we obtain the main result of this section.

Theorem 3.2.7 (Trace formula for compact quotient)**.** *Let G be a locally compact unimodular group, $\Gamma \subset G$ a discrete closed cocompact subgroup. Then*

$$\sum_{\pi} m_\Gamma(\pi) tr\, \pi(f \cdot dx) = \sum_{[\gamma] \in [\Gamma]} vol(G_\gamma / \Gamma_\gamma; dx_\gamma / d_{count}) \int_{x \in G/G_\gamma} f(x\gamma x^{-1}) dx/dx_\gamma,$$

for any function $f \in \mathcal{C}_c(G)$ of the form $f = f_1 * f_2$ with $f_1, f_2 \in \mathcal{C}_c(G)$, where dx_γ is an arbitrarily chosen Haar measure on G_γ and d_{count} is the counting measure on the discrete group Γ_γ.

Remark 3.2.8. The class of functions f of the form $f = f_1 * f_2$ may seem a bit unwieldy. Slightly more generally, we can allow finite sums of such functions. There are two special cases where one can say more precisely what functions f have this form.

(1) G is locally profinite, f is locally constant and compactly supported.

(2) G is a Lie group, f is smooth and compactly supported.

In case (1) we can take a sequence α_n of smooth compactly supported functions that approximate the Dirac delta distribution. That is, $\int_G \alpha_n = 1$ and $\mathrm{supp}(\alpha_n)$ forms a descending sequence of subsets of G with intersection equal to $\{1\}$. For any smooth compactly supported function f the sequence $f * \alpha_n$ converges uniformly to f. In fact, due to the disconnected nature of the topology of G, there exists n such that $f = f * \alpha_n$.

In case (2) we can apply the Dixmier–Malliavin theorem, which implies that $f = \sum_{n=1}^{N} f_n * g_n$ for some $f_n, g_n \in \mathcal{C}_c^\infty(G)$.

Remark 3.2.9. The quantity $\int_{x \in G/G_\gamma} f(x\gamma x^{-1}) dx/dx_\gamma$ is commonly referred to as the *orbital integral of f at x*, because it is the integral of f over the orbit through x for the conjugation action of G on itself.

Example 3.2.10. An example in which Theorem 3.2.7 applies is $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$. Then $\mathbb{R}/\mathbb{Z} = \mathbb{S}^1$ with action of $\mathbb{R}$ by translation, factoring through the quotient $\mathbb{R}/\mathbb{Z} = \mathbb{S}^1$. The $\mathbb{S}^1$-representation $L^2(\mathbb{S}^1)$ decomposes, by the Peter-Weyl theorem, as the sum of characters $e^{2\pi i n x}$ with multiplicity 1. The spectral side of the trace formula then becomes

$$\sum_n \mathrm{tr}\langle e^{2\pi i n x}, f \rangle = \sum_n \widehat{f}(n).$$

On the geometric side the abelianness of Γ makes the first sum go over Γ itself. The volume factor is trivial for the same reason. The orbital integral becomes the evaluation of f at γ. So the geometric side becomes $\sum_{n \in \mathbb{Z}} f(n)$. The trace formula thus recovers the Poisson summation formula.

4 The trace formula for anisotropic reductive groups

4.1 Basic notation

Let F be a number field. For any place v of F we denote by F_v the completion of F at v. If v is finite we denote by $P_v \subset O_v \subset F_v$ the maximal ideal and the ring of integers of the local field F_v, and $k_v = O_v/P_v$ the residue field.

Let $\mathbb{A} = \mathbb{A}_F = \prod_v' F_v$ be the ring of adeles of F. Thus $\mathbb{A} = \varinjlim_S \mathbb{A}_{(S)}$, where S runs over all finite sets of places of F containing all archimedean places and $\mathbb{A}_{(S)} = \prod_{v \in S} F_v \times \prod_{v \notin S} O_v$; we equip $\mathbb{A}_{(S)}$ with the product topology, and $\mathbb{A}$ with the final topology. The diagonal embedding of F into $\mathbb{A}$ has discrete image and compact cokernel, cf. [CF86, Chapter II, §14].

For a finite set of places S of F we will write $F_S = \prod_{v \in S} F_v$ and $\mathbb{A}^S = \{(a_v) \in \mathbb{A} | a_v = 0 \, \forall v \in S\}$, so that $\mathbb{A}_{(S)} = F_S \times \mathbb{A}^S$.

Let $\mathbb{A}^\times = \mathbb{A}_F^\times$ be the group of units in $\mathbb{A}$, thus the group of ideles. We have $\mathbb{A}^\times = \varinjlim \mathbb{A}_{(S)}^\times$ with $\mathbb{A}_{(S)}^\times = \prod_{v \in S} F_v^\times \times \prod_{v \notin S} O_v^\times$. Again we equip $A_{(S)}^\times$ with the product topology and $\mathbb{A}^\times$ with the final topology. Note that this topology is not the same as the subspace topology coming from the inclusion $\mathbb{A}^\times \subset \mathbb{A}$. The diagonal embedding of $F^\times$ into $\mathbb{A}^\times$ has discrete image, but the cokernel is not compact.

Let $|-|_{\mathbb{A}}\colon \mathbb{A}^\times \to \mathbb{R}_{>0}$ be the idele absolute value, given by $|(x_v)|_{\mathbb{A}} = \prod_v |x_v|_{F_v}$. It is a surjective continuous group homomorphism. Let $\mathbb{A}^1$ be the kernel of this group homomorphism. Then $F^\times \subset \mathbb{A}^1$ and $F^\times \backslash \mathbb{A}^1$ is compact, cf [CF86, Chapter II,§16].

Let G be a connected reductive F-group. We denote by A_G the maximal F-split torus in the center Z_G of G. Let $X^*(G) = \mathrm{Hom}_{\text{alg.grp}}(G, \mathbb{G}_m)$ be the abelian group of algebraic characters of G, and let $X^*(G)_F \subset X^*(G)$ be the subgroup of those characters that are defined over F. Note that $X^*(G)$ is a lattice (a finitely generated free $\mathbb{Z}$-module) with an action of the absolute Galois group Γ, and $X^*(G)_F = X^*(G)^\Gamma$. Note further that $X^*(A_G)_F = X^*(A_G)$, since A_G is a split torus.

Restriction along the inclusion $A_G \to G$ induces an inclusion of lattices $X^*(G)_F \to X^*(A_G)$ of F-rational characters, whose image is of finite index. Let $\mathfrak{a}_G = X_*(A_G) \otimes_{\mathbb{Z}} \mathbb{R}$. We have the continuous surjective group homomorphism

$$H_G\colon G(\mathbb{A}) \to \mathfrak{a}_G, \qquad \langle H_G(g), \chi\rangle = \log(|\chi(g)|_{\mathbb{A}}), \quad \forall\, \chi \in X^*(G)_F. \tag{4.1.1}$$

Define $G(\mathbb{A})^1 \subset G(\mathbb{A})$ to be the kernel of H_G. Note that, according to the product formula, $G(F) \subset G(\mathbb{A})^1$.

If $F = \mathbb{Q}$ then the surjective homomorphism $|-|_{\mathbb{A}}\colon \mathbb{A}^\times \to \mathbb{R}_{>0}$ has a distinguished splitting, given by the inclusion $\mathbb{R}_{>0} \to \mathbb{R}^\times \to \mathbb{A}^\times$. This leads to the natural direct product decomposition $\mathbb{A}^\times_{\mathbb{Q}} = \mathbb{R}_{>0} \times \mathbb{A}^1_{\mathbb{Q}}$, and more generally $G(\mathbb{A}) = A_G(\mathbb{R})^\circ \times G(\mathbb{A})^1$.

For general F we consider the embedding

$$\mathbb{R}_{>0} \to \prod_{v|\infty} F_v^\times, \quad r \mapsto (r^{c_v})_v, \quad c_v = ([F_v : \mathbb{R}] \cdot \#\{v|\infty\})^{-1}.$$

This embedding is again a section of $|-|_{\mathbb{A}} : \mathbb{A}^\times \to \mathbb{R}_{>0}$ and leads to the decomposition $\mathbb{A}_F^\times = \mathbb{R}_{>0} \times \mathbb{A}_F^1$. More generally $G(\mathbb{A}) = A_G^+ \times G(\mathbb{A})^1$, where we denote by A_G^+ the subgroup of $\prod_{v|\infty} A_G(F_v) = X_*(A_G) \otimes_{\mathbb{Z}} \prod_{v|\infty} F_v^\times$ isomorphic to $X_*(A_G) \otimes_{\mathbb{Z}} \mathbb{R}_{>0}$.

We could have also more simply considered the diagonal embedding $\mathbb{R}_{>0} \to \prod_{v|\infty} F_v^\times$, i.e. we could have taken $c_v = 1$ above. We would again obtain a direct product decomposition $\mathbb{A}^\times = \mathbb{R}_{>0} \times \mathbb{A}^1$, but we need to be mindful that the diagonal embedding is not a section of the idele norm map. We'd also get the product decomposition $G(\mathbb{A}) = G(\mathbb{A})^1 \times A_G^+$, with a slightly different A_G^+.

4.2 Decomposition of $L^2(G(F)\backslash G(\mathbb{A}_F))$ under the action of A_G^+

The elements of the Hilbert space $L^2(G(F)\backslash G(\mathbb{A}_F))$ are called *square-integrable automorphic forms*. The group A_G^+ acts on this space and decomposes it as the direct integral

$$\int L^2_\chi(G(F)\backslash G(\mathbb{A}_F))d\chi.$$

in analogy with Example 3.2.1, where χ runs over the space of unitary characters of A_G^+. Note that, non-canonically, $A_G^+ \cong \mathbb{R}^n$, where $n = \dim(A_G)$, and the space of unitary characters of A_G^+ is also non-canonically isomorphic to $\mathbb{R}^n$. More invariantly, the space of unitary characters of A_G^+ is identified with $i\mathfrak{a}_G^*$, with $\lambda \in i\mathfrak{a}_G^*$ serving as the restriction to $A_G^+ \subset G(\mathbb{A})$ of the unitary character of $G(\mathbb{A})$ given by

$$g \mapsto e^{\langle H_G(g),\lambda\rangle}. \tag{4.2.1}$$

Therefore we will write

$$L^2(G(F)\backslash G(\mathbb{A})) = \int_{i\mathfrak{a}_G^*} L^2_\lambda(G(F)\backslash G(\mathbb{A}_F))d\lambda. \tag{4.2.2}$$

The direct product decomposition $G(\mathbb{A}) = A_G^+ \times G(\mathbb{A})^1$ implies that restriction along the inclusion $G(\mathbb{A})^1 \to G(\mathbb{A})$ induces an isomorphism $L^2_\lambda(G(F)\backslash G(\mathbb{A})) = L^2(G(F)\backslash G(\mathbb{A})^1)$ of $G(\mathbb{A})^1$-representations, for all λ. If we transport the $G(\mathbb{A})$-action under this isomorphism, it will of course depend on λ, but this dependence is described very simply as follows. If we endow $L^2(G(F)\backslash G(\mathbb{A})^1)$ with the structure of a $G(\mathbb{A})$-representation by identifying it with $L^2(G(F)A_G^+\backslash G(\mathbb{A}))$, then the $G(\mathbb{A})$-structure obtained from the identification with $L^2_\lambda(G(F)\backslash G(\mathbb{A}))$ for an arbitrary $\lambda \in i\mathfrak{a}_G^*$ is given by twisting by the character (4.2.1). We will from now on write $L^2_\lambda(G(F)\backslash G(\mathbb{A})^1)$ for this action of $G(\mathbb{A})$ on the Hilbert space $L^2(G(F)\backslash G(\mathbb{A})^1)$.

4.3 Anisotropic groups

Recall that a connected reductive F-group G is called *anisotropic* if it does not contain a split torus. Recall further that an element $\gamma \in G(F)$ is called *elliptic* if it is contained in a maximal torus of G that is anisotropic modulo A_G.

Proposition 4.3.1. *The following are equivalent.*

1. *G/A_G is anisotropic.*
2. *G does not contain a proper parabolic subgroup.*
3. *Every element of $G(F)$ is elliptic.*

Example 4.3.2. 1. Let D/F be a central division algebra and let G be the connected reductive F-group such that $G(F) = D^\times$. Then $A_G = \mathbb{G}_m$ and G/A_G is anisotropic.

2. Let E/F be a quadratic extension and let (V, f) be an anisotropic Hermitian space over E. Then the unitary group $U(V, f)$ is anisotropic.

3. Let (V, f) be an anisotropic quadratic space over F. Then the special orthogonal group $SO(V, f)$ is anisotropic.

The trace formula for groups G for which G/A_G is anisotropic is much simpler than that for general reductive groups G. The reason is the following result of reduction theory.

Theorem 4.3.3 ([PR94, §5.3, Theorem 5.5]). *The space $G(F)\backslash G(\mathbb{A}_F)^1$ is compact if and only if G/A_G is anisotropic.*

4.4 The trace formula

Since the $G(\mathbb{A})$-representations $L^2_\lambda(G(F)\backslash G(\mathbb{A}_F)^1)$ for various λ are just character twists of each other, it is enough to study the case $\lambda = 0$, i.e. the space $L^2(G(F)\backslash G(\mathbb{A}_F)^1)$. Theorem 4.3.3 allows us to apply the trace formula for co-compact quotient, as stated in Theorem 3.2.7, and more generally the discussion of §3. It implies that the space of square-integrable automorphic forms decomposes discretely

$$L^2(G(F)\backslash G(\mathbb{A}_F)^1) = \widehat{\bigoplus_\pi} m(\pi) \cdot \pi.$$

The sum runs over all irreducible admissible representations of $G(\mathbb{A}_F)/A_G^+$. Those for which $m(\pi) > 0$ are called *automorphic*. Theorem 3.2.7 gives the formula

$$\sum_\pi m(\pi) \operatorname{tr} \pi(f) = \sum_{\gamma \in [G(F)]} \tau(G_\gamma) O_\gamma(f), \tag{4.4.1}$$

where

$$O_\gamma(f) = \int_{G(\mathbb{A})/G_\gamma(\mathbb{A})} f(x\gamma x^{-1}) dx/dx_\gamma.$$

We have used the canonical Tamagawa measures on $G(\mathbb{A})$ and $G_\gamma(\mathbb{A})$ reviewed in Appendix B, and $\tau(G_\gamma)$ is the Tamagawa number of G_γ.

Thus the spectral side of the trace formula consists of traces of automorphic representations weighted by the multiplicities of these representations, and the geometric side consists of orbital integrals at elliptic elements weighted by the Tamagawa numbers of their centralizers.

5 The problem of stability

A distribution d is called *invariant* if $d(f) = d({}^g f)$ for any test function $f \in \mathcal{C}_c^\infty(G(\mathbb{A}))$ and $g \in G(\mathbb{A})$, where ${}^g f(x) = f(g^{-1}xg)$. Both sides of the trace formula (4.4.1) are invariant distributions. However, for many applications a stronger invariance property is required, called *stable invariance*, or *stability*.

5.1 Stable conjugacy and stable distributions

The notion of stability of distributions is based on the notion of *stable conjugacy*. We introduce it first in a special case that will be sufficient for this exposition, and refer to [Kal24] for a more general discussion.

Let F be a field of characteristic zero, $\bar{F}$ a fixed algebraic closure, and Γ the absolute Galois group. Let G be a connected reductive F-group.

Definition 5.1.1. Let $\gamma \in G$ be a semi-simple element and let G_γ denote its centralizer. Then γ is called *regular* if G_γ° is a torus, and *strongly regular* if G_γ is a torus.

Thus, a strongly regular semi-simple element is a regular semi-simple element with connected centralizer. When the derived subgroup of G is simply connected, the centralizer of a semi-simple element is automatically connected (due to a theorem of Steinberg), hence the notions of "regular" and "strongly regular" coincide.

The study of stability begins with the following definition, which we give in the simplified context of connected centralizers, and refer to [Kot82, §3] for the general case.

Definition 5.1.2. Let $\gamma_1, \gamma_2 \in G(F)$ be semi-simple elements whose centralizers are connected. Then γ_1, γ_2 are called *stably conjugate* (sometimes denoted by $\gamma_1 \sim \gamma_2$) if there exists $g \in G(\bar{F})$ such that $g\gamma_1 g^{-1} = \gamma_2$.

It is clear that conjugate elements are stably conjugate. The most basic example of stably conjugate elements that are not conjugate is given by the elements

$$\gamma_1 = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}, \qquad \gamma_2 = \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix}$$

of $\mathrm{SL}_2(\mathbb{R})$. These elements are not conjugate in $\mathrm{SL}_2(\mathbb{R})$, but become conjugate in $\mathrm{SL}_2(\mathbb{C})$, because they have the same eigenvalues.

The set of elements of $G(F)$ that are stably conjugate to a given $\gamma \in G(F)$ is called the *stable (conjugacy) class* of γ. This stable class is a union of $G(F)$-conjugacy classes. The latter are sometimes called *rational* (conjugacy) classes, in order to distinguish them more easily in language from the stable (conjugacy) classes.

Recall that, for two F-groups A, B, an isomorphism $f\colon A_{\bar{F}} \to B_{\bar{F}}$ is called an *inner twisting* if for all $\sigma \in \Gamma$ the automorphism $f^{-1} \circ \sigma(f) := f^{-1} \circ \sigma_B \circ f \circ \sigma_A^{-1}$ of $A_{\bar{F}}$ is inner, where σ_A and σ_B denote the action of σ on $A_{\bar{F}}$ resp. $B_{\bar{F}}$. The following is easy to check.

Fact 5.1.3. *We continue with the setting of Definition 5.1.2.*

1. *If γ_1, γ_2 are strongly regular, the isomorphism $Ad(g)\colon G_{\gamma_1} \to G_{\gamma_2}$ depends only on γ_1 and γ_2, but not on the choice of g. It is defined over F. We shall call it $\varphi_{\gamma_1,\gamma_2}$.*

2. *In general, the isomorphism $Ad(g)\colon G_{\gamma_1} \to G_{\gamma_2}$ is an inner twisting. It induces an isomorphism $H^1(\Gamma, G_{\gamma_1}(\bar{F})) \to H^1(\Gamma, G_{\gamma_2}(\bar{F}))$ that sends the class of $z \in Z^1(\Gamma, G_{\gamma_1}(\bar{F}))$ to the class of $\sigma \mapsto gz(\sigma)\sigma(g)^{-1}$. It is independent of the choice of g and will be denoted by $\varphi_{\gamma_1,\gamma_2}$.*

3. *$\sigma \mapsto g^{-1}\sigma(g)$ belongs to $Z^1(\Gamma, G_{\gamma_1}(\bar F))$ and its cohomology class is independent of the choice of g. We shall call it inv(γ_1, γ_2).*

4. *The map $\gamma_2 \mapsto$ inv(γ_1, γ_2) is a bijection between the set of rational conjugacy classes inside of the stable class of γ_1 and the set ker$(H^1(\Gamma, G_{\gamma_1}(\bar F)) \to H^1(\Gamma, G(\bar F)))$.*

5. *Given γ_3 stably conjugate to γ_2, we have*

$$\mathit{inv}(\gamma_1, \gamma_3) = \varphi^{-1}_{\gamma_1,\gamma_2}(\mathit{inv}(\gamma_2, \gamma_3)).$$

The set of rational classes in a given stable class can well be infinite. This happens for example when F is a global field. On the other hand, when F is a local field, classical results in Galois cohomology ensure that this set is always finite, cf. [PR94, §6.4].

Consider now a local field F. A prototypical example of an invariant distribution the *orbital integral*

$$O_\gamma : \mathcal{C}^\infty_c(G(F)) \to \mathbb{C}, \qquad O_\gamma(f) = \int_{G(F)/G_\gamma(F)} f(x\gamma x^{-1})dx$$

for any $\gamma \in G(F)$. When γ is semi-simple, the conjugacy class of γ is closed in $G(F)$ and intersects the support of f in a compact subset, so the convergence of this integral is not problematic. These are the orbital integrals that occur in (4.4.1). For general γ the situation is not as transparent, but it has been established in [Rao72] that the integral does converge.

A prototypical example of a stable distribution is the *stable orbital integral* associated to a semi-simple element $\gamma \in G(F)$ that is *strongly regular*, i.e. its centralizer in G is a torus:

$$SO_\gamma(f) = \sum_{\substack{\gamma' \in [G(F)] \\ \gamma' \sim \gamma}} O_{\gamma'}(f), \tag{5.1.1}$$

where γ' runs over the set of rational classes in the stable class of γ. As discussed above, this set is finite.

One now defines a distribution to be stable if it is contained in the weak closure of the distributions SO_γ. More precisely, the definition is the following.

Definition 5.1.4. A distribution $d\colon \mathcal{C}^\infty_c(G(F)) \to \mathbb{C}$ is called *stable* if $d(f) = 0$ for all $f \in \mathcal{C}^\infty_c(G(F))$ such that $SO_\gamma(f) = 0$ for all strongly regular semi-simple $\gamma \in G(F)$.

This definition mirrors a theorem of Harish-Chandra, which states that a distribution is invariant if and only if it is contained in the weak closure of the distributions O_γ for all strongly regular semi-simple $\gamma \in G(F)$, cf. [Kal24, Theorem 2.1.7] for an exposition.

It is tautological that SO_γ is a stable distribution when γ is strongly regular semi-simple. However, if we formed the sum (5.1.1) in the case when γ is not strongly regular semi-simple, the result may not be a stable distribution. We will take this up again in §6.2.

We need to also consider adelic distributions, i.e. linear functionals on $\mathcal{C}_c^\infty(G(\mathbb{A}))$ when F is global. Such a distribution is to be considered invariant if it is invariant in each component f_v of a factorizable test function $f = \otimes_v f_v$, and stably invariant again if it is so in each component f_v. Again, it is tautological that the distribution

$$SO_\gamma(f) = \sum_{\substack{\gamma' \in [G(\mathbb{A})] \\ \gamma' \sim \gamma}} O_{\gamma'}(f), \tag{5.1.2}$$

is stable. Here we are summing over the adelic rational classes inside of the adelic stable class of γ. More precisely, Definition 5.1.2 and Fact 5.1.3 remain valid in the adelic context, provided we replace all occurrences of $\bar{F}$ with $\bar{\mathbb{A}}$, where $\bar{\mathbb{A}} = \mathbb{A} \otimes_F \bar{F} = \varinjlim_E \mathbb{A} \otimes_F E$.

We need to be mindful of the fact that the indexing set of the sum is generally infinite. But in the cases of interest all but finitely many summands will be zero. Indeed, we will assume that γ has the property that for all but finitely many places v of F the local component $\gamma_v \in G(F_v)$ is semi-simple and $1 - \alpha(\gamma_v) \in \bar{F}_v$ is either zero or a unit in $O_{\bar{F}_v}$ for every absolute root α. This is for example the case for semi-simple elements that are $G(\bar{\mathbb{A}})$-conjugate to an element of $G(\bar{F})$. We refer the reader to [Kot86, §7,§8] for more detailed discussion.

5.2 Rationality of conjugacy classes

According to Definition 5.1.2, the *stable conjugacy class* of a semi-simple element $\gamma \in G(F)$ with connected centralizer is the intersection with $G(F)$ of its conjugacy class in $G(\bar{F})$. Thus, one could define the notion of a stable class as being the intersection with $G(F)$ of a conjugacy class in $G(\bar{F})$ consisting of semi-simple elements (with connected centralizers), provided that this intersection is non-empty.

It is clear that a necessary condition for non-emptiness is that the conjugacy class in $G(\bar{F})$ be invariant under the Galois group Γ. A natural question is then whether or not this condition is also sufficient. This has been investigated by Kottwitz [Kot82], based on work of Steinberg, and has become a key tool in the stabilization of the trace formula, which we will use in §6.

Kottwitz's result [Kot82, Theorem 4.1] states that Γ-invariance is sufficient, provided G is quasi-split and its derived subgroup is simply connected.

5.3 Instability in the trace formula for anisotropic groups

Assume that G/A_G is anisotropic. For simplicity, we also assume that the derived subgroup is simply connected. Then G_γ is connected for any $\gamma \in G(F)$ according to [Ste68a, §8].

Consider the distribution $J(f) = \operatorname{tr} R(f)$. We can check if it is stable by looking either at the geometric or the spectral expansion. The geometric expansion is

$$J_{\text{geom}}(f) = \sum_{\gamma \in [G(F)]} \tau(G_\gamma) O_\gamma(f).$$

We can rewrite it as

$$\sum_{\gamma \in [[G(F)]]} \sum_{\substack{\gamma' \in [G(F)] \\ \gamma' \sim \gamma}} \tau(G_{\gamma'}) O_{\gamma'}(f),$$

where now $[[G(F)]]$ stands for the set of stable classes in $G(F)$, and γ' runs over the set of rational classes inside of the stable class of γ.

Given $\gamma' \sim \gamma$ there exists $g \in G(\bar{F})$ such that $g\gamma g^{-1} = \gamma'$. According to Fact 5.1.3, the isomorphism $\operatorname{Ad}(g)\colon G_\gamma \to G_{\gamma'}$, is an *inner twisting*. The main result of [Kot88], recalled here as Corollary B.0.5, shows $\tau(G_{\gamma'}) = \tau(G_\gamma)$. In fact, the proof of this identity used Kottwitz's work on the stabilization of the trace formula. The geometric expansion of J becomes

$$J_{\text{geom}}(f) = \sum_{\gamma \in [[G(F)]]} \tau(G_\gamma) \sum_{\substack{\gamma' \in [G(F)] \\ \gamma' \sim \gamma}} O_{\gamma'}(f). \tag{5.3.1}$$

Comparing with (5.1.1), we'd be led to believe that the above equals

$$\sum_{\gamma \in [[G(F)]]} \tau(G_\gamma) SO_\gamma(f),$$

and therefore the distribution J is stable. This is not so! There are in fact two issues at play here.

The first issue is that the inner sum runs over the set of rational classes inside of the stable class of γ in $G(F)$, and F is now global, while SO_γ would require the sum over the rational classes inside of the stable class of γ in $G(\mathbb{A})$, as in (5.1.2). The latter set is usually larger, and this discrepancy is the reason for the instability of the inner sum in (5.3.1) in the case of a strongly regular (automatically semi-simple) γ.

The second issue is posed by the fact that we have not investigated whether or not SO_γ is a stable distribution when γ is not strongly regular.

6 The stable trace formula for anisotropic groups

We continue with a global field F of characteristic zero, a fixed algebraic closure $\bar{F}$, Galois group Γ, and a connected reductive F-group G. We assume that the derived subgroup of G is simply connected; this is a simplifying condition which implies that the centralizer of any semi-simple element of G is connected, [Ste68a, §8].

In (4.4.1) we obtained a trace formula when G/A_G is anisotropic, and in §5.3 we discussed that this trace formula is not stable. In this section we give an indication of the stabilization process for this trace formula.

The stabilization of the geometric side involves roughly two steps. The first, called "prestabilization", rewrites the geometric side into a sum of so-called "kappa" terms. The second step, which we may called "transfer", interprets each "kappa" term in terms of a stable distribution on a smaller groups, a so-called "endoscopic group".

The stabilization of the spectral side involves the Arthur-Langlands conjectures for the automorphic spectrum of G, and is therefore only conditional.

6.1 Prestabilization assuming the Hasse principle

In this subsection we will assume for simplicity that G satisfies the Hasse principle, cf. [PR94, §6]. This makes the definition of a key cohomological object (the "obstruction") more transparent. In the general case the definition of the obstruction is more complicated and uses the fact that the simply connected cover G_{sc} of the derived subgroup of G does satisfy the Hasse principle, cf. [Kal24, §5.3] for an exposition of this more general case. We note that the groups considered in Example 4.3.2 do satisfy the Hasse principle.

We begin with the expression (5.3.1). As we discussed, the distribution

$$\sum_{\substack{\gamma' \in [G(F)] \\ \gamma' \sim \gamma}} O_{\gamma'}(f) \tag{6.1.1}$$

where γ' runs over the set of rational classes inside of the stable class of a fixed $\gamma \in G(F)$, is not stable, because the intersection with $G(F)$ of the stable class of $\gamma \in G(\mathbb{A})$ is generally larger than the stable class of $\gamma \in G(F)$. In order to move forward, we need to quantify this difference.

Fact 5.1.3 shows that the set of F-rational conjugacy classes in the F-stable class of γ is in natural bijection with

$$A := \ker(H^1(\Gamma, G_\gamma(\bar{F})) \to H^1(\Gamma, G(\bar{F}))).$$

The adelic version of this Fact shows that the set of $\mathbb{A}$-rational conjugacy classes in the $\mathbb{A}$-stable class of γ is in natural bijection with

$$B := \ker(H^1(\Gamma, G_\gamma(\bar{\mathbb{A}})) \to H^1(\Gamma, G(\bar{\mathbb{A}}))).$$

Since we now have two notions of rational conjugacy (F-rational and $\mathbb{A}$-rational), and also two notions of stable conjugacy (F-stable and $\mathbb{A}$-stable), we will temporarily write $\mathrm{inv}_F(\gamma, \gamma')$ and $\mathrm{inv}_{\mathbb{A}}(\gamma, \gamma')$ to avoid confusion. Eventually we will drop the subscript, since we will only use the $\mathbb{A}$-version.

The map

$$\alpha\colon A \to B$$

induced by $H^1(\Gamma, G_\gamma(\bar{F})) \to H^1(\Gamma, G_\gamma(\bar{\mathbb{A}}))$ sends $\mathrm{inv}_F(\gamma, \gamma')$ to $\mathrm{inv}_{\mathbb{A}}(\gamma, \gamma')$. This map is in general neither injective nor surjective. If $\gamma', \gamma'' \in [G(F)]$ are such that $\mathrm{inv}_{\mathbb{A}}(\gamma,\gamma') = \mathrm{inv}_{\mathbb{A}}(\gamma,\gamma'')$, then γ' and γ'' are conjugate in $G(\mathbb{A})$ and hence $O_{\gamma'}(f) = O_{\gamma''}(f)$. At the same time, given $\gamma' \in G(\mathbb{A})$ that is $\mathbb{A}$-stably conjugate to $\gamma \in G(F) \subset G(\mathbb{A})$, there exists an element of $G(F)$ inside of the $G(\mathbb{A})$-conjugacy class of γ' if and only if the element $\mathrm{inv}_{\mathbb{A}}(\gamma,\gamma') \in B$ lies in the image of α. Therefore the distribution (6.1.1) becomes

$$\sum_{\substack{\gamma' \in [G(\mathbb{A})] \\ \gamma' \sim \gamma \\ \mathrm{inv}_{\mathbb{A}}(\gamma,\gamma') \in \mathrm{im}(\alpha)}} |\alpha^{-1}(\mathrm{inv}_{\mathbb{A}}(\gamma,\gamma'))| \cdot O_{\gamma'}(f).$$

It is known that the fibers of α are finite; we will describe them more explicitly in Lemma 6.1.3. On the other hand, we have

$$SO_\gamma(f) = \sum_{\substack{\gamma' \in [G(\mathbb{A})] \\ \gamma' \sim \gamma}} O_{\gamma'}(f).$$

We have now quantified the difference between (6.1.1) and the stable adelic orbital integral in terms of the failure of the map α to be injective and surjective.

Our next goal is to understand the failure of surjectivity of the map α. For this we recall two useful bits of Galois cohomology: Borovoi's algebraic fundamental group $\pi_1(H)$, and Kottwitz's map $H^1(\Gamma, H(\bar{\mathbb{A}})) \to \pi_1(H)_{\Gamma,\mathrm{tor}}$, both associated to a connected reductive F-group H, which in our applications will be $H = G_\gamma$.

Borovoi's algebraic fundamental group $\pi_1(H)$ is a finitely generated abelian group equipped with a Γ-action and functorial in H, see [Bor98, §1] or [BGA14, §3]. For any maximal torus $T \subset H$ we have the free abelian group $X_*(T)$ and the subgroup $Q^\vee(T) \subset X_*(T)$ spanned by the coroots. If T_1, T_2 are two maximal tori, any element of $H(\bar{F})$ conjugating T_1 to T_2 induces the same isomorphism $X_*(T_1)/Q^\vee(T_1) \to X_*(T_2)/Q^\vee(T_2)$. In this way, $X_*(T)/Q^\vee(T)$ becomes a compatible system indexed by the set of maximal tori in H. Its limit is by definition $\pi_1(H)$. Since Γ acts on the system, it acts on $\pi_1(H)$.

Kottwitz's map $H^1(\Gamma, H(\bar{\mathbb{A}})) \to \pi_1(H)_{\Gamma,\mathrm{tor}}$, whose range is the torsion subgroup of the Γ-coinvariants in $\pi_1(H)$, is constructed in three steps, see [Kot86, §2].

1. When $H = T$ is a torus, then $\pi_1(T) = X_*(T)$ and therefore $\pi_1(T)_{\Gamma,\mathrm{tor}} = H^{-1}(\Gamma_{E/F}, X_*(T))$, where E/F is any finite Galois extension splitting T and $\Gamma_{E/F}$ is the Galois group of E/F. We begin with the the map $H^1(\Gamma, T(\bar{\mathbb{A}})) \to H^1(\Gamma, T(\bar{\mathbb{A}})/T(\bar{F}))$ induced by the projection $T(\bar{\mathbb{A}}) \to T(\bar{\mathbb{A}})/T(\bar{F})$. Due to the vanishing of $H^1(\Gamma_E, \bar{\mathbb{A}}^\times/\bar{F}^\times)$ inflation induces an isomorphism $H^1(\Gamma_{E/F}, T(\mathbb{A}_E)/T(E)) \to H^1(\Gamma, T(\bar{\mathbb{A}})/T(\bar{F}))$. We compose the inverse of this isomorphism with the inverse of the isomorphism $H^{-1}(\Gamma_{E/F}, X_*(T)) \to H^1(\Gamma_{E/F}, T(\mathbb{A}_E)/T(E))$ of Tate-Nakayama. The result is the desired map of Kottwitz, which in this case is a group homomorphism.

2. When the derived subgroup of H is simply connected, set $D = H/H_{\mathrm{der}}$. Then $\pi_1(H) = \pi_1(D)$ and the desired map is the composition of the natural map $H^1(\Gamma, H(\bar{\mathbb{A}})) \to H^1(\Gamma, D(\bar{\mathbb{A}}))$ and the homomorphism of step 1 applied to $T = D$.

3. When H is general one chooses a z-extension of H, i.e. an extension $1 \to K \to H_1 \to H \to 1$, where H_1 has simply connected derived subgroup, and K is an induced torus. Then Kottwitz shows that the map of step 2 applied to H_1 descends to a map for H that is independent of the choice of H_1.

The key result is now the following.

Theorem 6.1.1 ([Kot86, Proposition 2.6])**.** *The sequence of pointed sets*

$$H^1(\Gamma, H(\bar{F})) \to H^1(\Gamma, H(\bar{\mathbb{A}})) \to \pi_1(H)_{\Gamma,tor}$$

is exact.

Kottwitz's map applied in the case of $H = G_\gamma$ provides a map $B \to \pi_1(G_\gamma)_{\Gamma,\mathrm{tor}}$. Using the assumption that G satisfies the Hasse principle we obtain immediately the following result.

Corollary 6.1.2. *The sequence of pointed sets $A \to B \to \pi_1(G_\gamma)_{\Gamma,tor}$ is exact.*

Since $\pi_1(G_\gamma)$ is a finitely generated abelian group, the group $\pi_1(G_\gamma)_{\Gamma,\mathrm{tor}}$ is abelian and finite. We denote its Pontryagin dual by $\mathfrak{K}(G_\gamma)$. Then

$$|\mathfrak{K}(G_\gamma)|^{-1} \sum_{\kappa \in \mathfrak{K}(G_\gamma)} \kappa$$

is the characteristic function of the identity element of the group $\pi_1(G_\gamma)_{\Gamma,\mathrm{tor}}$, and pulling this back to B it becomes the characteristic function of the image of α. With this, (5.3.1) becomes

$$\sum_{\gamma \in [[G(F)]]} \tau(G_\gamma) \cdot |\mathfrak{K}(G_\gamma)|^{-1} \sum_{\kappa \in \mathfrak{K}(G_\gamma)} \sum_{\substack{\gamma' \in [G(\mathbb{A})] \\ \gamma' \sim \gamma}} |\alpha^{-1}(\mathrm{inv}_{\mathbb{A}}(\gamma, \gamma'))| \cdot \kappa(\mathrm{inv}_{\mathbb{A}}(\gamma, \gamma')) O_{\gamma'}(f).$$

We can refine this discussion slightly. Since elements of B map trivially to $H^1(\Gamma, G(\bar{\mathbb{A}}))$ by definition, the image of $B \to \pi_1(G_\gamma)_{\Gamma,\mathrm{tor}}$ lands in the kernel of the group homomorphism $\pi_1(G_\gamma)_{\Gamma,\mathrm{tor}} \to \pi_1(G)_{\Gamma,\mathrm{tor}}$. Let $\mathfrak{K}(\gamma)$ denote the Pontryagin dual of this kernel. Thus $\mathfrak{K}(\gamma)$ is a quotient of $\mathfrak{K}(G_\gamma)$, and is equal to the group $\mathfrak{K}(I/F)$ of [Kot86, §4.6]. The same discussion as above leads to

$$\sum_{\gamma \in [[G(F)]]} \tau(G_\gamma) \cdot |\mathfrak{K}(\gamma)|^{-1} \sum_{\kappa \in \mathfrak{K}(\gamma)} \sum_{\substack{\gamma' \in [G(\mathbb{A})] \\ \gamma' \sim \gamma}} |\alpha^{-1}(\mathrm{inv}_{\mathbb{A}}(\gamma, \gamma'))| \cdot \kappa(\mathrm{inv}_{\mathbb{A}}(\gamma, \gamma')) O_{\gamma'}(f). \tag{6.1.2}$$

We have thus fully explored the failure of surjectivity of α and have rewritten the geometric side of the trace formula accordingly. The reader might object that we have worked too hard, because the condition $\mathrm{inv}_{\mathbb{A}}(\gamma,\gamma') \in \mathrm{im}(\alpha)$, which we have now managed to remove, was also encoded in the fact that $|\alpha^{-1}(\mathrm{inv}_{\mathbb{A}}(\gamma,\gamma'))| = 0$ when it fails, so this condition was in fact redundant from the start. However, we will now quantify the failure of injectivity of α, and this quantification will only apply under the assumption $\mathrm{inv}_{\mathbb{A}}(\gamma,\gamma') \in \mathrm{im}(\alpha)$, i.e. $\gamma' \in G(F)$ up to $G(\mathbb{A})$-conjugacy.

Lemma 6.1.3. *For any $\gamma' \in G(F)$ that is stably conjugate to γ the following identity holds*

$$\tau(G_\gamma) \cdot |\mathfrak{K}(\gamma)|^{-1} \cdot |\alpha^{-1}(\mathit{inv}_{\mathbb{A}}(\gamma,\gamma'))| = \tau(G).$$

Proof. Write $\ker^1(F,H) = \ker(H^1(\Gamma, H(\bar{F})) \to H^1(\Gamma, H(\bar{\mathbb{A}}))$ for any connected linear algebraic group H. The Hasse principle for G is $\ker^1(F,G) = \{1\}$. Therefore we see that $\alpha^{-1}(1) = \ker^1(F,G_\gamma)$. This is known to be finite, cf. [PR94, Theorem 6.15]. Applying the method of Serre twisting (see [Ser97, §5.3]) we see $\alpha^{-1}(\mathrm{inv}_{\mathbb{A}}(\gamma,\gamma')) = \ker^1(F,G_{\gamma'})$. However, since $\ker^1(F,H)$ is invariant under inner twisting (see [Kot84, §4]), we are left with proving

$$\tau(G)\tau(G_\gamma)^{-1} = |\ker^1(F,G_\gamma)| \cdot |\mathfrak{K}(\gamma)|^{-1}.$$

According to Theorem B.0.4 the left hand side equals $|\pi_1(G)_{\Gamma,\mathrm{tor}}| \cdot |\pi_1(G_\gamma)_{\Gamma,\mathrm{tor}}|^{-1} \cdot |\ker^1(F,G_\gamma)|$. The map $\pi_1(G_\gamma) \to \pi_1(G)$ is surjective and remains so after taking Γ-coinvariants. The kernel of $\pi_1(G_\gamma)_\Gamma \to \pi_1(G)_\Gamma$ is finite, because G/A_G is anisotropic (in fact, it is enough that γ is contained in a maximal torus $T \subset G$ such that T/A_G is anisotropic, for then the kernel of $\pi_1(G_\gamma) \to \pi_1(G)$ can be described as $Q^\vee/Q^\vee_\gamma$, where $Q^\vee_\gamma \subset Q^\vee \subset X_*(T)$ are the lattices spanned by the coroots for $G_\gamma \subset G$, and we have $(Q^\vee)^\Gamma = \{0\}$), and therefore $\pi_1(G_\gamma)_{\Gamma,\mathrm{tor}} \to \pi_1(G)_{\Gamma,\mathrm{tor}}$ remains surjective. Since its kernel is the Pontryagin dual of $\mathfrak{K}(\gamma)$ the proof is complete. ☐

It is remarkable how the failure of injectivity of α fits perfectly into the stabilization process. With this lemma, (6.1.2) becomes

$$\tau(G) \sum_{\gamma \in [[G(F)]]} \sum_{\kappa \in \mathfrak{K}(\gamma)} \sum_{\substack{\gamma' \in [G(\mathbb{A})] \\ \gamma' \sim \gamma}} \kappa(\mathrm{inv}_{\mathbb{A}}(\gamma,\gamma')) O_{\gamma'}(f). \tag{6.1.3}$$

This is the first step of the stabilization of the geometric side of the trace formula for an anisotropic group, or more generally of the elliptic part of the geometric side of the trace formula for a general reductive group. The result is an expression in which the contribution of each stable class in $G(F)$ is decomposed into its κ-parts. One can think of these parts as the Fourier-coefficients of that contribution. The $\kappa = 1$ part equals $SO_\gamma(f)$, and is the stable part of the trace formula, which we will discuss in §6.2. The parts for $\kappa \neq 1$ will be discussed in §6.3.

The discussion of the next section will require one further modification of (6.1.3), whose utility will not be obvious at this stage. This modification is that, instead of summing over stable classes in $G(F)$, one should sum over stable classes in $G_0(F)$, where G_0 is the quasi-split inner form of G. Recall that there is an isomorphism $\xi\colon (G_0)_{\bar F} \to G_{\bar F}$ that is an inner twisting, i.e. for all $\sigma \in \Gamma$ the automorphism $\xi^{-1} \circ \sigma(\xi)$ of $(G_0)_{\bar F}$ is inner. It allows us to make the identification $G(\bar F) = G_0(\bar F)$, but we need to be mindful that the Galois-actions on both sides are not the same.

By Definition 5.1.2, a stable class in $G(F)$ is simply the intersection with $G(F)$ with a conjugacy class in $G(\bar F)$. The same holds for G_0. Any conjugacy class in $G(\bar F)$ that intersects $G(F)$ is Γ-stable. The result of Kottwitz discussed in §5.2 guarantees that the converse is true for G_0. This provides a canonical embedding $[[G(F)]] \to [[G_0(F)]]$. Given $\gamma \in [[G(F)]]$ let $\gamma_0 \in [[G_0(F)]]$ be its image. Going forward, we will write $\mathrm{inv}(\gamma,\gamma')$ for $\mathrm{inv}_{\mathbb{A}}(\gamma,\gamma')$.

We now exploit the fact that Kottwitz's map $H^1(\Gamma, H(\bar{\mathbb{A}})) \to \pi_1(H)_{\Gamma,\mathrm{tor}}$ factors through $H^1(\Gamma, H(\bar{\mathbb{A}})/Z_H(\bar F))$. Let $\overline{\mathrm{inv}}(\gamma,\gamma') \in H^1(\Gamma, G_\gamma(\bar{\mathbb{A}})/Z_G(\bar F))$ be the image of $\mathrm{inv}(\gamma,\gamma')$. Then $\kappa(\mathrm{inv}(\gamma,\gamma')) = \kappa(\overline{\mathrm{inv}}(\gamma,\gamma'))$. We can now define an element $\overline{\mathrm{inv}}(\gamma_0,\gamma') \in H^1(\Gamma, G_{0,\gamma_0}(\bar{\mathbb{A}})/Z_{G_0}(\bar F))$ as the class of the 1-cocycle $\sigma \mapsto g^{-1} z_\sigma \sigma(g)$, where $z_\sigma \in Z^1(\Gamma, G_0(\bar F)/Z_{G_0}(\bar F))$ is given by $\xi^{-1}\circ\sigma(\xi) = \mathrm{Ad}(z_\sigma)$ and $g \in G_0(\bar{\mathbb{A}}) = G(\bar{\mathbb{A}})$ is any element such that $g\gamma_0 g^{-1} = \gamma'$. Since both γ_0 and γ lie in $G_0(\bar F) = G(\bar F)$ and are conjugate in $G_0(\bar{\mathbb{A}}) = G(\bar{\mathbb{A}})$, they are also conjugate in $G_0(\bar F) = G(\bar F)$, which according to Fact 5.1.3 gives the inner twist $\varphi_{\gamma_0,\gamma}\colon G_{0,\gamma_0} \to G_\gamma$ and the corresponding bijections $H^1(\Gamma, G_{0,\gamma_0}(\bar F)) \to H^1(\Gamma, G_\gamma(\bar F))$ and $H^1(\Gamma, G_{0,\gamma_0}(\bar F)/Z_{G_0}(\bar F)) \to H^1(\Gamma, G_\gamma(\bar F)/Z_G(\bar F))$, as well as their analogs with $\bar{\mathbb{A}}$-coefficients, and finally the isomorphism $\pi_1(G_{0,\gamma_0}) \to \pi_1(G_\gamma)$, which in turn induces an isomorphism $\mathfrak{K}(\gamma_0) \to \mathfrak{K}(\gamma)$. If we let $\kappa_0 \in \mathfrak{K}(\gamma_0)$ corresponds to $\kappa \in \mathfrak{K}(\gamma)$ under $\varphi_{\gamma_0,\gamma}$, then the identity $\overline{\mathrm{inv}}(\gamma_0,\gamma') = \varphi^{-1}_{\gamma_0,\gamma}(\overline{\mathrm{inv}}(\gamma,\gamma'))$ of (the adelic version of) Fact 5.1.3 implies

$$\kappa(\mathrm{inv}(\gamma,\gamma')) = \kappa(\overline{\mathrm{inv}}(\gamma,\gamma')) = \kappa_0(\overline{\mathrm{inv}}(\gamma_0,\gamma')).$$

This allows us to rewrite the summand in (6.1.3) corresponding to γ in terms of γ_0. We claim that then (6.1.3) becomes

$$\tau(G) \sum_{\gamma_0 \in [[G_0(F)]]} \sum_{\kappa \in \mathfrak{K}(\gamma_0)} \sum_{\substack{\gamma' \in [G(\mathbb{A})] \\ \gamma' \sim \gamma_0}} \kappa(\overline{\mathrm{inv}}(\gamma_0,\gamma')) O_{\gamma'}(f). \tag{6.1.4}$$

The only thing remaining to prove is the following.

Lemma 6.1.4. *If $\gamma_0 \in [[G_0(F)]]$ doesn't come from $\gamma \in [[G(F)]]$, then $\overline{inv}(\gamma_0,\gamma')$ is a non-zero element of $\pi_1(G_{0,\gamma_0})_{\Gamma,tor}$ for any γ'.*

Proof. We will prove the contrapositive. Assume that there is some $\gamma' \in G(\mathbb{A})$ stably conjugate to γ_0 such that $\overline{\mathrm{inv}}(\gamma_0,\gamma')$ becomes zero in $\pi_1(G_{0,\gamma_0})_{\Gamma,\mathrm{tor}}$. We invoke a slight variation of Theorem 6.1.1, namely [Kot86, Theorem 2.2], which states that the sequence of pointed sets

$$H^1(\Gamma, H(\bar F)/Z_H(\bar F)) \to H^1(\Gamma, H(\bar{\mathbb{A}})/Z_H(\bar F)) \to \pi_1(H)_{\Gamma,\mathrm{tor}}$$

is exact for any connected reductive F-group H. Applied to $H = G_{0,\gamma_0}$ this shows the existence of $g \in G_0(\bar{\mathbb{A}})$ such that $g\gamma_0 g^{-1} = \gamma'$ (under the identification $G_0(\bar{\mathbb{A}}) = G(\bar{\mathbb{A}})$) and $g^{-1}\dot{z}_\sigma \sigma(g)$ takes values in $G_{0,\gamma_0}(\bar{F})$, where $\dot{z}_\sigma \in C^1(\Gamma, G_0(\bar{F}))$ is any lift of the cocycle $z_\sigma \in Z^1(\Gamma, G_0(\bar{F})/Z_{G_0}(\bar{F}))$.

We have been writing $\sigma(g)$ for the image of $g \in G_0(\bar{\mathbb{A}})$ under the action of the Galois element. We will now use the identification $(G_0)_{\bar{F}} = G_{\bar{F}}$ given by ξ as in the above discussion. In order to distinguish the actions of $\sigma \in \Gamma$ on these two groups we will write σ_0 for the action on $G_0(\bar{\mathbb{A}})$ and σ_1 for the action on $G(\bar{\mathbb{A}})$. By definition of z_σ we have $\sigma_1(x) = \dot{z}_\sigma \cdot \sigma_0(x) \cdot \dot{z}_\sigma^{-1}$ for any $x \in G_0(\bar{\mathbb{A}}) = G(\bar{\mathbb{A}})$. With this, we see that $g^{-1}\dot{z}_\sigma\sigma_0(g)\dot{z}_\sigma^{-1}$ belongs to $Z^1(\Gamma, G(\bar{F}))$ and is trivial in $H^1(\Gamma, G(\bar{\mathbb{A}}))$, because it equals $g^{-1}\sigma_1(g)$. According to the Hasse principle, this element is already trivial in $H^1(\Gamma, G(\bar{F}))$, which guarantees the existence of $h \in G_0(\bar{F})$ such that $1 = (gh)^{-1}\sigma_1(gh)$. In other words, $gh \in G(\mathbb{A})$. Thus $\gamma = (gh)^{-1}\gamma'(gh) \in G(\mathbb{A})$ is an element that is $G(\mathbb{A})$-conjugate to γ', while at the same time $\gamma = h^{-1}\gamma_0 h \in G(\bar{F}) \cap G(\mathbb{A}) = G(F)$. $\square$

6.2 The stable part of the geometric side of the trace formula

Our next goal is to express (6.1.4) in terms of stable distributions. Consider first a regular semi-simple $\gamma_0 \in G_0(F)$ and $1 = \kappa \in \mathfrak{K}(\gamma_0)$. Then the sum over γ' gives the stable adelic orbital integral $SO_{\gamma_0}(f)$, i.e. the integral over the stable adelic class indexed by γ_0. This distribution is stable, by definition.

Consider next an arbitrary semi-simple $\gamma_0 \in G_0(F)$ and $1 = \kappa \in \mathfrak{K}(\gamma_0)$. Again, the sum over γ' represents the integral of f over the stable class indexed by γ_0. Is this a stable distribution?

It turns out that the answer is no, in general. The problem reduces immediately to the local case. Let v be a place of F and consider a semi-simple element $\gamma_v \in G(F_v)$. The distribution $\sum_{\gamma'_v} O_{\gamma'_v}(f_v)$, where the sum runs over the set of $G(F_v)$-conjugacy classes in the stable class of γ_v, is in general not stable. This problem was examined by Kottwitz, who proved that in order to obtain a stable distribution, one must insert certain signs in the above sum. More precisely, it is shown in [Kot88, §3, Proposition 1] that

$$\sum_{\gamma'_v} e(G_{\gamma'_v}) O_{\gamma'_v}(f_v)$$

is a stable distribution, where $e(G_{\gamma'_v})$ is the so-called "Kottwitz sign" of the reductive group $G_{\gamma'_v}$ as defined in [Kot83]. We recall that for a connected reductive group H over a local field F_v the sign $e(H)$ equals $(-1)^{r(H^*)-r(H)}$ when F_v is non-archimedean, and $(-1)^{q(H^*)-q(H)}$ when F_v is archimedean, where H^* is the quasi-split inner form of H, $r(H)$ is the split rank of H, and $q(H)$ is one half of the dimension of the symmetric space for H_{sc}. The proof of the stability result proceeds by examining the Shalika germ expansion of orbital integrals at regular semi-simple elements near γ_v, using a result of Rogawski that relates the Shalika germs on elliptic maximal tori to Euler-Poincare measures, and showing that the Euler-Poincare measures on inner forms differ by the signs $e(H)$.

According to the last proposition in [Kot83], if H is defined over F then $\prod_v e(H_{F_v}) = 1$. Returning to (6.1.4), the centralizer $G_{\gamma'}$ is a connected reductive $\mathbb{A}$-group that is quasi-split at almost all places, so $e(G_{\gamma'}) = \prod_v e(G_{\gamma'_v})$ makes sense, and equals 1 if γ' is $G(\mathbb{A})$-conjugate to an element of $G(F)$. Therefore (6.1.4) can also be written as

$$J_{\text{geom}}(f) = \tau(G) \sum_{\gamma_0 \in [[G_0(F)]]} \sum_{\kappa \in \mathfrak{K}(\gamma_0)} \sum_{\substack{\gamma' \in [G(\mathbb{A})] \\ \gamma' \sim \gamma_0}} \kappa(\overline{\text{inv}}(\gamma_0, \gamma')) e(G_{\gamma'}) O_{\gamma'}(f). \quad (6.2.1)$$

Note that, while the summands in (6.2.1) are different from those in (6.1.4) for those γ' whose $G(\mathbb{A})$-class does not meet $G(F)$, the middle sum ensures that these summands do not contribute, because for those summands the term $\overline{\text{inv}}(\gamma_0, \gamma')$ is a non-trivial character of the finite abelian group $\mathfrak{K}(\gamma_0)$.

We have achieved that, for each γ_0, the summand for $1 = \kappa \in \mathfrak{K}(\gamma_0)$ of the middle sum of a stable distribution. We write it as

$$S_{\text{geom}}(f) = \tau(G) \sum_{\gamma_0 \in [[G_0(F)]]} SO_{\gamma_0}(f), \qquad SO_{\gamma_0}(f) = \sum_{\substack{\gamma' \in [G(\mathbb{A})] \\ \gamma' \sim \gamma_0}} e(G_{\gamma'}) O_{\gamma'}(f). \quad (6.2.2)$$

6.3 Transfer of κ-terms to endoscopic groups

We must now reinterpret the difference between (6.2.1) and (6.2.2). In vague terms, each κ will correspond to an endoscopic group, and the corresponding term of (6.2.1) will be reinterpreted as the stable trace formula, i.e. (6.2.2), for the endoscopic group. This is just a heuristic, since κ is an element of $\mathfrak{K}(\gamma_0)$ and this group depends on γ_0. Slightly more precisely, there is a bijection

$$(\gamma_0, \kappa) \leftrightarrow (H, \gamma_H), \quad (6.3.1)$$

where on the left are pairs consisting of an elliptic semi-simple element $\gamma_0 \in G_0(F)$ and $\kappa \in \mathfrak{K}(\gamma_0)$, taken up to stable conjugacy, and on the right are pairs of an elliptic endoscopic datum $(H, s, \mathcal{H}, \xi)$ of G_0, abbreviated simply as H, and an elliptic (G, H)-regular semi-simple element γ_H of $H(F)$ determined up to stable conjugacy, all taken up to isomorphism. Before we discuss this bijection we review the relevant notions.

6.3.1 Review of endoscopic data

The standard definition of an endoscopic datum is given in [LS87, §1.2], and involves the dual group $\widehat{G}$ of G. We present here a slight reformulation and take advantage of the simplification afforded by the Hasse principle that we are assuming holds for G. An *endoscopic datum* is a tuple $(H, s, \mathcal{H}, \xi)$, where H is a quasi-split F-group, $s \in Z(\widehat{H})^\Gamma$, $\mathcal{H}$ is a split extension of W_F by $\widehat{H}$

such that the homomorphism $\Gamma \to \mathrm{Out}(\widehat{H})$ induced by $\mathcal{H}$ coincides with the homomorphism $\Gamma \to \mathrm{Out}(H_{\bar{F}})$ induced by the rational structure of H under the canonical identification $\mathrm{Out}(\widehat{H}) = \mathrm{Out}(H_{\bar{F}})$, ξ is an L-embedding $\mathcal{H} \to {}^L G$ that identifies $\widehat{H}$ with the identity component of the centralizer of $\xi(s)$ in $\widehat{G}$. We have simplified [LS87, §1.2, Condition (a)] using

$$\ker^1(W_F, Z(\widehat{G})) = \ker^1(\Gamma, Z(\widehat{G})) = \ker^1(F, G)^* = \{1\},$$

where the first identity is [Kot84, Lemma 11.2.2], the second is [Kot84, (4.2.2)], and the third is the validity of the Hasse principle for G.

Other equivalent variants of the same notion can be considered. For example, one can ignore $\mathcal{H}$, and instead consider an embedding $\xi\colon \widehat{H} \to \widehat{G}$ that is an isomorphism onto $\mathrm{Cent}(\xi(s), \widehat{G})^\circ$ and whose $\widehat{G}$-conjugacy class is Γ-equivariant, where we are using the natural actions of Γ on $\widehat{H}$ and $\widehat{G}$. This leads to the notion of an *endoscopic triple* (H, s, ξ) from [Kot84, §7.4]. The group $\mathcal{H}$ can be recovered as

$$\mathcal{H} = \{x \in {}^L G \mid \mathrm{Ad}(x) \circ \xi = \xi \circ \sigma_x\} \cdot \xi(\widehat{H}),$$

where $\sigma_x \in \Gamma$ is the image of x under the natural projection ${}^L G \to \Gamma$.

In a different direction one can consider pairs $(s, \mathcal{H})$ consisting of a semi-simple element $s \in \widehat{G}$ and a subgroup $\mathcal{H} \subset {}^L G$ that maps onto Γ and centralizes s, and such that $\mathcal{H} \cap \widehat{G}$ is precisely the identity component of the centralizer of s in $\widehat{G}$. This is the notion of an *endoscopic pair*. One can recover H as the unique quasi-split F-group dual to $\widehat{H} = \mathcal{H} \cap \widehat{G}$ with F-structure given by the homomorphism $\Gamma \to \mathrm{Out}(\widehat{H}) = \mathrm{Out}(H)$ coming from the conjugation action of $\mathcal{H}$ on $\widehat{H}$. One also recovers ξ as the tautological inclusion.

Finally, one can also reinterpret an endoscopic datum without referring to the dual groups $\widehat{G}$ or $\widehat{H}$, as is for example suggested in [Lab04, §II.4]. Then s is seen as a character of $\pi_1(H)_\Gamma$ via the Γ-equivariant isomorphism $X^*(Z(\widehat{H})) = \pi_1(H)$, and in place of ξ one is given an isomorphism $T^G \to T^H$ from the universal maximal torus[2] of G to the universal maximal torus of H which identifies the Γ-action on T^H with a twist of the Γ-action on T^G by an element $\omega_\sigma \in Z^1(\Gamma, W^G(T^G))$, where $W^G(T^G) \subset \mathrm{Aut}(T^G)$ is the absolute Weyl group of G. It is required that when the character s is pulled back under $X_*(T^G) = \pi_1(T^G) \to \pi_1(T^H) \to \pi_1(T^H)_\Gamma \to \pi_1(H)_\Gamma$, the coroot system of H is the subset of the coroot system of G specified by $\langle s, \alpha^\vee \rangle = 1$. One can obtain an endoscopic pair from this by identifying s with an element of $\mathrm{Hom}(X_*(T^G), \mathbb{C}^\times) = \widehat{T}^G \subset \widehat{G}$ and letting $\mathcal{H} = \widehat{H} \cdot \mathcal{T}$, where $\widehat{H} = \mathrm{Cent}(s, \widehat{G})^\circ$ and

$$\mathcal{T} = \{x \in {}^L G \mid x \in \widehat{T}^G \cdot \omega_{\sigma_x} \cdot \sigma_x\}.$$

[2]The universal Borel pair of G is the limit of the diagram of all Borel pairs of $G_{\bar{F}}$ with transition maps $\mathrm{Ad}(g) : (T, B) \to (gTg^{-1}, gBg^{-1})$, $g \in G(\bar{F})$. It carries a natural F-structure, and the torus part of it is the universal maximal torus.

This concludes our brief review of the various equivalent notions of endoscopic data. When dealing with groups that are not quasi-split one needs a slight refinement of that notion, in which a lift of $s \in \widehat{G}$ is chosen to an element $\dot{s}$ of the universal cover $\widehat{\bar{G}}$ of the complex Lie group $\widehat{G}$. The resulting $(H, \dot{s}, \mathcal{H}, \xi)$ will be referred to as a *refined* endoscopic datum, cf. [Kal16b, §5.3].

Given an endoscopic datum we obtain a natural Γ-equivariant injection $Z(\widehat{G}) \to Z(\widehat{H})$. Noting that $X^*(Z(\widehat{G})) = \pi_1(G)$, the dual of this injection is a Γ-equivariant surjection $\pi_1(H) \to \pi_1(G)$. The endoscopic datum is called *elliptic* if the quotient $Z(\widehat{H})^\Gamma / Z(\widehat{G})^\Gamma$, equivalently the kernel of $\pi_1(H)_\Gamma \to \pi_1(G)_\Gamma$, is finite.

We now recall Kottwitz's result discussed in §5.2. It can be used to transfer stable semi-simple classes between G_0 and an endoscopic group H as follows. The Γ-stable $G_0(\bar{F})$-conjugacy classes in $G_0(\bar{F})$ are in bijection with the Γ-stable $W^G(T^G)$-orbits of elements of $T^G(\bar{F})$, again by intersection. The same is true for H in place of G_0. Via the identification $T^H = T^G$ a Γ-stable $W^H(T^H)$-orbit in T^H gives a Γ-stable $W^G(T^G)$-orbit in T^G. This leads to a *transfer* of stable classes in $H(F)$ to stable classes in $G_0(F)$. Since maximal tori are centralizers of strongly regular semi-simple elements, this also leads to transfer of tori: any maximal torus $T \subset H$ comes equipped with a stable class of embeddings into G_0, called *admissible*. In particular, we have a finite Γ-stable set $R(T, G) \subset X^*(T)$ containing $R(T, H)$. Moreover, H is elliptic if and only if the transfer of one, hence any, elliptic maximal torus of H to G is elliptic. We refer to [Kal19, §5.1] for more details.

A semi-simple element $\gamma_H \in H(F)$ is called (G, H)*-regular* if for one, hence any, maximal torus $T \subset H$ containing γ_H, we have $\{\alpha \in R(T, G) \,|\, \alpha(\gamma_H) = 1\} \subset R(T, H)$. This implies that if $\gamma \in G_0(F)$ is a transfer of γ_H then $H^0_{\gamma_H}$ and $G_{0,\gamma}$ are inner forms of each other, where $H^0_{\gamma_H}$ denotes the identity component of the centralizer of γ_H in H, see [Kot86, §3.1].

6.3.2 The bijection (6.3.1)

We can now state more precisely how the bijection (6.3.1) works, following [Kot86, Lemma 9.7].

Given a pair (H, γ_H) on the right hand side of (6.3.1), let $\gamma_0 \in G_0(F)$ represent the unique stable class that is the transfer of γ_H. Since G_{0,γ_0} and $H^0_{\gamma_H}$ are inner forms of each other we have the identification $\pi_1(G_{0,\gamma_0}) = \pi_1(H^0_{\gamma_H})$, and the latter surjects onto $\pi_1(H)$. Pulling back s under this surjection we obtain a character κ of $\mathfrak{K}(\gamma_0) = \ker(\pi_1(G_{0,\gamma_0})_{\Gamma,\mathrm{tor}} \to \pi_1(G)_{\Gamma,\mathrm{tor}})$.

Conversely, given a pair (γ_0, κ) choose an elliptic maximal torus $T \subset G_0$ containing γ_0. Then $T \subset G_{0,\gamma_0}$ and we have the map $\pi_1(T) \to \pi_1(G_{0,\gamma_0})$. Inside of $\pi_1(T) = X_*(T)$ we have the lattice $Q^\vee$ spanned by the set of coroots $R^\vee(T, G_0) \subset X_*(T)$. The ellipticity of T implies that $(Q^\vee)^\Gamma = \{0\}$, hence that $Q^\vee_\Gamma$ is torsion. This provides a homomorphism $Q^\vee_\Gamma \to \pi_1(T)_{\Gamma,\mathrm{tor}}$. Since the composition $Q^\vee \to \pi_1(T) \to \pi_1(G)$ is zero by definition, the composition of $Q^\vee_\Gamma \to \pi_1(T)_{\Gamma,\mathrm{tor}}$ with $\pi_1(T)_{\Gamma,\mathrm{tor}} \to \pi_1(G_{\gamma_0})_{\Gamma,\mathrm{tor}}$ takes image in $\mathfrak{K}(\gamma_0)$, so we can pull back the character κ to a character of $Q^\vee_\Gamma$. We let $R^\vee(T,H)$ be the subset of $R^\vee(T,G_0) \subset Q^\vee$ distinguished by the condition $\langle \kappa, \alpha^\vee \rangle = 1$, and we let $R(T,H) \subset R(T,G_0)$ be the set of roots whose coroots lie in $R^\vee(T,H)$. Then $(X^*(T), R(T,H), X_*(T), R^\vee(T,H))$ is a root datum. The natural Γ-action on T endows this root datum with a Γ-action, which need not preserve any basis of $R(T,H)$. We will modify it so that it does, in order to obtain a quasi-split group, as follows. Let $W^H(T) \subset W^G(T)$ be the subgroup generated by the reflections along all roots in $R(T,H)$. We fix arbitrarily a basis of $R(T,H)$ and let $\omega'_\sigma \in Z^1(\Gamma, W^H(T))$ be the unique element such that $\omega'_\sigma \cdot \sigma$ preserves that basis for all $\sigma \in \Gamma$. Let T^H be the torus obtained from T by twisting the Γ-action by ω'_σ. The root datum $(X^*(T^H), R(T^H,H), X_*(T^H), R^\vee(T^H,H))$, which is the same as the one above but now with a modified Γ-action, is the root datum of a quasi-split reductive group H with minimal Levi subgroup T^H. To obtain an identification of T^H with the universal maximal torus T^G of G we need to choose a Weyl chamber for $R(T,G)$ that is contained in the chosen chamber for $R(T,H)$. This identifies T with T^G, and hence T^H with T^G. Note that $\omega'_\sigma \in Z^1(\Gamma, W^H(T^H))$ obtained here is not the same as $\omega_\sigma \in Z^1(\Gamma, W^G(T^G))$ used in the definition of endoscopic datum above. The first measures the difference of Galois structures between T^H and T, while the second between T^H and T^G. Finally, $\gamma_0 \in T(F) \subset T(\bar{F}) = T^H(\bar{F})$ is an element whose $W^H(T^H)$-orbit is Γ-fixed, hence corresponds to a stable class in the quasi-split group $H(F)$. To check that this stable class is (G,H)-regular, consider $\alpha \in R(T,G_0)$ such that $\alpha(\gamma) = 1$. This is equivalent to $\alpha \in R(T, G_{0,\gamma_0})$, and hence to $\alpha^\vee \in R^\vee(T, G_{0,\gamma_0})$. The map $\pi_1(T) \to \pi_1(G_{0,\gamma_0})$ kills $R^\vee(T, G_{0,\gamma_0})$, so $\langle \kappa, \alpha^\vee \rangle = 1$, i.e. $\alpha \in R(T,H)$.

We have thus given the maps in the bijection (6.3.1) in both directions. We should now mention the equivalence under which both sides are taken. A pair (γ_0, κ) is taken up to stable conjugacy, meaning that another pair (γ'_0, κ') is equivalent if γ_0 and γ'_0 are stably conjugate and the canonical isomorphism $\mathfrak{K}(\gamma_0) \to \mathfrak{K}(\gamma'_0)$ transports κ to κ'. A pair (H, γ_H) is taken up to isomorphism, meaning that another pair (H', γ'_H) is equivalent if there exists an isomorphism $H \to H'$ (where the letters stand not just for the endoscopic groups, but for the full endoscopic data, and the isomorphism is one of endoscopic data) that identifies γ_H with γ'_H. Kottwitz moreover shows that such an isomorphism is unique up to inner automorphism, i.e. an element of $H_{\mathrm{ad}}(F)$.

6.3.3 Reducing global transfer to local transfer

The bijection (6.3.1) allows us to rewrite (6.2.1) as

$$J_{\mathrm{geom}}(f) = \tau(G) \sum_H \lambda_H^{-1} \sum_{\gamma_H \in [[H(F)^{(G,H)}_{\mathrm{ell}}]]} \sum_{\substack{\gamma' \in [G(\mathbb{A})] \\ \gamma' \sim \gamma_0}} \kappa(\overline{\mathrm{inv}}(\gamma_0, \gamma')) e(G_{\gamma'}) O_{\gamma'}(f), \tag{6.3.2}$$

where the first sum runs over (a set of representatives for) the set of isomorphism classes of elliptic endoscopic data for G and the second over the set of stable classes of elliptic (G,H)-regular semi-simple elements of $H(F)$. We have set λ_H for the cardinality of the finite quotient of the group of automorphisms of the endoscopic datum modulo inner automorphisms (this number comes about from the nature of the two sides of the bijection (6.3.1)). Note that the term $\kappa = 1$ corresponds to $G = H$ and hence $\lambda_H = 1$. Finally $\gamma_0 \in G_0(F)$ is any representative of the stable class in $G_0(F)$ that is the transfer of γ_H.

The final step in the stabilization process is the following major theorem.

Theorem 6.3.1. *There exists a function f^H on $H(\mathbb{A})$ with the property that*

$$\sum_{\substack{\gamma' \in [G(\mathbb{A})] \\ \gamma' \sim \gamma_0}} \kappa(\overline{inv}(\gamma_0, \gamma'))e(G_{\gamma'})O_{\gamma'}(f) = SO_{\gamma_H}(f^H).$$

The proof is the combined effort of many people and spans many papers. We give a brief summary of the various steps.

First, the problem is reduced to the case of local fields. While the right hand side is already of local nature, because the adelic stable orbital integral of a decomposable test function $f^H = \prod_v f_v^H$ is the product of the local stable orbital integrals, the left hand side is not yet of local nature due to the occurrence of the term $\kappa(\overline{\mathrm{inv}}(\gamma_0, \gamma'))$. For a moment let us simplify the exposition by assuming that G is quasi-split, i.e. $G = G_0$ (this essentially contradicts the earlier assumption that G is anisotropic, but the two assumptions are used for separate purposes; in particular, all of the structure we are using now is available without assuming that G is anisotropic, with the same definitions). This allows us to use the element $\mathrm{inv}(\gamma_0, \gamma') \in H^1(\Gamma, G_{\gamma_0}(\bar{\mathbb{A}}))$, rather than the more cumbersome $\overline{\mathrm{inv}}(\gamma_0, \gamma') \in H^1(\Gamma, G_{0,\gamma_0}(\bar{\mathbb{A}})/Z_{G_0}(\bar{F}))$. To form $\kappa(\mathrm{inv}(\gamma_0, \gamma'))$, the element $\mathrm{inv}(\gamma_0, \gamma')$ is then mapped to $\pi_1(G_{0,\gamma_0})_{\Gamma,\mathrm{tor}}$ via the Kottwitz homomorphism. Its image lands in the kernel of the map to $\pi_1(G_0)_{\Gamma,\mathrm{tor}}$, which is the dual of $\mathfrak{K}(\gamma_0)$.

The global Kottwitz homomorphism is equal to ([Kot86, 2.5 Corollary]) the composition of the total localization map $H^1(\Gamma, G_{\gamma_0}(\bar{\mathbb{A}})) \to \bigoplus_v H^1(\Gamma, G_{0,\gamma_0}(\bar{F}_v))$, the local Kottwitz homomorphisms $H^1(\Gamma, G_{\gamma_0}(\bar{F}_v)) \to \pi_1(G_{\gamma_0})_{\Gamma_v,\mathrm{tor}}$, and the summation map $\bigoplus \pi_1(G_{\gamma_0})_{\Gamma_v,\mathrm{tor}} \to \pi_1(G_{\gamma_0})_{\Gamma,\mathrm{tor}}$. Letting $\kappa_v : \pi_1(G)_{\Gamma_v,\mathrm{tor}} \to \mathbb{C}^\times$ be the pull-back of κ under the natural map $\pi_1(G_{\gamma_0})_{\Gamma_v,\mathrm{tor}} \to \pi_1(G_{\gamma_0})_{\Gamma,\mathrm{tor}}$ we obtain

$$\kappa(\mathrm{inv}(\gamma_0, \gamma')) = \prod_v \kappa_v(\mathrm{inv}(\gamma_{0,v}, \gamma'_v)).$$

It would appear that this reduces the problem to the local problem of finding for a test function f_v on $G(F_v)$ a test function f_v^H on $H(F_v)$ such that

$$\sum_{\substack{\gamma'_v \in [G(F_v)] \\ \gamma'_v \sim \gamma_0}} \kappa_v(\mathrm{inv}(\gamma_0, \gamma'_v))e(G_{\gamma'_v})O_{\gamma'_v}(f_v) = SO_{\gamma_H}(f_v^H).$$

But this is not the correct identity. Indeed, already considering the asymptotic behavior as γ_0 and γ_H approach the identity shows that both sides cannot possibly be equal, because each is asymptotic to the Weyl discriminant in the corresponding group G_0 or H, and these two discriminants have different behaviors due to the different number of roots in each group. Another problem is that the above local identity, unlike its global counterpart, does depend on the choice of γ_0, and is therefore not even well-defined.

This problem was solved in [LS87], where the notion of transfer factor is introduced. It is a function

$$\Delta\colon H(F_v)_{\text{sr}} \times G(F_v)_{\text{sr}} \to \mathbb{C},$$

where the subscript "sr" denotes the set of strongly regular semi-simple elements. This function is the product of a number of factors. One of them, called Δ_{III_1}, is the term $\kappa_v(\text{inv}(\gamma_0, \gamma_v'))$ that we see in the above identity. Another one, called Δ_{IV}, is the quotient of the two Weyl discriminants for G_0 and H and aligns the asymptotic behaviors of both sides. Yet another one, called Δ_{III_2}, compensates for the way the L-group structure changes as we pass from H to G, or from maximal tori of these groups to the groups themselves. Finally, two terms Δ_I and Δ_{II} synchronize the finer properties of harmonic analysis between G and H. For example, when $F_v = \mathbb{R}$ orbital integrals have certain discontinuities as the strongly regular orbits degenerate towards a singular orbit, and these pieces of the transfer factor compensate for the difference in discontinuity between G and H. This is the work of Shelstad, and we refer to [She08a] for an exposition on these matters, as well as to the original paper [She82]. When F_v is non-archimedean, the analog of these discontinuities is recorded in the Shalika germ expansion. The terms Δ_I and Δ_{II} align the germ expansions between G and H; the fact that this works for regular germs was verified in [LS87, §5].

The transfer factor defined in [LS87] has undergone a series of refinements. In its original definition, it was unambiguously constructed only when G is quasi-split, and depended on a choice of a pinning for G. When G is not quasi-split, the fact that one can only define $\overline{\text{inv}}(\gamma_0, \gamma') \in H^1(\Gamma_v, G_{\gamma_0}(\bar{F}_v)/Z_{G_0}(\bar{F}_v))$, rather than $\text{inv}(\gamma_0, \gamma') \in H^1(\Gamma_v, G_{\gamma_0}(\bar{F}_v))$ caused problems, which were handled in a minimalistic way, yielding a construction of $\Delta(\gamma_H, \gamma')$ that is ambiguous up to a uniform scalar constant. In [KS99, §5.3] a second way to normalize Δ was introduced, again only in the case when G is quasi-split, by fixing a Whittaker datum for G. This normalization has a spectral interpretation, see [Taï22, Conjectures 4.7,4.12], and is therefore preferable. Going beyond the case of quasi-split G, in [Kal11] it was shown how to normalize the transfer factor when G is a pure inner form of G_0. Since not all inner forms are pure, the problem of normalizing the transfer factor in general remained open, and was solved in [Kal16b] for local fields of characteristic zero, and in [Dil20] for local fields of positive characteristic. Another refinement of the transfer factor was brought by [KS], which suggest that it is better to invert the pieces Δ_I and Δ_{III_1}. We will not go into these refinements here, in order not to detract from the main thrust of our discussion, and refer the interested reader to Taïbi's article [Taï22] in these proceedings, as well as to [Kal16a] and [Kal24].

Remark 6.3.2. There is one further refinement that can be made. If $(H, s, \mathcal{H}, \xi)$ is an endoscopic datum, our assumption $G_{\text{der}} = G_{\text{sc}}$ implies in [Lan79] that there exists an isomorphism $\mathcal{H} \to {}^L H$. The transfer factor depends on a choice of such isomorphism, and generally there are many such choices. If one drops the assumption $G_{\text{der}} = G_{\text{sc}}$, then such an isomorphism may not exist, and one must choose a so-called z-pair (H_1, ξ_1), cf. [KS99, §2.2]. The transfer factor again depends on this choice, and moreover its first variable is now an element of $H_1(F)$, not of $H(F)$. However, matters can be made canonical, and in some sense simpler, in the following way. It turns out that there is a canonical non-algebraic double cover $H(F)_\pm$ of the topological group $H(F)$. It comes equipped with an L-group ${}^L H_\pm$ that satisfies an appropriate formulation of the local Langlands conjecture, provided this is true for algebraic groups closely related to H. The L-group ${}^L H_\pm$ has a canonical embedding into ${}^L G$. There is a canonical transfer factor $\Delta_\pm \colon H(F)_{\pm,\text{sr}} \times G(F)_{\text{sr}} \to \mathbb{C}$ that does not depend on any choices. It is the product of pieces $\Delta_{I,\pm}$, $\Delta_{III,\pm}$, and Δ_{IV}. Unlike the original definition, where the individual terms of the transfer factor depend on auxiliary choices, called a-data and χ-data, these terms are themselves canonical. The term $\Delta_{I,\pm}$ can be understood as an invariant measuring the relative position between a pinning of G_0 and an element of a cover of a maximal torus in G, while the term $\Delta_{III,\pm}$ can be understood as an invariant between various canonical L-embeddings. For more details we refer to [Kal22a].

The local transfer factors, suitably normalized, satisfy the following global product formula:

$$\prod_v \Delta(\gamma_H, \gamma'_v) = \kappa(\overline{\text{inv}}(\gamma_0, \gamma')),$$

where $\gamma_H \in H(F)$ is strongly regular, and $\gamma_0 \in G_0(F)$ and κ are obtained from (H, γ_H) via (6.3.1). This comes down to the fact that each of the pieces Δ_I, Δ_{II}, Δ_{III_2}, and Δ_{IV}, as well as the normalization constants used, satisfy global product formulas that result in 1, while the pieces Δ_{III_1} are the local components of the right hand side. For Δ_{II}, Δ_{III_2}, and Δ_{IV}, this is discussed in [LS87, Theorem 6.4.A], while for the normalizing constants it is the subject of [Kal18] and [Dil21]. In this way Theorem 6.3.1 finally does reduce to the following local statements.

Theorem 6.3.3. *There exists a smooth compactly supported function f_v^H on $H(F_v)$ with the property that*

$$\sum_{\substack{\gamma'_v \in [G(F_v)] \\ \gamma'_v \sim \gamma_H}} \Delta(\gamma_H, \gamma'_v) e(G_{\gamma'}) O_{\gamma'_v}(f_v) = SO_{\gamma_H}(f_v^H)$$

for all (G, H)-regular semi-simple elements $\gamma_H \in H(F_v)$.

Definition 6.3.4. A pair of functions (f, f^H) as in Theorem 6.3.1, or a pair of functions (f_v, f_v^H) as in Theorem 6.3.3, are called *matching*.

Theorem 6.3.5. *In the setting of above theorem, assume that H and G are unramified at v and that f_v is the characteristic function of a hyperspecial maximal compact subgroup. Then f_v^H can also be taken as the characteristic function of a hyperspecial maximal compact subgroup.*

Theorem 6.3.5 is famously known as the "Fundamental Lemma", or more precisely the "Fundamental Lemma for the unit element".

6.3.4 A discussion of the local transfer theorem

We continue with the brief summary of the proof of Theorem 6.3.1, which has now been reduced to Theorems 6.3.3 and 6.3.5. This is where the hard work really begins.

Let us first note that Theorem 6.3.3 involves values of the transfer factor at non-regular semi-simple elements, but in [LS87] it was only defined for strongly regular semi-simple elements. For general elements the value is obtained as a limit, but one has to show that this limit exists. Since the value at a regular element involves subtle data of the particular torus centralizing the regular element, and since a singular element can be approached through multiple different tori, the fact that this limit exists is highly non-trivial, and is the subject of [LS90]. There it is proved not only that the limit exists, but a descent step is performed which reduces the proof of Theorem 6.3.3 to the case when the elements are strongly regular and close to the identity, cf. [LS90, Theorem 2.3.A, Lemma 2.4.A]. With this, the proof of Theorems 6.3.3 and 6.3.5 is reduced to an analogous statement for the Lie algebras of G and H.

Initially it was believed that these two theorems, although clearly related, are to be proved independently of each other. But Waldspurger showed in [Wal97] that Theorem 6.3.5 implies Theorem 6.3.3, using a global argument based on the trace formula for the Lie algebra and its stabilization developed in [Wal95]. For details on this development we refer to [Cha11].

Theorem 6.3.5, the "Fundamental Lemma for the unit element", in the case of $G = \mathrm{SL}_2$ was stated in [LL79], where it was proved by a short argument. In its general form it was stated in [Lan83, §III.3], but no proof was given. It turned out to be one of the most difficult parts of the theory to establish. It was ultimately proved by Bao Châu Ngô [Ngô10a] for local fields of positive characteristic using algebraic geometry and analyzing the Hitchin fibration, a method used earlier by Laumon and Ngô to treat the special case of unitary groups [LN08]. The geometric approach to the fundamental lemma was introduced earlier by Goresky-Kottwitz-MacPherson [GKM04]. One can then "transfer" the validity of the Fundamental Lemma from local fields of positive characteristic to local fields of characteristic zero, either by a direct argument [Wal09] or by techniques from model theory [CHL11]. A result of Hales [Hal95] shows that the validity of Theorem 6.3.5 at almost all places implies the validity at all places of a stronger theorem, known as "the Fundamental Lemma for the spherical Hecke algebra", which shows that in the unramified situation the transfer map $f_v \mapsto f_v^H$ is realized by the homomorphism of unramified Hecke algebras $\mathcal{H}(G) \to \mathcal{H}(H)$ naturally obtained from the endoscopic datum. For exposition on these matters we refer to [Ngô10b] or [Hal12].

6.3.5 Completion of the stabilization of the geometric side

Applying Theorem 6.3.1 to (6.3.2) leads to

$$J_{\text{geom}}(f) = \tau(G) \sum_H \lambda_H^{-1} \sum_{\gamma_H \in [[H(F)_{\text{ell}}^{(G,H)}]]} SO_{\gamma_H}(f^H). \tag{6.3.3}$$

The inner sum seems rather close to (6.2.2) but for the group H in place of G. But there are two important differences. First, while the group G was anisotropic modulo A_G, the group H is quasi-split. This means that $H(F)$ will have elements that are not semi-simple, and semi-simple elements that are not elliptic. And second, even as far as elliptic semi-simple elements are concerned, those that happen to not be (G, H)-regular are missing from (6.3.3). For this reason, the inner sum in (6.3.3) is, for a general test function f^H, only a part of the stable trace formula for H (whatever that may be). But the test function we are using here is the endoscopic transfer of a test function on G, and for this particular test function the remainder of the stable trace formula for H will turn out to be zero. Admitting that, and setting $\iota(G, H) = \tau(G)\tau(H)^{-1}\lambda_H^{-1}$ we obtain

$$J_{\text{geom}}^G(f) = \sum_H \iota(G, H) S_{\text{geom}}^H(f^H), \tag{6.3.4}$$

where we have emphasized in the superscript the group for which the distribution is being taken.

This is the stabilization of the geometric side of the trace formula when G is anisotropic modulo A_G.

6.4 Stabilization of the spectral side

The "trace formula" for a connected reductive F-group G, such that G/A_G is anisotropic, is the identity (4.4.1), which expresses the trace of the operator $R(f)$ in two different ways: one more immediate, as the sum of traces of f on the various irreducible constituents π of R, and one less immediate, namely the sum of orbital integrals. It can be interpreted as giving a *formula* that expresses the sum of *traces* in geometric terms.

We have now defined a new geometric distribution S_{geom} by (6.2.2), again assuming G/A_G is anisotropic. But as of yet we do not have an analog of the identity (4.4.1) that expresses this distribution in spectral terms.

In this subsection we will perform a stabilization of the spectral side of (4.4.1), following [Kot84], and arrive at an equation similar to (6.3.4), but where on the right the summands will be certain stable distributions S_{spec} defined in spectral terms. It would then be natural to conjecture that

$$S_{\text{geom}}(f) = S_{\text{spec}}(f),$$

and this can then be seen as the stable form of the trace formula (4.4.1).

A big difference between the geometric and spectral stabilization is that the geometric stabilization is unconditional, given that the transfer theorem 6.3.3 and the Fundamental Lemma 6.3.5 have been established, while the spectral stabilization is based on Arthur's conjecture [Art89b, Conjecture 8.1] and the local spectral transfer conjecture [Taï22, Conjectures 4.7, 4.12], and hence is conditional. But one can turn things around and use the stabilization identity in order to prove these conjectures. This is the approach taken in [Art13], which we will review in §8.

Recall that Arthur's conjecture gives a decomposition

$$L^2(G(F)A_G^+\backslash G(\mathbb{A})) = \bigoplus_\psi \bigoplus_\pi m(\psi,\pi)\pi.$$

The first sum is over $\widehat{G}$-conjugacy classes[3] of "global Arthur parameters", which are L-homomorphisms $\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$ involving the hypothetical Langlands group $\mathcal{L}_F$ of the global field F that are continuous on $\mathcal{L}_F$, algebraic on $\mathrm{SL}_2(\mathbb{C})$, and do not factor through a proper Levi subgroup. The group $\mathcal{L}_F$ should come equipped with homomorphisms $\mathcal{L}_{F_v} \to \mathcal{L}_F$, where the Langlands group $\mathcal{L}_{F_v}$ of the local field F_v is unconditionally defined in terms of the absolute Weil group W_{F_v} of F_v as

$$\mathcal{L}_{F_v} = \begin{cases} W_{F_v}, & \text{if } F/\mathbb{R}, \\ W_{F_v} \times \mathrm{SL}_2(\mathbb{C}), & \text{else.} \end{cases}$$

Composing ψ with such a homomorphism provides a local Arthur parameter $\psi_v \colon \mathcal{L}_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$. The local Arthur conjecture (cf. [Taï22, Conjectures 4.1, 4.3, 4.11] for the tempered case) predicts the existence of a local Arthur packet $\Pi_{\psi_v}(G)$, equipped with a map $\Pi_{\psi_v}(G) \to \mathrm{Irr}(\pi_0(S_{\psi_v}^+))$; here S_{ψ_v} is the centralizer of ψ_v in $\widehat{G}$, and $S_{\psi_v}^+$ is the preimage of S_{ψ_v} in the universal cover of $\widehat{G}$. Write $\langle \pi_v, \dot{s}\rangle$ for the value at $\dot{s} \in S_{\psi_v}^+$ of the character of the irreducible representation of $\pi_0(S_{\psi_v}^+)$ associated by this map to $\pi_v \in \Pi_{\psi_v}(G)$. The second sum runs over adelic representations $\pi = \otimes'_v \pi_v$, where $\pi_v \in \Pi_{\psi_v}(G)$ for all v, and $\langle \pi_v, -\rangle = 1$ for almost all v. The latter condition should imply that π_v is unramified for almost all v, so that π is an admissible representation of $G(\mathbb{A})$. Write $\langle \pi, \dot{s}\rangle = \prod_v \langle \pi_v, \dot{s}\rangle$ for $s \in S_\psi^+$. Then $\langle \pi, z\rangle = 1$ for $z \in [Z(\widehat{\widehat{G}})]^+$, so $\langle \pi, -\rangle$ descends to $\mathcal{S}_\psi = S_\psi^+/[Z(\widehat{\widehat{G}})]^+ = S_\psi/Z(\widehat{G})^\Gamma$, cf. [Kal18, Propositions 4.1, 4.2]. The integer $m(\psi,\pi)$ is then defined as (cf. [Art89b, (8.5)][4])

$$m(\psi,\pi) = \mathrm{mult}(\epsilon_\psi, \langle \pi, -\rangle) = |\mathcal{S}_\psi|^{-1} \sum_{s \in \mathcal{S}_\psi} \epsilon_\psi(s)\langle \pi, s\rangle, \tag{6.4.1}$$

[3]We are using again the Hasse principle here. When it doesn't hold, the notion of equivalence of global parameters is slightly more complicated, cf. [Kot84, §10.4].

[4]The superscript $+$ in S_ψ^+ in loc. cit. has a different meaning from the superscript $+$ used here: in loc. cit. it refers to the possibility of G being a disconnected reductive group, while for us it refers to the universal cover of $\widehat{G}$ that is needed to treat general non-quasi-split groups.

where $\epsilon_\psi \colon \mathcal{S}_\psi \to \{\pm 1\}$ is a certain explicit but rather subtle sign character, defined in [Art89b, (8.4)]. Recalling it would be beyond the scope of this note. While in general this character can be non-trivial, conjecturally it is trivial at least when all local components of π are tempered, so the reader is welcome to focus on that case initially. For a motivation on the definition of ϵ_ψ we refer to [Art90, Proposition 5.1].

Remark 6.4.1. When the group $\mathcal{S}_\psi$, as well as its local analogs $\pi_0(S_{\psi_v})$, are abelian, which happens for example when G is a classical group, then the above sum is either 0 or 1, and it is 1 precisely when the character $\langle \pi, - \rangle \colon \mathcal{S}_\psi \to \mathbb{C}^\times$ equals the character ϵ_ψ. Therefore Arthur's conjecture can be stated in the following shorter form

$$L^2(G(F)A_G^+ \backslash G(\mathbb{A})) = \bigoplus_\psi \bigoplus_{\pi \colon \langle \pi, - \rangle = \epsilon_\psi} \pi.$$

Assuming these conjectures, the spectral side of (4.4.1) becomes

$$\sum_\psi \sum_\pi |\mathcal{S}_\psi|^{-1} \sum_{s \in \mathcal{S}_\psi} \epsilon_\psi(s) \langle \pi, s \rangle \mathrm{tr}\, \pi(f), \tag{6.4.2}$$

where again ψ runs over the set of $\widehat{G}$-conjugacy classes of discrete Arthur parameters, and π runs over all members of the global A-packet

$$\Pi_\psi = \{\pi = \otimes'_v \pi_v \mid \pi_v \in \Pi_{\psi_v},\ \langle \pi_v, - \rangle = 1 \text{ for almost all } v\}.$$

Consider the element

$$S_\psi \ni s_\psi = \psi\left(1, \begin{bmatrix} -1 & \\ & -1 \end{bmatrix}\right). \tag{6.4.3}$$

If $\psi|_{\mathrm{SL}_2} = 1$ then ψ is called "generic", or "tempered". In that case the element s_ψ is trivial, but for general ψ it can be non-trivial. We use a factorizable test function $f = \prod_v f_v$, shift the summation by s_ψ, and switch the sums over s and π, to turn (6.4.2) into

$$\sum_\psi |\mathcal{S}_\psi|^{-1} \sum_{s \in \mathcal{S}_\psi} \epsilon_\psi(s \cdot s_\psi) \prod_v \sum_{\pi_v} \langle \pi_v, \dot{s} \cdot \dot{s}_\psi \rangle \mathrm{tr}\, \pi_v(f_v). \tag{6.4.4}$$

There is a spectral analog of (6.3.1) that takes the form

$$(\psi, s) \leftrightarrow (H, s, \mathcal{H}, {}^L\xi, \psi^H). \tag{6.4.5}$$

On the left we have pairs consisting of an Arthur parameter ψ and an element $s \in \mathcal{S}_\psi$, while on the right we have tuples consisting of an elliptic endoscopic datum $(H, s, \mathcal{H}, \xi)$, an extension[5] of ξ to an L-isomorphism ${}^L\xi \colon \mathcal{H} \to {}^LH$ and

[5] We are using here the assumption that G_{der} is simply connected, which guarantees the existence of ${}^L\xi$, cf. [Lan79]

an Arthur parameter ψ^H for H. The correspondence between both sides is obtained as follows. Given $(H, s, \mathcal{H}, {}^L\xi, \psi^H)$ we set $\psi = {}^L\xi \circ \psi^H$.

According to [Taï22, Conjecture 4.12] we have

$$\sum_{\pi_v} \langle \pi_v, \dot{s} \cdot \dot{s}_\psi \rangle \operatorname{tr} \pi_v(f_v) = S\Theta_{\psi_v^H}(f_v^H)$$

for a stable distribution $S\Theta_{\psi_v^H}$ on $H(F_v)$ associated to the parameter ψ_v^H, called the *stable character of* ψ_v^H; here f_v and f_v^H are matching functions as in Definition 6.3.4, and $\dot{s}$ is a certain refinement of s on which the normalization of the transfer factor, and hence the notion of matching, depends.

In addition, it is expected that the identity

$$\epsilon_\psi(s \cdot s_\psi) = e_{\psi^H}(s_{\psi^H})$$

holds, and this identity is known by [Art13, Lemma 4.4.1] for the case of classical groups. With this, (6.4.4) becomes

$$\sum_H \iota(G, H) \sum_{\psi^H} |\mathcal{S}_{\psi^H}|^{-1} e_{\psi^H}(s_{\psi^H}) S\Theta_{\psi^H}(f^H). \tag{6.4.6}$$

We have not discussed here how the quantity $\iota(G, H)|\mathcal{S}_{\psi^H}|^{-1}$ arises from the quantity $|\mathcal{S}_\psi|^{-1}$. The two are not equal, and their discrepancy accounts for the failure of the correspondence (6.4.5) to be bijective. We are also being vague about the equivalence up to which the parameters ψ^H are to be taken, which as we have already mentioned is more subtle than $\widehat{H}$-conjugacy when H does not satisfy the Hasse principle. Details can be found in [Kot84, §11], especially Proposition 11.2.1 there. We will content ourselves with understanding the special case when H does satisfy the Hasse principle (in addition to G, which has been assumed earlier), which happens for example when G is a classical group, for then H is itself a product of classical groups. In that special case, ψ^H runs over the set of $\widehat{H}$-conjugacy classes of discrete Arthur parameters. The quantity $\iota(G, H) = \tau(G)\tau(H)^{-1}|\mathrm{Out}(H, s, \mathcal{H}, \xi)|^{-1}$ can be computed using the formulas $\tau(G) = |\pi_1(G)_{\Gamma,\mathrm{tor}}|$ and $\tau(H) = |\pi_1(H)_{\Gamma,\mathrm{tor}}|$, which hold by Theorem B.0.4 and the assumed Hasse principle for G and H. Since $\pi_1(G)_{\Gamma,\mathrm{tor}} = \pi_0(Z(\widehat{G})^\Gamma)^*$ and $\mathcal{S}_\psi = S_\psi/Z(\widehat{G})^\Gamma$, we see that $\tau(G)\tau(H)^{-1}$ accounts for the difference in size between $\mathcal{S}_\psi$ and $\mathcal{S}_{\psi^H}$ (by ellipticity of H we have $Z(\widehat{G})^{\Gamma,0} = Z(\widehat{H})^{\Gamma,0}$), while $\mathrm{Out}(H, s, \mathcal{H}, \xi)$ accounts for the different $\widehat{H}$-conjugacy classes ψ^H that become the same $\widehat{G}$-conjugacy class ψ.

Comparing (6.3.4) and (6.4.6) we arrive at the following.

Conjecture 6.4.2.

$$S^G_{geom}(f) = \sum_\psi |\mathcal{S}_\psi|^{-1} e_\psi(s_\psi) S\Theta_\psi(f).$$

This is a simplified version of Arthur's "stable multiplicity formula", cf. [Art13, Theorem 4.1.2] in the case of classical groups. The main simplification comes from the assumption that G/A_G is anisotropic. This implies that only discrete Arthur parameters ψ contribute. In particular, the group $\bar{S}_{\psi}^{0}$, i.e. the identity component of $\bar{S}_{\psi} = S_{\psi}/Z(\widehat{G})^{\Gamma}$, is trivial. Since G is also assumed to satisfy the Hasse principle, the sum runs over $\widehat{G}$-conjugacy classes of discrete Arthur parameters.

The argument that we just summarized shows that Conjecture 6.4.2 follows from the local and global Arthur conjectures for G, as well as the validity of Conjecture 6.4.2 for all proper endoscopic groups of G, which can be assumed by induction on the dimension of G. This argument is reversible, in the sense that one can obtain (a version of) the global Arthur conjecture from knowing Conjecture 6.4.2 for all endoscopic groups of G, and the local Arthur conjectures. This was done in [Taï19] for certain inner forms of classical groups.

It is however clear that this argument alone cannot prove both Conjecture 6.4.2 for G and Arthur's conjectures for G, even if all of these are known for all proper endoscopic subgroups of G. In that situation one needs additional input. In the case of classical groups this input comes by considering the twisted trace formula for GL_N and its stabilization, and will be discussed in §8 (as well as various supplementary local and global results), see the proof of Proposition 8.2.9 and the argument in §8.4 in particular.

7 THE STABLE TRACE FORMULA FOR A GENERAL REDUCTIVE GROUP

We now consider a connected reductive F-group G whose derived subgroup is not anisotropic. For simplicity, we still assume that the derived subgroup is simply connected, and that G satisfies the Hasse principle.

The trace formula for a reductive group whose derived subgroup is not anisotropic is vastly more complicated. On the geometric side, the group $G(F)$ now contains elements that need not be semi-simple and elliptic. The usual terms in the trace formula – orbital integrals and volumes of centralizers – no longer make sense. On the spectral side, the space $L^2(G(F)\backslash G(\mathbb{A}))$ need not decompose as a discrete Hilbert direct sum any more, and the operator $R(f)$ need not be of trace class. Handling these difficulties takes considerable effort. The survey [Art05] gives a detailed account of this, and we encourage the reader to consult it carefully. In this chapter, we have tried to give a survey that is as disjoint from [Art05] as possible while maintaining readability. For example, we have omitted many technical details, while adding the running example of $G = \mathrm{SL}_2$ as well as various further remarks and clarifications. We hope that the reader will find it fruitful to consult this text alongside [Art05].

The derivation of the trace formula goes through a number of stages, beginning with the larval stage of the non-invariant trace formula, mutating through the pupal stage of the invariant trace formula, and ending with the adult stage of the stable trace formula. Each of these stages consists of a distribution that comes along with two expressions for it, that could loosely be described as a "geometric" and "spectral" expression.

We will use the notation J for the distribution of the non-invariant trace formula, I for the distribution of the invariant trace formula, and S for the distribution of the stable trace formula. It will be often convenient to emphasize the group G we are working with, because in the course of working other groups, such as Levi subgroups of G, or endoscopic groups of G, will enter the picture, and their trace formulas will also play a role. For this purpose, we will write J^G, I^G, and S^G, in place of just J, I, and S, respectively. We will also use this notational device for other distributions associated with the trace formula, such as the discrete part $I_{\text{disc}} = I^G_{\text{disc}}$, or the weighted orbital integrals/weighted characters $J_M = J^G_M$ and ther invariant analogs $I_M = I^G_M$, which will be introduced in the course of this section.

The rough order of development is the following. One first obtains the distribution J by integrating along the diagonal a modification of the original automorphic kernel, the so-called truncated kernel. Two expansions of the kernel, one geometric and one spectral, lead to the first form of the corresponding expansions for J, which takes the form

$$\sum_{\mathfrak{o}} J_{\mathfrak{o}}(f) = \sum_{\chi} J_\chi(f)$$

and is called the "coarse expansion", see (7.3.1) below. Developing explicit formulas for the geometric distributions $J_{\mathfrak{o}}$ and spectral distributions J_χ takes significant effort, and leads to the "fine expansion", in which the weighted orbital integrals $J_M(\gamma, f)$ and the weighted characters $J_M(\pi, f)$, both indexed by Levi subgroups M of G, are the main terms.

The distribution J and its various pieces J_M are generally not invariant under conjugation by the group $G(\mathbb{A})$. This necessitates a further development, in which the distribution J is replaced by a distribution I that is $G(\mathbb{A})$-invariant and has two expansions, one in terms of invariant distributions I_M of geometric nature, and one in terms of invariant distributions I_M of spectral nature, each an invariant analog of the distribution J_M. The relationship between $J = J^G$ and $I = I^G$ is roughly that $J^G = \sum_M I^M$, the sum being over all Levi subgroups of G containing the fixed minimal Levi subgroup M_0, see (7.7.3) for the more precise statement.

The distribution I and its pieces I_M are invariant under conjugation by $G(\mathbb{A})$, but for many applications a stronger invariance, called stable invariance and already discussed in §6 in the setting of anisotropic groups, is required. This leads to replacing I by yet another distribution, called S, which is stably invariant, and again has two expansions, one in terms of stable distributions S_M of geometric nature, and another in terms of stable distributions S_M of spectral nature. The relationship between the distributions $I = I^G$ and $S = S^G$ is roughly that $I^G = \sum_{G'} S^{G'}$, where the sum runs over the elliptic endoscopic groups G' for G, see (7.8.1) for the more precise statement.

7.1 Non-compactness and the continuous spectrum

As we just mentioned, there are two key difficulties in deriving the trace formula for the isotropic group G. On the geometric side, the group $G(F)$ contains elements that need not be semi-simple and elliptic, and the usual terms in the trace formula – orbital integrals and volumes of centralizers – no longer make sense. On the spectral side, the space $L^2(G(F)\backslash G(\mathbb{A}))$ need not decompose as a discrete Hilbert direct sum any more, and the operator $R(f)$ need not be of trace class.

We will now discuss this in a bit more detail and consider the example of $G = \mathrm{SL}_2$. One has the preliminary orthogonal decomposition

$$L^2(G(F)\backslash G(\mathbb{A})^1) = L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A})^1) \oplus L^2_{\text{cts}}(G(F)\backslash G(\mathbb{A})^1),$$

where $L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A})^1)$ is the so-called *discrete spectrum*, the sum of all irreducible subrepresentations of $L^2(G(F)\backslash G(\mathbb{A})^1)$, and $L^2_{\text{cts}}(G(F)\backslash G(\mathbb{A})^1)$ is the *continuous spectrum*. By definition, we have the Hilbert direct sum decomposition

$$L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A})^1) = \widehat{\bigoplus}_\pi m(\pi) \cdot \pi,$$

where π runs again over all admissible irreducible representations of $G(\mathbb{A})$, and those with $m(\pi) > 0$ are called *discrete automorphic representations*. A major problem in the theory of automorphic forms is to describe $m(\pi)$ for all irreducible admissible representations π of $G(\mathbb{A})$.

In contrast, the continuous spectrum $L^2_{\text{cts}}(G(F)\backslash G(\mathbb{A})^1)$ does not contain any closed irreducible subrepresentation. It is described by Langlands' work on Eisenstein series. We give a brief summary in Appendix A.3.

The problem is that, while the restriction of $R(f)$ to $L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A})^1)$ is of trace class and can be used to compute the numbers $m(\pi)$ in the decomposition of this space, the restriction of $R(f)$ to $L^2_{\text{cts}}(G(F)\backslash G(\mathbb{A})^1)$ is not of trace class.

We now give simple examples for the various issues in the trace formula using the group $G = \mathrm{SL}_2/\mathbb{Q}$.

1. Consider the semi-simple but non-elliptic element $\gamma = \mathrm{diag}(a, a^{-1}) \in \mathrm{SL}_2(\mathbb{Q})$. Then the orbital integral at γ still converges. However, $G_\gamma = \mathbb{G}_m$, and $G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A}) = \mathbb{Q}^\times\backslash\mathbb{A}^\times$ has infinite volume.

2. Consider an upper-triangular unipotent element γ. Its centralizer is the product $G_\gamma = ZN$, where $Z = \{\pm 1\}$ and $N = \mathbb{G}_a$. Then $G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A}) = (\mathbb{Q}\backslash\mathbb{A}) \times (\{\pm1\}\backslash \prod_v \{\pm 1\})$ is compact, in particular has finite volume. But the problem is with the orbital integral. While at each place the local orbital integral does converge (a special case of the general result of Ranga Rao, cf [Rao72]), the adelic orbital integral does not.

   Let us consider the special case where the test function is the characteristic function of $G(\mathbb{Z}_p)$. Then the value of the local orbital integral at a finite place p is $(1 - 1/p)^{-1} = \zeta_p(1)$, provided the measures are chosen appropriately, see Theorem B.0.6 and Definition B.0.7. Therefore the product over all p, which together with the local orbital integral at ∞ would be the adelic orbital integral, diverges.

3. The Hilbert space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ decomposes as the direct sum of three pieces $L^2_{\text{cts}}(G(\mathbb{Q})\backslash G(\mathbb{A})) \oplus \mathbb{C} \oplus L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. The middle part is spanned by the constant function, which is square-integrable because the quotient $G(\mathbb{Q})\backslash G(\mathbb{A})$ has finite volume. The space $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is spanned by the cusp forms. It decomposes as a Hilbert direct sum of irreducible representations. The operator $R(f)$ is of trace class on both of these summands. The space $L^2_{\text{cts}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is the orthocomplement of the other two. It is spanned by Eisenstein series. It does not have any irreducible subquotient (as an analogy, recall Example 3.2.1), and in fact decomposes as a continuous direct integral. The restriction of $R(f)$ to this factor is not of trace class.

7.2 Truncation

The way to obtain a trace formula despite these difficulties is to perform a truncation, as explained in [Art05, §6] and [Cha22, §3.1.3]. One replaces the kernel k_f of the operator $R(f)$ (cf. (3.2.1)) with a truncated kernel k_f^T, which is absolutely integrable along the diagonal and one can develop a geometric and spectral expansion for its integral. This process takes time and goes through various steps. We will review the cornerstones here, referring the reader to [Art05] and [Cha22] for more details, as well as to [Gar] for a discussion of SL_2.

The definition of the truncated kernel is given in [Art05, (6.1)] in the setting of the standard trace formula discussed here, and in [Cha22, (3.1.3.8)] in the more general setting of the relative trace formula. The truncation is controlled by a parameter T, which is a point in the positive cone $\mathfrak{a}_0^+ = \mathfrak{a}_{P_0}^+$ associated to a minimal parabolic pair (M_0, P_0) of G, and is supposed to be sufficiently regular, i.e. lie sufficiently deep in the cone. We will discuss the linear algebra of these cones in more detail in the next subsection, and give the definition of the cone in (7.3.6). Integrating this kernel along the diagonal produces a distribution J^T. The trace formula is obtained by providing geometric and spectral descriptions of the distribution J^T. Unlike the case of anisotropic (modulo center) groups, the spectral expansion of $J^T(f)$ does not give the trace of the operator $R(f)$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$, since the latter doesn't have a trace. But the usage of the words "trace formula" is still justified, because the spectral expansion does contain the trace of $R(f)$ on the discrete part of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. It also contains other terms, coming from the continuous part of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. The difference between the geometric expansion and the contribution of the continuous part of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ to the spectral expansion does give a formula for the trace of $R(f)$ on the discrete part of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$.

The dependence of the distribution J^T on T is resolved by setting T equal to a specific preferred value T_0. While for many groups T_0 is simply equal to 0, this is not true for all groups. The precise definition of T_0 is given in [Art05, (9.4)]. Note that T_0 will usually not be a sufficiently regular point in the cone $\mathfrak{a}_0^+$, and in fact may not even lie in that (open) cone. Therefore a preliminary result to establish is that the function $T \mapsto J^T(f)$, a-priori defined for a sufficiently deep subcone of $\mathfrak{a}_0^+$, is polynomial in T, and therefore can be evaluated at any $T \in \mathfrak{a}_0$, cf. [Art05, Theorem 9.1].

7.3 The coarse expansions and fine expansions

The first step in developing a geometric and spectral expansion for the distribution $J = J^{T_0}$ takes the form

$$\sum_{\mathfrak{o}} J_{\mathfrak{o}}(f) = \sum_{\chi} J_{\chi}(f). \tag{7.3.1}$$

The left sum runs over the set $\mathcal{O}$ of equivalence classes of elements of $G(\mathbb{Q})$, where two elements are deemed equivalent if their semi-simple parts are conjugate. The right sum is over conjugacy classes of pairs (M, σ) where M is a Levi subgroup of G and σ is a cuspidal automorphic representation of $M(\mathbb{A})^1$.

The left sum is called the "coarse geometric expansion". It is obtained (cf. [Art05, §10]) by decomposing the truncated kernel as a sum $k^T = \sum_{\mathfrak{o}} k_{\mathfrak{o}}^T$ over the set $\mathcal{O}$ of equivalence classes of elements of $G(\mathbb{Q})$ and letting $J_{\mathfrak{o}}$ be the integral of $k_{\mathfrak{o}}^T$ over the diagonal.

The right sum is called the "coarse spectral expansion". It is obtained (cf. [Art05, §14]) by writing $k^T = \sum_{\chi} k_{\chi}^T$ and defining J_{χ} as the integral of k_{χ}^T over the diagonal. This process is more involved than for the geometric expansion, because the interplay of truncation with the spectral data is less transparent than with the geometric data. It involves a variation of the truncation operator, reviewed in [Art05, §13].

The distribution $J_{\mathfrak{o}}$ is of geometric nature, and the distribution J_{χ} is of spectral nature. The "coarse expansion" (7.3.1) is a first step towards the trace formula. It must now be refined by providing more useful formulas for the distributions $J_{\mathfrak{o}}$ and J_{χ}. These formulas constitute the "fine expansions".

The simplest cases of the fine expansions are those that we have already encountered in the case of an anisotropic (modulo center) group. Namely, on the geometric side, if $\mathfrak{o}$ is the conjugacy class of a semi-simple element γ that is not contained in a proper Levi subgroup, then

$$J_{\mathfrak{o}}(f) = \mathrm{vol}(G_{\gamma}(\mathbb{Q})\backslash G_{\gamma}(\mathbb{A})^1) \int_{G_{\gamma}(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x)dx. \tag{7.3.2}$$

On the spectral side, if $\chi = (G, \sigma)$ and $m_{\text{cusp}}(\sigma)$ is the multiplicity of σ in $L^2_{\text{cusp}}(G(F)\backslash G(\mathbb{A})^1)$, then

$$J_{\chi}(f) = m_{\text{cusp}}(\sigma) \mathrm{tr}\, \sigma(f). \tag{7.3.3}$$

However, the general distributions $J_{\mathfrak{o}}$ and J_{χ} are not as easy to describe. We will now attempt to describe some of them.

On the geometric side, consider the case where $\mathfrak{o}$ is the conjugacy class of a semi-simple element γ that is contained in a Levi subgroup $M \subset G$ but in no smaller Levi subgroup, and such that $G_{\gamma} \subset M$. Then (cf. [Art05, Theorem

11.2])

$$\begin{aligned} J_{\mathfrak{o}}(f) &= \operatorname{vol}(M_\gamma(\mathbb{Q})\backslash M_\gamma(\mathbb{A})^1)\cdot J_M(\gamma,f) \qquad (7.3.4)\\ J_M(\gamma,f) &= \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x)v_M(x)dx. \end{aligned}$$

The distribution $J_M(\gamma,f)$ is an example of a weighted orbital integral, and v_M is a scalar function called the "weight factor", defined as the volume of a certain polytope in the real vector space, $\mathfrak{a}_M = X_*(A_M)\otimes_{\mathbb{Z}}\mathbb{R}$, whose dual we denote by $\mathfrak{a}_M^* = X^*(A_M)\otimes_{\mathbb{Z}}\mathbb{R}$.

On the spectral side, consider $\chi = (M,\sigma)$ such that the stabilizer of σ in $N_G(M)(F)$, which necessarily contains $M(F)$, equals exactly $M(F)$. Then (cf. [Art05, Theorem 15.4])

$$\begin{aligned} J_\chi(f) &= m_{\text{cusp}}(\sigma)J_M(\sigma,f) \qquad (7.3.5)\\ J_M(\sigma,f) &= \int_{i\mathfrak{a}_M^*} \operatorname{tr}(\mathcal{M}_P(\sigma_\lambda)\mathcal{I}_P(\sigma_\lambda,f))d\lambda. \end{aligned}$$

The distribution $J_M(\sigma,f)$ is an example of a weighted character, and the "weight" here is the operator $\mathcal{M}_P(\sigma_\lambda)$, constructed in terms of intertwining operators, on the space of the parabolically induced representation $\mathcal{I}_P(\sigma_\lambda)$, where $\sigma_\lambda = \sigma\otimes e^{\langle H_M(-),\lambda\rangle}$, see (4.2.1) for the character $e^{\langle H_M(-),\lambda\rangle}$ and Appendix A.1 for parabolic induction.

The scalar weight function v_M and the operator weight function $\mathcal{M}_P$ are instances of an abstract combinatorial construction, called a (G,M)-family, and discussed in [Art05, §17]. We will now very briefly recall the relevant definitions.

First, we need to recall the hyperplane structure in the vector spaces $\mathfrak{a}_M$ and $\mathfrak{a}_M^*$, following [Art05, §5] and [GKM97, §5]. The adjoint action of A_M on Lie(G) decomposes the latter vector space into a direct sum of weight spaces (with multiplicities), and we write $\Phi_M\subset X^*(A_M)$ for the set of non-zero weights. In general Φ_M is not a root system, but we can nonetheless introduce some concepts familiar from root systems. For example, for each $\alpha\in\Phi_M$ we can consider its vanishing hyperplane in $\mathfrak{a}_M$. The complement of the union of these hyperplanes is a union of cones in $\mathfrak{a}_M$, called *chambers*. The set of chambers is in bijection with the set of parabolic subgroups P which admit M as a Levi factor. Writing $\mathfrak{a}_P^+$ for the chamber corresponding to P, we have

$$\mathfrak{a}_P^+ = \{x\in\mathfrak{a}_M \mid \langle x,\alpha\rangle > 0,\ \alpha\in\Phi_P^+\}, \qquad (7.3.6)$$

where $\Phi_P^+\subset\Phi_M$ is the set of weights of the adjoint action of A_M on Lie(N_P), with N_P the unipotent radical of P. More generally, if Q is a parabolic subgroup that contains M, there is a unique Levi factor L of Q containing M, and we have $\mathfrak{a}_L\subset\mathfrak{a}_M$, so we can consider $\mathfrak{a}_Q^+\subset\mathfrak{a}_L\subset\mathfrak{a}_M$. This leads to the disjoint union decomposition

$$\mathfrak{a}_M = \coprod_Q \mathfrak{a}_Q^+,$$

where Q runs over the set of parabolic subgroups that contain M.

At this point it is useful to recall that the natural inclusion $\mathfrak{a}_L \subset \mathfrak{a}_M$ has a natural retraction. This comes from the fact that $A_L \to L \to L/[L,L]$ is an isogeny of tori, hence leads to an isomorphism after applying $X_*(-) \otimes_{\mathbb{Z}} \mathbb{R}$. The inclusion $\mathfrak{a}_L \to \mathfrak{a}_M$ comes from the inclusion $A_L \to A_M$, while its retraction comes from the inclusion $M \to L$. The kernel of the retraction is denoted by $\mathfrak{a}_M^L$ and we have the direct sum decomposition $\mathfrak{a}_M = \mathfrak{a}_L \oplus \mathfrak{a}_M^L$ and the corresponding dual direct sum decomposition. The set Φ_M lies in $(\mathfrak{a}_M^G)^* = X^*(A_M/A_G) \otimes_{\mathbb{Z}} \mathbb{R}$.

A hyperplane structure similar to that of $\mathfrak{a}_M$ is also present in the dual space $\mathfrak{a}_M^*$. For each "root" $\alpha \in \Phi_M \subset \mathfrak{a}_M^*$ one can define a "coroot" $\alpha^\vee \in \mathfrak{a}_M$. We have placed quotation marks to remind ourselves that Φ_M is generally not a root system, and as a result there is no classically defined dual root system. Nonetheless, we can argue as follows. Choose a minimal Levi subgroup $M_0 \subset M$. Then $\Phi_{M_0} \subset \mathfrak{a}_{M_0}^* = \mathfrak{a}_0^*$ is an actual root system and α is the image of some $\alpha_0 \in \Phi_{M_0}$ under the restriction map $\mathfrak{a}_{M_0}^* \to \mathfrak{a}_M^*$. We have a dual root system $\Phi_{M_0}^\vee \subset \mathfrak{a}_{M_0}$, hence a (bona fide) coroot $\alpha_0^\vee \in \mathfrak{a}_{M_0} = \mathfrak{a}_0$, which we can map under the retraction map $\mathfrak{a}_{M_0} \to \mathfrak{a}_M$. The result is independent of the choices of M_0 and α_0. We can now define the analogous hyperplane structure. In particular, we have the chamber

$$(\mathfrak{a}_P^*)^+ = \{x \in \mathfrak{a}_M^* | \langle x, \alpha^\vee \rangle > 0,\ \alpha \in \Phi_P^+\}.$$

In the formulas for $\mathfrak{a}_P^+$ and $(\mathfrak{a}_P^*)^+$ one can replace Φ_P^+ with the subset $\Delta_P \subset \Phi_P^+$ of "simple roots", which can be defined either as the subset of elements of Φ_P that are not non-trivial sums of other elements, or as the image under $\mathfrak{a}_{M_0}^* \to \mathfrak{a}_M^*$ of an actual set of simple roots in Φ_{M_0} corresponding to the chamber for a minimal parabolic subgroup P_0 contained in P with Levi factor M_0.

We can now return to the short review of the definition of a (G, M)-family. It is a collection $(c_P(\lambda))_P$, indexed by the set of parabolic subgroups P that admit M as a Levi factor. The term $c_P(\lambda)$ is a smooth function of the real vector space $i\mathfrak{a}_M^*$. In the case of the weight factor, the values of this function are complex numbers. For the (G, M)-families that occur in the spectral side, the values will be operators on the space of the induced representation $\mathcal{I}_P(\sigma_\lambda)$. The condition that makes the collection $(c_P(\lambda))_P$ into a (G, M)-family is that, when the chambers $(\mathfrak{a}_{P_1}^*)^+$ and $(\mathfrak{a}_{P_2}^*)^+$ in $\mathfrak{a}_M^*$ are adjacent and $i\lambda$ lies in the hyperplane separating these chambers, then $c_{P_1}(\lambda) = c_{P_2}(\lambda)$.

Given a (G, M)-family $(c_P(\lambda))_P$ one defines the function

$$c_M(\lambda) = \sum_P c_P(\lambda) \mathrm{vol}(\mathfrak{a}_M^G / \mathbb{Z}[\Delta_P^\vee]) \prod_{\alpha \in \Delta_P} \lambda(\alpha^\vee)^{-1}.$$

We recall that $\mathfrak{a}_M^G = X_*(A_M/A_G) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\Delta_P \subset (\mathfrak{a}_M^G)^*$ is the set of "simple roots" associated to P, cf. [Art05, §5]. A basic but important result ([Art05, Lemma 17.1]) is that the function $c_M(\lambda)$ is smooth on $i\mathfrak{a}_M^*$, i.e. it does not have singularities at the hyperplanes $\lambda(\alpha^\vee) = 0$. In particular, one can define the constant

$$c_M = c_M(0).$$

One sees easily that the product $(c_P(\lambda)c'_P(\lambda))_P$ of two (G,M)-families $(c_P(\lambda))_P$ and $(c'_P(\lambda))_P$ is again a (G,M)-family. The resulting function $(cc')_M(\lambda)$ can then be expressed in terms of the functions $c_M(\lambda)$ and $c'_M(\lambda)$, and such an expression is called a "splitting formula". We refer the reader to [Art05, Lemmas 17.4, 17.6] for a precise statement. Splitting formulas are important in writing the global distributions in the trace formula in terms of local distributions.

Besides taking a product we will need to describe another operation on (G,M)-families, called *restriction*. It will not be needed for the description of the term (7.3.5), but will be needed for the more general term (7.3.9) to be described below. If $(c_P(\lambda))_P$ is a (G,M)-family, and $M \subset L \subset G$ is a larger Levi subgroup, we can consider the collection $(d_Q(\lambda))_Q$ indexed by the set of parabolic subgroups that admit L as a Levi factor, and defined as

$$d_Q(\lambda) = c_P(\lambda),$$

where P is any parabolic subgroup of G that admits M as a Levi factor and is contained in Q, and $\lambda \in i\mathfrak{a}_L^* \subset i\mathfrak{a}_M^*$. One can show that this is well-defined, i.e. independent of the choice of P, and that $(d_Q(\lambda))_Q$ is a (G,L)-family.

After reviewing the basic definitions surrounding a (G,M)-family we return to the discussion of the weight factors v_M and $\mathcal{M}_P$. The scalar weight factor is obtained as the constant c_M, now seen as a function of the variable $x \in G(\mathbb{A})$, associated to the (G,M)-family

$$(e^{-\lambda(H_P(x))})_P,$$

where H_P is the function

$$H_P\colon G(\mathbb{A}) \to \mathfrak{a}_M,\ H_P(nmk) = H_M(m),\ n \in N_P(\mathbb{A}), m \in M(\mathbb{A}), k \in K, \tag{7.3.7}$$

defined in terms of the Iwasawa decomposition $G(\mathbb{A}) = N_P(\mathbb{A})M(\mathbb{A})K$ (see [GH24, Appendix A]) and H_M is the function (4.1.1) for the group M. One can show that $v_M(x)$ is the volume of the convex hull of the set of points $\{-H_P(x)\}$ in the real vector space $\mathfrak{a}_M^G$, where P runs over the set of parabolic subgroups with Levi factor M, cf. [Art05, Lemma 17.2] and the discussion thereafter.

The operator weight factor $\mathcal{M}_P(\sigma_\lambda)$ is obtained as the "constant" c_M of the operator-valued (G,M)-family

$$\mathcal{M}_Q(\Lambda, \sigma_\lambda, P) = M_{Q|P}(\sigma_\lambda)^{-1} M_{Q|P}(\sigma_{\lambda+\Lambda}),$$

where $M_{Q|P}$ denotes the global unnormalized intertwining operator discussed in Appendix A.2. The notation can be a little confusing here. The parabolic subgroup P and the vector λ are treated as constant parameters for this (G,M)-family. The role of the parabolic subgroup denoted by P in the definition of a (G,M)-family is played here by the parabolic subgroup Q, and the role of the variable λ in the definition of a (G, M)-family is played here by Λ. Therefore, the resulting "constant" $c_M = \mathcal{M}_P(\sigma_\lambda)$ depends on the parameters P and λ.

It is important to note that the (G, M)-family for the global weight factor $v_M(x)$ can be described as the product of the (G, M)-families for the local weight factors at all places. The splitting formulas then describe the global weight factor as a sum of local weight factors, and this leads to an expression of the global weighted orbital integrals in terms of local weighted orbital integrals, again called a "splitting formula". We will see an example of this behavior below.

On the other hand, the (G, M)-family for the global weight factor $\mathcal{M}_P(\sigma_\lambda)$ is not a-priori a product of local (G, M)-families. Therefore the global weighted character is not readily expressed in terms of local weighted characters. This problem will be rectified in §7.5 by using normalized intertwining operators.

The cases of the distributions $J_\mathfrak{o}$ and J_χ treated so far can be termed (G, M)-*regular*, in the sense that the centralizer of the M-object (being either a conjugacy class or a representation) in G is equal to the centralizer in M. In [Art05] Arthur uses the term "unramified" instead, but this word already has a well-established meaning of a different nature, so we prefer to avoid it.

The description of the remaining, non-(G, M)-regular cases, is more complicated. On the geometric side, we now consider an arbitrary equivalence class $\mathfrak{o}$. The most (G, M)-singular case is that of the class consisting of all unipotent elements, and the general case can be reduced to it. In order to treat this case, Arthur defines a generalization $J_M(\gamma, f)$ of the concept of a weighted orbital integral, for elements $\gamma \in M(F)$ that are not (G, M)-regular. The definition, which is reviewed in [Art05, §18], is not the same as that for the (G, M)-regular case. Indeed, that definition would not make sense, since the weight factor v_M is not a well-defined function on $G_\gamma(\mathbb{A})\backslash G(\mathbb{A})$, but only on $M_\gamma(\mathbb{A})\backslash G(\mathbb{A})$, but the integrand may not converge on $M_\gamma(\mathbb{A})\backslash G(\mathbb{A})$. Instead, the definition of $J_M(\gamma, f)$ in this more general case is obtained from the (G, M)-regular case by a limiting process. One examines the limit $\lim_{a\to 1} J_M(a\gamma, f)$, where $a \in Z_M(F_S)$ is an element in general position, so that $a\gamma \in M(F_S)$ is (G, M)-regular; here S is a finite set of places. It turns out that this limit does not converge, but that by combining the various distributions $J_L(a\gamma, f)$ for all Levi subgroups L containing M and forming a suitable linear combination, the limit does converge and provides a well-defined distribution $J_M(\gamma, f)$, cf. [Art05, Theorem 18.2].

Once the general weighted orbital integrals $J_M(\gamma, f)$ have been defined, the distribution $J_\mathfrak{o}(f)$ for a general equivalence class $\mathfrak{o} \in \mathcal{O}$ takes the following form, cf. [Art05, Theorem 19.2].

$$J_\mathfrak{o}(f) = \sum_M |W_0^M| \cdot |W_0^G|^{-1} \sum_\gamma a^M(S, \gamma) J_M(\gamma, f). \tag{7.3.8}$$

Here M runs over the set of standard Levi subgroups, $W_0^M = N_M(M_0)/M_0$, $W_0^G = N_G(M_0)/M_0$, and γ runs over the set of elements of $M(F)$ that lie in $\mathfrak{o}$, up to conjugation by $M(F_S)$, where S is a finite but sufficiently large set of places of F. The global factor $a^M(S,\gamma)$ is defined in [Art05, (19.5)] and is generally of inductive nature. It vanishes unless the semi-simple part of γ is $M(F)$-elliptic, and equals vol$(M_\gamma(F)\backslash M_\gamma(\mathbb{A})^1)$ if γ is semi-simple (we are using here that G_{der} is simply connected).

Consider now a possibly non-(G,M)-regular contribution χ to the spectral side. This means that $\chi=(M,\sigma)$ for a cuspidal automorphic representation σ of $M(\mathbb{A})^1$. Then (cf. [Art05, Theorem 21.6])

$$\begin{aligned} J_\chi(f) &= \sum_{M',L,\pi,s} |W_0^{M'}|\cdot|W_0^G|^{-1} \\ &\cdot\quad |\det(s-1|\mathfrak{a}_{M'}^L)|^{-1}\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \operatorname{tr}\left(\mathcal{M}_L(\lambda,P)M_P(s,0)\mathcal{I}_{P,\chi,\pi}(\lambda,f)\right)d\lambda. \end{aligned} \tag{7.3.9}$$

There is a good amount of new notation in this expression, which we will now briefly explain. Before we do that, we alert the reader that there is a typo in [Art05, Theorem 21.6], where $\mathfrak{a}_{M'}^L$ is replaced by $\mathfrak{a}_{M'}^G$.

The sum runs over standard Levi subgroups M' (we have placed a prime to distinguish from the Levi subgroup M in the datum $\chi=(M,\sigma)$), Levi subgroups L containing M', $s\in W^L(M')_{\text{reg}}$, and unitary representations π of $M'(\mathbb{A})^1$, where

$$W^L(M')_{\text{reg}}=\{s\in W^L(M')|\ker(s-1|\mathfrak{a}_{M'})=\mathfrak{a}_L\}.$$

The representation $\mathcal{I}_{P,\chi,\pi}(\lambda)$ is the restriction of $\mathcal{I}_P(\lambda)$ (see Appendix A.2) on the following invariant subspace $\mathcal{H}_{P,\chi,\pi}$ of $\mathcal{H}_P=\mathcal{I}_P^G(L^2_{\text{disc}}(M(F)\backslash M(\mathbb{A})^1))$. We consider the σ-isotypic subspace $L^2_{\text{disc}}(M(F)\backslash M(\mathbb{A})^1)[\sigma]$ of $L^2_{\text{disc}}(M(F)\backslash M(\mathbb{A})^1)$ and write $\mathcal{H}_{P,\chi}=\mathcal{I}_P^G(L^2_{\text{disc}}(M(F)\backslash M(\mathbb{A})^1)[\sigma])\subset\mathcal{H}_P$. Recall from §A.2 that this space consists of functions $\phi\colon N_P(\mathbb{A})M(F)\backslash G(\mathbb{A})\to\mathbb{C}$. We let $\mathcal{H}_{P,\pi}\subset\mathcal{H}_P$ be the subspace consisting of those functions ϕ for which the function $\phi_x\in L^2_{\text{disc}}(M(F)\backslash M(\mathbb{A})^1)$, $\phi_x(m)=\phi(mx)$, is a matrix coefficient of π for all $x\in G(\mathbb{A})$. We let $\mathcal{H}_{P,\chi,\pi}=\mathcal{H}_{P,\chi}\cap\mathcal{H}_{P,\pi}$.

The operator $M_P(s,0)=M_{P|P}(s,0)$ is a special case of the operator $M_{Q|P}(s,\lambda)$ discussed in Appendix A.2.

Finally, the operator $\mathcal{M}_L(\lambda,P)$ is defined similarly to the operator $\mathcal{M}_P(\sigma_\lambda)$ defined above that was part of (7.3.5). Namely, one considers the (G,M)-family

$$\mathcal{M}_Q(\Lambda,\lambda,P):=M_{Q|P}(\lambda)^{-1}M_{Q|P}(\lambda+\Lambda),$$

where as above P and λ are fixed parameters, the role of the parabolic subgroup denoted by P in the definition of a (G,M)-family is played by Q, and the role of the variable λ in that definition is played by Λ. But, unlike in the definition of $\mathcal{M}_P(\sigma_\lambda)$ above, we will now not directly take the associated constant function called c_M above. Rather, we will first restrict the (G,M)-family to a (G,L)-family, by the restriction process described above, and then take the corresponding constant function, denoted now by $\mathcal{M}_L(\lambda,P)$. This is possible, because $\lambda\in i\mathfrak{a}_L^*$.

Let us note that, when χ is (G, M)-regular, the sum over (M', L) in term (7.3.9) collapses to the single summand $L = M' = M$. The reason for this is implicit in Langlands' description of the discrete spectrum via residues of cuspidal Eisenstein series and a brief discussion is given following [Art05, (15.14)]. The sum over s then collapses to the term $s = 1$ and the term $|\det(s - 1|\mathfrak{a}_M^M)|^{-1}$ is to be interpreted as 1. The sum over π can be absorbed into the full induction $\mathcal{H}_{P,\chi}$, which is the parabolic induction (see Appendix A.1) of the σ-isotypic component of $L^2_{\text{disc}}(M(F)\backslash M(\mathbb{A})^1)$. That component is isomorphic to $\sigma^{\oplus m_{\text{cusp}}(\sigma)}$. Using these arguments one can identify the terms (7.3.9) and (7.3.5).

We return to a general cuspidal data χ. It is possible to slightly condense the presentation of the fine spectral expansion. At the moment, it consists of a sum $\sum_\chi J_\chi$, and then each J_χ is given by (7.3.9), which is a further sum over (M', L, π, s). One can condense the pair (χ, π) in this sum into a single real number $t \geq 0$ as follows. Consider a unitary representation σ of $M(\mathbb{A})^1$. Its infinite component σ_∞ has an infinitesimal character ν_σ, an element of the complex vector space $i\mathfrak{t}^*_{\mathbb{C}}/i\mathfrak{a}^*_G$, where $\mathfrak{t}_{\mathbb{C}} = X_*(T) \otimes \mathbb{C}$ for an arbitrary maximal torus $T \subset G$. This vector space has a Weyl-invariant real structure $X_*(T) \otimes \mathbb{R}$. Using that we can write $\nu_\sigma = \text{Re}(\nu_\sigma) + i\text{Im}(\nu_\sigma)$, where $\text{Re}(\nu_\sigma), \text{Im}(\nu_\sigma)$ are $\mathbb{R}$-valued linear forms on $X_*(T/A_G) \otimes \mathbb{R}$. Fixing a Weyl-invariant norm on the vector space $\mathfrak{t}_{\mathbb{C}}$ we can consider the non-negative real number $|\text{Im}(\nu_\sigma)|$. Then we can define for $t \in \mathbb{R}_{\geq 0}$ the distribution

$$J_t = \sum_{\substack{\chi=(M,\sigma)\\ |\text{Im}(\nu_\sigma)|=t}} J_\chi.$$

We also define

$$\mathcal{H}_{P,t}(\lambda) = \bigoplus_{\substack{\chi=(M,\sigma),\pi\\ |\text{Im}(\nu_\sigma)|=t}} \mathcal{H}_{P,\chi,\pi}$$

and denote by $\mathcal{I}_{P,t}(\lambda)$ the restriction of the representation $\mathcal{I}_P(\lambda)$ of $G(\mathbb{A})$ on $\mathcal{H}_P(\lambda)$ to the invariant subspace $\mathcal{H}_{P,t}(\lambda)$. It can be shown that the summand for (χ, π) vanishes unless $|\text{Im}(\nu_\pi)| = t$. Then we obtain from the above discussion the spectral decomposition $J = \sum_{t\geq 0} J_t$ with

$$\begin{aligned} J_t(f) &= \sum_{M,L,s} |W_0^M| \cdot |W_0^G|^{-1} \\ &\cdot \quad |\det(s-1|\mathfrak{a}_M^L)|^{-1} \int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \text{tr}\,(\mathcal{M}_L(\lambda, P) M_P(s,0) \mathcal{I}_{P,t}(\lambda, f)) d\lambda, \end{aligned} \tag{7.3.10}$$

cf. [Art05, Corollary 21.7]. We have dropped here the prime decoration from M, since we are not using the cuspidal datum χ of which M denoted a component.

7.4 The discrete part of the spectral side

We have seen that the spectral side of the trace formula is the sum over χ of the contributions (7.3.9), of which the contributions (7.3.5) are a special case.

We have further condensed the sum over χ to a sum over $t \in \mathbb{R}_{\geq 0}$ of the terms (7.3.10).

Thus, the spectral side is an explicit sum of integrals. Each integral has a "continuous" nature, except for the special case $L = G$, where the domain of integration is a single point. Therefore, the part of the spectral side that is of "discrete" nature is the part where $L = G$, thus

$$\begin{aligned} I^G_{\text{disc}}(f) &:= \sum_{t \in \mathbb{R}_{\geq 0}} I^G_{\text{disc},t}(f) \\ I^G_{\text{disc},t}(f) &:= \sum_{M,s} |W_0^M| \cdot |W_0^G|^{-1} \cdot |\det(s-1|\mathfrak{a}^G_M)|^{-1} \mathrm{tr}\,(M_P(s,0)\mathcal{I}_{P,t}(0,f)), \end{aligned} \tag{7.4.1}$$

where we recall that the second sum runs over the set of Levi subgroups M contained in a fixed minimal Levi subgroup M_0, and the set $s \in W^G(M)_{\text{reg}}$ consisting of those elements of $N_G(M)(F)/M(F)$ whose action on $\mathfrak{a}_M$ fixes precisely $\mathfrak{a}_G$. This distribution contains the contributions to the spectral side of the discrete spectrum, as well as contributions from the continuous spectrum coming from non-(G,M)-regular points. Strictly speaking it is not yet established that the sum over t converges, but one can work with an individual t-summand $I_{\text{disc},t}$.

The fact that this is a discrete distribution means that it can be written as a sum

$$I^G_{\text{disc}}(f) = \sum_{\pi} a^G_{\text{disc}}(\pi) \mathrm{tr}\,\pi(f),$$

where π runs over the set of admissible unitary representations of $G(\mathbb{A})^1$ (with infinitesimal character constrained by t if we are considering $I^G_{\text{disc},t}$), and the coefficients $a^G_{\text{disc}}(\pi)$ are certain complex numbers. The sum is finite for any fixed f. When π is cuspidal and regular then $a^G_{\text{disc}}(\pi)$ equals the multiplicity of π in the discrete spectrum, which we denoted previously by $m_{\text{cusp}}(\pi)$ (see the discussion after (7.3.9)).

7.5 The refined non-invariant trace formula

The fine expansions discussed so far provide a trace formula which is an equality of a geometric and spectral side, and on each side we have some information about the distributions that occur. However, the two sides have somewhat different structures, with the geometric side being a sum over Levi subgroups M and over certain elements γ of $M(\mathbb{Q})$, see the general term (7.3.8), while the spectral side is a sum over quadruples (t, M, L, s) as well as an integral, see the general term (7.3.9). Another problem is that, while the distributions on the geometric side are of local nature – they are weighted orbital integrals and satisfy splitting formulas coming from the splitting formulas satisfied by the weight factors v_M, the distributions on the spectral side do not satisfy similar splitting formulas, because the spectral weight terms $\mathcal{M}_L(\lambda, P)$ do not. This in turn comes from the fact that the standard global intertwining operators do not generally factor as products of standard local intertwining operators.

A further refinement remedies these problems, by rewriting the fine spectral expansion in a different way. The resulting trace formula takes the following form, cf. [Art05, Corollary 22.1],

$$\sum_M |W_0^M||W_0^G|^{-1}\sum_\gamma a^M(\gamma)J_M(\gamma,f) = \sum_M |W_0^M||W_0^G|^{-1}\int_\pi a^M(\pi)J_M(\pi,f)d\pi. \tag{7.5.1}$$

Here both sums are over Levi subgroups containing in a fixed minimal Levi subgroup M_0. The geometric side is the one discussed in the previous subsection, but the spectral side has been rewritten.

The integral over π runs over the set $\Pi(M)^T$ consisting of representations induced from discrete automorphic representations of Levi subgroups of M, with infinitesimal character bounded by T. This set stratifies according to pairs (L,σ) consisting of a Levi subgroups L of M and a discrete automorphic representation σ of L, and on each stratum one has the Lebesgue measure coming from $i\mathfrak{a}_L^*/i\mathfrak{a}_M^*$, normalized by $|W_0^M||W_0^L|^{-1}$. The coefficient $a^M(\pi)$, evaluated on the stratum corresponding to (L,σ), is the product of the number $a^L_{\text{disc}}(\sigma)$ coming from the expression (7.4.1) for I^L_{disc} and a function $r_L^M(\sigma_\lambda)$, while the weighted character $J_M(\pi,f)$ is defined as

$$\int_{i\mathfrak{a}_M^*} \text{tr}\,(R_M(\pi_\lambda,P)\mathcal{I}_P(\pi_\lambda,f))d\lambda.$$

The two functions $r_L^M(\sigma_\lambda)$ and $R_M(\pi_\lambda,P)$ come from (G,M)-families involving normalized intertwining operators, which we now briefly review.

The global standard intertwining operator $J_P(w,\lambda)$ acting on the space of the induced representation $\mathcal{I}_P(\pi_\lambda)$, was reviewed in Appendix A.1. It is defined by an integral that converges absolutely when λ is sufficiently positive. One of Langlands' fundamental results is that this operator has an analytic continuation to a function of all λ. The analogous definition for $J_P(w,\lambda)$ can also be made locally. When λ is sufficiently positive, the local integral again converges absolutely. Moreover, the global integral decomposes as the product of the local integrals. In other words, the global operator $J_P(w,\lambda)$ is of local nature, in that it equals the product of the local operators. But unlike the global operator, its local counterpart does not have analytic continuation to all λ. It has a meromorphic continuation, but this meromorphically continued operator may well have poles at certain values of λ. These poles reflect the reducibility of the induced representation. In particular, the global standard intertwining operator $J_P(w,\lambda)$, evaluated at $\lambda=0$, is not necessarily the product of local standard intertwining operators.

A further study of the properties of local standard intertwining operators reveals that their poles are of scalar nature. In other words, one can find a scalar function $r_{Q|P}(\pi_\lambda)$ whose poles mirror those of the operator-valued function $J_{Q|P}(\pi_\lambda)\colon \mathcal{I}_P(\pi_\lambda)\to\mathcal{I}_Q(\pi_\lambda)$ that is the main part of the self-intertwining operator $J_P(w,\lambda)$, in the sense that the product

$$R_{Q|P}(\pi_\lambda) = r_{Q|P}(\pi_\lambda)^{-1}J_{Q|P}(\pi_\lambda) \tag{7.5.2}$$

has analytic continuation to a function of all λ. The operator $R_{Q|P}(\pi_\lambda)$ is called a "normalized intertwining operator", and the scalar function $r_{Q|P}(\pi_\lambda)$ is called a "normalizing factor". In addition to canceling the poles of the standard intertwining operator, the normalizing factor can be (and is) chosen so that the product behavior $R_{P_3|P_1} = R_{P_3|P_2} \circ R_{P_2|P_1}$ holds for all triples of parabolic subgroups with common Levi factor M. The analogous product behavior holds for the standard intertwining operator only when the triple of parabolic subgroups satisfies the property that the number of hyperplanes that separate the chambers of P_3 and P_1 is the sum of the analogous numbers for P_3, P_2 and P_2, P_1. We refer to [Art05, Theorem 21.4] for a more detailed discussion.

The product over all places of the local normalized intertwining operators is called the "global normalized intertwining operator". It is of course just the product of the global standard intertwining operator and the global normalizing factor obtained as the product of the local normalizing factors. The global normalizing factor is an analytic function, because the same is true for the global standard intertwining operator. The global normalized intertwining operator has the local property that it is (by definition) equal to the product of the local normalized intertwining operators, for all values of λ, in particular for $\lambda = 0$.

We return to the discussion of the functions $r_L^M(\sigma_\lambda)$ and $R_M(\pi_\lambda, P)$ that enter the description of the fine spectral expansion. Given global normalizing factors $r_{Q|P}(\pi_\lambda)$ and corresponding global normalized intertwining operators $R_{Q|P}(\pi_\lambda)$ we can form the (G, M)-families

$$r_Q(\Lambda, \pi_\lambda, P) = r_{Q|P}(\pi_\lambda)^{-1} r_{Q|P}(\pi_{\lambda+\Lambda})$$

and

$$\mathcal{R}_Q(\Lambda, \pi_\lambda, P) = R_{Q|P}(\pi_\lambda)^{-1} R_{Q|P}(\pi_{\lambda+\Lambda}).$$

We recall that, just as in the case of the (G, M)-family $\mathcal{M}(\Lambda, \sigma_\lambda, P)$ encountered earlier, the parabolic subgroup P and the value of λ are fixed parameters, while the variable parabolic subgroup denoted by P in the definition of a (G, M)-family is now the parabolic subgroup Q, and the variable denoted by λ in that definition is now Λ. We form the constant c_M for each of these two (G, M)-families and denote the results as $r_G^M(\pi_\lambda, P)$ and $R_M(\pi_\lambda, P)$. Both of these constants now become functions of λ. The first of them is independent of P and can thus be denoted by $r_G^M(\pi_\lambda)$. We can of course apply this discussion with G replaced by any Levi subgroup L of G that contains M and obtain the function $r_L^M(\pi_\lambda)$.

7.6 The example of $\mathbf{SL}_2$

We will now describe in some detail the distributions occurring in the non-invariant trace formula in the example of $G = \mathrm{SL}_2/\mathbb{Q}$.

Consider first the geometric side. There are two expressions for it. The first expression is the left hand side of (7.3.1), which is the sum $\sum_{\mathfrak{o}} J_{\mathfrak{o}}(f)$, where in this specific case $\mathfrak{o}$ is either a regular semi-simple elliptic conjugacy class, a regular semi-simple hyperbolic conjugacy class, the set of all unipotent elements, or the set of all unipotent elements multiplied by the non-trivial central element -1.

The second expression is the left hand side of (7.5.1), which in this specific case is

$$\frac{1}{2} \sum_{\gamma \in T(\mathbb{Q})/T(F_S)-conj} a^T(S,\gamma) J_T(\gamma, f) + \sum_{\gamma \in G(\mathbb{Q})/G(F_S)-conj} a^G(S,\gamma) J_G(\gamma, f),$$

where we have fixed here a sufficiently large finite set S of places of $\mathbb{Q}$. Let us discuss the various summands in both expressions and how they relate to each other.

1. Consider an equivalence class $\mathfrak{o}$ that is a conjugacy class of regular semi-simple elliptic elements in $G(\mathbb{Q})$, i.e. elements with two distinct non-rational eigenvalues. The conjugacy class $\mathfrak{o}$ is associated to a quadratic field extension $E/\mathbb{Q}$, and $G_\gamma(\mathbb{Q})$ is the subgroup of norm-1 elements in $E^\times$, while $G_\gamma(\mathbb{A})$ is the subgroup of norm-one elements in $\mathbb{A}_E^\times = (\mathbb{A} \otimes_\mathbb{Q} E)^\times$. Then $J_\mathfrak{o}(f) = a^G(S,\gamma) J_G(\gamma, f)$ with $a^G(S,\gamma) = \mathrm{vol}(G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A})^1)$ and $J_G(\gamma, f) = \int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(x^{-1}\gamma x) dx$. The orbital integral is an invariant distribution. Moreover, it is local, in the sense that when $f = \prod_v f_v$ is a factorizable test-function, the orbital integral is the product of the corresponding local orbital integrals. This is an example of (7.3.2).

2. Consider an equivalence class $\mathfrak{o}$ that is a conjugacy class of regular semi-simple hyperbolic elements in $G(\mathbb{Q})$, i.e. elements with two distinct rational eigenvalues. Then $J_\mathfrak{o}(f) = a^T(S,\gamma) J_T(\gamma, f)$, where $a^T(S,\gamma) = \mathrm{vol}(T(\mathbb{Q}) \backslash T(\mathbb{A})^1)$ depends on γ only through its centralizer T, and $J_T(\gamma, f) = \int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(x^{-1}\gamma x) v_T(x) dx$. This is an example of (7.3.4), and J_T here is a "weighted orbital integral" in the simple case of a (G, M)-regular element. The weight factor $v_T(x)$ is the volume of the convex hull of the set of points $\{-H_B(x), -H_{\bar{B}}(x)\}$, where B is the standard Borel subgroup of G and $\bar{B}$ is its opposite. We have $H_{\bar{B}}(x) = w^{-1} H_B(wx) = -H_B(wx)$, where
$$w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$
This volume is thus $-H_B(x) - (-H_{\bar{B}}(x)) = -(H_B(x) + H_B(wx))$, which one can check to be non-negative by a direct matrix calculation. With this, we see
$$J_T(\gamma, f) = -\int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(x^{-1}\gamma x)(H(x) + H(wx)) dx.$$
The weighted orbital integral is again of "local nature", albeit in a somewhat different sense than its unweighted cousin. While the latter is the product of local unweighted orbital integrals when the test function f is

factorizable, the weighted integral has a slightly more complicated behavior. It comes from the identity $v(x) = \sum_w v(x_w)$, where the sum runs over the set of places of $\mathbb{Q}$. This leads to the following "splitting formula" for the adelic weighted orbital integral: When $f = \prod_w f_w$, $J_T(\gamma, f)$ equals

$$\sum_w \left(\int_{G_\gamma(\mathbb{Q}_w)\backslash G(\mathbb{Q}_w)} f_w(x_w^{-1}\gamma_w x_w) v(x_w) dx_w \right) \prod_{w' \neq w} \left(\int_{G_\gamma(\mathbb{Q}_{w'})\backslash G(\mathbb{Q}_{w'})} f_{w'}(x_{w'}^{-1}\gamma_{w'} x_{w'}) dx_{w'} \right).$$

3. There are only two equivalence classes $\mathfrak{o}$ that remain, indexed by the two non-regular semi-simple elements, namely the central elements, of $G(\mathbb{Q})$. Thus, one equivalence class consist of all unipotent elements, and the other is the product of that with the non-trivial central element. Fix $a \in \{\pm 1\}$. Then the corresponding equivalence class is the union of the conjugacy classes represented by the matrices

$$z = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \qquad \gamma = \begin{bmatrix} a & u \\ 0 & a \end{bmatrix}, u \in \mathbb{Q}^\times/\mathbb{Q}^{\times,2}.$$

The contribution of $\mathfrak{o}$ is

$$J_{\mathfrak{o}}(f) = \mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})) f(z) + \sum_u f.p._{s=0} Z_\gamma(s).$$

The first summand is $a^G(S, z) J_G(z, f)$. The second summand is the "finite part" (that is, the constant term) of the Laurent expansion at $s = 0$ of the complex valued function

$$Z_\gamma(s) = \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x) e^{-sH_B(x)} dx,$$

where the integral converges for $\mathrm{Re}(s) > 0$. Notice that the "value" at $s = 0$, should it exist, would be equal to the adelic orbital integral of f at γ. However, this integral does not converge. In order to make it converge, the convergence factor $e^{-sH_B(x)}$ is added.

This finite part can be expressed more explicitly as follows. The local counterpart of the adelic orbital integral does converge. When the local test function at a finite place p is the characteristic function of $G(\mathbb{Z}_p)$, this integral has the value $\zeta_p(1) = (1 - 1/p)^{-1}$. This shows that the divergence behavior of the adelic integral, being the product of the local integrals, mirrors exactly the divergence behavior of the Euler product expansion of $\zeta(1)$. Therefore the function

$$\theta[\gamma, f](s) = \zeta(1+s)^{-1} Z_\gamma(s)$$

is regular at $s = 0$. Using the Laurent expansion

$$\zeta(1+s) = s^{-1} + \lambda_0 + \cdots$$

the finite part above can then be expressed as

$$\theta[\gamma, f]'(0) + \lambda_0 \theta[\gamma, f](0).$$

We can relate this to the second expression of the geometric side of the trace formula as follows. Instead of multiplying $Z_\gamma(s)$ by $\zeta(1+s)^{-1}$ to compute the finite part, we can take the derivative at $s = 0$ of $s \cdot Z_\gamma(s)$. Let S be a sufficiently large finite set of places of F. Decompose $Z_\gamma(s) = Z_S(s) \cdot Z^S(s)$ according to the decomposition $\mathbb{A} = F_S \times \mathbb{A}^S$. Note that $Z_S(s)$ is regular at 0, being the product of finitely many local unipotent orbital integrals, while $Z^S(s)$ has a pole at $s = 0$. Applying the product rule, the derivative at 0 of $s \cdot Z_\gamma(s)$ equals

$$\Big(Z_S\Big|_{s=0}\Big) \cdot \Big(\frac{d}{ds}\Big|_{s=0}(sZ^S(s))\Big) + \Big(\frac{d}{ds}\Big|_{s=0} Z_S\Big) \cdot \Big((sZ^S(s))\Big|_{s=0}\Big).$$

Now $Z_S(0) = J_G(\gamma, f)$ is the usual unipotent orbital integral of f_S at γ in the group $G(F_S)$. The number $\frac{d}{ds}|_{s=0}(sZ^S_\gamma(s))$ is the global coefficient $a^G(S,\gamma)$; it does not depend on f because S is chosen so large that for any place outside of S the components of f is the unit in the spherical Hecke algebra. The term $\frac{d}{ds}|_{s=0} Z_S$ equals $J_T(z, f)$, while $\frac{d}{ds}|_{s=0}(sZ^S_\gamma(s))$ is the global factor $a^T(S,1) = \mathrm{vol}(\mathbb{Q}^\times\backslash\mathbb{A}^1)$. For more details on these calculations we refer the reader to the work of Chaudouard [Cha17] (in particular Theorem 8.5.1 and §12.7), [Cha18] (in particular Theorems 5.1.1 and 6.2.1).

We have thus discussed all equivalence classes $\mathfrak{o}$ in the left hand side of (7.3.1) for $G = \mathrm{SL}_2/\mathbb{Q}$, and have seen how they contribute to summands in the left hand side of (7.5.1). In this way we have accounted for all summands of the latter. Indeed, the elliptic $\mathfrak{o}$ correspond to the elliptic regular semi-simple elements in $G(\mathbb{Q})/G(\mathbb{Q}_S)$-conj, while two classes $\mathfrak{o}$ consisting of unipotent (up to center) elements correspond to the central and regular unipotent elements in $G(\mathbb{Q})/G(\mathbb{Q}_S)$-conj, as well as to the G-central elements in $T(\mathbb{Q})$. The regular hyperbolic equivalence classes $\mathfrak{o}$ correspond to the non-G-central elements in $T(\mathbb{Q})$. The regular hyperbolic elements of $G(\mathbb{Q})/G(\mathbb{Q}_S)$-conj do not contribute, because the global factor $a^G(S,\gamma)$ vanishes for them.

Next, we turn to the spectral side of (7.3.1). It is indexed by pairs $\chi = (M, \sigma)$ of a Levi subgroup M and an irreducible cuspidal automorphic representation σ of $M(\mathbb{A})^1$.

1. When $\chi = (G, \pi)$ with π cuspidal, this is the special case (7.3.3) and

$$J_\chi(f) = m_{\text{cusp}}(\pi) \cdot \operatorname{tr} \pi(f).$$

2. When $\chi = (T, \mu)$, where T is the group of diagonal elements of G, this $T(\mathbb{A}) = \mathbb{A}^\times$, and $\mu\colon \mathbb{A}^1 \to \mathbb{C}^\times$ is a character. Since the Weyl group of (G, T) is $\mathbb{Z}/2\mathbb{Z}$ and the non-trivial element acts on T by inversion, the

datum (T,μ) is (G,T)-regular if and only if $\mu \neq \mu^{-1}$, i.e. $\mu^2 \neq 1$. We shall assume this for now and deal with the case $\mu^2 = 1$ later. Then (ignoring positive normalization constants)

$$J_\chi(f) = -\int_{-\infty}^{\infty} \mathrm{tr}\,(M_{\bar{B}|B}(it)^{-1} M'_{\bar{B}|B}(it)\mathcal{I}_B(\mu_{it}, f))dt.$$

Here $\mu_z = \mu \cdot |-|_{\mathbb{A}}^z$, $z \in \mathbb{C}$. The induced representation $\mathcal{I}_B(\mu_z)$ can be realized on a vector space $\mathcal{H}(\mu)$ independent of z, cf. Appendix A.1. For a second Borel subgroup B' containing T we have the standard intertwining operator $M_{B'|B}(z)\colon \mathcal{H}_B(\mu) \to \mathcal{H}_{B'}(\mu)$ that intertwines the representations $\mathcal{I}_B(\mu_z)$ and $\mathcal{I}_{B'}(\mu_z)$, as recalled in Appendix A.2. It can be seen as a meromorphic function of z, and Langlands has proved that it has analytic continuation to all $z \in \mathbb{C}$. We have denoted by $M'_B(z)$ the derivative of this function, which is again an operator $\mathcal{H}_B(\mu) \to \mathcal{H}_{B'}(\mu)$. Therefore $M_{B'|B}(z)^{-1} M'_{B'|B}(z)$ is a self-operator on $\mathcal{H}_B(\mu)$. Another such operator is $\mathcal{I}_B(\mu_s, f)$, and we can take the trace of the composition of these two operators.

To see that this is the weighted character (7.3.5), we need to check that the operator $-M_{\bar{B}|B}(it)^{-1} M'_{\bar{B}|B}(it)$ is the result of the (G,M)-family

$$\mathcal{M}_{B'}(\Lambda, \mu_{it}, B) = M_{B'|B}(it)^{-1} M_{B'|B}(it + i\Lambda).$$

This family has the two members, one for $B' = B$ and one for $B' = \bar{B}$. When $B' = B$ the operator equals the identity, and when $B' = \bar{B}$ the operator equals the composition of the self-intertwining operators $M_{\bar{B}|B}(it)^{-1} M_{\bar{B}|B}(it + i\Lambda)$. Ignoring normalizations and volumes, the operator valued "constant" c_M associated to the (G,M)-family is by definition

$$\begin{aligned} & \lim_{\Lambda\to 0} (i\Lambda)^{-1}(\mathrm{id} - M_{\bar{B}|B}(it)^{-1} M_{\bar{B}|B}(it + i\Lambda)) \\ = \quad & M_{\bar{B}|B}(it)^{-1} \lim_{\Lambda\to 0} (i\Lambda)^{-1}(M_{\bar{B}|B}(it) - M_{\bar{B}|B}(it + i\Lambda)) \\ = \quad & -M_{\bar{B}|B}(it)^{-1} M'_{\bar{B}|B}(it). \end{aligned}$$

To see how this relates to the weighted characters defined in terms of normalized intertwining operators we recall the identity $M_{\bar{B}|B}(it) = r_{\bar{B}|B}(it) R_{\bar{B}|B}(it)$ from (7.5.2), where again we have abbreviated μ_{it} by simply it. Differentiating in it we obtain

$$M'_{\bar{B}|B}(it) = r'_{\bar{B}|B}(it) R_{\bar{B}|B}(it) + r_{\bar{B}|B}(it) R'_{\bar{B}|B}(it)$$

and hence

$$M_{\bar{B}|B}(it)^{-1} M'_{\bar{B}|B}(it) = \frac{r'_{\bar{B}|B}(it)}{r_{\bar{B}|B}(it)}\mathrm{id} + R_{\bar{B}|B}(it)^{-1} R'_{\bar{B}|B}(it).$$

This leads to

$$J_\chi(f) = -\int_{\mathbb{R}} \frac{r'_{\bar{B}|B}(it)}{r_{\bar{B}|B}(it)} \text{tr}\,(\mathcal{I}_B(\mu_{it}, f))dt - \int_{-\infty}^{\infty} \text{tr}\,(R_{\bar{B}|B}(it)^{-1} R'_{\bar{B}|B}(it)\mathcal{I}_B(\mu_{it}, f))dt.$$

The second term is the weighted character defined in terms of the global normalized intertwining operator R_B. Using the fact that the global normalized intertwining operator decomposes as a product of local normalized intertwining operators $R_{\bar{B}|B}(it) = \otimes_v R_{\bar{B}|B}(it, v)$, we obtain

$$R_{\bar{B}|B}(it)^{-1} R'_{\bar{B}|B}(it) = \sum_v R_{\bar{B}|B}(it, v)^{-1} R'_{\bar{B}|B}(it, v) \otimes \bigotimes_{w \neq v} \text{Id}_w,$$

the sum running over all places v of F. This leads to the expression

$$-\sum_v \int_{\mathbb{R}} \text{tr}(R_{\bar{B}|B}(it, v)^{-1} R'_{\bar{B}|B}(it, v) \mathcal{I}_B^G(\mu_{it}, f_v) \cdot \prod_{w \neq v} \text{tr}(\mathcal{I}_B(\mu_{it}, f_w)))dt$$

for the normalized weighted character.

3. Consider now $\chi = (T, \mu)$ with $\mu^2 = 1$ and $\mu \neq 1$; this is a non-(G, T)-regular term. Then (again ignoring positive normalization constants , $J_\chi(f)$ equals

$$-\int_{-\infty}^{\infty} \text{tr}\,(M_{\bar{B}|B}(it)^{-1} M'_{\bar{B}|B}(it)\mathcal{I}_B(\mu_{it}, f))dt + \frac{1}{4}\text{tr}\,(M_{\bar{B}|B}(w, 0)\mathcal{I}_B(\mu, f)).$$

   To see that this is the weighted character (7.3.9), we note that the first summand corresponds to the pair $(M', L) = (T, T)$, where it is derived as in the previous point, while the second summand corresponds to the pair $(M', L) = (T, G)$ and is the contribution of χ to I^G_{disc}. There is no term for $(M, L) = (G, G)$, (T, μ) does not contribute to the discrete spectrum of G.

4. Consider finally $\chi = (T, 1)$. Then $J_\chi(f)$ equals

$$-\int_{-\infty}^{\infty} \text{tr}\,(M_{\bar{B}|B}(it)^{-1} M'_{\bar{B}|B}(it)\mathcal{I}_B(\mu_{it}, f))dt + \frac{1}{4}\text{tr}\,(M_{\bar{B}|B}(w, 0)\mathcal{I}_B(\mu, f)) + \int_{G(\mathbb{A})} f(x)dx.$$

   The argument is as above, except that now we also have a term corresponding to the pair $(M, L) = (G, G)$, because (T, μ) does contribute to the discrete spectrum of G via a residue of an Eisenstein series. This contribution is the unique residual automorphic representation of $G = \text{SL}_2$, namely the trivial representation. The terms for $(M, L) = (T, G)$ and $(M, L) = (G, G)$ are the contributions of χ to I^G_{disc}.

7.7 The invariant trace formula

While (7.5.1) provides a useful trace formula, it still has one major flaw: the distributions that occur in it are not invariant under conjugation by $G(\mathbb{A})$. In fact, it is not just the individual distributions that often fail to be non-invariant, but both the geometric and the spectral side of the trace formula are non-invariant distributions. The reason this is a problem is that often times the test function f is not explicitly given as a function, but is rather described abstractly by some process of local harmonic analysis that specifies it only up to conjugation.

The fact that neither the geometric nor the spectral side are invariant distributions means that, in order to obtain a trace formula that consists of invariant distributions, one has to mix geometric and spectral terms in an effort to cancel the non-invariance behavior. The result will be an identity in which at least one of the sides will not be of purely geometric or purely spectral nature, but rather a mix of both. Of course, the task is to do this mixing in a minimalistic way, in order to preserve as much as possible the original structure. Indeed, if we brought both sides of the trace formula to the same side of the equation sign, we would trivially obtain an invariant formula, but it will not be of much use.

The basis of the invariant trace formula are the "variation formulas" which quantify the non-invariance of weighted orbital integrals and weighted characters. These formulas take the following form. Given a test function f on $G(\mathbb{A})$ and $y \in G(\mathbb{A})$ we have (cf. [Art05, Lemma 18.1])

$$J_M^G(\gamma, f^y) = \sum_Q J_M^{M_Q}(\gamma, f_{Q,y}), \tag{7.7.1}$$

where $f^y(x) = f(yxy^{-1})$ and we have now used superscripts on the distributions J_M in order to denote the ambient group, which varies in this formula. The sum runs over all parabolic subgroups containing M, M_Q is the Levi factor of Q containing M, and $f \mapsto f_{Q,y}$ is a continuous linear map from smooth compactly supported test functions on $G(\mathbb{A})$ to those on $M_Q(\mathbb{A})$ that is given by an explicit formula, cf. the discussion surrounding [Art05, Theorem 9.4].

In the above sum on the right, the summand for $Q = G$ is simply $J_M^G(\gamma, f)$. The formula can thus be seen as expressing the difference $J_M^G(f^y) - J_M^G(f)$, which measures the failure of invariance of the distribution J_M^G, in terms of the distributions $J_M^{M_Q}$ associated to proper Levi subgroups of G.

Note that the failure of invariance of J_M^G gets worse the smaller M becomes, because then the sum on the right has more terms. The simplest case is when $M = G$, in which case the distribution J_M^G is actually invariant. This is not surprising, since this distribution is then a usual, i.e. unweighted, orbital integral. One proves the above formula first for (G, M)-regular γ by examining the non-invariance of the weight factor v_M, which is captured by the general splitting formulas for (G, M)-families. One then reduces the case of general γ to the (G, M)-regular case.

In principle, the weighted character satisfies the exact analog of this formula, namely

$$J_M^G(\pi, f^y) = \sum_Q J_M^{M_Q}(\pi, f_{Q,y}). \tag{7.7.2}$$

The reason this is not literally true is that weighted characters are distributions on a slightly smaller space of test functions, namely the space $\mathcal{H}(G)$ of smooth compactly supported functions on $G(\mathbb{A})$ that are K-finite, i.e. their left as well as right translates under the fixed maximal compact subgroup K span a finite-dimensional subspace. Since $\mathcal{H}(G)$ is not invariant under conjugation by arbitrary elements of $G(\mathbb{A})$, the above formula does not make literal sense. It can be made sense of by replacing conjugation by $G(\mathbb{A})$ with convolution by $\mathcal{H}(G)$, cf. [Art05, (22.11), (22.12)], but we will ignore this technical point here.

We have already discussed that, since neither of the geometric and spectral side of the trace formula are invariant, we'll have to achieve invariance by mixing them. The parallel form of the variation formulas (7.7.1) and (7.7.2) suggests that one should be able to modify a weighted orbital integral by weighted characters, or vice versa, in order to cancel the failure of invariance. There are two possible ways to do this: modify the geometric side by subtracting spectral terms, or modify the spectral side by subtracting geometric terms. Both have been used in the literature, and each comes with analytic technical difficulties stemming from the asymptotic properties of the respective distributions. Arthur has found that, for general groups, subtracting spectral terms from the geometric side presents more manageable difficulties. This is the way he has obtained the invariant trace formula. For more discussion on this we refer to [Art05, §22].

To explain the process in a bit more detail, we first formulate the goal. We want to modify the non-invariant distribution $J_M^G(\gamma, f)$ in such a way as to obtain an invariant distribution $I_M^G(\gamma, f)$, and we want to modify the non-invariant distribution $J_M^G(\pi, f)$ in such a way as to obtain an invariant distribution $I_M^G(\pi, f)$, and both modifications need to be parallel, in the sense that the identity relating the various $J_M^G(\gamma, f)$ to the various $J_M^G(\pi, f)$, i.e. the trace formula, continues to hold with the new distributions $I_M^G(\gamma, f)$ and $I_M^G(\pi, f)$.

The process involves a space $\mathcal{I}(G(F_S))$ that we will briefly describe. Roughly speaking, it is the space of orbital integrals of test functions, and at the same time the space of tempered characters of test functions. More precisely, given a test function f on $G(F_S)$, we can form the set of its strongly regular orbital integrals $\{f_G(\gamma) \mid \gamma \in G_{\mathrm{sr}}(F_S)\}$, as well as the set of its tempered character values $\{f_G(\pi) \mid \pi \in \Pi_{\mathrm{temp}}(G(F_S))\}$. Here G_{sr} denotes the set of strongly regular elements of G, Π_{temp} denotes the set of isomorphism classes of tempered representations, and we have the formulas

$$f_G(\gamma) = \int_{G_\gamma(F_S)\backslash G(F_S)} f(x^{-1}\gamma x) dx,$$

and

$$f_G(\pi) = \mathrm{tr}\, \pi(f).$$

Those two sets determine each other, cf [Kaz86, Theorem 0] for an argument in the p-adic case. In particular, one of the sets consists entirely of zeroes if and only if the other does. We can define $\mathcal{I}(G(F_S))$ as the quotient of $\mathcal{H}(G(F_S))$ by the subspace of functions for which $f_G(\gamma)$, or equivalently $f_G(\pi)$, is identically zero. We can then interpret this space as a subspace of the space of functions on strongly regular semi-simple conjugacy classes of $G(F_S)$, or as a subspace of the space of functions on the set of tempered representations of $G(F_S)$. For the latter interpretation, the Paley-Wiener theorem ([BDK86], [CD90]) describes the subspace by precise conditions on the functions on $\Pi_{\text{temp}}(G(F_S))$.

A distribution on $G(F_S)$ that factors through the quotient $\mathcal{I}(G(F_S))$ is said to be *supported on characters*. It is clear that any such distribution is invariant. The opposite implication is conjectured, and has been established for p-adic groups in [Kaz86], but is not yet known for real groups.

Returning to the process of making the trace formula invariant, the analytic difficulties that arise can be overcome by replacing (yet again) the space of test functions with a suitable modification, called $\mathcal{H}_{\text{ac}}(G(F_S))$ in [Art05, §23]. Since we are trying to get the basic idea across, we will ignore this point, and instead formulate the following result, which is a key tool in the process, but becomes correct (cf. [Art05, Proposition 23.1]) only when using the appropriate spaces of test functions.

Proposition 7.7.1. *The map $\phi_M^G\colon f \mapsto J_M^G(\pi, f)$ is a continuous linear transformation from the space of test functions $\mathcal{H}(G(F_S))$ to the space $\mathcal{I}(M(F_S))$.*

We are using here the interpretation of $\mathcal{I}(M(F_S))$ as a space of functions on the set of tempered characters of $M(F_S)$. That is, we are thinking of $J_M^G(\pi, f)$ as a test function $\phi_M^G(f)$ on $M(F_S)$ specified by the identity $\operatorname{tr}\pi(\phi_M^G(f)) = J_M^G(\pi, f)$ for all $\pi \in \Pi_{\text{temp}}(M(F_S))$.

The process of making the trace formula invariant is then encapsulated in the following result.

Theorem 7.7.2 ([Art05, Theorems 23.2, 23.3])**.** *There exist invariant distributions $I_M^G(\gamma, f)$ and $I_M^G(\pi, f)$, that are supported on characters, such that*

$$I_M^G(\gamma, f) = J_M^G(\gamma, f) - \sum_L I_M^L(\gamma, \phi_L^G(f))$$

and

$$I_M^G(\pi, f) = J_M^G(\pi, f) - \sum_L I_M^L(\pi, \phi_L^G(f)).$$

In both cases the sum runs over the set of proper Levi subgroups of G containing M. The existence of the distributions is immediate, because the right-hand sides provide the definition by induction on $\dim(G)$. The content of the theorem is that they are invariant, which follows easily from the variance formulas above, and that they are supported on characters, which takes more work to show.

Our notation is slightly simpler than that of [Art05, Theorem 23.3]. We have not placed hats on I_M^L, which Arthur uses to indicate the factorization of I^L_M through the quotient $\mathcal{I}(M(F_S))$ whose existence is guaranteed by the claim that I^L_M is supported on characters. We have also not used the variable X, which is tied to the modified space of test functions.

The invariant distributions $I^G_M(\gamma, f)$ and $I^G_M(\pi, f)$ satisfy splitting formulas that are analogous, and implied by, those of their non-invariant analogs $J^G_M(\gamma, f)$ and $J^G_M(\pi, f)$. In this sense, they are local in nature.

The invariant trace formula now follows from its non-invariant analog. Let us decorate the distribution J by a superscript denoting the ambient group, just as we did with the distributions J_M.

Theorem 7.7.3 ([Art05, Theorem 23.4])**.** *The distribution*

$$I^G(f) := J^G(f) - \sum_{L \neq G} |W_0^L| \cdot |W_0^G|^{-1} I^L(\phi^G_L(f))$$

is invariant and supported on characters. It has the "geometric" expansion

$$I^G(f) = \sum_M |W_0^M| \cdot |W_0^G|^{-1} \sum_\gamma a^M(\gamma) I^G_M(\gamma, f)$$

and the "spectral" expansion

$$I^G(f) = \sum_M |W_0^M| \cdot |W_0^G|^{-1} \int_\pi a^M(\pi) I^G_M(\pi, f) d\pi.$$

The term "invariant trace formula" can refer to both the distribution $I^G(f)$, as well as to the equality of the two expressions given for it by Theorem 7.7.3. The first identity in that theorem can be rewritten as

$$J^G(f) = \sum_L |W_0^L| \cdot |W_0^G|^{-1} I^L(\phi^G_L(f)), \tag{7.7.3}$$

where now L runs over all Levi subgroups of G containing the fixed minimal Levi subgroup M_0. This identity expresses the non-invariant trace formula $J^G(f)$ as a sum of invariant trace formulas of all Levi subgroups of G. We will see a very analogous identity connecting the invariant trace formula and the stable trace formula in (7.8.1).

We return to our running example of the group $G = \mathrm{SL}_2/\mathbb{Q}$. The relevant variance formulas were established in [LL79, Lemma 3.2, 3.3, 3.4]. The regular hyperbolic weighted orbital integral is denoted in this reference by $A_1(\gamma, f)$ and Lemma 3.2 of loc. cit. expresses the difference $A_1(\gamma, f^y) - A_1(\gamma, f)$ as the difference $\check{H}(\gamma) - \check{H}({}^w\gamma)$, where w is the non-trivial Weyl element. Here $\check{H}$ is the function on the split adelic torus $T(\mathbb{A}) = \mathbb{A}^\times$ that is the Fourier transform of the function $H(\eta)$ defined on the character group of $T(\mathbb{A})$ by $H(\eta) = \mathrm{tr}(\mathcal{I}^G_B(\eta, f) \circ N(g))$, where $N(g)$ is the operator on the space of the induced representation $\mathcal{I}^G_B(\eta)$ defined by $(N(g)\phi)(k) = \langle \alpha, H_B(kg) \rangle \cdot \phi(k)$, where $\alpha \in X^*(T)$ is the unique B-positive root. Explicitly, $(N(g)\phi)(k) = \beta(kg) \cdot \phi(k)$ for $g \in G(\mathbb{A})$ and

$k \in K$, where $n_1 a_1(k,g) k_1$ is the Iwasawa decomposition of kg, we identify $a_1(k,g) \in T(\mathbb{A}) = \mathbb{A}^\times$, and set $\beta(kg) = \log |a_1(k,g)|^2$. The Fourier transform is defined as $\check{H}(\gamma) = \int_\eta H(\eta)\overline{\eta(\gamma)} d\eta$.

According to Lemma 3.4 of loc. cit. the variance of the normalized weighted character is exactly parallel to that, namely (note $\eta^w = \eta^{-1}$)

$$\mathrm{tr}\,(R_B(\eta)^{-1} R'_B(\eta)\mathcal{I}_B^G(\eta, f^y)) - \mathrm{tr}\,(R_B(\eta)^{-1} R'_B(\eta)\mathcal{I}_B^G(\eta, f)) = H(\eta) - H(\eta^{-1}).$$

In other words, the variance of the normalized weighted character is the Fourier transform of the variance of the regular hyperbolic weighted orbital integral. The Poisson summation formula for the locally compact group $T(\mathbb{A}) = \mathbb{A}^\times$ and its discrete subgroup $T(\mathbb{Q}) = \mathbb{Q}^\times$ then implies that if we combine the regular hyperbolic weighted orbital integrals and the normalized weighted characters, the result will be an invariant distribution. To be more explicit, the function H on $T(\mathbb{A})$ and its Fourier transform are related by the Poisson summation formula (Theorem 2.1.3)

$$\sum_{\gamma \in T(\mathbb{Q})} \check{H}(\gamma) = \sum_{\eta \in (T(\mathbb{A})^1/T(\mathbb{Q}))^*} H(\eta).$$

Note however the following asymmetry. If we take the sum of all normalized weighted characters, then this sum runs over all $\eta \in (T(\mathbb{A})^1/T(\mathbb{Q}))^*$, and the variance formula for this sum will produce the full right hand side of the above identity. However, if we take the sum of all regular weighted hyperbolical integrals, then this sum runs over all $\gamma \in T(\mathbb{Q})$ *except* $\gamma \in \{\pm 1\}$, because these are exactly the non-regular elements. Therefore, the variance formula for this sum produces the left hand side of the above identity up to the two missing summands for $\gamma \in \{\pm 1\}$. As a result, we cannot apply the Poisson summation formula and obtain the required cancellation before we have supplied the missing summands for $\gamma \in \{\pm 1\}$.

It turns out that the missing summands for $\gamma \in \{\pm 1\}$ will be contributed by the variance formula for the unipotent orbital integrals, i.e. the distributions $J_\mathfrak{o}$ for the equivalence classes $\mathfrak{o} = (T, 1)$ and $\mathfrak{o} = (T, -1)$. This is the content of Lemma 3.3 of loc. cit. Thus, combining the regular hyperbolic weighted orbital integrals, the normalized weighted characters, and part of the unipotent contribution, results in an invariant trace formula. For further exposition we refer to [Kal24], §6.4, especially §6.4.3.

7.8 The stable trace formula

The discussion of this section is philosophically analogous to that of the previous section, and follows closely [Art05, §29]. Initially we had obtained a trace formula for the group G, by which we meant a distribution $J^G(f)$ together with two different expressions for it, one of geometric nature and one of spectral nature. We discussed the problem that this distribution was not invariant, and we modified it so that we obtained an invariant distribution I^G, together with two different expressions for it, again one of geometric and one of spectral nature, at least in some sense. The two distributions J^G and I^G were related by the identity (7.7.3), which we recall expresses the non-invariant distribution J^G as a sum

$$J^G(f) = \sum_L |W_0^L| \cdot |W_0^G|^{-1} I^L(\phi_L^G(f)),$$

of the invariant distributions I^L over the various Levi subgroups L of G.

The main motivation to replace J^G by I^G is that often times the test function to be used in the distribution is not given explicitly as a function, i.e. an element of $\mathcal{H}(G)$, but only via its orbital integrals or character values, i.e. only as an element of $\mathcal{I}(G)$.

In fact, many applications of the trace formula require an even stronger invariance, namely stable invariance, that was discussed in §5. Such applications involve the computation of the Hasse-Weil zeta function of Shimura varieties [Kot90], or the comparison of spectra of inner forms [LL79], or the classification of representations of classical groups [Art13].

The outcome we desire is formally analogous to that of making the trace formula invariant. We would like to construct a stable distribution S^G by modifying the invariant distribution I^G by terms "of lower order". Thus we are going to have

$$I^G(f) = \sum_{G'} \iota(G, G') S^{G'}(f'), \tag{7.8.1}$$

where the sum runs over the set of elliptic endoscopic data G', $\iota(G, G')$ are certain positive coefficients, described in §6.3, and $S^{G'}$ is the stable distribution on the group G'.

We are using here Arthur's notation G' for endoscopic data, instead of Kottwitz's notation H. This allows the systematic usage of "prime" for endoscopic objects. The function f' is the "endoscopic transfer" of the function f. The map $f \mapsto f'$ can be seen as an analog of the map $\phi_L^G \colon \mathcal{H}(G) \to \mathcal{I}(L)$ used in the process of making the trace formula invariant. It is the map $\mathcal{H}(G) \to \mathcal{SI}(G')$ determined by the transfer theorem (Theorem 6.3.1), where the target space $\mathcal{SI}(G')$ is a stable analog of the space $\mathcal{I}(G')$ and will be discussed below. It is a quotient of $\mathcal{I}(G')$, and the property of $S^{G'}$ being a stable distribution means that it factors through this quotient.

The above stabilization identity was already obtained in the case when G is anisotropic modulo center in (6.3.4). These arguments still apply to an arbitrary group G, but only treat the part of I^G that consists of orbital integrals at elliptic elements. The general case is, as with the trace formula itself, significantly more complicated. We'll give just a broad outline of the process, and discuss the example of SL_2.

The desired identity (7.8.1) provides an inductive definition of the distribution $S^G(f)$ when G is quasi-split, namely as

$$S^G(f) := I^G(f) - \sum_{G' \neq G} S^{G'}(f').$$

The restriction to G quasi-split comes from the fact that the summands G' in (7.8.1) are all quasi-split, by definition.

The question then becomes to show that the invariant distribution $S^G(f)$ defined as above is stable. When G is not quasi-split, it does not occur as a summand on the right hand side of (7.8.1); what occurs is its quasi-split inner form. Therefore, both sides of the identity are now defined, and the identity needs to be proved.

Neither of these tasks seems possible by a direct approach. Instead, one defines "endoscopic" and "stable" versions of each of the terms of the two expansions of the invariant trace formula, by a process similar to the one that defines the distribution S^G. Assembling these terms, one obtains an "endoscopic" distribution $I^{G,\mathcal{E}}$ and a "stable" distribution S^G, and the task is again to show that $I^{G,\mathcal{E}} = I^G$ and that the identity (7.8.1) holds. Here the words "endoscopic" and "stable" are used just as labels to refer to the distributions; they do not (yet) have mathematical content. Of course, this is just a reformulation of the original problem, and is completely formal. But what it allows for is the use of the structure of these distributions, namely as a sum over Levi subgroups, which opens the possibility for an inductive argument.

Before we go deeper into this process we need to discuss two technical points. The first concerns a stable analog $\mathcal{SI}(G)$ of the space $\mathcal{I}(G)$ of orbital integrals. It is defined as the quotient of the space of test functions by the subspace consisting of those test functions whose strongly regular stable orbital integrals vanish. By definition, a stable distribution is one that factors through the space $\mathcal{SI}(G)$. That space has an obvious interpretation as a space of functions on the set of strongly regular semi-simple stable orbits. Just like $\mathcal{I}(G)$, it should also have a spectral interpretation, namely as the space of tempered stable characters. According to the local Langlands conjecture, the latter space should be indexed by tempered Langlands parameters. Since this is only conjectural, but the spectral interpretation of $\mathcal{SI}(G)$ is needed in the arguments of the stabilization of the trace formula, Arthur has produced [Art96] a formal replacement of the space of tempered Langlands parameters that does provide a basis for $\mathcal{SI}(G)$ and has enough formal properties to support that process.

The second technical point concerns the endoscopic transfer of functions, and transfer factors. The Langlands–Shelstad transfer factor Δ is defined as a function that takes a pair of strongly regular semi-simple elements, one of G and one of the endoscopic group G', or rather a z-extension of it. This in turn provides a characterization of a transfer of functions $f \mapsto f'$ via matching of orbital integrals. While this transfer is not a map between spaces of functions, it does induce a well-defined map $\mathcal{I}(G) \to \mathcal{SI}(G')$. Arthur puts these transfer maps for the various endoscopic groups together into a total transfer map

$$\mathcal{I}(G) \to \prod_{G'} \mathcal{SI}(G'), \tag{7.8.2}$$

where the product runs over the (finite) set of isomorphism classes of endoscopic data. He then proves that this map is injective, and denotes its image by $\mathcal{I}^{\mathcal{E}}(G)$, so that it induces an isomorphism $\mathcal{I}(G) \to \mathcal{I}^{\mathcal{E}}(G)$. In fact, $\mathcal{I}^{\mathcal{E}}(G)$ can be characterized as the subspace of $\prod_{G'} \mathcal{SI}(G')$ consisting of tuples that are compatible with parabolic descent (a Levi subgroup M' of an endoscopic group G' of G is itself an endoscopic group of G, and the transfer from G to M' is the composition of the transfer to G' and the usual parabolic descent from G' to M').

This transfer map can in turn be used to extend the definition of the transfer factor beyond the setting of strongly regular semi-simple elements. Indeed, any such element provides an invariant distribution via its orbital integral, and a stable distribution via its stable orbital integral. The total transfer map induces dually a surjective map

$$\bigoplus_{G'} \mathcal{SI}(G')^* \to \mathcal{I}(G)^*, \tag{7.8.3}$$

with the same indexing set as above. Since the transfer of functions and the transfer factor contain the same information, this map can be interpreted as a formal extension of the Langlands–Shelstad transfer factor; therefore Arthur denotes it by the same letter Δ.

We now return to the process of stabilization, which is developed in [Art02]. Let us say a bit more about the definitions of $I^{G,\mathcal{E}}$ and S^G. Consider first the geometric expansion

$$I^G(f) = \sum_M |W_0^M| W_0^G|^{-1} \sum_\gamma a^M(\gamma) I_M^G(\gamma, f).$$

One defines inductively the "endoscopic" and "stable" analogs $a^{M,\mathcal{E}}(\gamma)$ and $b^M(\delta)$ of $a^M(\gamma)$, as well as the corresponding analogs $I_M^{G,\mathcal{E}}(\gamma, f)$ and $S_M^G(\delta, f)$ of $I_M^G(\gamma, f)$. The definition of $S_M^G(\delta, f)$ is similar to that for S^G given above: Assuming that G is quasi-split, one sets

$$S_M^G(\gamma, f) = I_M^G(\gamma, f) - \sum_{G' \neq G} \iota_{M'}(G, G') S_{M'}^{G'}(f'),$$

where G' runs over the set of elliptic endoscopic data for G that admit a Levi subgroup M' that is elliptic endoscopic for M, and the coefficients $\iota_{M'}(G, G')$ are analogs of $\iota(G, G')$. Assuming G is not quasi-split, one sets

$$I_M^{G,\mathcal{E}}(\gamma, f) = \sum_{G'} \iota_{M'}(G, G') S_{M'}^{G'}(\delta', f').$$

The goal is to show that $S_M^G(\gamma, f)$ is a stable distribution, and that $I_M^{G,\mathcal{E}}(\gamma, f) = I_M^G(\gamma, f)$.

In a similar manner, one defines endoscopic and stable analogs of the global coefficient $a^M(\gamma)$ (that occurs in the refined geometric expansion, see (7.3.8) and note that the coefficient $a^M(S, \gamma)$ stabilizes for large S). The endoscopic analog of the coefficient a^G is denoted by $a^{G,\mathcal{E}}(\gamma)$ and is defined trivially as $a^{G,\mathcal{E}}(\gamma) = a^G(\gamma)$ if G is quasi-split, and via

$$a^{G,\mathcal{E}}(\gamma) = \sum_{G'} \sum_{\delta'} \iota(G, G') b^{G'}(\delta') \Delta(\delta', \gamma)$$

if G is not quasi-split, where G' runs over the set of elliptic endoscopic data and δ' over the set of stable regular semi-simple classes. This presupposes that the coefficients $b^{G'}$ have already been defined for all the quasi-split groups G' occurring in the sum, one of which is the quasi-split inner form of G.

The stable analog of the coefficient a^G is denoted by $b^G(\delta)$, and is defined when G is quasi-split. Informally we would like to think of it as a function on the set of stable regular semi-simple conjugacy classes. However, it is first defined as a function on an a-priori larger set, namely the image of (7.8.3) via the identity

$$\sum_{\delta} b^G(\delta) \Delta(\delta, \gamma) = a^G(\gamma) - \sum_{G' \neq G} \sum_{\delta'} \iota(G, G') b^{G'}(\delta') \Delta(\delta', \gamma), \quad \forall \gamma,$$

where on the left the sum runs over an (somewhat arbitrarily chosen) basis $\Delta^{\mathcal{E}}(G)$ for that image, while on the right G' runs over the set of proper elliptic endoscopic data of G, and δ' runs over the set of stable classes in G'.

The claims to prove are now that $a^{G,\mathcal{E}}(\gamma) = a^G(\gamma)$ for all γ, and $b^G(\delta) = 0$ for all $\delta \in \Delta^{\mathcal{E}}(G)$ that do not come from the set $\Delta(G)$ of stable regular semi-simple classes in G via the map $\Delta(G) \to \mathcal{SI}(G)^* \to \prod_{G'} \mathcal{SI}(G')^* \to \mathcal{I}(G)^*$.

In a similar fashion, one defines endoscopic and stable analogs of the spectral terms of the expansion of I^G, and asserts similar statements.

With all these definitions in place, an inductive argument can be undertaken whose ultimate purpose is to prove all the statements made so far. The main induction is on the rank of the derived subgroup of G, although further supplementary inductions are taking place, such as on the dimension of the split center of G. The main induction allows to treat all statements as known for all proper Levi subgroups of G, as well as all proper endoscopic groups of G. This argument is carried out in the papers [Art01] and [Art03].

The base of the induction is the term $M = G$ of the invariant trace formula. This argument is still very complicated, and is one of the main tasks of the manuscript [Art03]. It is a combination of endoscopic considerations together with the analytic considerations involved in the comparison of the invariant trace formulas of inner forms of GL_N, that is the subject of [AC89] and is reviewed in [Art05, §25]. We encourage the reader to study [Art05, §25] before going to [Art03].

Here we will only give an example in the setting of the group SL_2. The term $M = G$ of the invariant trace formula, which we denote by I_{orb} following Arthur, consists of the elliptic regular term, whose stabilization is very similar to the discussion of §6, and the parts of the unipotent term that are left over after the process of making the trace formula invariant.

In §6 we had established the stabilization of the geometric side of the trace formula under the assumption that G/A_G is an anisotropic group. The same arguments apply to a general group and provide a stabilization of the *elliptic part* of I^G, i.e. the distribution $I^G_{\text{ell}} = J^G_{\text{ell}}$ that is obtained by taking in the definition of J^G only those summands that are indexed by elliptic conjugacy classes. We can thus interpret (6.3.4) as the idenity

$$I^G_{\text{ell}}(f) = S^G_{\text{ell}}(f) + \sum_{G' \neq G} \iota(G, G') S^{G'}_{\text{G-reg}}(f'),$$

where the notation is as follows. Consider the inner sum of (6.3.3), i.e. the sum of stable orbital integrals of (G, G')-regular elliptic elements. When $G = G'$, the condition of (G, G')-regularity is vacuous, and we will denote this sum by S^G_{ell}. In the current special case of $G = \text{SL}_2$, when $G \neq G'$, then G' is a torus, and (G, G')-regular is the same as G-regular, and we denote the corresponding sum by $S^{G'}_{\text{G-reg}}$. The function f' is what was denoted by f^H in (6.3.3).

In the setting of (6.3.4) the group G/A_G was anisotropic, so $I^G_{\text{ell}} = I^G = J^G$ and we had mentioned that the properties of the transfer f' of f implied that all the complicated terms of $S^{G'}$ vanished on f'. For a general group this is not the case, and can be seen in the special example of $G = \text{SL}_2$ as follows. The fact that G' is a torus when $G \neq G'$ means that both f' and $S^{G'}$ must be very simple, because the conjugation action of G' on itself is trivial. In fact, since $SO_{\gamma'}(f') = f'(\gamma')$ we see that the transfer theorem 6.3.3 fully determines the function f' on the set of G-regular elements in G'. But this theorem also stipulates that f' be smooth, which means that the values of f' on the non-regular elements of $G'(\mathbb{A})$ are determined by continuity. It turns out that those values are generally non-zero and do contribute to $S^{G'}(f')$.

This turns out, as it must, to be related to the difference between I^G_{ell} and I^G. More precisely, the answer lies in understanding how regular semi-simple orbital integrals degenerate towards singular elements. This is governed by the unipotent orbital integrals of f. If we were treating an anisotropic inner form of SL_2, the lack of unipotent orbits would imply that f' is zero at $\{\pm 1\}$, and the above identity is already the full stabilization of the trace formula. But in the setting of SL_2, the value of f' at $\{\pm 1\}$, which a-fortiori is determined by continuity of f', equals the unipotent κ-orbital integral of f. For more details on this point we refer to [Kal24, §6.4.1]. This observation leads to the identity

$$I_{\text{orb}}(f) = \sum_{G'} \iota(G, G') S^{G'}_{\text{orb}}(f').$$

7.9 The stable multiplicity formula

As a result of the stabilization process, one obtains not just the identity (7.8.1), but also an identity that gives two equal expression for the stable distribution S^G, namely the equality of

$$\sum_M |W_0^M||W_0^G|^{-1} \sum_\delta b^M(\delta) S^G_M(\delta, f)$$

and

$$\sum_M |W_0^M||W_0^G|^{-1} \int_\phi b^M(\phi) S_M^G(\phi, f) d\phi.$$

In both expansions the first sum runs over Levi subgroups of G containing the fixed minimal Levi subgroup M_0. In the first expansion δ runs over the set of stable classes, while in the second expansion ϕ runs over the set of formal parameters.

We could regard these as "geometric" and "spectral" sides, and their identity as a stable analog of the trace formula. However, while the "geometric" side does indeed have some geometric nature, the nature of the "spectral" side is not clear. After all, the set of formal parameters is only conjecturally related to the actual Langlands parameters, and $S_M^G(\phi, f)$ is only conjecturally related to stable characters.

Recall the distribution I_{disc}^G of (7.4.1). The above "spectral" expansion of S^G does at least allow us to define in a similar fashion the distribution S_{disc}^G, as the $M = G$ summand. The identity (7.8.1) then implies the identity

$$I_{\text{disc}}^G(f) = \sum_{G'} \iota(G, G') S_{\text{disc}}^{G'}(f'). \tag{7.9.1}$$

This is the key identity that is used in many applications of the trace formula, such as the classification of representations of classical groups that we will discuss in §8. But it has to be supplemented with an expression for $S_{\text{disc}}^G(f)$ that is more explicit than its formal definition. This expression is given by the following important conjecture.

Conjecture 7.9.1 (Stable multiplicity formula, cf. [Art13, Theorem 4.1.2])**.**

$$S_{\mathit{disc}}^G(f) = \sum_\psi |\mathcal{S}_\psi|^{-1} \sigma(\bar{S}_\psi^0) \epsilon^G(\psi) f^G(\psi).$$

This is the general version of Conjecture 6.4.2. The sum is over conjectural global Arthur parameters ψ. We are using Arthur's notation $f^G(\psi)$ for the associated stable distribution, i.e. the stable character of the associated Arthur packet, evaluated at f (which was denoted by $S\Theta_\psi(f)$ in (6.4.6)). We have written $\epsilon^G(\psi) = \epsilon_\psi(s_\psi)$ with ϵ_ψ and s_ψ as in (6.4.1) and (6.4.3), following Arthur's notation. The new term, that did not appear in Conjecture 6.4.2, is the complex number $\sigma(\bar{S}_\psi^0)$. It is a non-conjectural quantity, defined more generally for any connected complex reductive group, cf. [Art13, Proposition 4.1.1].

Not only the claimed identity in this formula is conjectural, but in fact the objects ψ and $f^G(\psi)$ are themselves conjectural. Thus, in order to prove it, one must first define what these terms mean. This can only be done once sufficient understanding of the situation has been established, as was done for example in the setting of classical groups in [Art13], and will be discussed in §8.

8 CLASSIFICATION OF REPRESENTATIONS OF QUASI-SPLIT CLASSICAL GROUPS

In this section we will give a rough summary of [Art13]. Due to the length of this reference, it is clear that we cannot reproduce the arguments precisely. Our summary will necessarily be impressionistic, with many subtle points ignored[6]. It will also be uneven, with some arguments provided in detail, while others only alluded to or entirely ignored. We hope that our summary will provide a map that will help the interested reader navigate [Art13], and possibly provide some complementary information.

8.1 Summary of results

The results of the endoscopic classification involve local and global Arthur parameters. These are defined as certain kinds of homomorphisms $\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$ for a local or global field F, where $\mathcal{L}_F$ is the Langlands group of F. When F is local archimedean, $\mathcal{L}_F = W_F$, and when F is local non-archimedean, $\mathcal{L}_F = W_F \times \mathrm{SL}_2(\mathbb{C})$. Here W_F is the Weil group of F. The fact that there are two copies of $\mathrm{SL}_2(\mathbb{C})$ in the source of a local non-archimedean Arthur parameter provides a symmetry that will be important in the treatment of non-tempered packets.

When F is global, the group $\mathcal{L}_F$ is not yet known to exists, so this definition of Arthur parameters is only conjectural. In §8.1.1 we will review Arthur parameters for classical groups, admitting the existence of $\mathcal{L}_F$. This will make their structure clear without burdening the discussion. In §8.1.2 we will add the additional layer needed to define global parameters unconditionally.

8.1.1 Local and global parameters for classical groups

Let F be a local or global field of characteristic zero. Arthur defines sets

$$\begin{array}{l} \Psi_{\text{cusp}}(N) \subset \Psi_2(N) = \Psi_{\text{sim}}(N) \subset \Psi(N), \\ \quad \cap \\ \Phi_{\text{cusp}}(N) = \Phi_2(N) = \Phi_{\text{sim}}(N) \subset \Phi(N), \end{array} \tag{8.1.1}$$

of parameters for GL_N, as well as subsets of self-dual parameters

$$\begin{array}{l} \tilde\Psi_{\text{cusp}}(N) \subset \tilde\Psi_2(N) = \tilde\Psi_{\text{sim}}(N) \subset \tilde\Psi_{\text{ell}}(N) \subset \tilde\Psi(N), \\ \quad || \\ \tilde\Phi_{\text{cusp}}(N) = \tilde\Phi_2(N) = \tilde\Phi_{\text{sim}}(N) \subset \tilde\Phi_{\text{ell}}(N) \subset \tilde\Phi(N), \end{array} \tag{8.1.2}$$

[6] For example, we will often ignore a case called "N even and $\eta_\psi = 1$", which causes additional technical complications. We will also ignore, both in the arguments and in the notation, the complications caused by the outer automorphism of SO_{2n}.

and finally the corresponding sets

$$\begin{array}{l} \Psi_{\mathrm{sim}}(G) \subset \Psi_2(G) \subset \Psi_{\mathrm{ell}}(G) \subset \Psi_{\mathrm{disc}}(G) \subset \Psi(G), \\ \quad || \\ \Phi_{\mathrm{sim}}(G) \subset \Phi_2(G) \subset \Phi_{\mathrm{ell}}(G) \subset \Phi_{\mathrm{disc}}(G) \subset \Phi(G). \end{array} \tag{8.1.3}$$

of parameters for a quasi-split symplectic or special orthogonal group G.

We begin with (8.1.1). The set $\Psi(N)$ consists of continuous complex N-dimensional representations $\psi\colon \mathcal{L}_F \times \mathrm{SU}_2 \to \mathrm{GL}_N(\mathbb{C})$ that are *unitary*, i.e. have bounded image. These are taken up to isomorphism, i.e. up to conjugation in $\mathrm{GL}_N(\mathbb{C})$. Note that SU_2 is the maximal compact subgroup of $\mathrm{SL}_2(\mathbb{C})$ and that restriction along the inclusion $\mathrm{SU}_2 \to \mathrm{SL}_2(\mathbb{C})$ induces a bijection between the irreducible continuous representations of SU_2 and the irreducible algebraic representations of $\mathrm{SL}_2(\mathbb{C})$. Therefore, using SU_2 is equivalent to using $\mathrm{SL}_2(\mathbb{C})$, but SU_2 is more convenient here because it accommodates the unitarity hypothesis.

The set $\Psi_2(N)$ consists of the irreducible representations in $\Psi(N)$. Any $\psi \in \Psi_2(N)$ has the form $\psi = \mu \boxtimes \nu$, with an irreducible unitary representation $\mu\colon \mathcal{L}_F \to \mathrm{GL}_M(\mathbb{C})$ and an irreducible (automatically unitary) representation $\nu\colon \mathrm{SU}_2 \to \mathrm{GL}_{N/M}(\mathbb{C})$, for some $M|N$. The set $\Psi_{\mathrm{cusp}}(N)$ consists of those $\psi = \mu \boxtimes \nu \in \Psi_2(N)$ for which ν is the trivial representation, equivalently $M = N$.

The set $\Phi(N)$ consists of continuous complex N-dimensional representations $\phi\colon \mathcal{L}_F \to \mathrm{GL}_N(\mathbb{C})$ that *need not be unitary*. Inside we have the subset $\Phi_{\mathrm{cusp}}(N) = \Phi_2(N) = \Phi_{\mathrm{sim}}(N)$ of irreducible representations. The intersection $\Phi(N) \cap \Psi(N)$ is the set of unitary representations $\phi\colon \mathcal{L}_F \to \mathrm{GL}_N(\mathbb{C})$, called *generic* Arthur parameters.

Next we turn to (8.1.2). We have the duality operation on $\Psi(N)$ that sends any ψ to its contragredient $\psi^\vee$, and

$$\tilde{\Psi}(N) = \{\psi \in \Psi(N) \,|\, \psi^\vee = \psi\}$$

consists of the self-dual elements in $\Psi(N)$. Intersecting $\tilde{\Psi}(N)$ with the various sets in (8.1.1) produces the corresponding member of (8.1.2). Finally, $\tilde{\Psi}_{\mathrm{ell}}(N)$ consists of those $\psi \in \tilde{\Psi}(N)$ that are multiplicity-free direct sums of irreducible self-dual representations, i.e. $\psi = \psi_1 \oplus \cdots \oplus \psi_n$ with ψ_m pairwise inequivalent self-dual representations of $\mathcal{L}_F \times \mathrm{SU}_2$. Note that a self-dual member of $\Phi_{\mathrm{sim}}(N)$ is automatically unitary.

We now turn to (8.1.3). The set $\Psi(G)$ is defined as the set of $\widehat{G}$-conjugacy classes of continuous homomorphisms $\psi\colon \mathcal{L}_F \times \mathrm{SU}_2 \to {}^L G$ with bounded image. Note that, when F is global, G satisfies the Hasse principle, hence $\widehat{G}$-conjugacy is the correct notion of equivalence of global parameters. Given such $\psi \in \Psi(G)$ one can define the centralizer

$$S_\psi = \mathrm{Cent}(\psi, \widehat{G}),$$

of the homomorphism ψ, equivalently of its image. Then

$$\Psi_2(G) = \{\psi \in \Psi(G) \,|\, |S_\psi| < \infty\}, \tag{8.1.4}$$

$$\Psi_{\text{ell}}(G) = \{\psi \in \Psi(G) \,|\, \exists s \in (S_\psi)_{\text{ss}} : |\text{Cent}(s, S_\psi)| < \infty\}, \tag{8.1.5}$$

$$\Psi_{\text{disc}}(G) = \{\psi \in \Psi(G) \,|\, |Z(S_\psi)| < \infty\}. \tag{8.1.6}$$

The elements of $\Psi_2(G)$ are often called *discrete* Arthur parameters, and the elements of $\Psi_{\text{ell}}(G)$ *elliptic* Arthur parameters (we have written $(S_\psi)_{\text{ss}}$ for the set of semi-simple elements of the complex group S_ψ). We caution the reader that some authors use the word elliptic in place of discrete to denote the set $\Psi_2(G)$, and do not give a name to the set $\Psi_{\text{ell}}(G)$. The elements of $\Psi_{\text{disc}}(G)$ do not usually have a specific name attached to them. Note ψ lies in $\Psi_{\text{ell}}(G)$ if and only if it comes from $\psi' \in \Psi_2(G')$ for an elliptic endoscopic group G' of G.

We note here that Arthur mainly uses the notation $\tilde{\Psi}(G)$ in place of $\Psi(G)$. This is to accommodate the action of the outer automorphism of SO_{2n}. We will ignore this complication and work with $\Psi(G)$ instead.

For Arthur's arguments it is very important to understand the relationship between (8.1.3) and (8.1.2). This relationship arises from the standard representation ${}^LG \to \text{GL}_N(\mathbb{C})$. When $G = \text{Sp}_{2n}$ then $\widehat{G} = {}^LG = \text{SO}_{2n+1}(\mathbb{C}) \subset \text{GL}_N(\mathbb{C})$ with $N = 2n+1$; when $G = \text{SO}_{2n+1}$ then $\widehat{G} = {}^LG = \text{Sp}_{2n}(\mathbb{C}) \subset \text{GL}_N(\mathbb{C})$ with $N = 2n$; when $G = \text{SO}_{2n}$ then $\widehat{G} = \text{SO}_{2n}(\mathbb{C})$ and ${}^LG = \text{SO}_{2n}(\mathbb{C}) \rtimes \Gamma_{E/F} = \text{O}_{2n}(\mathbb{C}) \subset \text{GL}_N(\mathbb{C})$ with $N = 2n$, where E/F is a quadratic extension that splits G (if G is split we take an arbitrary quadratic extension).

Composing with the standard representation induces a map

$$\Psi(G) \to \tilde{\Psi}(N) \subset \Psi(N).$$

When $G = \text{Sp}_{2n}$ or $G = \text{SO}_{2n+1}$ this map is injective. When $G = \text{SO}_{2n}$ this map is *not* injective. That's because elements in the source are taken up to $\widehat{G}$-conjugacy, while those in the target up to $\text{GL}_N(\mathbb{C})$-conjugacy, and in the first two cases the stabilizer of $\widehat{G}$ in $\text{GL}_N(\mathbb{C})$ equals $\widehat{G}$, while in the third case it equals ${}^LG$, which contains $\widehat{G}$ of index 2. The image $\tilde{\Psi}(G) \subset \Psi(N)$ of this map represents parameters for the disconnected group O_{2n} (cf. [Kal22b]), or equivalently orbits of parameters under the outer automorphism of G. Arthur's arguments can only describe $\tilde{\Psi}(G)$, and not $\Psi(G)$, which makes his results slightly less precise than desired. In this summary we will ignore this subtlety to streamline the exposition, and thus drop the tilde from $\tilde{\Psi}(G)$ and pretend that the map $\Psi(G) \to \Psi(N)$ is always injective.

Next we explore some more structure connected to $\psi \in \tilde{\Psi}(N)$ which will give more precise information about (8.1.3) in terms of linear algebra, in particular a description of S_ψ for $\psi \in \Psi(G)$. Consider the decomposition of ψ into irreducible representations, $\psi = \bigoplus_{k \in K_\psi} \ell_k \psi_k$ with all $\psi_k \in \Psi_{\text{sim}}(M_k)$ irreducible and pairwise inequivalent, and ℓ_k the multiplicity of ψ_k. Then $\psi_k \mapsto \psi_k^\vee$ induces an involution on the set K_ψ. Set $I_\psi = \{k \in K_\psi \,|\, \psi_k^\vee = \psi_k\}$ and choose arbitrarily a set of representatives $J_\psi \subset K_\psi \setminus I_\psi$ for the orbits of that involution. Then $K_\psi = I_\psi \sqcup J_\psi \sqcup J_\psi^\vee$, resulting in

$$\bigoplus_{i \in I_\psi} \ell_i \psi_i \oplus \bigoplus_{j \in J_\psi} \ell_j (\psi_j \oplus \psi_j^\vee),$$

where $\psi_i = \psi_i^\vee$ are irreducible self-dual, and $\psi_j \neq \psi_j^\vee$ are irreducible non-self dual.

To any $\psi \in \tilde{\Psi}_2(N)$ one can associate a type, which is either *symplectic*, or *orthogonal*. This type comes from taking an isomorphism $f\colon \psi \to \psi^\vee$ of representations and noting that its dual $f^\vee$ is also an isomorphism $\psi \to \psi^\vee$, hence $f^\vee = c \cdot f$ for some $c \in \mathbb{C}^\times$ (Schur's lemma). Now $(f^\vee)^\vee = f$ implies $c^2 = 1$, and thus $c \in \{\pm 1\}$, which is the sign of self-duality of ψ. We say that ψ is symplectic if $c = -1$, and orthogonal if $c = +1$. Note that, if $\psi = \mu \boxtimes \nu$ with $\mu \in \Phi_{\text{sim}}(M)$ and ν an irreducible representation of SU_2 of dimension N/M, then both μ and ν have a type defined in the same way (one can check that ν is of symplectic type if and only if N/M is even), and the type of ψ is orthogonal if and only if μ and ν have the equal type (both symplectic or both orthogonal), while the type of ψ is symplectic if and only if μ and ν have opposite types (one symplectic and one orthogonal).

The type of ψ corresponds to the type of the complex group $\widehat{G}$ such that ψ comes from $\Psi(G) \to \tilde{\Psi}(N)$, with G classical. For example, if N is odd, the sign is automatically $+1$ and $\tilde{\Psi}_2(N) \subset \Psi(G)$ for $G = \text{Sp}_{2n}$, $N = 2n+1$, in which case $\widehat{G} = \text{SO}_{2n+1}(\mathbb{C}) \subset \text{GL}_N(\mathbb{C})$. If N is even, the sign can be either -1 or $+1$, and ψ comes from $\Psi(G)$ where G is either the split group SO_{2n+1}, in which case $\widehat{G} = \text{Sp}_{2n}(\mathbb{C}) \subset \text{GL}_N(\mathbb{C})$, or G is a (split or quasi-split) SO_{2n}, in which case $\widehat{G} = \text{SO}_{2n}(\mathbb{C}) \subset \text{GL}_N(\mathbb{C})$; in both cases $N = 2n$.

Consider now $\psi \in \Psi(G) \subset \tilde{\Psi}(N)$. We can further decompose $I_\psi = I_\psi^+ \sqcup I_\psi^-$, where I_ψ^+ (resp. I_ψ^-) is the set of all $i \in I_\psi$ for which ψ_i is of the same (resp. opposite) type (symplectic or orthogonal) as $\widehat{G}$. Then

$$S_\psi = \Big(\prod_{i \in I_\psi^+} \text{O}_{\ell_i}(\mathbb{C}) \Big)_\psi^+ \times \prod_{i \in I_\psi^-} \text{Sp}_{\ell_i}(\mathbb{C}) \times \prod_{j \in J_\psi} \text{GL}_{\ell_j}(\mathbb{C}), \tag{8.1.7}$$

where in the first factor $(-)_\psi^+$ denotes the kernel of the character $\xi_\psi^+ : \prod_{i \in I_\psi^+} g_i \mapsto \prod_i (\det(g_i))^{\dim \psi_i}$. The multiplicity ℓ_i for $i \in I_\psi^-$ must be even.

With this we can give the following description of (8.1.3). For an element $\psi \in \Psi(G)$ we have

$$\begin{array}{lllll}
\psi \in \Psi_{\text{sim}}(G) & \Leftrightarrow & S_\psi = Z(\widehat{G}) & \Leftrightarrow & \psi = \psi_1, \\
\psi \in \Psi_2(G) & \Leftrightarrow & |S_\psi| < \infty & \Leftrightarrow & \psi = \psi_1 \oplus \cdots \oplus \psi_m, \\
\psi \in \Psi_{\text{ell}}(G) & \Leftrightarrow & \exists s \in (S_\psi)_{\text{ss}}, |S_{\psi,s}| < \infty & \Leftrightarrow & (*), \\
\psi \in \Psi_{\text{disc}}(G) & \Leftrightarrow & |Z(S_\psi)| < \infty & \Leftrightarrow & J_\psi = \varnothing,
\end{array}$$

where on the right each $\psi_k \in \tilde{\Psi}_{\text{sim}}(N_k)$ for suitable N_k, and in the first three lines the type of each ψ_k is the same as that of $\widehat{G}$; we have denoted by $(*)$ the following condition:

$$\psi = 2\psi_1 \oplus \cdots \oplus 2\psi_q \oplus \psi_{q+1} \oplus \cdots \oplus \psi_m,$$

and either $q \neq m$ or $2|\#\{i : 2 \nmid N_i\}$. We denoted by $S_{\psi,s}$ the centralizer of s in S_ψ.

Note that now we recognize the set $\tilde{\Psi}_{\text{ell}}(N)$ as the union of $\Psi_2(G)$, as G runs over the elliptic twisted endoscopic groups of GL_N, where such a group is a product of at most two factors, each of which is a symplectic or special orthogonal group.

When F is global and v is a place of F, there should be a homomorphism $\mathcal{L}_{F_v} \to \mathcal{L}_F$. Restriction along this homomorphism provides maps $\tilde{\Psi}(N) \to \tilde{\Psi}_v(N)$ and $\Psi(G) \to \Psi_v(G)$, where the sets on the left are for the field F, and the sets on the right are for the field F_v. These are called localization maps, and written as $\psi \mapsto \psi_v$. Note that localization need not respect the various subsets in (8.1.2) and (8.1.3). For example, if $\psi \in \Psi_2(G)$, then ψ_v need not (and usually will not) lie in $\Psi_{v,2}(G)$.

8.1.2 Formal global parameters

In §8.1.1 we reviewed Arthur parameters for classical groups, but this review was based on the assumption that $\mathcal{L}_F$ exists. When F is local this assumption holds, but when F is global it does not (yet). To circumvent the current non-existence of $\mathcal{L}_F$, Arthur defines [Art13, §1.4] a formal replacement of these parameters. The idea is that, conjecturally, the set of irreducible unitary N-dimensional representations of $\mathcal{L}_F$, which by definition is the set of discrete generic Arthur parameters for GL_N, should be in bijective correspondence with the set of unitary cuspidal automorphic representations of $\text{GL}_N(\mathbb{A})$. The latter set is certainly mysterious, but at least its existence is not conjectural. Therefore, it can be used as a replacement for discrete generic global Arthur parameters valued in $\text{GL}_N(\mathbb{C})$.

Thus, Arthur defines $\Psi_{\text{cusp}}(N)$ to be the set of isomorphism classes of unitary cuspidal automorphic representations of $\text{GL}_N(\mathbb{A})$.

As discussed in §8.1.1, a conjectural element ψ of $\Psi_2(N)$ should decompose (uniquely) as $\psi = \mu \boxtimes \nu$, with $\mu \in \Psi_{\text{cusp}}(M)$ and ν an irreducible representation of SU_2 of dimension N/M. Led by this, Arthur defines $\Psi_2(N)$ to be the set of pairs (μ, ν), with μ a unitary cuspidal automorphic representation of $\text{GL}_M(\mathbb{A})$ and ν an irreducible representation of SU_2 of dimension N/M, and uses the notation $\mu \boxtimes \nu$ for such a pair.

A conjectural element ψ of $\Psi(N)$ decomposes as a direct sum $\psi = \psi_1 \oplus \cdots \oplus \psi_m$ with $\psi_k \in \Psi_2(M_k)$. This leads to a definition of $\Psi(N)$ in the current context, in which the direct sum is again taken formally, i.e. ψ is identified with the tuple $(\psi_1, \ldots, \psi_m)$, but the direct sum notation is used in a formal manner. With this, the sets in (8.1.1) have been unconditionally defined.

Note that we have a tautological bijection between $\Psi_{\text{cusp}}(N)$ and the set of unitary cuspidal automorphic representations of $\text{GL}_N(\mathbb{A})$. We also have a bijection between $\Psi_2(N)$ and the set of unitary discrete automorphic representations of $\mathrm{GL}_N(\mathbb{A})$, as well as between $\Psi(N)$ and the set of all unitary automorphic representations of $\mathrm{GL}_N(\mathbb{A})$ that contribute to the L^2-spectrum. The latter two bijections are not tautological, but rely on results of Mœglin–Waldspurger for the discrete automorphic spectrum, and Langlands for the continuous spectrum. More precisely, Moeglin–Waldspurger show [MgW89] that a unitary discrete automorphic representation of $\mathrm{GL}_N(\mathbb{A})$ is the unique irreducible quotient of the parabolic induction of the representation $(\mu\otimes|\det|^{a})\boxtimes(\mu\otimes|\det|^{a-1})\boxtimes\cdots\boxtimes(\mu\otimes|\det|^{-a})$ of the Levi subgroup $(\mathrm{GL}_M)^{N/M}$, where $M|N$, μ is a unitary cuspidal automorphic representation of $\mathrm{GL}_M(\mathbb{A})$, and $a=(N/M-1)/2$. Thus, it corresponds bijectively to the pair (μ,ν), where ν is the unique irreducible representation of SU_2 of dimension N/M. Moreover, a unitary automorphic representation of $\mathrm{GL}_N(\mathbb{A})$ contributing to $L^2(\mathrm{GL}_N(F)\backslash\mathrm{GL}_N(\mathbb{A}))$ is obtained as the parabolic induction of a representation $\pi_1\boxtimes\cdots\boxtimes\pi_m$, where each π_i is a unitary discrete automorphic representation of $\mathrm{GL}_{N_i}(\mathbb{A})$, and the product $\prod_i \mathrm{GL}_{N_i}$ is understood as a standard Levi subgroup of GL_N. This parabolic induction is irreducible [Ber84]. Such a representation corresponds bijectively to the tuple $(\psi_1,\dots,\psi_m)$, where $\psi_i\in\Psi_2(N_i)$ corresponds to π_i. We will denote π_ψ the automorphic representation corresponding to $\psi\in\Psi(N)$.

Next we come to the sets in (8.1.2). They rely on the duality operation $\psi\mapsto\psi^\vee$ on the set $\Psi(N)$. It is easy to define this operation in the current formal context. It sends $\mu\in\Psi_{\text{cusp}}(N)$ to the contragredient cuspidal automorphic representation $\mu^\vee$, it sends $\mu\boxtimes\nu\in\Psi_2(N)$ to $\mu^\vee\boxtimes\nu^\vee$ (where we again note that $\nu^\vee=\nu$), and it sends $\psi_1\oplus\cdots\oplus\psi_m$ to $\psi_1^\vee\oplus\cdots\oplus\psi_m^\vee$. This defines all sets in (8.1.2).

Finally we come to the sets in (8.1.3), where G is a quasi-split symplectic or special orthogonal group. We recall that, conjecturally, the standard representation ${}^LG\to\mathrm{GL}_N(\mathbb{C})$ provides a map $\Psi(G)\to\tilde\Psi(N)$ and that we identify $\Psi(G)$ with its image, even though when $G=\mathrm{SO}_N$ this map will not be injective. Thus, the elements of $\Psi(G)$ are understood as those elements of $\tilde\Psi(N)$ which factor through the standard representation.

The obvious difficulty with this definition is that it relies on elements of $\Psi(N)$ being homomorphisms valued in $\mathrm{GL}_N(\mathbb{C})$, which is not what the formal definition of $\Psi(N)$ supplies. Arthur resolves this difficulty by creating for each $\psi\in\tilde\Psi(N)$ a group L_ψ, that is a crude approximation of $\mathcal{L}_F$ tailored to the specific ψ, and a homomorphism $\tilde\psi\colon L_\psi\times\mathrm{SU}_2\to\mathrm{GL}_N(\mathbb{C})$. The set $\Psi(G)$ is then defined as the set of those $\psi\in\tilde\Psi(N)$ for which the corresponding $\tilde\psi$ factors through the standard representation of ${}^LG$. This is done via the following three-step procedure.

First, Arthur creates a relationship between $\Psi(N)$ and $\mathrm{GL}_N(\mathbb{C})$. Given $\psi\in\Psi(N)$ consider the automorphic representation π_ψ of $\mathrm{GL}_N(\mathbb{A})$ associated to it by the above discussion. This representation is unramified outside of a finite set S of places. For $v\notin S$ let $c(\psi)_v\in\mathrm{GL}_N(\mathbb{C})$ be the Satake parameter of the unramified representation $\pi_{\psi,v}$ of $\mathrm{GL}_N(F_v)$, well-defined up to conjugation. Thus, we have the family $(c(\psi)_v)$ of conjugacy classes in $\mathrm{GL}_N(\mathbb{C})$, defined for almost all places v.

Second, recall from §8.1.1 that each $\psi \in \tilde{\Psi}_{\text{sim}}(N)$, defined in terms of the conjectural group $\mathcal{L}_F$, has a type (symplectic or orthogonal), which is the type of $\widehat{G}$ for the unique ${}^L G \subset \text{GL}_N(\mathbb{C})$ through which it factors. Arthur defines an analog of this notion for the formal definition of the set $\tilde{\Psi}_{\text{sim}}(N)$ via the following "seed" theorem [Art13, Theorem 1.4.1]. For its statement, we recall that $G_1 = \text{SO}_{2n+1}$ and $G_2 = \text{SO}_{2n}$ are precisely the two kinds of *simple twisted endoscopic groups* for $\text{GL}_N(\mathbb{C})$ for $N = 2n$, and $G = \text{Sp}_{2n}$ is the unique *simple twisted endoscopic group* for $\text{GL}_N(\mathbb{C})$ for $N = 2n + 1$.

Theorem 8.1.1. *Let ψ be an element of $\tilde{\Psi}_{sim}(N)$, thus a simple generic formal parameter (i.e. a self-dual unitary cuspidal automorphic representation of $GL_N(\mathbb{A})$). Then there is a unique simple twisted endoscopic group G that admits a unitary discrete automorphic representation π with $c(\pi)_v = c(\psi)_v$.*

Here $c(\pi)_v \in \widehat{G} \rtimes \text{Frob}_v$ is the Satake parameter of the unramified representation π_v of $G(F_v)$, a semi-simple element of $\widehat{G} \rtimes \text{Frob}_v$ well-defined up to $\widehat{G}$-conjugation, regarded as an element of $\text{GL}_N(\mathbb{C})$ via the embedding ${}^L G \to \text{GL}_N(\mathbb{C})$. The actual seed theorem is in fact stronger, and distinguishes G not just among the simple, but even among the elliptic endoscopic groups, but we can ignore this for the purposes of the current discussion. We note that this theorem is significant, and its proof is part of the long induction argument that stretches the entire length of [Art13].

This seed theorem allows us to define the notion of type for elements of $\tilde{\Psi}_{\text{sim}}(N)$ as the type (symplectic or orthogonal) of the group $\widehat{G}$ dual to the group G in the theorem; it leads, as discussed in §8.1.1, to the notion of type for elements of $\tilde{\Psi}_2(N)$, namely the type of $\psi = \mu \boxtimes \nu$ is orthogonal if μ and ν have equal types, and symplectic if μ and ν have opposite types. With this we already have a characterization of $\Psi_2(G)$.

Third, Arthur defines the approximation L_ψ of $\mathcal{L}_F$ and the L-homomorphism

$$\tilde{\psi}\colon L_\psi \times \text{SU}_2 \to \text{GL}_N(\mathbb{C})$$

as follows. As in §8.1.1, the duality operation on $\Psi(N)$ endows the set K_ψ of irreducible constituents of ψ with the disjoint union decomposition

$$K_\psi = I_\psi \sqcup J_\psi \sqcup J_\psi^\vee,$$

where $I_\psi = I_\psi^\vee$ and $J_\psi \cap J_\psi^\vee = \emptyset$. To each $\mu_i \boxtimes \nu_i$ for $i \in I_\psi$ with $\mu_i \in \tilde{\Phi}_{\text{sim}}(m_i)$ the seed theorem assigns a classical group G_i, and we have the standard embedding $\tilde{\mu}_i\colon {}^L G_i \to \text{GL}_{m_i}(\mathbb{C})$. To each $\mu_j \boxtimes \nu_j$ for $j \in J_\psi$ with $\mu_j \in \Phi_{\text{sim}}(m_j)$ set $G_j = \text{GL}_{m_j}$ and let $\tilde{\mu}_j\colon \text{GL}_{m_j}(\mathbb{C}) \to \text{GL}_{2m_j}(\mathbb{C})$ be given by $h \mapsto (h, h^{-t})$, where h^{-t} is the transpose-inverse of h. Let L_ψ be the fiber product over Γ of ${}^L G_k$ for all $k \in I_\psi \sqcup J_\psi$. Then $\tilde{\psi}$ is defined as the product

$$\tilde{\psi} : L_\psi \times \text{SL}_2(\mathbb{C}) \to \text{GL}_N(\mathbb{C}) \times \Gamma, \qquad \tilde{\psi} = \bigoplus_{i \in I_\psi} \tilde{\mu}_i \otimes \nu_i \oplus \bigoplus_{i \in J_\psi} \tilde{\mu}_j \otimes \nu_j.$$

This completes the formal construction of the set $\Psi(G)$. One can define $\Phi(G)$ as before, namely the set of those ψ in whose decomposition all representations of SU_2 are trivial. To each $\psi \in \Psi(G)$ Arthur defines S_ψ as the centralizer in $\widehat{G}$ of the image of the homomorphism $\tilde{\psi}$. Using the decomposition of K_ψ and the notion of type of an element of $\tilde{\Psi}_2(N)$, this group can be computed in the same way as in §8.1.1 and the result is (8.1.7). This in turn allows for the definitions of the various sets in (8.1.3).

We have thus reviewed the formal definitions of the parameter sets in (8.1.1), (8.1.2), and (8.1.3). But there is one more issue to discuss, namely localization of parameters. These are maps $\Psi(N) \to \Psi_v(N)$ and $\Psi(G) \to \Psi_v(G)$, which, in the conjectural framework of §8.1.1, arise by composition with the conjectural homomorphism $\mathcal{L}_{F_v} \to \mathcal{L}_F$.

Let us begin with $\phi \in \Psi_{\mathrm{sim}}(N) = \Phi_{\mathrm{sim}}(N)$, thus $\phi = \mu$ with μ a unitary cuspidal automorphic representation of $\mathrm{GL}_N(\mathbb{A})$. Then $\mu = \otimes'_v \mu_v$, with μ_v an irreducible admissible unitary representation of $\mathrm{GL}_N(F_v)$. Via the local Langlands correspondence for GL_N, which has been established by Harris–Taylor and Henniart, there is an associated local parameter $\phi_v \colon \mathcal{L}_{F_v} \to \mathrm{GL}_N(\mathbb{C})$, a not necessarily irreducible N-dimensional representation of $\mathcal{L}_{F_v}$. However, it is not known that this representation is unitary. This would be a consequence of the generalized Ramanujan conjecture, which asserts that the local components μ_v are tempered, but this is not known in full generality. Therefore, ϕ_v has to be taken in the larger set $\Phi_v^+(N)$ of semi-simple representations $\mathcal{L}_{F_v} \to \mathrm{GL}_N(\mathbb{C})$ that are not assumed unitary.

The extension of the map $\Phi_{\mathrm{sim}}(N) \to \Phi_v^+(N)$ to a map $\Psi(N) \to \Psi_v^+(N)$ is now a formal matter: a formal tensor product $\psi = \phi \boxtimes \nu$ is sent to the actual tensor product $\psi_v := \phi_v \boxtimes \nu$, and a formal direct sum $\psi_1 \oplus \cdots \oplus \psi_m$ is sent to the actual direct sum $\psi_{1,v} \oplus \cdots \oplus \psi_{v,m}$.

We now come to the localization map $\Psi(G) \to \Psi_v^+(G)$. Again we are forced to work with the set $\Psi_v^+(G)$ of not necessarily bounded homomorphisms $\mathcal{L}_{F_v} \times \mathrm{SU}_2 \to {}^LG$, taken up to conjugation by the full ${}^LG$. The formal nature of the elements of $\Psi(G)$ now presents a second obstacle: given $\psi \in \Psi(G) \subset \Psi(N)$, the roundabout definition of $\psi_v \in \Psi_v^+(N)$ makes it unclear why it should factor through the standard representation of ${}^LG$. This obstacle is significant enough to warrant the formulation of a second "seed" theorem ([Art13, Theorem 1.4.2]), whose proof is again part of the long induction argument of [Art13].

Theorem 8.1.2. *Let $\phi \in \Phi_{sim}(G)$. Then the element $\phi_v \in \Phi_v^+(N)$ lies in the subset $\Phi_v^+(G)$.*

Assuming the validity of this theorem, the localization map $\Psi(G) \to \Psi_v^+(G)$ is now constructed formally using the structure of the elements of $\Psi(G)$ as formal direct sums of formal tensor products.

8.1.3 Statements of the main results

We will now discuss the statements of the main results proved in [Art13]. These statements involve many details, and in an effort to make the presentation clearer and shorter, we will suppress some of them. For example, there is an integer $m(\psi)$ defined for each $\psi \in \Psi(G)$ as the number of orbits of $\widehat{G}$ in the ${}^L G$-conjugacy class of the homomorphism $\tilde{\psi}\colon L_\psi \times \mathrm{SU}_2 \to {}^L G$ discussed in §8.1.2. Since this is an artifact of the formal substitutes of global parameters, we will ignore it, just like we are ignoring the distinction between $\widehat{G}$ and ${}^L G$-conjugacy of parameters.

The main theorems proved in [Art13] are the following.

First, we have the local classification result. Assume F is local. Given $\psi \in \Psi(N)$ there is an associated representation π_ψ of $\mathrm{GL}_N(F)$, by applying the local Langlands correspondence to the Langlands parameter ϕ_ψ obtained from ψ by composing with the inclusion $\mathcal{L}_F \to \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C})$ given by $w \mapsto (w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2}))$. If $\psi \in \tilde{\Psi}(N)$, then the representation π_ψ is self-dual and there is a natural extension to the semi-direct product $\mathrm{GL}_N(F) \rtimes \langle\theta\rangle$ where θ is a pinned outer automorphism. This extension is denoted by $\tilde{\pi}_\psi$. Given a test function $\tilde{f}$ on $\mathrm{GL}_N(F) \rtimes \langle\theta\rangle$ one can consider the distribution character of $\tilde{\pi}_\psi$ evaluated at $\tilde{f}$. Arthur denotes this distribution by $\tilde{f} \mapsto \tilde{f}^N(\psi)$, and specializes it to functions supported on the coset $\tilde{G}(N, F) := \mathrm{GL}_N(F) \rtimes \theta$.

Theorem 8.1.3 (Local classification result: Theorems 1.5.1 and 2.2.1 of [Art13])**.** *Let $\psi \in \Psi(G)$.*

1. *There exists a stable linear form $f \mapsto f^G(\psi)$ on the space of test functions that is the twisted transfer of the distribution $\tilde{f}^N(\psi)$, that is, $f^G(\psi) = \tilde{f}^N(\psi)$ for any test function $\tilde{f}$ on $\tilde{G}(N, F)$ with twisted transfer f, a test function on $G(F)$.*

2. *There is a finite set $\Pi_\psi = \Pi_\psi(G)$ of irreducible unitary representations of $G(F)$ (with possible repetitions) and a map $\Pi_\psi \to (\mathcal{S}_\psi)^*$ such that for any semi-simple element $s \in S_\psi$ and any test function f on $G(F)$*

$$f'(\psi') = \sum_{\pi \in \Pi_\psi} \langle s_\psi s, \pi\rangle f_G(\pi).$$

Let us explain the notation. As before $\mathcal{S}_\psi = \pi_0(S_\psi / Z(\widehat{G})^\Gamma)$ and the hat denotes the group of characters of this finite abelian group. The pairing $\langle -, \pi\rangle$ is the character of $\mathcal{S}_\psi$ associated to π under the map $\Pi_\psi \to (\mathcal{S}_\psi)^*$ asserted in the theorem. The element s_ψ was reviewed in (6.4.3). The map $f \mapsto f_G(\pi)$ is the distribution character of π. The pair (s, ψ) leads to a tuple (G', ψ') consisting of an endoscopic datum G' and a parameter ψ' via the bijection (6.4.5). The function f' is the endoscopic transfer of f to G' (as usual only well-defined in terms of its stable orbital integrals), and $f'(\psi')$ is the value of the stable linear form associated to G' and the parameter ψ' by the first part of the theorem: indeed G' is a product of classical and linear groups.

The set Π_ψ is called the *Arthur-packet* (A-packet for short) associated to ψ. When ψ is generic, i.e. lies in $\Phi(G) \cap \Psi(G)$, the theorem further asserts that Π_ψ consists of tempered representations and has no repetitions, that every tempered representation lies in exactly one such packet, and that the map

$\Pi_\psi \to (\mathcal{S}_\psi)^*$ is injective when F is archimedean, and bijective when F is non-archimedean. The packet Π_ψ is then also-called an *L-packet*; thus A-packets and L-packets are the same thing for generic ψ. When ψ is not generic, Mœglin has proved [Mg11] that Π_ψ has no repetitions when F is non-archimedean. When F is archimedean, Adams–Arancibia-Robert–Mezo have shown [AARM24] that the packets defined by Arthur coincide with those constructed many years earlier by Adams–Barbasch–Vogan using entirely different methods.

The statement of this theorem in [Art13] allows also products of classical groups and has an additional part about compatibility with such products, but we have omitted it for clarity.

Next we come to the global classification result, for which we take F global. It uses the local classification result and the localization maps $\Psi(G) \to \Psi_v^+(G)$ to obtain from a parameter $\psi \in \Psi(G)$ a global packet

$$\Pi_\psi = \{\otimes'_v \pi_v \mid \pi_v \in \Pi_{\psi_v}\},$$

where the notation $\otimes'_v$ denotes a restricted tensor product: We have fixed an integral model of G away from a finite set of places of F, which endows for each such place the group $G(F_v)$ with a hyperspecial compact subgroup $G(O_v)$. Then for each $\pi \in \Pi_\psi$ and all but finitely many places v, the local component π_v is the unique $G(O_v)$-spherical representation in Π_{ψ_v}, and π is generated by simple tensors $\otimes_v f_v$, where $f_v \in \pi_v$ is the unique (up to scalar) vector in $\pi_v^{G(O_v)}$ for almost all places.

The reader will note that the local theorem applies as stated only to parameters in the set $\Psi_v(G)$, rather than the larger set $\Psi_v^+(G)$. According to the generalized Ramanujan conjecture ψ_v should lie in $\Psi_v(G)$, but the lack of knowledge of its validity necessitates the use of $\Psi_v^+(G)$. The local theorem can easily be extended to the set $\Psi_v^+(G)$ in a formal manner and this is done in [Art13, §1.5]. We will not review this here.

Theorem 8.1.4 (Global classification result: Theorem 1.5.2 of [Art13])**.** *The discrete spectrum of a quasi-split symplectic or special orthogonal group G decomposes as*

$$L^2_{\mathit{disc}}(G) = \bigoplus_\psi \bigoplus_\pi m(\psi,\pi)\pi,$$

where ψ runs over the set $\Psi_2(G)$, π runs over Π_ψ, and

$$m(\psi,\pi) = |\mathcal{S}_\psi|^{-1} \sum_{s \in \mathcal{S}_\psi} \epsilon_\psi(s)\langle s,\pi\rangle.$$

Here $\langle s,\pi\rangle = \prod_v \langle s,\pi_v\rangle$ and $\epsilon_\psi \colon \mathcal{S}_\psi \to \{\pm 1\}$ is a character defined in [Art13, (1.5.6)]. Since $\mathcal{S}_\psi = S_\psi/Z(\widehat{G})^\Gamma$ is abelian, in fact a 2-group, the number $m(\psi,\pi)$ is 1 if $\langle \pi, s\rangle = \epsilon_\psi(s)$, and 0 otherwise, and one can state the decomposition equivalently as

$$L^2_{\text{disc}}(G) = \bigoplus_\psi \bigoplus_{\pi \in \Pi_\psi(\epsilon_\psi)} \pi,$$

where $\Pi_\psi(\epsilon_\psi) = \{\pi \in \Pi_\psi \mid \langle -, \pi\rangle = \epsilon_\psi\}$, cf. §6.4.

In addition to the two main theorems, one local and one global, there are a number of additional theorems proved in [Art13]. They can be regarded as supplementary, but are essential for the proof of the two main theorems, and are also of independent interest.

We have already stated two of them, namely the two seed theorems 8.1.1 and 8.1.2, which enable the use of the formal global parameters. The next supplementary theorem is related to the seed theorems, and gives an alternative way to detect the classical group G through which a given parameter $\psi \in \tilde{\Psi}_{\text{sim}}(N)$ factors. It is used in the handling of compound parameters in the trace formula. More precisely, it is used in the two key sign calculations states as [Art13, Lemma 4.3.1, 4.4.1] and proved in [Art13, §4.6], which in turn flow into the proofs of [Art13, Lemma 4.3.2, 4.4.2], stated here as Theorems 8.2.3, 8.2.5. The latter, which will be discussed below, make the core of the trace formula comparison that Arthur calls the "standard model".

Theorem 8.1.5 (Theorem 1.5.3 of [Art13])**.**

1. *Let $\phi \in \tilde{\Phi}_{sim}(N)$ and let G be the unique classical group such that $\phi \in \Phi_{sim}(G)$ according to Theorem 8.1.1. Then $\widehat{G}$ is orthogonal if the symmetric square L-function $L(s, \phi, S^2)$ has a pole at $s = 1$, and $\widehat{G}$ is symplectic if the exterior square L-function $L(s, \phi, \bigwedge^2)$ has a pole at $s = 1$.*

2. *Given $\phi_i \in \tilde{\Phi}_{sim}(G_i)$ for $i = 1, 2$ the Rankin–Selberg factor $\epsilon(1/2, \phi_1 \times \phi_2)$ equals 1 if $\widehat{G}_1$ and $\widehat{G}_2$ are of the same type.*

The next supplementary theorem has both a local and a global form, and goes by the name of the "intertwining relation". We will state only the local form, the global form [Art13, Corollary 4.2.1] taking the analogous form and being a direct corollary of the local form. For this, one begins with a parabolic subgroup $P = MN \subset G$ and a parameter $\psi \in \Psi(M)$. Let $\widehat{M} \subset \widehat{G}$ be the dual Levi subgroup. Then $A_{\widehat{M}} = Z(\widehat{M})^{\Gamma,\circ}$ is a torus contained in $S_\psi = S_\psi(G)$. The analog $S_\psi(M)$ of S_ψ relative to M is then equal to $\text{Cent}(A_{\widehat{M}}, S_\psi(G))$. Let $N_\psi(M) = \text{Norm}(A_{\widehat{M}}, S_\psi(G))$. Let $n \in N_\psi(M) \subset S_\psi(G)$.

The local intertwining relation is an identity of two distributions, one related to G and one to an endoscopic group G' of G, like the identities of Theorem 8.1.3. In fact, one of the main purposes of the local intertwining relation is to reduce the proof of Theorem 8.1.3 to the case of discrete parameters. But unlike the case of Theorem 8.1.3, the G-distribution is more complicated and involves normalized intertwining operators.

More precisely, let $w \in W_{\widehat{M}}(\widehat{G}) = W_M(G)$ be the image of n. One can construct a normalized self-intertwining operator $R_P(\psi, \pi_M, w)$ on the parabolically induced representation $\mathcal{I}_P^G(\pi_M)$ for $\pi_M \in \Pi_\psi(M)$. This construction is rather intricate and performed in [Art13, §2.3]; we will not review it. One can also extends the pairing $\langle -, \pi_M\rangle$ from $S_\psi(M)$ to its product with n and thus define a complex number $\langle n, \pi_M\rangle$. There is some ambiguity in the construction of both $R_P(\psi, \pi_M, w)$ and $\langle n, \pi_M\rangle$, but Arthur argues that their product

$\langle n, \pi_M \rangle R_P(\psi, \pi_M, w)$, considered in [Art13, (2.4.4)], is unambiguously defined. The G-distribution is then

$$f_G(\psi, n) = \sum_{\pi_M \in \Pi_\psi(M)} \langle n, \pi_M \rangle \mathrm{tr}\big(R_P(\psi, \pi_M, w)\mathcal{I}_P^G(\pi_M, f)\big).$$

For the endoscopic distribution, the pair (n, ψ) leads via (6.4.5) to an endoscopic datum G' for G and a parameter ψ' for G', hence a stable linear form $f' \mapsto f'(\psi')$ by Theorem 8.1.3. Arthur denotes by $f'_G(\psi, s)$ this distribution, where s is the image of n in $\mathcal{S}_\psi(G)$.

Theorem 8.1.6 (Local intertwining relation, Theorem 2.4.1 of [Art13])**.** *For every test function f on $G(F)$ with transfer f' to $G'(F)$ the following identity holds*

$$f_G(\psi, n) = f'_G(\psi, s_\psi \cdot s).$$

The final supplementary theorem that we will mention is the stable multiplicity formula, which was already stated as Conjecture 7.9.1, and is [Art13, Theorem 4.2.1].

8.1.4 Assumptions on which the results are conditional

The results of [Art13] are not completely unconditional, but some assumptions on which they were based have been established since the book was written. The main tool employed in [Art13] is the ordinary (i.e. untwisted) and twisted trace formula, and their stabilization. The stabilization of the untwisted trace formula was obtained by Arthur, culminating in [Art02], [Art01], [Art03]. The stabilization of the twisted trace formula was not yet complete at the time of writing of [Art13], and was listed as a forthcoming manuscript [A24] in the bibliography. It has since been obtained by Mœglin–Waldspurger [MW16a], [MW16b]. This work itself is conditional on the weighted fundamental lemma and its twisted variant [Art02, Conjecture 5.1]. Just like the ordinary fundamental lemma, the twisted fundamental lemma is not a "lemma", but rather a deep conjecture. While the unweighted untwisted fundamental lemma was proved by Ngô [Ngô10a] and its twisted variant was derived from that by Waldspurger [Wal08], the weighted and twisted weighted fundamental lemmas have not been proved in general, despite initial work by Chaudouard–Laumon [CL10], [CL12].

At the time [Art13] appeared, the results were further conditional on a number of technical statements that are involved in key parts of the arguments of [Art13], and whose proofs had been relegated to further forthcoming papers, referred to as [A25], [A26], [A27] in [Art13]. These statements have since been proved in [AGI$^+$24].

8.2 Applying the trace formula

8.2.1 A remark about the twisted trace formula

The main vehicle used by Arthur to prove both the local and global results is the trace formula and its stabilization. However, as we have reviewed in §6.4, this vehicle can either transport one from knowledge of the stable multiplicity formula to knowledge of the global Arthur conjecture, or vice versa, but cannot establish both at once. The additional input needed for this is provided by the twisted trace formula for the group GL_N and its stabilization.

The twisted trace formula arises from the fact that an automorphism θ of a connected reductive group G acts on all data functorially associated to G, such as the automorphic spectrum. In the case we are discussing here, the reductive group is GL_N and θ is the pinned automorphism that is in the same inner class as the transpose-inverse automorphism. One considers test functions $\tilde{f}$ on the non-identity coset $\mathrm{GL}_N(\mathbb{A}) \rtimes \theta$ in the disconnected group $\mathrm{GL}_N(\mathbb{A}) \rtimes \langle\theta\rangle$ lets them act on the automorphic spectrum of $\mathrm{GL}_N(\mathbb{A})$, and derives a geometric and spectral expansion for that action, eventually resulting in an invariant distribution $\tilde{f} \mapsto I^N(\tilde{f})$ with two expressions for it.

We will not review the derivations of these expansions. While the broad strokes are generally similar to the untwisted case, the notation, and many technical considerations, becomes even more cumbersome. Therefore, we will use the twisted trace formula symbolically, and focus on the applications of the untwisted trace formula.

The distribution $I^N(\tilde{f})$ has a stabilization identity formally analogous to (7.8.1), which Arthur writes as

$$I^N(\tilde{f}) = \sum_G \tilde{\iota}(N,G) S^G(\tilde{f}^G). \tag{8.2.1}$$

The coefficients $\tilde{\iota}(N,G)$ are analogous to the coefficients $\iota(G,G')$ of (7.8.1). The function $\tilde{f}^G$ is the (twisted) endoscopic transfer of $\tilde{f}$ to G. The summation runs over the set of elliptic twisted endoscopic data of the pair (GL_N, θ). These data are equivalent to tuples (N_S, N_O, η), where N_S and N_O are non-negative integers with $N_S + N_O = N$, N_S is even, and $\eta\colon \Gamma \to \{\pm 1\}$ is a quadratic (possibly trivial) character of the Galois group Γ of F, assumed trivial when $N_O = 0$, non-trivial when $N_O = 2$, and no assumption for other values of N_O. The endoscopic group associated to this tuple is the group $G_S \times G_O$, where $G_S = \mathrm{SO}(N_S+1)$ is the (automatically split) odd orthogonal group, and G_O, depending on the parity of N_O, is the (automatically split) symplectic group $\mathrm{Sp}(N_O - 1)$ when N_O is odd, and the even orthogonal group $\mathrm{SO}(N_O)$ that splits over the (trivial or quadratic) extension E_η/F specified by the character η when N_O is even.

The product structure for an elliptic twisted endoscopic group facilitates an inductive argument, of which the *simple* endoscopic data become the focal point. These correspond to tuples (N_S, N_O, η) in which one of the integers

N_S and N_O is zero. When N is odd, necessarily $N_S = 0$, and the only simple twisted endoscopic group is the symplectic group $\mathrm{Sp}(N_O - 1)$. It occurs infinitely many times as a simple twisted endoscopic datum $(0, N, \eta)$ with η varying, and η is used to embed the L-group $\mathrm{SO}_N(\mathbb{C}) \times \Gamma$ into $\mathrm{GL}_N(\mathbb{C}) \times \Gamma$ by mapping $1 \times \sigma$ to $\eta(\sigma)I \times \sigma$, where I is the non-trivial element of the center of $\mathrm{O}_N(\mathbb{C})$. When N is even, the simple twisted endoscopic data are $(N, 0, 1)$ and $(0, N, \eta)$, with $(N, 0, 1)$ corresponding to the split odd orthogonal group SO_{N+1}, and $(0, N, \eta)$ corresponding to the quasi-split even orthogonal group SO_N with splitting field E_η.

8.2.2 Decomposition of the discrete part of the trace formula

For simplicity assume $F = \mathbb{Q}$ and G quasi-split. Let $T_0 \subset G$ be a minimal Levi subgroup, and $K = K_\infty \cdot K^\infty$ a maximal compact subgroup of $G(\mathbb{A})$ in good relative position to T_0.

The discrete part of the trace formula is the spectral distribution $I_{\mathrm{disc}} = I^G_{\mathrm{disc}}$ defined in (7.4.1), which we recall here

$$I^G_{\mathrm{disc}}(f) = \sum_M |W_0^M| \cdot |W_0^G|^{-1} \sum_{s \in W^G(M)_{\mathrm{reg}}} |\det(s - 1|\mathfrak{a}^G_M)|^{-1} \mathrm{tr}\,(M_P(s, 0)\mathcal{I}_P(0, f)), \tag{8.2.2}$$

where $\mathcal{I}_P(0, f)$ is the action of the operator of averaging against f on the induced representation $\mathcal{H}_P$ and $M_P(s, 0)$ is the intertwining operator for the Weyl element w acting on the space space; see Appendices A.1 and A.2.

We remark that we have presented the distribution I^G_{disc} in terms of the Hilbert space $L^2(G(F)\backslash G(\mathbb{A})^1)$, which is a-priori a representation of $G(\mathbb{A})^1$ but can be extended to $G(\mathbb{A})$ using the decomposition $G(\mathbb{A}) = G(\mathbb{A})^1 \times A^+_G$ and imposing a character on A^+_G, see §4.2. An alternative presentation is to fix a subgroup $\mathfrak{X}_G \subset Z_G(\mathbb{A})$ such that $Z_G(F) \cdot \mathfrak{X}_G \backslash Z_G(\mathbb{A})$ is compact, and consider functions on $G(F)\backslash G(\mathbb{A})$ that transform under $\mathfrak{X}_G$ by a fixed character. The two approaches are equivalent. Note that when G is a symplectic or special orthogonal group then Z_G is finite, so $A^+_G = \{1\}$ and moreover one can take $\mathfrak{X}_G = \{1\}$, so the two approaches coincide.

Arthur gives a decomposition of the distribution $I^G_{\mathrm{disc}}(f)$ as a sum of pieces indexed by formal global parameters. It is based on two invariants that can be associated to an automorphic representation π of G, one coming from the archimedean place, and another coming from the finite places.

The archimedean component π_∞ has an infinitesimal character, which is an element $\mu_\pi \in \mathfrak{t}^*_{\mathbb{C}}$. The complex vector space $\mathfrak{t}_{\mathbb{C}} = X_*(T) \otimes \mathbb{C}$ has a Weyl-invariant real structure $X_*(T) \otimes \mathbb{R}$. Using that we can write $\mu_\pi = \mathrm{Re}(\mu_\pi) + i\mathrm{Im}(\mu_\pi)$, where $\mathrm{Re}(\mu_\pi), \mathrm{Im}(\mu_\pi)$ are $\mathbb{R}$-valued linear forms on $X_*(T) \otimes \mathbb{R}$. Fixing a Weyl-invariant norm on the vector space $\mathfrak{t}_{\mathbb{C}}$ we can extract from μ the non-negative real number $t_\pi := |\mathrm{Im}(\mu_\pi)|$, which is the first invariant. The second invariant is the collection of Satake parameters $c_\pi = (c_{\pi,v})_v$, for all finite places v where π is unramified. Here $c_{\pi,v}$ is a semi-simple conjugacy class in $\widehat{G} \rtimes$

Frob$_v$, and the collection c_π is taken for almost all v in the sense that two such collections are deemed equal if they are equal at all but finitely many places.

Consider a formal global parameter $\psi \in \tilde{\Psi}(N)$. For now we do not impose that ψ lies in $\Psi(G)$. We can extract analogously a non-negative real number t_ψ and a collection c_ψ. Then, for any unitary representation π of $M(\mathbb{A})^1$, we can ask if $t_\pi = t_\psi$ and $c_\pi \mapsto c_\psi$, where the meaning of the arrow is given by the standard representation $\widehat{M} \to \widehat{G} \to \mathrm{GL}_N(\mathbb{C})$. We then write $\mathcal{I}_{P,\psi}(\chi)$ to be the sum of $\pi = \mathcal{I}_P(\pi_M)$, where π_M runs over the irreducible constituents of $L^2_{\text{disc}}(M(F)\backslash M(\mathbb{A})^1)$ and we are requiring that the central character of π_M restricts to χ on $\mathfrak{X}_G$ and that $t_\pi = t_\psi$ and $c_\pi \mapsto c_\psi$. We then define $I_{\text{disc},\psi}$ in analogy with the definition of $I_{\text{disc},\psi}$ in (7.4.1), namely

$$I^G_{\text{disc},\psi}(f) := \sum_{M,s} |W_0^M| \cdot |W_0^G|^{-1} \cdot |\det(s-1|\mathfrak{a}_M^G)|^{-1} \mathrm{tr}\,(M_{P,\psi}(s,0)\mathcal{I}_{P,\psi}(0,f)), \tag{8.2.3}$$

with $M_{P,\psi}(s,0)$ being the restriction of $M_P(s,0)$ to the invariant subspace $\mathcal{H}_{P,\psi}$ of $\mathcal{H}_P$ on which $\mathcal{I}_{P,\psi}$ acts.

This leads to the following decompositions, the first one being a refinement of the decomposition $I^G_{\text{disc}} = \sum_{t\geq 0} I^G_{\text{disc},t}$ obtained by separating the sum in (7.4.1) into an outer sum over t and an inner sum over M, s.

Proposition 8.2.1 ([Art13, Corollaries 3.4.2, 3.4.3])**.**

$$\begin{aligned} I^G_{\textit{disc}}(f) &= \sum_{\psi\in\tilde{\Psi}(N)} I^G_{\textit{disc},\psi}(f), \\ L^2_{\textit{disc}}(G(F)\backslash G(\mathbb{A})) &= \bigoplus_{\psi\in\tilde{\Psi}(N)} L^2_{\textit{disc},\psi}(G(F)\backslash G(\mathbb{A})). \end{aligned}$$

In the second decomposition, the direct summand $L^2_{\text{disc},\psi}(G(F)\backslash G(\mathbb{A}))$ of $L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A}))$ is defined analogously, namely as the direct sum of those irreducible constituents π of $L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A}))$ whose central character restricts to χ on $\mathfrak{X}_G$ and which satisfy $t_\pi = t_\psi$ and $c_\pi \mapsto c_\psi$. We will write $R_{\text{disc},\psi}$ for the right regular representation of $G(\mathbb{A})$ on the summand $L^2_{\text{disc},\psi}(G(F)\backslash G(\mathbb{A}))$ of $L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A}))$. Thus, the right regular representation R_{disc} of $G(\mathbb{A})$ on $L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A}))$ decomposes as the direct sum of $R_{\text{disc},\psi}$ over all $\psi \in \tilde{\Psi}(N)$.

Of course our eventual goal is to reduce both sums in Proposition 8.2.1 to the subset $\Psi(G)$, i.e. to show that the summands for the other ψ are zero, but this is one of the main global theorems and will be achieved only at the end of the argument.

8.2.3 Stabilization identities

The stabilization of the usual trace formula takes the form

$$I^G(f) = \sum_{G'} \iota(G,G') S^{G'}(f').$$

This results in an analogous stabilization identity

$$I^G_{\text{disc}}(f) = \sum_{G'} \iota(G,G') S^{G'}_{\text{disc}}(f').$$

The decomposition of Proposition 8.2.1 induces a decomposition of the entire stabilization identity.

Proposition 8.2.2 ([Art13, Lemma 3.3.1]). *For each $\psi \in \tilde{\Psi}(N)$ the following identity holds*

$$I^G_{disc,\psi}(f) = \sum_{G'} \iota(G,G') S^{G'}_{disc,\psi}(f').$$

The identity stated above holds, suitably interpreted, also for the twisted trace formula for the group GL_N. At the time [Art13] was written, the stabilization of the twisted trace formula was not fully established, which is why the assumption of [Art13, Lemma 3.3.1] is stated as "Assume that G satisfies (3.2.3)". The stabilization of the twisted trace formula has since been established by [MW16a], [MW16b]. Thus, [Art13, Lemma 3.3.1] now holds, conditional only on the validity of the twisted weighted fundamental lemma, which at the time of writing of this text has not been established.

Let us say a word about the content of Proposition 8.2.2. If G is a quasi-split group, then it is its largest-dimensional endoscopic group, and the identity amounts to an inductive definition of $S^G_{\text{disc}}(f)$, as well as its ψ-component, and the assertion that this definition provides a stable distribution. When G is not quasi-split, then its largest-dimensional endoscopic group is its quasi-split form. The right hand side in Proposition 8.2.2 has then been completely defined, and one has to prove that it equals the left hand side. Finally, when G is twisted GL_N, then again the right hand side has been completely defined, with all G' being products of quasi-split classical groups, and again one has to prove that it equals the left hand side.

Finally, by combining the two identities obtained from Proposition 8.2.2, one for a classical group G, and one for twisted GL_N, one obtains additional information about the distribution S^G_{disc}. It is this information that can be exploited to finally obtain the desired theorems.

8.2.4 The standard model: getting to the payload in $I^G_{\text{disc}}(f)$

Recall from (8.2.2) that the distribution $I^G_{\text{disc}}(f)$ contains not just the quantity $\mathrm{tr}\, R_{\text{disc}}(f)$, which we are most interested in, but also contributions from the continuous spectrum. More precisely, $|\kappa_G|^{-1}\mathrm{tr}\, R_{\text{disc}}(f)$ is the summand for $M = G$ in (8.2.2) (the constant $|\kappa_G|$ is recalled below), while the continuous spectrum contributes the summands $M \neq G$. By induction we assume as known all local and global theorems for the groups $M \neq G$. This observation alone can already establish some part of the local and global theorems for G itself. In order to realize this idea, we need to reinterpret the summands $M \neq G$ of $I^G_{\text{disc}}(f)$ in a different way and then compare them to summands in the stabilization identity of Proposition 8.2.2.

Recall the summands $I_{\text{disc},\psi}$ and $R_{\text{disc},\psi}$ introduced in Proposition 8.2.1 and the paragraph thereafter. We begin by rewriting the difference $I_{\text{disc},\psi}(f) - |\kappa_G|^{-1}\text{tr}\, R_{\text{disc},\psi}(f)$. The following theorem is valid both for a classical group G as well as for twisted GL_N.

Theorem 8.2.3 ([Art13, Lemma 4.3.2]). *Let G be either a quasi-split classical group or twisted GL_N. Let $\psi \in \tilde{\Psi}(N)$.*

1. *If $\psi \in \Psi_2(M)$ for a proper Levi subgroup $M \subset G$, then the difference $I_{disc,\psi}(f) - |\kappa_G|^{-1} tr\, R_{disc,\psi}(f)$ equals*

$$\frac{1}{|\kappa_G| \cdot |\mathcal{S}_\psi|} \sum_{x \in \mathcal{S}_\psi} \epsilon_\psi^G(x) |W_\psi^0|^{-1} \sum_{u \in \mathfrak{N}_{\psi,reg}(x)} s_\psi^0(w_u) |\det(w_u - 1)|^{-1} f_G(\psi, u).$$

2. *Otherwise $I_{disc,\psi}(f) = |\kappa_G|^{-1} tr\, R_{disc,\psi}(f)$.*

Let us summarize the notation.

1. $|\kappa_G| = 1$ when G is classical and $|\kappa_G| = 2$ when G is twisted GL_N. We note that in [Art13], there is also the constant m_ψ that we have omitted here, due to our convention that we are ignoring the complications caused by the outer automorphism of SO_{2n}.

2. $\epsilon_\psi^G(x) = \epsilon_\psi(x)$ is the sign character associated to ψ that enters the global multiplicity formula (Theorem 8.1.4).

3. W_ψ^0 is the part of the relative Weyl group $N_G(M)(F)/M(F)$ consisting of elements that can be realized within the identity component S_ψ^0 of S_ψ. We will also need below the part W_ψ of the relative Weyl group consisting of elements that can be realized within S_ψ.

4. $\mathfrak{N}_\psi = \pi_0(N_\psi(M)/Z(\widehat{G})^\Gamma)$, where we recall from §8.1.3 that $N_\psi(M) = \text{Norm}(A_{\widehat{M}}, \widehat{G})$. Note that the assumption $\psi \in \Psi_2(M)$ implies that $A_{\widehat{M}}$ is a maximal torus of S_ψ^0. Moreover, $\mathfrak{N}_{\psi,\text{reg}}$ consists of those elements whose image in the Weyl group is regular[7], and $\mathfrak{N}_\psi(x)$ is the subset of those elements that map to x under $\mathfrak{N}_\psi \to \mathcal{S}_\psi$. We then set $\mathfrak{N}_{\psi,\text{reg}}(x) = \mathfrak{N}_\psi(x) \cap \mathfrak{N}_{\psi,\text{reg}}$.

5. $w_u \in W_\psi$ is the image of u.

6. s_ψ^0 is the sign function on the Weyl group W_ψ of the disconnected reductive group S_ψ.

[7] Arthur's formulation involves the subset $\mathfrak{N}'_{\text{reg}}$ of elements whose image in the Weyl group is non-trivial. The only way in which a regular Weyl element can be trivial is when $M = G$, so the prime decoration is just used to exclude the case $M = G$, where M is the standard Levi subgroup dual to $\widehat{M} = \text{Cent}(A_{\widehat{M}}, \widehat{G})$. This is equivalent to our condition $\psi \notin \Psi_2(G)$.

7. $\det(w_u - 1)$ is taken for the action on the real vector space $\mathfrak{a}_M^G$, equivalently the real vector space $X_*(\bar{T}_\psi)\otimes_{\mathbb{Z}}\mathbb{R}$, where $\bar{T}_\psi = A_{\widehat{M}}/A_{\widehat{G}} \subset S_\psi/Z(\widehat{G})^\Gamma$ is a maximal torus.

8. $f_G(\psi, u)$ is the linear form that enters the global intertwining relation, and equals the product over all places of the linear forms that enter the local intertwining relation Theorem 8.1.6.

The proof of Theorem 8.2.3 is obtained by applying the induction hypothesis that all local and global results are known for the groups $M \neq G$. More precisely, recall from (7.4.1) that $I_{\text{disc}}^G(f)$ is a sum over Levi subgroups M of G of terms of the form $\text{tr}(M_P(w)\mathcal{I}_P(f))$. We have the corresponding versions where we control the infinitesimal character at ∞ by a real parameter t, or where we use a formal Arthur parameter ψ to control both the infinitesimal character at ∞ as well as the Satake parameters at unramified finite places, see (8.2.3), and we use the subscripts t or ψ to indicate that. The main global theorem, which we assume inductively as known for all $M \neq G$, expresses $L^2_{\text{disc}}(M)$ as a direct sum over discrete Arthur parameters ψ for M and the multiplicity of each representation in the corresponding adelic packet is provided by the multiplicity formula. The intertwining operator $M_{P,\psi}(w, 0)$ is expressed as the product of a normalized operator $R_{P,\psi}$ and a normalizing factor $r_{P,\psi}$. Regrouping terms one obtains from this a description ([Art13, Corollary 4.2.4]) for the term $\text{tr}(M_{P,\psi}(w)\mathcal{I}_{P,\psi}(f))$ that is the main contribution to the summand for M of $I_{\text{disc},\psi}(f)$. This description still contains the normalizing factor $r_{P,\psi}$ and the sign character ϵ_ψ^M. A key computation of signs ([Art13, Lemma 4.3.1]) shows that their product equals the product of the sign characters ϵ_ψ^G and s_ψ^0.

We now make a slight refinement of Theorem 8.2.3, by considering what we expect the value of $\text{tr}\, R_{\text{disc},\psi}(f)$ to be. When $\psi \notin \Psi_2(G)$, we expect that this value is zero. When $\psi \in \Psi_2(G)$, we expect this value to be the contribution of the (as of yet to be defined) global A-packet for ψ to the discrete spectrum of G. We remind the reader that this is not a tautology at all, since the actual definition of $R_{\text{disc},\psi}$, as recalled in Proposition 8.2.1 and the paragraph thereafter, is by imposing conditions on the infinitesimal character of the archimedean component and the Satake parameters of the unramified non-archimedean components of irreducible constituents of $L^2_{\text{disc}}(G)$. The contribution of the A-packet for ψ to $\text{tr}\, R_{\text{disc}}(f)$ is given by the conjectural formula in the global classification theorem (Theorem 8.1.4), which is based on the existence of the local A-packets Π_{ψ_v} and their pairing with $\mathcal{S}_{\psi_v}$. So let us put ourselves in the hypothetical position that these local objects have been defined. This allows us to define the (rescaled) difference ${}^0r^G_{\text{disc},\psi}$ between $\text{tr}\, R_{\text{disc}}(f)$ and its expected value as

$$
{}^0r^G_{\text{disc},\psi} = \frac{1}{|\kappa_G|}\begin{cases}\text{tr}\, R_{\text{disc},\psi}(f), & \psi \notin \Psi_2(G)\\ \text{tr}\, R_{\text{disc},\psi}(f) - \frac{1}{|\mathcal{S}_\psi|}\sum\limits_{x\in\mathcal{S}_\psi}\epsilon_\psi^G(x)\sum\limits_{\pi\in\Pi_\psi}\langle x,\pi\rangle f_G(\pi), & \psi \in \Psi_2(G).\end{cases}
$$

With this notation, we obtain the following result, which puts us one step closer to proving that $\text{tr}\, R_{\text{disc},\psi}(f)$ is equal to its expected value, i.e. that ${}^0r^G_{\text{disc},\psi} = 0$.

Corollary 8.2.4 ([Art13, Corollary 4.3.3]). *Let G be either a quasi-split classical group or twisted GL_N. Let $\psi \in \Psi(G)$.*

1. *If $\psi \in \Psi_2(M)$ for a proper Levi subgroup $M \subset G$, assume that the linear form $f_G(\psi, u)$ depends only on the image $x \in \mathcal{S}_\psi$ of u.*

2. *If $\psi \in \Psi_2(G)$, assume that the local packet $\Pi_{\psi_v}(G)$ and its pairing with $\mathcal{S}_{\psi_v}$ have been defined for each place v, and set $f_G(\psi, x) = \sum_{\pi \in \Pi_\psi(G)} \langle x, \pi \rangle f_G(\pi)$.*

Then $I_{disc,\psi}(f) - {}^0 r^G_{disc,\psi}(f)$ equals

$$\frac{1}{|\kappa_G| \cdot |\mathcal{S}_\psi|} \sum_{x \in \mathcal{S}_\psi} i_\psi(x) \epsilon^G_\psi(x) f_G(\psi, x),$$

where

$$i_\psi(x) = |W^0_\psi|^{-1} \sum_{w \in W_{\psi,reg}(x)} s^0_\psi(w) |\det(w-1)|^{-1}.$$

The proof of this corollary is immediate from the theorem, upon noting that the map $\mathfrak{N}_{\psi,\mathrm{reg}}(x) \to W_{\psi,\mathrm{reg}}(x)$ is bijective, and that if $\psi \in \Psi_2(G)$ then $i_\psi(x) = 1$.

The output of Corollary 8.2.4 will be compared against the stabilization identity of Proposition 8.2.2. In order to facilitate this comparison, we now rework the latter identity in a way similar to what we just did to $I^G_{\mathrm{disc}}(f)$. The goal this time is to compute the difference $I^G_{\mathrm{disc},\psi}(f) - S^G_{\mathrm{disc},\psi}(f)$ when G is a classical group, or the difference $I^G_{\mathrm{disc},\psi}(f) - \sum_{G'} \iota(G, G') S^{G'}_{\mathrm{disc},\psi}(f')$ when G is twisted GL_N and G' runs over the simple twisted endoscopic data, i.e. the various classical groups that occur as twisted endoscopic groups and that are not products of smaller groups; thus when N is even we are considering the various quasi-split special orthogonal groups in N and $N+1$ variables, and when N is odd we are considering the symplectic group in $N-1$ variables. Note that $S^{G'}_{\mathrm{disc},\psi}$ is defined for all $\psi \in \tilde{\Psi}(N)$, just like $I^G_{\mathrm{disc},\psi}$, and not just for those ψ that lie in $\Psi(G')$ resp. $\Psi(G)$.

For uniformity, write $s^G_{\mathrm{disc},\psi}(f) = S^G_{\mathrm{disc},\psi}(f)$ when G is a classical group, and $s^G_{\mathrm{disc},\psi}(f) = \sum_{G'} \iota(G, G') S^{G'}_{\mathrm{disc},\psi}(f')$ when G is twisted GL_N. Then

Theorem 8.2.5 ([Art13, Lemma 4.4.2]). *Let $\psi \in \tilde{\Psi}(N)$ and let G be either a quasi-split classical group or twisted GL_N. Then $I^G_{disc,\psi}(f) - s^G_{disc,\psi}(f)$ equals*

$$\frac{1}{|\kappa_G| \cdot |\mathcal{S}_\psi|} \sum_{x \in \mathcal{S}_\psi} \epsilon^G_\psi(x \cdot s_\psi) \sum_{s \in \mathcal{E}'_{\psi,ell}(x)} |\pi_0(\bar{S}_{\psi,s})|^{-1} \sigma(\bar{S}^0_{\psi,s}) f'_G(\psi, s).$$

Let us summarize the new notation.

1. As remarked above, we continue to ignore the constant m_ψ.

2. $\bar{S}_\psi = S_\psi / Z(\widehat{G})^\Gamma$ and $\bar{S}^0_\psi = (\bar{S}_\psi)^0$ is the identity component.

3. The set $\bar{S}_{\psi,\text{ell}}$ consists of semi-simple elements of $\bar{S}_\psi$ for which the centralizer $\bar{S}_{\psi,s}$ of s in $\bar{S}_\psi$ has finite center, cf. [Art13, (4.1.7)]. This is precisely the set for which the endoscopic datum associated to (ψ, s) as in (6.4.5) is elliptic. We introduce the subset $\bar{S}'_{\psi,\text{ell}}$ consisting of those s for which the endoscopic datum is not simple.

4. The set $\mathcal{E}_{\psi,\text{ell}}$ consists of the orbits under the action of $\bar{S}^0_\psi$ by conjugation on $\bar{S}_{\psi,\text{ell}}$, and the analogous version decorated with a prime. The subset $\mathcal{E}_{\psi,\text{ell}}(x)$ is the preimage of x under the natural map $\mathcal{E}_{\psi,\text{ell}} \to \mathcal{S}_\psi$; we also have the analogous notation decorated with a prime.

5. The group $\bar{S}^0_{\psi,s}$ is the identity component in the usually disconnected reductive group $\bar{S}_{\psi,s}$ that is the centralizer of s in $\bar{S}_\psi$, and the quantity $\sigma(\bar{S}^0_{\psi,s})$ is a complex number defined inductively in [Art13, Proposition 4.1.1].

6. $f'_G(\psi, s)$ is (the global version of) the endoscopic linear form that enters the local intertwining relation (Theorem 8.1.6).

The proof of Theorem 8.2.5 is based on the bijection (6.4.5) and the inductive assumption that the local and global theorems, in particular the endoscopic character identities and the stable multiplicity formula, are valid for all G'. The stable multiplicity formula relates $S^{G'}_{\text{disc},\psi}$ to the linear form $f'_G(\psi, s)$. The key sign computation $\epsilon^{G'}_{\psi'}(s_{\psi'}) = \epsilon^G_\psi(s_\psi \cdot s)$, which allows one to convert the sign $\epsilon^{G'}_{\psi'}(s_{\psi'})$ from the stable multiplicity formula for (G', ψ') into the sign $\epsilon^G_\psi(s_\psi \cdot s)$ that appears in the above statement, comes from [Art13, Lemma 4.4.1].

As with Theorem 8.2.3, we will reinterpret Theorem 8.2.5 by considering what we expect the value of $s^G_{\text{disc},\psi}(f)$ to be. The stable multiplicity formula (Conjecture 7.9.1) gives the expected value for $S^G_{\text{disc},\psi}(f)$. We let ${}^0S^G_{\text{disc},\psi}(f)$ be the difference between $S^G_{\text{disc},\psi}(f)$ and its expected value, i.e.

$$ {}^0S^G_{\text{disc},\psi}(f) = S^G_{\text{disc},\psi}(f) - |\mathcal{S}_\psi|^{-1}\sigma(\bar{S}^0_\psi)\epsilon(\psi)f(\psi), $$

where we recall that $\epsilon(\psi) = \epsilon_\psi(s_\psi)$, and keeping with previous notation we set

$$ {}^0S^{G'}_{\text{disc},\psi}(f') = \sum_{\psi' \mapsto \psi} {}^0S^{G'}_{\text{disc},\psi'}(f'). $$

Finally, we set

$$ {}^0s^G_{\text{disc},\psi}(f) = \sum_{G' \in \tilde{\mathcal{E}}_{\text{sim}}(G)} \iota(G, G')^0 S^{G'}_{\text{disc},\psi}(f'), $$

where the sum over G' runs over the set $\tilde{\mathcal{E}}_{\text{sim}}(G)$ of simple endoscopic data; thus it has just the term $G' = G$ when G is a classical group, but has multiple terms when G is twisted GL_N.

To rewrite the result of Theorem 8.2.5 we add some assumptions that will be in force when we make the comparison with the result of Corollary 8.2.4. These assumptions concern the linear form $f'_G(\psi, s)$. According to our induction hypothesis it is defined whenever (ψ, s) leads via (6.4.5) to an endoscopic datum that is not simple, and is hence a product of smaller classical groups. In that case we make the assumption that this linear form depends only on the image x of s in $\mathcal{S}_\psi$. This will allow the term $f'_G(\psi, x)$ in Theorem 8.2.5 to be pulled out of the inner sum. The second assumption will be that the linear form $f'_G(\psi, s)$ is also defined when (ψ, s) leads to a simple endoscopic datum. Just as in the discussion of Corollary 8.2.4, this assumption is local, since that linear form is the product of local distributions at all places. Assuming that $f'_G(\psi, s)$ is defined, as well as that it only depends on the image $x \in \mathcal{S}_\psi$ of s, we obtain from Theorem 8.2.5 the following result.

Corollary 8.2.6 ([Art13, Corollary 4.4.3])**.** *Let $\psi \in \tilde{\Psi}(N)$ and let G be either a quasi-split classical group or twisted GL_N. Given $x \in \mathcal{S}_\psi$ and $s \in \mathcal{E}_\psi(x)$, assume that the linear form $f'_G(\psi, s)$ is defined and depends on s only through x. Then $I^G_{disc,\psi}(f) - {}^0 s^G_{disc,\psi}(f)$ equals*

$$\frac{1}{|\kappa_G| \cdot |\mathcal{S}_\psi|} \sum_{x \in \mathcal{S}_\psi} \epsilon_\psi(x) \epsilon^G_\psi(x) f'_G(\psi, x \cdot s_\psi),$$

where

$$\epsilon_\psi(x) = \sum_{s \in \mathcal{E}_{\psi,ell}(x)} |\pi_0(\bar{S}_{\psi,s})|^{-1} \sigma(\bar{S}^0_{\psi,s}).$$

8.2.5 Reduction of the global theorems to non-elliptic non-exceptional parameters via comparison of trace formulas

Considering Corollaries 8.2.4 and 8.2.6 side by side, it is apparent that their structure is very similar. In this section we will obtain the proofs of the global theorems for a large class of global parameters by virtue of comparing the expressions in Corollaries 8.2.4 and 8.2.6, once with G being a quasi-split classical group, and once with G being twisted GL_N. The key assumptions that are needed for this comparison are the following crucial global result.

Theorem 8.2.7 (Global intertwining relation)**.** *The linear forms $f_G(\psi, s)$ and $f'_G(\psi, s)$ depend only on the image $x \in \mathcal{S}_\psi$ of s and*

$$f_G(\psi, x) = f'_G(\psi, x \cdot s_\psi).$$

This is the global analog of, and a direct consequence of, the local intertwining relation (Theorem 8.1.6). Just like its local counterpart, it is established in its entirety only at the end of the long induction argument.

A small part of it, namely the dependence of $f'_G(\psi, s)$ on s can be treated by a simple and direct argument as follows.

Lemma 8.2.8. *The linear form $f'_G(\psi, s)$ depends only on x.*

Proof. This is a fairly simple and general argument that uses the fact that this linear form satisfies parabolic descent. See [Art13, pp. 204-205]. □

Proposition 8.2.9 (Discussion in the beginning of [Art13, §4.5])**.** *Let $\psi \in \tilde{\Psi}(N) \smallsetminus \tilde{\Psi}_{ell}(N)$. Assume that the global intertwining relation (Theorem 8.2.7) holds for ψ. Then Theorem 8.1.4 and Conjecture 7.9.1 hold for ψ.*

Proof. The assumption $\psi \notin \tilde{\Psi}_{\mathrm{ell}}(N)$ implies that both linear forms $f_G(\psi, s)$ and $f'_G(\psi, s)$ are defined. By assumption they depend only on the image $x \in \mathcal{S}_\psi$ of s, and are equal to each other. A combinatorial result ([Art13, Proposition 4.1.1]) shows that $i_\psi(x) = \epsilon_\psi(x)$ and this implies that the right hand sides of Corollaries 8.2.4 and 8.2.6 are equal. This equates their left hand sides, leading to ${}^0r_{\mathrm{disc},\psi}(f) = {}^0s_{\mathrm{disc},\psi}(f)$. Recalling definitions and using the fact that $\psi \notin \tilde{\Psi}_{\mathrm{ell}}(N) \supset \Psi_2(G)$ this identity becomes

$$|\kappa_G|^{-1} \mathrm{tr}\, R_{\mathrm{disc},\psi}(f) = \sum_{G'} \iota(G, G')^0 S^{G'}_{\mathrm{disc},\psi}(f'), \tag{8.2.4}$$

where the sum runs over the simple endoscopic data. In fact, we have two such identities: one for the classical group G, and one for twisted GL_N. When G is the classical group the identity (8.2.4) becomes

$$\mathrm{tr}\, R_{\mathrm{disc},\psi}(f) = {}^0S^G_{\mathrm{disc},\psi}(f), \tag{8.2.5}$$

because $|\kappa_G| = 1$ and $G' = G$ is the only summand on the right. This implies in particular that $\mathrm{tr}\, R_{\mathrm{disc},\psi}(f)$ is a stable distribution. Of course we expect it to be zero, since $\psi \notin \Psi_2(G)$, but at the moment we don't know that.

When G is twisted GL_N we do know that $\mathrm{tr}\, R_{\mathrm{disc},\psi}(f) = 0$. This is a consequence of the theorem of Jacquet–Shalika that isobaric automorphic representations of GL_N are determined by their Satake parameters, together with our assumption that $\psi \notin \tilde{\Psi}_2(N)$. Therefore the identity (8.2.4) becomes

$$0 = \sum_{G \in \tilde{\mathcal{E}}_{\mathrm{sim}}(N)} \tilde{\iota}(N, G)^0 S^G_{\mathrm{disc},\psi}(f^G),$$

where the sum runs over the set $\tilde{\mathcal{E}}_{\mathrm{sim}}(N)$ of simple twisted endoscopic data for GL_N. Combining this with (8.2.5) applied to each G in the above sum we arrive at

$$0 = \sum_{G \in \tilde{\mathcal{E}}_{\mathrm{sim}}(N)} \tilde{\iota}(N, G) \mathrm{tr}\, R^G_{\mathrm{disc},\psi}(f^G).$$

Since the constants $\tilde{\iota}(N, G)$ are positive real numbers, and $R^G_{\mathrm{disc},\psi}$ decomposes as a Hilbert direct sum of irreducible representations, we see that $R^G_{\mathrm{disc},\psi}(f^G) = 0$ for each G (the actual argument is a bit more involved, see [Art13, Proposition 3.5.1]). This is the first key conclusion, namely that ψ does not contribute to the discrete spectrum of G.

The second key conclusion is obtained by combining this vanishing with (8.2.5). This implies that ${}^0S^G_{\text{disc},\psi}(f^G) = 0$, which by definition is the validity of the stable multiplicity formula Conjecture 7.9.1 for ψ. □

In order to be able to apply Proposition 8.2.9 we must know that the global intertwining relation holds. It can be proved directly for all those $\psi \in \tilde{\Psi}(N)$ which do not lie in $\tilde{\Psi}_{\text{ell}}(N)$, nor in $\Psi_{\text{ell}}(G)$ for any simple twisted endoscopic group G, nor fall in one of the following two exceptional families (cf. [Art13, (4.5.11), (4.5.12)]).

$$\begin{cases} \psi = 2\psi_1 \boxplus \psi_2 \boxplus \cdots \boxplus \psi_r \\ S_\psi = \mathrm{Sp}(2,\mathbb{C}) \times (\mathbb{Z}/2\mathbb{Z})^{r'-1} \end{cases} \tag{8.2.6}$$

$$\begin{cases} \psi = 3\psi_1 \boxplus \psi_2 \boxplus \cdots \boxplus \psi_r \\ S_\psi = O(3,\mathbb{C}) \times (\mathbb{Z}/2\mathbb{Z})^{r'}, \end{cases} \tag{8.2.7}$$

where $r' = r$ if the character ξ^+_ψ defined just below Equation (8.1.7) is trivial, and $r' = r - 1$ otherwise.

Proposition 8.2.10. *Let $\psi \in \tilde{\Psi}(N)$. Assume that ψ does not lie in $\tilde{\Psi}_{ell}(N)$, or in $\Psi_{ell}(G)$ for any simple twisted endoscopic group, or in one of the two exceptional families* (8.2.6), (8.2.7)*. Then the global intertwining relation holds.*

Proof. This is the first half of the proof of [Art13, Proposition 4.5.1], which is a direct argument that involves the explicit structure of the centralizer S_ψ, which allow one to reduce all these cases to Levi subgroups and appeal to the induction hypothesis. □

Corollary 8.2.11. *Let $\psi \in \tilde{\Psi}(N)$. Assume that ψ does not lie in $\tilde{\Psi}_{ell}(N)$ or $\Psi_{ell}(G)$ for any simple twisted endoscopic group G, or in one of the two exceptional families* (8.2.6),(8.2.7)*. Then the global Theorem 8.1.4 and Conjecture 7.9.1 hold.*

8.3 On the local theorems

In the previous subsection we obtained the global theorems for non-elliptic non-exceptional parameters by comparison of trace formulas and the stabilization identities, taking advantage of induction on dimension of the group. The case of elliptic or exceptional parameters is more difficult. In that case, one obtains the local theorems first, again using the trace formula and its stabilization as a main tool, but supplementing it with a finer analysis and using additional tools, such as the orthogonality relations of elliptic tempered characters. Once the local theorems have been established, they can be used to supplement the analysis of the trace formula and to ultimately obtain the remaining cases of the global theorems.

In this subsection we will sketch Arthur's arguments that lead to the proof of the local theorems. We will first deal with generic parameters, and then turn to non-generic parameters.

The method of proving the local theorems can be described as "peeling an onion". All the relevant information is contained in the trace formula, but extracting it is challenging, because there is too much information mixed together. The approach is to carefully remove it, layer by layer, starting from the outside and moving inwards. The direction inwards is measured by the number and multiplicities of the simple constituents of the local parameter ϕ. One first discusses non-discrete parameters by proving the local intertwining relation, which in turn implies the local classification theorem for such parameters. Having established the local intertwining relation, it is used to remove from the trace formula the contributions of non-discrete parameters, thus giving access to the information contained in the discrete parameters. The next step is to consider discrete parameters that are not simple. Once the non-simple discrete parameters are handled, their information can also be removed from the trace formula, revealing the information contained in the simple parameters.

Since the parameters are local, but the trace formula is global, one needs a local-to-global passage. This is provided by the globalization results of §8.3.2. One must then control the various places that are away from the place of interest. The archimedean places play a key role in this process, because much of the desired local information is known for them.

8.3.1 Reduction of local results

The local theorems can be reduced by local means certain core cases. The core case for the local intertwining relation is that when the parameter ψ_M is discrete for M and the endoscopic element has regular image in the endoscopic Weyl group.

Proposition 8.3.1. *Let G be a classical group. Assume that the local intertwining relation holds for any proper Levi subgroup $M \subset G$, any $\psi \in \Psi_2(M)$, and any $s \in N_\psi(M)$ whose image in W_ψ is regular. Then it holds for any proper Levi subgroup $M \subset G$ and any $\psi \in \Psi(M)$, and any $s \in N_\psi(M)$.*

Proof. First, Arthur reduces from the case of general $\psi \in \Psi(M)$ to the case of $\psi \in \Psi_2(M)$. This is [Art13, Lemma 2.4.2], whose proof comes at the end of §2.4, and it's a simple matter of verifying that the normalized intertwining operators are compatible with induction in stages. Indeed, if $\psi \notin \Psi_2(M)$ there exists a Levi subgroup $M' \subset M$ with $\psi \in \Psi_2(M')$. The packet $\Pi_\psi(M)$ is then induced from the packet $\Pi_\psi(M')$, cf. proof of Proposition 8.3.2 below, and the assumed validity of the local intertwining relation for (ψ, M') and induction in stages completes this reduction.

We now assume $\psi \in \Psi_2(M)$. The reduction to elliptic s, i.e. those with regular image in W_ψ, is analogous. If the image is not regular, then s centralizes a torus in $S_\psi(G)^0$ of positive dimension. The centralizer of this torus is a Levi subgroup $\widehat{M_1} \subset \widehat{G}$ that contains $\widehat{M}$ (possibly up to conjugation by $S_\psi(G)^0$), because the assumption $\psi \in \Psi_2(M)$ implies that $A_{\widehat{M}}$ is a maximal torus of $S_\psi(G)^0$. Then one shows that both sides of the identity of Theorem 8.1.6 can be reduced from G to M_1, after which their identity follows from the inductive hypothesis on dimension of G. The reduction of the endoscopic linear form $f'_G(\psi, s)$ is a general argument that this form only depends on the image of s in $\mathcal{S}_\psi(G)$. It is given in the paragraphs below [Art13, (4.5.8)] and stems from the compatibility of endoscopic transfer with parabolic induction. The reduction of the linear form $f_G(\psi, s)$ relies on the statement that the intertwining operator $R_P^G(w)$ is given by $\mathcal{I}_P^G(R_{P\cap M_1}^{M_1}(w))$ whenever w lies in the Weyl group for M_1. □

The local intertwining relation in turn implies the local classification theorem for non-discrete parameters.

Proposition 8.3.2. *Let G be a classical group, $M \subset G$ a proper Levi subgroup, and $\psi \in \Psi_2(M)$. Assume that the local classification theorem 8.1.3 holds for M and the local intertwining relation 8.1.6 holds for ψ, M, and G. Then the local classification theorem holds for ψ.*

Proof. This is [Art13, Proposition 2.4.3]. The local classification theorem provides a packet $\Pi_\psi(M)$ together with a map $\Pi_\psi(M) \to \widehat{\mathcal{S}}_\psi(M)$, which we temporarily denote by $\pi^M \mapsto \rho_\pi^M$. One forms the representation

$$\Pi_\psi^M := \bigoplus_{\pi^M \in \Pi_\psi(M)} \pi^M \boxtimes \rho_\pi^M$$

of the group $M(F)\times\mathcal{S}_\psi(M)$. Recall the group $\mathfrak{N}_\psi(M)$ of components of $N_\psi(M)/Z(\widehat{G})^\Gamma$, where $N_\psi(M)$ is the normalizer of $A_{\widehat{M}}$ in $S_\psi(G)$. The group $\mathfrak{N}_\psi(M)$ acts on M as follows. An element $u \in \mathfrak{N}_\psi(M)$ maps to an element $w_u \in W^G(M)$, and the latter has a canonical Tits lift to an element of $N_G(M)$ coming from the natural pinning of the classical group G. We can then form $M(F) \rtimes \mathfrak{N}_\psi(M)$.

There is a natural extension of Π_ψ^M to the group $M(F) \rtimes \mathfrak{N}_\psi(M)$. Applying parabolic induction $\mathcal{I}_P^G$ to the $M(F)$-part of this representation, and the normalized intertwining operators to the $\mathfrak{N}_\psi(M)$-part, one obtains a representation Π_ψ^G of $G(F) \times \mathfrak{N}_\psi(M)$.

The assumption that ψ is discrete for M implies that $A_{\widehat{M}}$ is a maximal torus in $S_\psi(G)^\circ$, and the conjugacy of maximal tori implies that the natural map $\mathfrak{N}_\psi(M) \to \mathcal{S}_\psi(G)$ is surjective. The local intertwining relation implies that Π_ψ^G descends to a representation of $G(F) \times \mathcal{S}_\psi(G)$. Decomposing this representation one obtains

$$\Pi_\psi^G = \bigoplus \pi \boxtimes \rho_\pi$$

and one defines the packet $\Pi_\psi(G)$ to be the set of all π that occur in this decomposition. A-priori $\pi \leftrightarrow \rho_\pi$ is a correspondence, not necessarily a map. One can make it into a map formally by allowing repetitions in the set $\Pi_\psi(G)$. Assuming that ψ is generic, the Harish-Chandra basis theorem implies that the correspondence is in fact a bijection. The character identity that is part of Theorem 8.1.3 follows again from the local intertwining relation. □

Finally, the first assertion of the local classification theorem, namely the factorization of $\tilde{f}^N(\psi)$ through endoscopic transfer, can be reduced to the case of generic parameters.

Lemma 8.3.3. *Assume that Theorem 8.1.3(1) holds for all generic parameters ψ. Then it holds for all parameters ψ.*

Proof. This is [Art13, Lemma 2.2.2]. We will not reproduce the proof. □

8.3.2 Globalization results for generic parameters

Since the arguments are again global, based on the trace formula, the first step is to embed the local situation into a global one. The required "globalization results" are again a consequence of the trace formula, but here the trace formula is applied to a single group and the necessary information is obtained by calculating its terms. We present two of them, the first globalizing a representation, and the second globalizing a parameter. Both pertain to the case of tempered representations and generic parameters.

Proposition 8.3.4 ([Art13, Lemma 6.2.2, Corollary 6.2.3])**.** *Let G be a quasi-split classical group over a local field F and let π be an irreducible square-integrable representation of G. There exists a totally real number field $\dot{F}$, a connected reductive quasi-split $\dot{F}$-group $\dot{G}$, a place u of $\dot{F}$, and a discrete automorphic representation $\dot{\pi}$ of $\dot{G}$, such that*

1. *$\dot{F}_u = F$, $\dot{G}_u = G$, $\dot{\pi}_u = \pi$.*
2. *$\dot{\pi}_v$ is spherical at any finite place $v \neq u$.*
3. *$\dot{\pi}_v$ is a discrete series representation with infinitesimal character in general position at any real place $v \neq u$.*
4. *The unique parameter $\dot{\psi}$ such that $\dot{\pi} \in L^2_{disc,\dot{\psi}}(G(F)\backslash G(\mathbb{A}))$ according to Proposition 8.2.1, is generic.*
5. *For any $v \neq u$, the localization $\dot{\pi}_v$ lies in the L-packet $\Pi_{\dot{\psi}_v}(\dot{G}_v)$.*
6. *The parameter $\dot{\psi}_u$ lies in $\tilde{\Phi}_{ell}(N)$.*

Moreover, one can achieve that $\dot{F}$ has as many real places as is desired.

Proof. The constructions of $\dot{F}$ and $\dot{G}$ are elementary, cf. [Art13, Lemma 6.2.1], and beginning of proof of [Art13, Lemma 6.2.2]. To obtain the automorphic representation $\dot{\pi}$ one applies the trace formula to a special test function $\dot{f} = \prod_v f_v$. Here f_u is a pseudocoefficient for π. For example, if π were supercuspidal one could just take a matrix coefficient of π. In the general case such a matrix coefficient would not be compactly supported, but the theory of pseudo-coefficients provides a suitable substitute, cf. [BDK86, CD90]. At all finite places $v \neq u$ we take f_v to be the unit in the spherical Hecke algebra. At all real places $v \neq u$

we take f_v to be the sum of pseudo-coefficients for the discrete series in a fixed L-packet with sufficiently regular infinitesimal character.

The test function f_v for $v|\infty$ is thus what Arthur calls *stable cuspidal*, cf. [Art89a, §4]. We apply the trace formula with this test function. The fact that the test function is stable cuspidal at one infinite place, and cuspidal at more than one, simplifies the trace formula significantly [Art88, Theorem 7.1]: on the geometric side all terms vanish that are indexed by a conjugacy class that is either not semi-simple, or semi-simple but not elliptic at all real places. Using estimates of orbital integrals due to Harish-Chandra one deduces that the geometric side of the trace formula is non-zero, provided the infinitesimal character at one fixed real place $v \neq u$ is sufficiently large.

The spectral side of the trace formula is therefore also non-zero for this function. Due to the nature of the test function (see again the references listed above), this spectral side is simply $\mathrm{tr}\, R_{\mathrm{disc}}(\dot{f})$. This implies the existence of an automorphic representation $\dot{\pi}$ with $\mathrm{tr}\, \dot{\pi}(\dot{f}) \neq 0$. The precise form of the test function implies that $\dot{\pi}$ is of the desired form, although some additional arguments are required to substantiate the claim that $\dot{\pi}_u = \pi$. The problem comes from the fact that a pseudocoefficient distinguishes π among all tempered representations of $G(F)$, but a-priori we do not know that $\dot{\pi}_u$ is tempered. This is what needs to be shown. Since twisted endoscopic transfer preserves the notion of temperedness, it will be enough to know that the transfer of $\dot{\pi}_u$ to $\mathrm{GL}(N)$ is tempered, which we will discuss in a moment. Modulo temperedness of $\dot{\pi}_u$ we have thus proved (1), (2), and (3).

(4) Consider now the parameter $\dot{\psi}$. To show that it is generic it is enough to show that any of its localizations is generic. We will consider the localization at an archimedean place. To obtain the desired information, we will consider the three identities provided by the ψ-component of (8.2.2) and Proposition 8.2.2 applied once to $\dot{G}$ and once to twisted GL_N. The fact that $\dot{f}_v$ is a pseudocoefficient for a discrete series L-packet implies that the terms for $\dot{M} \neq \dot{G}$ in (8.2.2), and the terms for $G' \neq \dot{G}$ in Proposition 8.2.2 for $\dot{G}$, vanish. Choosing a test function $\dot{f}^N$ on twisted GL_N appropriately, the summands on the right hand side of Proposition 8.2.2 applied to twisted GL_N also vanish for all G' except $G' = \dot{G}$. This leads to

$$\iota(N, \dot{G})^{-1} I^N_{\mathrm{disc},\dot{\psi}}(\dot{f}^N) = S^{\dot{G}}_{\mathrm{disc},\dot{\psi}}(\dot{f}) = I^{\dot{G}}_{\mathrm{disc},\dot{\psi}}(\dot{f}) = \mathrm{tr}\, R^{\dot{G}}_{\mathrm{disc},\dot{\psi}}(\dot{f}). \qquad (8.3.1)$$

We know that the right-hand side is non-zero (it contains the representation $\dot{\pi}$), so the left-hand side is non-zero. The archimedean infinitesimal characters on both side correspond, and are in general position. This forces the localization $\dot{\psi}_v$ at archimedean places to be tempered, since non-tempered representations of real reductive groups do not have infinitesimal characters in general position. We have concluded that $\dot{\psi}_v$ is generic, hence $\dot{\psi}$ is generic, hence also $\dot{\psi}_u$ is generic. This completes the proof of (4).

(5) We use again (8.3.1), but we use a slightly different function $\dot{f}$, where we allow $\dot{f}_v$ for finite $v \neq u$ to vary through the entire spherical Hecke algebra,

while we fix $\dot{f}_u$ to be a pseudocoefficient for π. This identity then implies

$$\dot{f}^N(\dot{\psi}) = c(\dot{\psi}) \mathrm{tr}\, R^{\dot{G}}_{\mathrm{disc},\dot{\psi}}(\dot{f}), \tag{8.3.2}$$

for a non-zero constant $c(\dot{\psi})$. For each place v this identity implies the analogous local identity up to some unspecified non-zero constant. Since for every $v \neq u$ the local packet is characterized by the fact that $\sum_{\pi_v \in \Pi_{\dot{\psi}_v}} \dot{f}_v(\pi_v) = c_v(\dot{\psi}) \dot{f}_v^N(\dot{\psi}_v)$, the claim follows.

(6) We note that the right hand side of (8.3.2) is non-zero for a test function $\dot{f}$ that is cuspidal at u, namely the pseudocoefficient of π. Therefore the left hand side is non-zero for a test function $\dot{f}^N$ that is cuspidal at u. This implies that $\dot{\psi}_u$ is elliptic, because the local linear forms associated to non-elliptic parameters vanish on cuspidal functions.

Finally, we observe that the ellipticity of $\dot{\psi}_u$ implies that $\dot{\psi}_u$ is tempered, being the direct sum of mutually inequivalent self-dual, hence unitary, representations of $\mathcal{L}_F$. Thus the transfer of $\dot{\pi}_u$ to GL_N is tempered, which completes the proof of (1). □

Proposition 8.3.5 ([Art13, Proposition 6.3.1]). *Let G be a quasi-split classical group, $M \subset G$ a Levi subgroup, $\phi_M \in \Phi_2(M)$. Let $\phi = \ell_1\phi_1 \oplus \cdots \oplus \ell_r\phi_r$ be the decomposition of the parameter of G obtained from ϕ_M. Assume that $\phi \notin \Phi_{sim}(G)$. There exist*

1. *a totally real number field $\dot{F}$, with sufficiently many real places*
2. *a quasi-split classical $\dot{F}$-group $\dot{G}$ and a Levi subgroup $\dot{M}$*
3. *a global generic parameter $\dot{\phi}_M$ for $\dot{M}$*
4. *a place u of $\dot{F}$*

such that

1. *$\dot{F}_u = F$, $\dot{G}_u = G$, $\dot{M}_u = M$, $\dot{\phi}_{M,u} = \phi_M$.*
2. *The canonical maps $\mathcal{S}_{\dot{\phi}_M} \to \mathcal{S}_{\phi_M}$ and $\mathcal{S}_{\dot{\phi}} \to \mathcal{S}_{\phi}$ are isomorphisms, where $\dot{\phi}$ and ϕ are the parameters of $\dot{G}$ and G obtained from $\dot{\phi}_M$ and ϕ_M.*
3. *For any finite place $v \neq u$, $\dot{\phi}_v = \ell_1\dot{\phi}_{1,v} \oplus \cdots \oplus \ell_r\dot{\phi}_{r,v}$ is a sum of characters, and $\dot{\phi}_{1,v}, \ldots, \dot{\phi}_{r,v}$ contains at at most one of which is ramified.*
4. *For a real place $v \neq u$, $\dot{\phi}_{i,v}$ is a discrete series parameter of $\dot{G}_{i,v}$ in relative general position.*
5. *If V is the set of real places $v \neq u$, the canonical mapping $\Pi_{\dot{\phi}_M,V} \to \mathcal{S}^*_{\dot{\phi}_M} = \mathcal{S}^*_{\phi_M}$ is surjective.*
6. *If $M = G$ there exists a real place $v \neq u$ such that if $\dot{\phi}_v$ lies in $\Phi(G^*_v)$ for some $G^*_v \in \dot{\tilde{\mathcal{E}}}_{sim,v}(N)$, then $\widehat{G}^*_v = \widehat{G}$.*

7. *If $M \neq G$ there is a real place $v \neq u$ such that the kernel of the composed map $\mathcal{S}_{\dot{\phi}} \to \mathcal{S}_{\dot{\phi}_v} \to R_{\dot{\phi}_v}$ contains no element whose image in the global R-group $R_{\dot{\phi}}$ is regular.*

Proof. We give a brief outline. The assumption that ϕ is not simple allows us to use the induction hypothesis to globalize each of the pieces ϕ_i separately. That is, we let G_i be the unique classical group for which ϕ_i is a simple parameter, and consider a member π_i of the L-packet $\Pi_{\phi_i}(G_i)$. Proposition 8.3.4 provides an automorphic representation $\dot{\pi}_i$ of a global version $\dot{G}_i$ of G_i. This representation corresponds to a global parameter $\dot{\phi}_i$. We set $\dot{\phi} = \ell_1\dot{\phi}_1 \oplus \cdots \oplus \ell_r\dot{\phi}_r$. The construction of $\dot{M}$ follows analogously.

The condition that for all finite places $v \neq u$ the parameter $\dot{\phi}_v$ is almost unramified follows from the fact that in Proposition 8.3.4 the representation $\dot{\pi}_v$ is spherical. One just has to arrange that the globalized groups $\dot{G}_i$ be as unramified as possible. This is only an issue with non-split even orthogonal groups, where one needs to choose the corresponding idele character more carefully.

The condition that the archimedean components of $\dot{\phi}$ have infinitesimal characters in general position is already built into Proposition 8.3.4 for the individual $\dot{\phi}_i$. One must just take care to impose this regularity condition step by step to ensure that it remains valid for the full parameter $\dot{\phi}$.

For the condition that the canonical mapping $\Pi_{\dot{\phi}_M,V} \to \mathcal{S}^*_{\dot{\phi}_M} = \mathcal{S}^*_{\phi_M}$ is surjective, where V is the set of real places $v \neq u$, we use the fact from the real local theory that for each such v the image of the canonical mapping $\Pi_{\dot{\phi}_M,v} \to \mathcal{S}^*_{\dot{\phi}_M,v}$ is generating (cf. [Art13, Lemma 6.1.2]). If we knew that the canonical mapping $\prod_{v\in V} \mathcal{S}^*_{\dot{\phi}_M,v} \to \mathcal{S}^*_{\dot{\phi}_M}$ is surjective, we would see that the image of $\Pi_{\dot{\phi}_M,V}$ generates $\mathcal{S}^*_{\dot{\phi}_M}$. Increasing the number of real places V if necessary, by replacing $\dot{F}$ with a larger global field, we can convert this generation statement into surjectivity. So it remains to show that the dual natural map $\mathcal{S}_{\dot{\phi}_M} \to \prod_{v\in V} \mathcal{S}_{\dot{\phi}_M,v}$ can be arranged to be injective. This is equivalent to the claim that $S_{\dot{\phi}_M} \cap \bigcap_{v\in V} S^0_{\dot{\phi}_v} = \{1\}$. But the latter can be checked directly using the explicit description of the centralizer groups.

The remaining conditions are also checked in a direct way using the explicit description of the centralizer groups. □

8.3.3 The local intertwining relation for generic parameters

With the two globalization results established, we turn to the proofs of the local theorems in the case of generic parameters. The first one we will attack is the local intertwining relation. For this, we begin with a quasi-split classical F-group G, a proper Levi subgroup $M \subset G$, and a generic parameter $\phi_M \in \Phi(M)$. Due to the reduction afforded by Proposition 8.3.1 we may assume that ϕ_M is discrete, i.e. $\phi_M \in \Phi_2(M)$. We let $\phi \in \Phi(G)$ denote the image of ϕ_M under the natural map $\Phi(M) \to \Phi(G)$. We recall that the global intertwining relation, and hence also the local intertwining relation, has been established in all but the following three cases: $\phi \in \Phi_{\mathrm{ell}}(G)$, ϕ is of type (8.2.6), ϕ is of type (8.2.7). We recall their explicit structure here:

1. $$\begin{cases} \psi = 2\psi_1 \boxplus \cdots \boxplus 2\psi_q \boxplus \psi_{q+1} \boxplus \cdots \boxplus \psi_r, \\ S_\psi \subset O(2,\mathbb{C})^q \times O(1,\mathbb{C})^{r-q} \end{cases}$$

2. $$\begin{cases} \psi = 2\psi_1 \boxplus \psi_2 \boxplus \cdots \boxplus \psi_r \\ S_\psi = \mathrm{Sp}(2,\mathbb{C}) \times (\mathbb{Z}/2\mathbb{Z})^{r'-1} \end{cases}$$

3. $$\begin{cases} \psi = 3\psi_1 \boxplus \psi_2 \boxplus \cdots \boxplus \psi_r \\ S_\psi = O(3,\mathbb{C}) \times (\mathbb{Z}/2\mathbb{Z})^{r'} \end{cases}$$

Recall further that ϕ does not lie in $\Phi_2(G)$, because it comes from a parameter ϕ_M of a *proper* Levi subgroup. Therefore, the number q in 1. above is not equal to zero.

The essential global input that the trace formula provides us is the following weaker version of the global intertwining relation.

Proposition 8.3.6. *Let $(\dot{G}, \dot{\phi}, \dot{M}, \dot{\phi}_M)$ be a globalization of (G, ϕ, M, ϕ_M) as in Proposition 8.3.5. Then for every test function $\dot{f}$ on $\dot{G}(\mathbb{A})$ we have*

$$\sum_{x \in \mathcal{S}_{\phi,\mathit{ell}}} \big(\dot{f}_{\dot{G}}(\dot{\phi}, \dot{x}) - \dot{f}'_{\dot{G}}(\dot{\phi}, \dot{x})\big) = 0.$$

Moreover,

$$L^2_{\mathit{disc},\dot{\phi}}(\dot{G}(\mathbb{Q})\backslash \dot{G}(\mathbb{A})) = 0.$$

Proof. This argument spans [Art13, Proposition 3.5.1, Corollaries 3.5.3, 4.5.2, Lemmas 5.2.1, 5.2.2, 5.4.3, 5.4.4].

To ease notation in the proof, we will drop the dots and keep the understanding that all objects are global, and the local objects are found at the place u.

The case where N is even and $\eta_G = 1$ is more complicated, so we focus on the remaining cases. The first claim is that then

$$\sum_{x \in \mathcal{S}_{\phi,\mathrm{ell}}} \Big(f_G(\phi, x) - f'_G(\phi, x)\Big) = C \cdot \mathrm{tr}(R_{\mathrm{disc},\phi})(f) \tag{8.3.3}$$

for some positive constant C, cf. [Art13, Lemma 5.2.1]. This is consistent with the statement being proved, because we expect $\mathrm{tr}(R_{\mathrm{disc},\phi})(f)$ to equal zero, since $\phi \notin \Phi_2(G)$. In particular, the discrepancy ${}^0r_{\mathrm{disc},\phi}$ between $\mathrm{tr}(R_{\mathrm{disc},\phi})(f)$ and its expected value equals $\mathrm{tr}(R_{\mathrm{disc},\phi})(f)$ itself. We then have

$$\begin{aligned} \mathrm{tr}(R^G_{\mathrm{disc},\phi})(f) - {}^0S^G_{\mathrm{disc},\phi}(f) &= {}^0r^G_{\mathrm{disc},\phi}(f) - {}^0s^G_{\mathrm{disc},\phi}(f) \\ &= (I^G_{\mathrm{disc},\phi}(f) - {}^0s^G_{\mathrm{disc},\phi}(f)) - (I^G_{\mathrm{disc},\phi}(f) - {}^0r^G_{\mathrm{disc},\phi}(f)) \\ \text{(apply Cor. 8.2.4, 8.2.6)} &= C_\phi \sum_{x \in \mathcal{S}_\phi} i_\phi(x)\Big(f'_G(\phi, x) - f_G(\phi, x)\Big), \end{aligned}$$

with C_ϕ positive. The coefficient $i_\phi(x)$ vanishes unless x is elliptic, and is positive otherwise. To finish the proof of (8.3.3) it remains to be shown that ${}^0S^G_{\mathrm{disc},\phi}(f) = 0$. The argument, given in [Art13, §5.1], is in fact rather similar, but now applied to the twisted group (GL_N, θ) rather than to the group G. In that case the local and global intertwining relations are already known, so the final term in the above chain of equalities vanishes, yielding ${}^0\tilde{r}^N_{\mathrm{disc},\phi}(\tilde{f}) = {}^0\tilde{s}^N_{\mathrm{disc},\phi}(\tilde{f})$. At the same time ${}^0\tilde{r}^N_{\mathrm{disc},\phi}(\tilde{f}) = 0$ because the expected formula for $\mathrm{tr}(R^N_{\mathrm{disc},\phi})(\tilde{f})$ is known, and this leads to

$$\sum_{G^* \in \tilde{\mathcal{E}}_{\mathrm{sim}}(N)} \tilde{\iota}(N, G^*)^0 S^{G^*}_{\mathrm{disc},\phi}(\tilde{f}^{G^*}) = 0.$$

According to Remark 8.3.11 the term ${}^0S^{G^*}_{\mathrm{disc},\phi}(\tilde{f}^{G^*})$ vanishes unless $\eta_{G^*} = \eta_\phi$, and since $\eta_\phi = \eta_G$ the sum extends only over those G^* with $\eta_{G^*} = \eta_G$. This sum contains G, and it contains a second group when N is even and $\eta_G = 1$. This is the case that we are suppressing from the discussion in order to simplify the exposition. With this, our review of the proof of (8.3.3) is complete.

The next step is to leverage the fact that a weak version of the local intertwining relation is known at Archimedean places, following the proof of [Art13, Lemma 5.4.3]. Let v be an Archimedean place as in part 7. of Proposition 8.3.5. We may take a decomposable test function $f = f_v f^v$. This decomposes both linear forms $f_G(\phi, x)$ and $f'_G(\phi, x)$ into a product of the term at v and the term away from v.

Consider the term at v. Despite the mature state of the theory over $\mathbb{R}$, the local intertwining relation is not known, and will be discussed in Lemma 8.3.9 below. However, a weaker version of it can be extracted from the known theory, namely

$$f'_{v,G_v}(\phi_v, x_v) = \sum_{\pi_{M_v} \in \Pi_{\phi_v}(M_v)} \epsilon_v(w_v)\langle n, \pi_{M_v}\rangle \mathrm{tr}\big(R_{P_v}(\phi_v, \pi_{M_v}, w_v)\mathcal{I}^{G_v}_{P_v}(\pi_{M_v}, f_v)\big), \tag{8.3.4}$$

for a sign character $\epsilon_v \colon W_{\phi_v} \to \{\pm 1\}$ which descends to the quotient $R_{\phi_v} = W_{\phi_v}/W^0_{\phi_v}$. In other words, the local intertwining relation (Theorem 8.1.6) holds up to the possible error given by ϵ_v. Recall here that $w_v \in W_{\phi_v}$ is the image of x_v. Therefore one sees that $\sum_{x \in \mathcal{S}_{\phi,\mathrm{ell}}} \Big(f_G(\phi, x) - f'_G(\phi, x) \Big)$ equals

$$\sum_{x \in \mathcal{S}_{\phi,\mathrm{ell}}} \sum_{\pi_{M_v} \in \Pi_{\phi_v}(M_v)} \Big(f^v_G(\phi, x) - \epsilon(x_v) f'^v_G(\phi, x) \Big) \langle n, \pi_{M_v}\rangle f_{v,G}(M_v, \pi_{M_v}, x_v), \tag{8.3.5}$$

where

$$f_{v,G}(M_v, \pi_{M_v}, x_v) = \mathrm{tr}\big(R_{P_v}(\phi_v, \pi_{M_v}, w_v)\mathcal{I}^{G_v}_{P_v}(\pi_{M_v}, f_v)\big).$$

We can rewrite the above double sum as

$$\sum_{\tau_v} d(\tau_v, f^v) f_{v,G}(\tau_v),$$

with

$$d(\tau_v, f^v) = \sum_{x, \pi_{M_v}} \Big(f^v_G(\phi, x) - \epsilon(x_v) f'^v_G(\phi, x) \Big) \langle n, \pi_{M_v} \rangle.$$

Here the sum over τ_v runs over the set of conjugacy classes of triples $(M'_v, \pi_{M'_v}, r_v)$, with $M'_v \subset G_v$ a Levi subgroup, $\pi_{M'_v}$ an essentially discrete series representation of $M'_v(F_v)$, r_v an element of the Harish-Chandra R-group $R(\pi_{M'_v})$, and $f_{v,G}(\tau_v) = f_{v,G}(M'_v, \pi_{M'_v}, r_v)$. The double sum in the definition of $d(\tau_v, f^v)$ runs over the same set as that in (8.3.5), but with the condition that (M_v, π_{M_v}, x_v) is conjugate to τ_v. Thus, in the end, the τ_v over which the sum runs are strongly constrained, with their Levi component M'_v being pinned down to be M_v.

Combining this with (8.3.3) we obtain

$$C \cdot \sum_{\pi} m(\pi) f_G(\pi) = \sum_{\tau_v} d(\tau_v, f^v) f_{v,G}(\tau_v),$$

where the sum on the left runs over the discrete automorphic representations in $L^2_{\text{disc},\phi}(G)$ and $m(\pi)$ are their multiplicities. The conclusion from this is that both $m(\pi) = 0$ for all π, i.e. that $L^2_{\text{disc},\phi}(G) = 0$, and $d(\tau_v, f^v) = 0$ for all τ_v, implying in particular the desired weak form of the global intertwining relation. This conclusion is derived in [Art13, Proposition 3.5.1, Lemma 3.5.2, Corollary 3.5.3].

Roughly speaking, the key point is the following bit of representation theory over the local field F_v. The vector space of characters of $G(F_v)$ has three different bases: the set of characters of irreducible representations, the set of characters of standard representations (representations that are parabolic inductions of essentially discrete series representations of Levi subgroups), and the set of distributions $f \mapsto f_{v,G}(\tau_v)$ for triples $\tau_v = (M'_v, \pi_{M'_v}, r_v)$ as above. The matrix that translates between irreducible characters and triples τ_v has the property that for every irreducible representation π_v there is a triple τ_v with $r_v = 1$ such that the corresponding coefficient is non-zero. On the other hand, the complex number $d(\tau_v, f^v)$ has the property of being zero for any such triple, due to the fact that the sum over x in the definition of $d(\tau_v, f^v)$ runs only over the elliptic $x \in \mathcal{S}_\phi$, which are those whose image in the global R-group R_ϕ, and part 7. of Proposition 8.3.5 ensures that the image of x in the local R-group R_{ϕ_v} is non-trivial. This forces all the numbers $m(\pi)$ to be zero (this is a more refined version of the idea that if a sum of non-negative numbers equals zero, then all summands equal zero, and is given in [Art13, Proposition 3.5.1]). In turn, this forces all the numbers $d(\tau_v, f^v)$ to be zero, using linear independence of the distributions $f_{v,G}(\tau_v)$. □

In order to extract the local intertwining relation from this global information we need to know that it holds at all places away from the place of interest. Let us assume this for now and see how this completes the proof of the local intertwining relation at u.

Proposition 8.3.7. *Assume that the local intertwining relation holds for $(\dot{G}, \dot{\phi}, \dot{M}, \dot{\phi}_M)$ at all places $v \neq u$. Then it also holds at $v = u$, i.e. it holds for (G, ϕ, M, ϕ_M).*

Proof. Consider a factorizable test function $\dot{f} = \prod_v f_v$. Then $\dot{f}_{\dot{G}}(\dot{\phi}, \dot{x}) = \prod_v f_{v,\dot{G}_v}(\dot{\phi}_v, \dot{x})$, and the analogous statement holds for $\dot{f}'_{\dot{G}}(\dot{\phi}, \dot{x})$. Using the assumption that the local intertwining relation holds at all places $v \neq u$ the identity of Proposition 8.3.6 becomes

$$\sum_{x \in \mathcal{S}_{\phi,\mathrm{ell}}} \Big(\prod_{v \neq u} f_{v,\dot{G}_v}(\dot{\phi}_v, \dot{x}) \Big) \cdot \big(\dot{f}_{u,\dot{G}_u}(\dot{\phi}_u, \dot{x}) - \dot{f}'_{u,\dot{G}_u}(\dot{\phi}_u, \dot{x}) \big) = 0. \tag{8.3.6}$$

The local classification theorem is known at all archimedean places by the work of Shelstad [She08a], [She10], [She08b]. Therefore for each such place v we have

$$f_{v,\dot{G}_v}(\dot{\phi}_v, \dot{x}) = \sum_{\pi_v \in \Pi_{\dot{\phi}_v}(\dot{G}_v)} \langle \pi_v, \dot{x} \rangle f_{v,\dot{G}_v}(\pi_v).$$

The following cute lemma, whose short and amusing proof we leave to the reader, implies that the set of linear forms $\{ f_{V,\dot{G}_V}(\dot{\phi}_V, \dot{x}) \,|\, \dot{x} \in \mathcal{S}_{\phi,\mathrm{ell}} \}$ is linearly independent, where we recall that V is the set of real places $v \neq u$.

Lemma 8.3.8. *Let W be a complex vector space, $D \subset W$ a finite linearly independent subset, A a finite abelian group, and $p\colon D \to A^*$ a surjective map. Then the set of vectors $\{ \sum_{w \in D} p(w)(x) \cdot w \,|\, x \in A \}$ is linearly independent.*

This lemma is applied with W the vector space of distributions on $G(\mathbb{A})$, D the set of distributions $f_{v,\dot{G}_v}(\pi_v)$ for all real places $v \neq u$ and all $\pi_v \in \Pi_{\dot{\phi}_v}(\dot{G}_v)$, and $A = \mathcal{S}_{\dot{\phi}_M}$. The surjectivity statement is part (5) of Proposition 8.3.5. But there is a slight discrepancy here: the sum we are considering in our problem is over $\mathcal{S}_{\phi,\mathrm{ell}} = \mathcal{S}_{\dot{\phi},\mathrm{ell}}$, and not over $\mathcal{S}_{\dot{\phi}_M}$. Arthur gives a brief indication how to handle this discrepancy, namely using "the disjointness of constituents of tempered representations". Another way would be to check explicitly that for the types of parameters being considered (elliptic non-discrete, or of exceptional type (8.2.6) or (8.2.7)), we either have an equality $S_\phi = S_{\dot{\phi}_M}$, or $S_{\phi,\mathrm{ell}}$ is a torsor under $S_{\dot{\phi}_M}$.

The linear independence of $\{ f_{V,\dot{G}_V}(\dot{\phi}_V, \dot{x}) \,|\, \dot{x} \in \mathcal{S}_{\phi,\mathrm{ell}} \}$ converts (8.3.6) into

$$\Big(\prod_{v \neq u} f_{v,\dot{G}_v}(\dot{\phi}_v, \dot{x}) \Big) \cdot \big(\dot{f}_{u,\dot{G}_u}(\dot{\phi}_u, \dot{x}) - \dot{f}'_{u,\dot{G}_u}(\dot{\phi}_u, \dot{x}) \big) = 0 \qquad \forall x \in \mathcal{S}_{\phi,\mathrm{ell}}.$$

Since for each $v \neq u$ the linear form $f_{v,\dot{G}_v}(\dot{\phi}_v, \dot{x})$ is non-zero, we infer that the local intertwining relation holds at the place u. □

In order to apply Proposition 8.3.7 we need to know the local intertwining relation at all places $v \neq u$.

Lemma 8.3.9. *Assume that $F = \mathbb{R}$ and that ϕ_M is a discrete series parameter in general position. Then the local intertwining relation holds for (G, ϕ, M, ϕ_M).*

Proof. Despite the mature state of the theory over $\mathbb{R}$, this fundamental theorem is not known. It would be nice to have an intrinsic proof, that even holds without assuming general position. As things stand, Arthur is forced to give an adhoc argument, based on the available local theory over $\mathbb{R}$ and supplemented by a global argument that works in the setting of classical groups established thus far. We will not repeat this argument, and refer the reader to [Art13, Lemmas 6.4.2, 6.4.3]. $\square$

Lemma 8.3.10 (incomplete, based on unpublished reference [A27])**.** *In the setting of Proposition 8.3.6, the local intertwining relation holds at all finite places $v \neq u$.*

Proof. Consider such a finite place. According to Part (3) of Proposition 8.3.5, $\dot{\phi}_v$ is a sum of characters. If one of them is not self-dual, this forces the element $\dot{x}_v$ to have non-regular image in the Weyl group, and the claim follows by Proposition 8.3.1. If all of them are self-dual, there are at most three distinct characters (the trivial and sign characters are the only unramified self-dual characters, and Part (3) of of Proposition 8.3.5 ensures that there is at most one ramified character). Examining the three types of parameters we are considering, listed as 1., 2., 3., above, we conclude that the Levi subgroup M is either abelian or its derived subgroup is SL_2. At this point, the proof is deferred to an unpublished paper [A27], cf. proof of [Art13, Lemma 6.4.1]. $\square$

With this, the proof of the local intertwining relation for generic (i.e. tempered parameters) ϕ is complete, modulo the missing reference [A27]. From this, Arthur derives [Art13, Corollary 6.4.5], which states that the normalized local self-intertwining operator $R_P(w, \tilde{\pi}_M, \phi)$ acts by the scalar 1 when $w \in W^0_\phi$. We caution the reader that this is a misprint. The correct statement, which follows from the proof that is provided in loc. cit., should state that the product $\langle \tilde{u}, \tilde{\pi}_M \rangle R_P(w_u, \tilde{\pi}_M, \phi)$ equals 1, where the pairing $\langle \tilde{u}, \tilde{\pi}_M \rangle$ is the one used in [Art13, (2.4.5)]. One could define $R_P(u, \pi_M, \phi) = \langle \tilde{u}, \tilde{\pi}_M \rangle R_P(w_u, \tilde{\pi}_M, \phi)$, in which case the statement of [Art13, Corollary 6.4.5] would become correct (with w there now replaced by u). This is the form in which the result is stated in [KMSW14, Theorem 2.6.2(1)]. The operator $R_P(u, \pi_M, \phi)$ has the advantage of not depending on an arbitrary choice of extension $\tilde{\pi}_M$ of π_M.

8.3.4 The local classification theorem for generic parameters

We next turn to the local classification theorem for generic (i.e. tempered) parameters. This is the material in [Art13, §6.6,§6.7]. When the local field F is archimedean, the desired results have been established by Shelstad, cf. [She10], [She08b]. We will therefore assume that F is a non-archimedean local field. The case of $\phi \in \Phi(G) \smallsetminus \Phi_2(G)$ is essentially resolved by Proposition 8.3.2, with the exception of the identification of the representation-theoretic and endoscopic R-groups. For this argument, which involves notation that we do not want to introduce here, we refer the reader to the discussion in the beginning of [Art13, §6.6].

We focus now on the main case, which is $\phi \in \Phi_2(G)$. Decompose

$$\phi = \phi_1 \oplus \cdots \oplus \phi_r. \tag{8.3.7}$$

We will first deal with the case $r > 1$, i.e. the case of a "non-simple", or "composite" parameter, which is the material in [Art13, §6.6]. We will denote the subset of $\Phi_2(G)$ consisting of composite ϕ by $\Phi_2^{\text{sim}}(G)$.

Apply Proposition 8.3.5 (here $M = G$) to obtain a number field $\dot{F}$, a $\dot{F}$-group $\dot{G}$, and a global generic parameter $\dot{\phi}$, so that $(\dot{F}, \dot{G}, \dot{\phi})$ specializes to (F, G, ϕ) at the place u.

Remark 8.3.11. An elliptic Arthur parameter $\psi \in \Psi_{\text{ell}}(N)$ has a quadratic character $\eta_\psi : \Gamma \to \{\pm 1\}$ associated to it. An elliptic endoscopic group $G \in \mathcal{E}_{\text{ell}}(N)$ also has a quadratic character $\eta_G : \Gamma \to \{\pm 1\}$ associated to it.

We first review η_G. An elliptic endoscopic datum consists of a group of the form $G = G_1 \times G_2$ where $\widehat{G}_1 = \text{Sp}(N_-, \mathbb{C})$ and $G_2 = \text{SO}(N_+, \mathbb{C})$, as well as an embedding of ${}^L G = \widehat{G}_1 \times \widehat{G}_2 \rtimes \Gamma_{E/F}$ into GL_N. Here E/F is either trivial or quadratic, and splits G_2. The image of $\widehat{G}_1$ is isomorphic to $\text{Sp}(N_-, \mathbb{C})$. The image of $\widehat{G}_2 \rtimes \Gamma_{E/F}$ lands in $\text{O}(N_+, \mathbb{C})$ and contains $\text{SO}(N_+, \mathbb{C})$. The character η_G is obtain by restricting this embedding to $\Gamma_{E/F}$ and composing with the projection to $\text{O}(N_+, \mathbb{C})/\text{SO}(N_+, \mathbb{C})$.

We now review η_ψ. Write $\psi = \psi_1 \oplus \cdots \oplus \psi_n$ into a direct sum of mutually inequivalent simple parameters. Each ψ_i has a sign $\epsilon_i \in \{\pm 1\}$ associated to it. The sign is $\epsilon_i = -1$ if and only if ψ_i factors through $\text{Sp}(N_i, \mathbb{C})$, and $\epsilon = +1$ if ψ_i factors through $\text{O}(N_i, \mathbb{C})$. We arrange the decomposition so that $\epsilon_1 = \cdots = \epsilon_k = -1$ and $\epsilon_{k+1} = \cdots = \epsilon_n = +1$ and set $\psi_- = \psi_1 \oplus \cdots \oplus \psi_k$ and $\psi_+ = \psi_{k+1} \oplus \cdots \oplus \psi_n$. Then ψ_- factors through $Sp(N_-, \mathbb{C})$ and ψ_+ factors through $O(N_+, \mathbb{C})$, with $N = N_- + N_+$. The composition of ψ_+ with $O(N_+, \mathbb{C}) \to O(N_+, \mathbb{C})/SO(N_+, \mathbb{C}) = \mathbb{Z}/2\mathbb{Z}$ extends to a quadratic character of Γ_F, which is η_ψ.

Note that if the datum G is simple then it is uniquely determined by η_G unless N is even and $\eta_G = 1$, in which case there are two possible such data, namely $G = \text{SO}(N)$ and $G = \text{SO}(N+1)$, both split. If $\psi \in \Psi_2(G)$, then $\eta_\psi = \eta_G$.

Recall from (7.8.2) and (7.8.3) the total transfer map between test functions and its dual on the level of distributions. We will now use the global analogs of these maps, obtained by taking the products of the local maps over all places. The next two results are valid even when $r = 1$, i.e. when ϕ is simple, and will be used later for that case as well.

We consider the linear form $\dot{f}_N \mapsto \dot{f}_N(\dot{\phi})$ that evaluates the character of the automprphic representation $\dot{\phi}$ of $\text{GL}_N(\mathbb{A})$ on a test function $\dot{f}_N$.

Proposition 8.3.12. *1. The linear form $\dot{f}_N(\dot{\phi})$ is the image of a unique stable linear form $\dot{f}^{\dot{G}}(\dot{\phi})$, interpreted as an element of $\bigoplus_{\dot{G}^*} \mathcal{SI}(\dot{G}^*)^*$ with only non-zero coordinate for $\dot{G}^* = \dot{G}$.*

2. *The stable multiplicity formula holds for $\dot\phi$. That is,*

$$S^{\dot G}_{disc,\dot\phi}(\dot f) = |\mathcal{S}_{\dot\phi}|^{-1}\cdot \dot f^G(\dot\phi).$$

Proof. In this proof we will use the fact that a simple twisted endoscopic datum G is determined by η_G and $\widehat{G}$ unless N is even and $\eta_G = 1$, in which case there are the two possibilities of split SO$(N+1)$ and split SO(N), with dual groups Sp(N) and SO(N), respectively. The latter disambiguation is very technical, so we will not discuss it. To lighten notation we set $\psi = \dot\phi$ and drop the dot on G.

(1) This is [Art13, Lemma 5.4.2]. We use the stabilization identity

$$I^N_{\text{disc},\psi}(f) = \sum_{G^*}\tilde\iota(N,G^*)S^{G^*}_{\text{disc},\psi}(f^{G^*})$$

the sum being over elliptic endoscopic groups. The spectral expansion (7.4.1) of $I^N_{\text{disc},\psi}(f)$ shows that it is a non-zero constant multiple of the linear form $f_N(\psi)$, and our goal is to show that in the above sum $S^{G^*}_{\text{disc},\psi}(f^{G^*}) = 0$ for all $G^* \neq G$.

If ψ is simple, then the above term vanishes automatically unless G^* is simple. It also vanishes if $\eta_{G^*} \neq \eta_\psi$. The tuple $(\eta_G, \widehat{G})$ uniquely pins down G. To pin down $\widehat{G}$ we use part (3) of Proposition 8.3.4, which implies that the local parameter ψ_v factors through the L-group of G_v only; since $\widehat{G} = \widehat{G}_v$ we are done.

Assume now that ψ is not simple. We use Proposition 8.3.5 to pin down $\widehat{G}$, specifically part (6). Let V be the set of real places not equal to u, and let $v \in V$ be a places satisfying part (6) of that proposition. The identification of the image of the total twisted transfer map and the trace Paley-Wiener theorem allow us to choose the local function f_V so that $f_V(\psi_V) \neq 0$, but the image of f_v in $\mathcal{SI}(G^*_v)$ vanishes for all simple G^* with $\widehat{G}^*_v \neq \widehat{G}_v$. Letting the local parts of f vary at all other places one deduces that $S^{G^*}_{\text{disc},\psi}(f^{G^*}) = 0$ for all $G^* \neq G$, proving $I^N_{\text{disc},\psi}(f) = \tilde\iota(N,G)S^G_{\text{disc},\psi}(f^G)$.

(2) This is [Art13, Corollary 5.1.3, Lemma 5.1.4, Lemma 5.4.5]. In the previous point we have already shown that $S^{G^*}_{\text{disc},\psi}(f^{G^*}) = 0$ for all $G^* \neq G$. Since $\psi \notin \Psi(G^*)$ we have $S^{G^*}_{\text{disc},\psi}(f^{G^*}) = {}^0S^{G^*}_{\text{disc},\psi}(f^{G^*})$. The twisted stabilization identity (8.2.1) implies that

$$\sum_{G^*}\tilde\iota(N,G^*)^0S^{G^*}_{\text{disc},\psi}(f^{G^*})$$

equals the difference between $I^N_{\text{disc},\psi}(f)$ and its "expected value" $|\kappa_G|^{-1}\text{tr}\, R^N_{\text{disc},\psi}(f)$. But the latter difference is zero, due to the known spectral expansion of $I^N_{\text{disc},\psi}(f)$. Therefore ${}^0S^G_{\text{disc},\psi}(f^G) = 0$, as claimed. □

Corollary 8.3.13 ([Art13, Lemma 6.6.3])**.** *The linear form $f_N(\phi)$ is the image of a unique stable linear form $f^G(\phi)$, interpreted as an element of $\bigoplus_{G^*}\mathcal{SI}(G^*)^*$ with only non-zero coordinate for $G^* = G$.*

Proof. This is clearly the local analog of Proposition 8.3.12(1). Since the global linear form $\dot{f}_N(\dot{\phi})$ is a product of local linear forms over all places, the claim at the place u of interest follows from Proposition 8.3.12(1) and the validity of the local claim at all places $v \neq u$, which we now have to establish.

If $v \neq u$ is an archimedean place, this follows from the results on twisted endoscopy for real groups of Mezo [Mez13]. If $v \neq u$ is a non-archimedean place, Part (3) of Proposition 8.3.5 asserts that the parameter at v has a very simple form. If $S_{\dot{\phi}_v}$ is infinite, i.e. the parameter is not discrete at v, the claim follows from Proposition 8.3.2. If $S_{\dot{\phi}_v}$ is finite, this forces $N \leq 3$ and $\dot{\phi}_v$ is the direct sum of N-many inequivalent characters that are either trivial or a sign character. This small case can be handled directly using the results of Labesse-Langlands [LL79], cf. [Art13, Lemma 6.6.2]. □

The stable linear form $f^G(\phi)$ will turn out to be the stable character of the L-packet $\Pi_\phi(G)$ associated to the parameter ϕ. We can obtain the set $\Pi_\phi(G)$ using a piece of local harmonic analysis that generalizes the well-known duality between conjugacy classes and irreducible characters of a finite group. Namely, for each discrete series representation π of $G(F)$ we have the distribution $f_G(\pi)$. More generally, consider a triple $\tau = (M, \pi, r)$ consisting of a Levi subgroup $M \subset G$, a discrete series representation π of $M(F)$, and a regular element $r \in R_{\pi,\text{reg}}$ in the R-group of π. Then we have the distribution

$$f_G(\tau) = \text{tr}(R_P(r, \tilde{\pi}_M, \phi)\mathcal{I}_P(\pi_M, f)),$$

where R_P is the normalized intertwining operator, and we are using the inductive assumption that the local theorems are known for M in order to have the desired normalization. The result from local harmonic analysis that we need is that the space $\mathcal{I}_{\text{cusp}}(G)$ of orbital integrals of cuspidal functions is isomorphic, via the map $f \mapsto (\tau \mapsto f_G(\tau))$, to the space of functions of finite support on the set of triples $\tau = (M, \pi, r)$, cf. [Kaz86, Theorem A] or [Art13, §6.5]. The dual statement to that is that the space of invariant distributions, upon restriction to cuspidal test functions, has as basis the set of triples $\tau = (M, \pi, r)$, where each τ identified with the distribution $f \mapsto f_G(\tau)$. This leads to the expression

$$f^G(\phi) = \sum_\tau c_\phi(\tau) \cdot f_G(\tau), \tag{8.3.8}$$

valid for all cuspidal test functions f, where $c_\phi(\tau)$ are unknown complex numbers, non-zero for only finitely many τ. We will show that $c_\phi(\tau) = 0$ unless $\tau = (G, \pi, 1)$ for some discrete series representation π of G. Then $\Pi_\phi(G) = \{\pi \,|\, c_\phi(G, \pi, 1) \neq 0\}$.

But our goal is not just to construct $\Pi_\phi(G)$, but also the pairing between it and $\mathcal{S}_\phi$. With this in mind, we apply the above strategy more generally, not just to the linear form $f^G(\phi)$, but also to the linear form $f'(\phi')$ for any $x \in \mathcal{S}_\phi$. We recall here that x leads via (6.4.5) to an endoscopic datum and a parameter ϕ' for it, and we denote by f' the transfer of f to this endoscopic datum. As above we obtain

$$f'(\phi') = \sum_{\tau} c_\phi(x,\tau) \cdot f_G(\tau), \tag{8.3.9}$$

again valid for all cuspidal test functions f on G, where $c_\phi(x,\tau)$ are unknown complex numbers, non-zero for only finitely many τ, and $c_\phi(\tau) = c_\phi(1,\tau)$ (for the finite support property see [Art96, Lemma 5.2]).

Again we will want to show that $c_\phi(x,\tau) = 0$ unless $\tau = (G,\pi,1)$. In addition, we will want to show that $x \mapsto c_\phi(x,(G,\pi,1))$ is either zero or a character of the finite abelian group $\mathcal{S}_\phi$ and that, as π runs over $\Pi_\phi(G)$, the character $c_\phi(-,(G,\pi,1))$ runs over all characters of $\mathcal{S}_\phi$. In other words, we will have $\langle x,\pi\rangle = c_\phi(x,(G,\pi,1))$.

Note that the constructions of $\Pi_\phi(G)$ and $\langle -,\pi\rangle$ are local in nature. The global input from the trace formula will only be used to justify that they work.

Lemma 8.3.14. $c_\phi(x,\tau) = 0$ *unless* $\tau = (G,\pi,1)$.

Proof. This is a consequence of the orthogonality relations (cf. [Art13, §6.5]) satisfied by the constants $c_\phi(x,\tau)$, and the fact that for $\tau = (M,\pi,r)$ with $M \subset G$ proper, the already proved local intertwining relation determines that $c_\phi(x,\tau)$ is closely related to the pairing $\langle \pi, -\rangle$ on $\mathcal{S}_{\phi_M}$. □

We will henceforth write $c_\phi(x,\pi)$ instead of $c_\phi(x,(G,\pi,1))$. To examine the coefficients $c_\phi(x,\pi)$ with $\pi \in \Pi_\varphi(G)$ we return to the global setting.

Lemma 8.3.15. *Write* $R_{disc,\phi} = \bigoplus n_{\dot\phi}(\dot\pi_G)\dot\pi_G$*, where* $\dot\pi_G$ *runs over the admissible representations of* $\dot G(\mathbb{A})$ *and* $n_{\dot\phi}(\dot\pi_G)$ *are non-negative integers. Then*

$$\sum_{\dot\pi_G} n_{\dot\phi}(\dot\pi_G)\dot f_{\dot G}(\dot\pi_G) = |\mathcal{S}_{\dot\phi}|^{-1} \sum_{\dot x \in \mathcal{S}_{\dot\phi}} \dot f'(\dot\phi').$$

Proof. According to Proposition 8.3.12(2), the stable multiplicity formula holds for $\dot G$ and $\dot\phi$. By inductive assumption it also holds for each proper endoscopic group $\dot G'$. We combine this with the stabilization identity of Proposition 8.2.2 and after a small reformulation, which converts the product of $\iota(\dot G,\dot G')$, $|\mathcal{S}_{\dot\phi'}|^{-1}$, and a few more supplementary terms, into $|\mathcal{S}_{\dot\phi}|^{-1}$, we obtain

$$I^{\dot G}_{\text{disc},\dot\phi}(\dot f) = |\mathcal{S}_{\dot\phi}|^{-1} \sum_{\dot x \in \mathcal{S}_{\dot\phi}} \dot f'(\dot\phi').$$

On the other hand, the global theorem holds for all proper Levi subgroups $\dot M$ of $\dot G$ by induction, and describes their discrete spectrum in terms of global parameters that are in the complement of $\tilde\Phi_{\text{ell}}(\dot N)$. Therefore, the parameter $\dot\phi$ does not contribute to the discrete spectrum of any proper Levi subgroup $\dot M$ of $\dot G$, which conversely means that no proper Levi subgroup $\dot M$ contributes to the expression (8.2.2) of $I^{\dot G}_{\text{disc},\dot\phi}$. Therefore,

$$I^{\dot G}_{\text{disc},\dot\phi}(\dot f) = \sum_{\dot\pi_G} n_{\dot\phi}(\dot\pi_G)\dot f_{\dot G}(\dot\pi_G).$$

□

The following result completes the proof of the main local theorem for the parameter $\phi \in \Phi_2^{\mathrm{sim}}(G)$. It is enough to focus on the non-archimedean setting, since the archimedean setting is already known and used as input.

Proposition 8.3.16 ([Art13, Proposition 6.6.5])**.** *For each $\pi \in \Pi_\phi(G)$ the function $x \mapsto c_\phi(x,\pi)$ is a character of $\mathcal{S}_\phi$ and the map $\pi \mapsto c_\phi(-,\pi)$ is a bijection between $\Pi_\phi(G)$ and the group $\mathcal{S}_\phi^*$. As ϕ ranges over $\Phi_2^{sim}(G)$, the sets $\Pi_\phi(G)$ are disjoint.*

Proof. We apply the identity of Lemma 8.3.15 and use a factorizable function $\dot f = \prod_v f_v$. As was already remarked and used in the proof of Corollary 8.3.13, the local theorem is known at all places $v \neq u$. We apply it to the right hand side, and use Equation (8.3.9) at $v = u$ together with Lemma 8.3.14. This leads to

$$\sum_{\dot\pi_G} n_{\dot\phi}(\dot\pi_G)\dot f_{\dot G}(\dot\pi_G) = |\mathcal{S}_{\dot\phi}|^{-1}\sum_{\dot x\in\mathcal{S}_{\dot\phi}}\sum_{\dot\pi^u,\pi}\langle \dot x, \dot\pi^u\rangle c_\phi(x,\pi)\dot f_{\dot G}(\dot\pi^u\otimes\pi),$$

where on the right hand side $\dot\pi^u$ runs over $\otimes'_{v\neq u}\Pi_{\dot\phi_v}$ and π runs over Π_ϕ.

Consider a character $\xi \in \mathcal{S}_\phi^*$. According to part (5) of Proposition 8.3.5 there exists a representation $\pi_\infty(\xi) \in \Pi_{\dot\phi_\infty}(\dot G_\infty) = \otimes_{v|\infty}\Pi_{\dot\phi_v}(\dot G_v)$ such that $\langle \dot x, \pi_\infty(\xi)\rangle = \xi(x)^{-1}$ under the surjection $\mathcal{S}^*_{\dot\phi_\infty} \to \mathcal{S}^*_{\dot\phi}$. Let $\pi^{\infty,u} = \otimes'_{v\neq\infty,u}\pi_v$ be such that $\pi_v \in \Pi_{\dot\phi_v}$ with $\langle -,\pi_v\rangle = 1$ on $\mathcal{S}_{\dot\phi_v}$. Choosing f_∞ and $f^{\infty,u}$ appropriately to isolate $\pi_\infty(\xi)$ and $\pi^{\infty,u}$, and choosing the test function $f = f_u$ to be cuspidal, the above identity becomes

$$\sum_\pi n_{\dot\phi}(\pi_\infty(\xi)\otimes\pi^{\infty,u}\otimes\pi)f_G(\pi) = |\mathcal{S}_{\dot\phi}|^{-1}\sum_{\dot x\in\mathcal{S}_{\dot\phi}}\sum_\pi \xi^{-1}(\dot x)c_\phi(\dot x,\pi)f_G(\pi),$$

where on both sides the sum runs over $\pi \in \Pi_\phi(G)$. Using that characters of discrete series representations are linearly independent even after restriction to cuspidal functions, and the identity $\mathcal{S}_{\dot\phi} = \mathcal{S}_\phi$ of Proposition 8.3.5, we conclude

$$n_{\dot\phi}(\pi_\infty(\xi)\otimes\pi^{\infty,u}\otimes\pi) = |\mathcal{S}_\phi|^{-1}\sum_{x\in\mathcal{S}_\phi}\xi^{-1}(x)c_\phi(x,\pi). \tag{8.3.10}$$

We may abbreviate $n_{\dot\phi}(\pi_\infty(\xi)\otimes\pi^{\infty,u}\otimes\pi) =: n_\phi(\pi,\xi)$, noting from the above identity that it depends only on the three quantities ϕ,π,ξ. The power of (8.3.10) comes from the fact that the left hand side is a non-negative integer, while the coefficients of the right hand side satisfy orthogonality relations: the term ξ because it is a character of a finite abelian group, and the term $c_\phi(x,\pi)$ due to the orthogonality relations coming from local harmonic analysis already used in the proof of Lemma 8.3.14. Applying the orthogonality relations first, one obtains

$$\sum_{\pi\in\Pi_2(G)} n_{\phi_1}(\pi,\xi_1)n_{\phi_2}(\pi,\xi_2) = \begin{cases}1, & (\phi_1,\xi_1) = (\phi_2,\xi_2)\\ 0, & \text{else},\end{cases}$$

where $\Pi_2(G)$ denotes the set of isomorphism classes of irreducible discrete series representations of $G(F)$. Taking $(\phi_1, \xi_1) = (\phi_2, \xi_2) = (\phi, \xi)$, and using the fact that $n_\phi(\pi, \xi)$ is a non-negative integer, we conclude that it must equal 1 for a unique $\pi = \pi(\xi) \in \Pi_2(G)$, which by definition of $\Pi_\phi(G)$ lies in $\Pi_\phi(G)$. Taking $\phi_1 = \phi_2 = \phi$ and $\xi_1 \neq \xi_2$ we see $\pi(\xi_1) \neq \pi(\xi_2)$. We have thus established an injective map

$$\mathcal{S}_\phi^* \to \Pi_\phi(G), \qquad \xi \mapsto \pi(\xi). \tag{8.3.11}$$

Taking $\phi_1 \neq \phi_2$, we see that the sets $\Pi_{\phi_1}(G)$ and $\Pi_{\phi_2}(G)$ are disjoint.

To see that the map (8.3.11) is surjective, take $\pi \in \Pi_\phi(G)$. By definition, the function $x \mapsto c_\phi(x, \pi)$ is non-zero. Therefore, it has a non-zero Fourier coefficient, i.e. there exists a character ξ of $\mathcal{S}_\phi$ such that the right hand side of (8.3.10) is non-zero. Thus $n_\phi(\pi, \xi)$ is non-zero, but the above argument shows that it then equals 1 and $\pi = \pi(\xi)$. □

We have thus completed the construction of L-packets for parameters $\phi \in \Phi_2^{\text{sim}}(G)$ and proved their endoscopic character identities, the latter being (8.3.9) with the identification of the coefficients $c_\phi(x, \tau)$ given by Lemma 8.3.14 and Proposition 8.3.16. However, we must be careful to note that these identities are so far proved only for *cuspidal* test functions on G. The extension of these identities to all test functions requires an independent argument, which we will not give here, and for which we refer to [Art13, Corollary 6.7.4].

We now consider the case where the decomposition (8.3.7) consists of a single summand, i.e. $r = 1$. Thus ϕ is a "simple" parameter, and we write $\Phi_{\text{sim}}(G) \subset \Phi_2(G)$ for the set of those. This is discussed in [Art13, §6.7]. The logic here flows in the opposite direction, namely from π to ϕ, rather than from ϕ to π. For this, we define $\Pi_{\text{sim}}(G)$ to be the complement in $\Pi_2(G)$ of the union of packets $\Pi_\phi(G)$ for all $\phi \in \Phi_2^{\text{sim}}(G)$, as just defined. Then we fix a representation $\pi \in \Pi_{\text{sim}}(G)$. Our goal is to match it with a parameter $\phi \in \tilde{\Psi}(N)$ and then prove that $\phi \in \Phi_{\text{sim}}(G)$. For each $\phi \in \Phi_{\text{sim}}(G)$ the L-packet should be a singleton. Thus we want to provide a bijection $\Pi_{\text{sim}}(G) \to \Phi_{\text{sim}}(G)$.

The reason that the logic flows in the opposite direction is that Proposition 8.3.5 is not available for simple parameters. Instead, we apply Proposition 8.3.4 to (F, G, π) and obtain global objects $(\dot{F}, \dot{G}, \dot{\pi})$, and let $\dot{\phi} \in \tilde{\Phi}(N)$ be the unique parameter such that $\dot{\pi} \in L^2_{\text{disc},\dot{\phi}}(G(F)\backslash G(\mathbb{A}))$. The parameter we attach to π is the localization $\phi = \dot{\phi}_u$. We already know that it is generic. What we need to show is that it is simple, and depends only on π, rather than the full globalization $\dot{\pi}$.

Lemma 8.3.17. *We have $\dot{\phi} \in \Phi_{sim}(\dot{G}) \subset \tilde{\Phi}_{sim}(\dot{N})$ and $\phi \in \Phi_{sim}(G) \subset \Phi_{sim}(N)$.*

Proof. This argument is given on pages 369-370 in [Art13, §6.7].

We consider first the local parameter ϕ. The argument of the proof of Proposition 8.3.4(6) shows that $\phi \in \tilde{\Phi}_{\text{ell}}(N)$. Moreover, any cuspidal test function $\tilde{f}$ on $\mathrm{GL}_N(F)$ whose transfer to G is a pseudocoefficient of π must have zero transfer to any other elliptic endoscopic group. This means that the linear form $f_N(\phi)$ on $\mathrm{GL}_N(F)$ is the endoscopic transfer of a stable linear form $f_G(\phi)$ on

$G(F)$. Recall that this characterizes G uniquely, a fact that will be used in the next paragraph.

Next we argue that if ϕ were not simple, then $\phi \in \Phi_2(G)$. Indeed, in that case we could apply Corollary 8.3.13 to ϕ and to the unique elliptic endoscopic datum G^* through which ϕ factors as a discrete parameter, and show that the linear form $f_N(\phi)$ on $\mathrm{GL}_N(F)$ is the endoscopic transfer of a stable linear form $f_{G^*}(\phi)$ on $G^*(F)$. But this implies $G^* = G$.

Thus, assuming that ϕ is not simple we have arrived at $\phi \in \Phi_2(G)$. Proposition 8.3.16 then expresses $f_G(\phi)$ as $\sum_{\pi^* \in \Pi_\phi} f_G(\pi^*)$, where Π_ϕ is the L-packet associated to the discrete but (allegedly) non-simple parameter ϕ according to the discussion of the case $r > 1$. Applying this identity with f being a pseudo-coefficient of π and using the fact that $f^N(\phi) \neq 0$, we see $\pi \in \Pi_\phi$, which is a contradiction to $\pi \in \Pi_{\mathrm{sim}}(G)$. Therefore, ϕ is indeed simple.

We now consider the global parameter $\dot\phi$. It represents a generic automorphic representation of GL_N. We have just shown that its local component $\dot\phi_u$ is a discrete series representation of $\mathrm{GL}_N(F_u)$. This implies that $\dot\phi$ represents a cuspidal automorphic representation, hence $\dot\phi \in \tilde\Phi_{\mathrm{sim}}(\dot N)$. Moreover, at all unramified places the localization of $\dot\phi$ matches the Satake parameter of $\dot\pi$. This implies $\dot\phi \in \Phi_{\mathrm{sim}}(\dot G)$. Strictly speaking, at this point in the argument the set $\Phi(\dot G)$ has been redefined according to a provisional definition made in [Art13, §5.1], so an additional argument is required, but we will not review it here. □

The analog of Proposition 8.3.12 holds in this setting, and so does Corollary 8.3.13. We thus obtain the stable linear form $f^G(\phi)$. After restriction to cuspidal functions we decompose it as (8.3.8) and use the argument of Lemma 8.3.14 to obtain

$$f^G(\phi) = \sum_{\pi^* \in \Pi_2(G)} c_\phi(\pi^*) \cdot f_G(\pi^*) \tag{8.3.12}$$

with some complex coefficients $c_\phi(\pi^*)$, only finitely many of which are nonzero. We have used the index π^*, because π is reserved for the fixed element of $\Pi_{\mathrm{sim}}(G)$.

Lemma 8.3.18. *Write $R_{disc,\phi} = \bigoplus n_{\dot\phi}(\dot\pi_G)\dot\pi_G$, where $\dot\pi_G$ runs over the admissible representations of $\dot G(\mathbb{A})$ and $n_{\dot\phi}(\dot\pi_G)$ are non-negative integers. Then*

$$\sum_{\dot\pi_G} n_{\dot\phi}(\dot\pi_G)\dot f_{\dot G}(\dot\pi_G) = \dot f(\dot\phi).$$

Proof. The proof is the same as for Lemma 8.3.15, with the additional information that, since $\dot\phi$ is simple according to Lemma 8.3.17, we have $\mathcal{S}_{\dot\phi} = \{1\}$. □

Proposition 8.3.19 ([Art13, Proposition 6.7.2])**.**

1. *The parameter ϕ depends only on π, and not on $\dot\pi$.*
2. *The map $\pi \mapsto \phi$ is a bijection $\Pi_{sim}(G) \to \Phi_{sim}(G)$, and this bijection is characterized by $f^G(\phi) = f_G(\pi)$ for all cuspidal test functions f.*

Proof. Most of the argument follows the lines of the proof of Proposition 8.3.16. We apply the identity of Lemma 8.3.18 to a factorizable function $\dot{f} = \prod_v f_v$, apply the known local theorems at all places $v \neq u$, and use Equation (8.3.12) at $v = u$, to obtain

$$\sum_{\dot{\pi}_G} n_{\dot{\phi}}(\dot{\pi}_G)\dot{f}_{\dot{G}}(\dot{\pi}_G) = \sum_{\dot{\pi}^u,\pi} c_\phi(\pi)\dot{f}_{\dot{G}}(\dot{\pi}^u \otimes \pi),$$

where on the right hand side $\dot{\pi}^u$ runs over $\otimes'_{v\neq u}\Pi_{\dot{\phi}_v}$ and π runs over Π_ϕ.

We fix the representation $\pi^u = \otimes'_{v\neq u}\pi_v$ where $\pi_v \in \Pi_{\dot{\phi}_v}$ corresponds to $\langle -, \pi_v\rangle = 1$. We choose again f^u to isolate this representation, and f_u to be cuspidal, and obtain from the above identity

$$\sum_{\pi^*} n_{\dot{\phi}}(\pi^u \otimes \pi^*)f_u(\pi^*) = \sum_{\pi^*} c_\phi(\pi^*)f_u(\pi^*),$$

where on both sides the sum runs over $\pi^* \in \Pi_2(G)$, from which we conclude that $c_\phi(\pi^*)$ is a non-negative integer, namely equal to the automorphic multiplicity $n_{\dot{\phi}}(\pi^u \otimes \pi^*)$.

The orthogonality relations show that

$$\sum_{\pi^*} c_{\phi_1}(\pi^*)c_{\phi_2}(\pi^*) = \begin{cases} 1, & \phi_1 = \phi_2 \\ 0, & \text{else} \end{cases}$$

Here ϕ_1, ϕ_2 are any discrete local parameters that are either not simple, or are simple and arise from some simple π via the preceding construction (for a non-simple ϕ the number $c_\phi(\pi^*)$ was defined by (8.3.8)). Since both $c_{\phi_1}(\pi^*)$ and $c_{\phi_2}(\pi^*)$ are non-negative integers, we conclude by setting $\phi_1 = \phi_2 = \phi$ that $c_\phi(\pi^*)$ is either 0 or 1. Moreover, we conclude that $c_\phi(\pi) = 1$, since the global representation $\dot{\pi}$ with $\dot{\pi}_u = \pi$ does occur in the automorphic spectrum of $\dot{G}$. But then $c_\phi(\pi^*) = 0$ for all $\pi^* \neq \pi$. This collapses the sum in Equation (8.3.12), which now becomes $f^G(\phi) = f_G(\pi)$.

Applying the above orthogonality relation with $\phi_1 = \phi \neq \phi_2$ we see that $c_{\phi'}(\pi) = 0$ for any $\phi' \neq \phi$, concluding that ϕ is uniquely determined by π. This proves (1) and provides the map $\pi \mapsto \phi$ of (2), which is injective by the above paragraph, and has the desired characterization.

It remains to prove that the map $\pi \mapsto \phi$ is surjective. Let $\Phi^c_{\text{sim}}(G)$ denote the image of this map, and let $\Phi^c_2(G) = \Phi^c_{\text{sim}}(G) \cup \Phi^{\text{sim}}_2(G)$. We can rephrase our goal as showing that $\Phi^c_2(G) = \Phi_2(G)$. The basic idea is to show that $\Phi^c_2(G)$ provides a basis of the space $\mathcal{S}_{\text{cusp}}(G)$ of stable orbital integrals of cuspidal functions, which leaves no room for the existence of discrete parameters in $\Phi_2(G) \smallsetminus \Phi^c_2(G)$.

To implement this idea, Arthur introduces subspaces $\mathcal{I}_{\text{sim}}(G) \subset \mathcal{I}_{\text{cusp}}(G)$ and $\mathcal{S}_{\text{sim}}(G) \subset \mathcal{S}_{\text{cusp}}(G)$. The first consists of those $f_G \in \mathcal{I}_{\text{cusp}}(G)$ such that $f_G(\tau) = 0$ for all $\tau = (M, \pi, r)$ except $\tau = (G, \pi, 1)$ with $\pi \in \Pi_{\text{sim}}(G)$. The second consists of those $f^G \in \mathcal{S}_{\text{cusp}}(G)$ such that $f^G(\phi) = 0$ for all $\phi \in \Phi^{\text{sim}}_2(G)$. We then need the following result.

Lemma 8.3.20 ([Art13, Lemma 6.7.1]). *The total endoscopic transfer isomorphism*

$$\mathcal{I}_{cusp}(G) \to \bigoplus_{G'} \mathcal{S}_{cusp}(G')^{Aut_G(G')}$$

maps $\mathcal{I}_{sim}(G)$ isomorphically onto $\mathcal{S}_{sim}(G)$.

For each $\pi \in \Pi_{\text{sim}}(G)$, the pseudocoefficient f_π lies in $\mathcal{I}_{\text{sim}}(G)$, and the functions $(f_\pi)_\pi$ obtained this way form a basis of this space. By the above lemma these functions are transported to a basis of $\mathcal{S}_{\text{sim}}(G)$. According to the characterization of the map $\pi \mapsto \phi$, the basis vector f_π of $\mathcal{I}_{\text{sim}}(G)$ is transported to the basis vector f^ϕ of $\mathcal{S}_{\text{sim}}(G)$ having the property that if ϕ^* is in the image of the map $\pi \to \phi$, then $f^\phi(\phi^*) = 0$ unless $\phi^* = \phi$, and $f^\phi(\phi) = 1$. The orthogonal complement of $\mathcal{S}_{\text{sim}}(G)$ in $\mathcal{S}_{\text{cusp}}(G)$ (with respect to the scalar product [Art13, (6.5.12)]) can be equipped with the basis $f^\phi = |\mathcal{S}_\phi|^{-1} \sum_{\pi \in \Pi_\phi(G)} f_\pi$, where ϕ runs over $\Phi_2^{\text{sim}}(G)$ and f_π is a pseudocoefficient of π.

Consider now an arbitrary parameter $\phi^* \in \Phi_{\text{sim}}(G) \subset \tilde{\Phi}_{\text{sim}}(N)$. Associated is a linear form $f^N(\phi^*)$ on $\mathcal{H}(N)$. Via the total twisted transfer map

$$\mathcal{I}_{\text{cusp}}(N) \to \bigoplus_{G^*} \mathcal{S}_{\text{cusp}}(G^*),$$

where G^* runs over the twisted elliptic endoscopic data of twisted GL_N, this linear form induces stable linear forms on each $\mathcal{S}_{\text{cusp}}(G^*)$, and we consider the component for $G^* = G$ and call it $f^G(\phi^*)$. The definition of the subset $\Phi_{\text{sim}}(G) \subset \Phi_{\text{sim}}(N)$ is that $f^G(\phi^*)$ is non-zero. It is then enough to show that $f^{\phi,G}(\phi^*) = 0$ for any $\phi \in \Phi_2^c(G)$ that is not equal to ϕ^*. But this statement is true after twisted transfer to GL_N. □

8.3.5 Co-tempered packets

We have so far discussed the proof of the local theorems (the local classification and the local intertwining relation) for generic parameters. The approach was that of embedding the local situation in a global situation using the globalization results of §8.3.2, controlling the places away from the place of interest, and peeling away the layers of information from the trace formula, as discussed in the beginning of §8.3. A key part of this process was the fact that much of the desired local results were already available for the archimedean places due to the work of Langlands and Shelstad, as well as further work by Mezo.

The first problem one is faced with when dealing with non-generic parameters is that the endoscopic results of Shelstad, which only apply to tempered representations, are no longer available. This leaves the archimedean places as mysterious as the non-archimedean places, and they cannot be used to control the global argument.

The resolution to this problem is provided by an additional symmetry that is afforded by the non-archimedean places. On the side of parameters it is provided by the two copies of the group $\text{SL}_2(\mathbb{C})$ in the source of a general

Arthur parameter $\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$. Arthur defines $\widehat{\psi}$ to be the composition of ψ with the automorphism of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ that switches the two copies of $\mathrm{SL}_2(\mathbb{C})$. Note that $S_{\widehat{\psi}} = S_\psi$ as subgroups of $\widehat{G}$. On the side of representations, one has the Aubert–Zelevinski duality operator $\pi \mapsto \widehat{\pi}$. Thus, given any Arthur packet $\Pi_\psi(G)$, one can define the set $\widehat{\Pi}_\psi(G) = \{\widehat{\pi} \mid \pi \in \Pi_\psi(G)\}$.

The expectation is that $\Pi_{\widehat{\psi}}(G) = \widehat{\Pi}_\psi(G)$. At the moment both sides are not defined in general, so this is only a guiding principle. But there is a special class of parameters for which both sides are defined, namely those ψ which are trivial on the first copy of $\mathrm{SL}_2(\mathbb{C})$. This is equivalent to saying $\psi = \widehat{\phi}$ for a generic Arthur parameter ϕ. One can call these parameters *co-tempered*.

Given a co-tempered parameter $\psi = \widehat{\phi}$, Arthur defines $\Pi_\psi(G) = \widehat{\Pi}_\phi(G)$. This definition must now be supplemented with the proofs of the local theorems for the packet $\Pi_\psi(G)$. Once this is done, the co-tempered parameters can be used to control the global argument, in the same way that archimedean parameters were used in the discussion of the generic case.

Let us discuss the local classification theorem (Theorem 8.1.3) for $\Pi_\psi(G)$. It's first statement, asserting that the distribution $\tilde{f}^N(\psi)$ factors through endoscopic transfer from G, and is thus given as $f^G(\psi) = \tilde{f}^N(\psi)$ for a uniquely determined stable distribution $f^G(\psi)$ on $G(F)$, has already been established, because it was reduced by local means to the case of generic parameters in Lemma 8.3.3, and the case of generic parameters is at this point known. It is the second statement that now needs to be proved, namely that, for any $s \in S_\psi$, the identity $f'(\psi') = \sum_{\pi \in \Pi_\psi(G)} \langle s_\psi s, \pi \rangle f_G(\pi)$ holds. Note that there are natural bijections $\Pi_\psi(G) \leftrightarrow \Pi_\phi(G)$ and $(\mathcal{S}_\psi)^* \leftrightarrow (\mathcal{S}_\phi)^*$, as well as an already constructed bijection $\Pi_\phi(G) \to (\mathcal{S}_\phi)^*$, which combine to give the bijection $\Pi_\psi(G) \to (\mathcal{S}_\psi)^*$ that is implicit in the above identity. It will however turn out that this naïve definition of the bijection $\Pi_\psi(G) \to (\mathcal{S}_\psi)^*$ is not the one that makes the above identity hold, so part of the work is to modify the above bijection. The precise modification is given in [AGI$^+$24, Theorem 5.4.1], which is where this identity is proved.

A key input in the proof of the endoscopic character identity is the work of Hiraga [Hir04] on the compatibility of endoscopic transfer with the Aubert–Zelevinski involution. Hiraga's result is that ordinary and twisted endoscopic character identities commute with the involution up to a certain explicit sign, called $\beta(\phi)$. On the other hand, the Aubert–Zelevinski involution involves a second explicit sign, called $\beta(\pi)$. Hiraga's work and elementary manipulations using Fourier theory on the finite abelian group $(\mathcal{S}_\psi)^*$ now reduce the problem to the following sign identity ([Art13, (7.1.8)]) for any $\pi \in \Pi_\phi(G)$

$$\beta(\phi)\beta(\pi) = \langle s_\psi, \pi \rangle. \tag{8.3.13}$$

Despite its explicit nature, this identity is not immediate. In fact, it takes a fair bit of work to prove.

Consider now the second major local theorem, the local intertwining relation (Theorem 8.1.6). It is the identity of two distributions $f_G(\psi, n) = f'_G(\psi, s_\psi \cdot n)$. Hiraga's result again shows that $f'_G(\psi, s_\psi \cdot n) = \beta(\phi)(Df_G)'(\phi, s_\psi x)$, where D is the operator on test functions that is dual to Aubert–Zelevinski duality on representations. The validity of the local intertwining relation for ϕ now reduces the same for ψ to the identity $f_G(\psi, n) = \beta(\phi)(Df_G)(\phi, s_\psi \cdot n)$. The definition of both sides as sums of terms and linear independence of characters reduces this to the corresponding identity between the individual summands, namely

$$\langle n, \widehat{\pi}_M\rangle \mathrm{tr}\Big(R_P(\psi, \widehat{\pi}_M, w)\mathcal{I}_P^G(\widehat{\pi}_M, f)\Big) = \langle s_\psi \cdot n, \pi_M\rangle \mathrm{tr}\Big(R_P(\phi, \pi_M, w)\mathcal{I}_P^G(\pi_M, f)\Big). \tag{8.3.14}$$

This identity is considerably more subtle than (8.3.13), as it involves the action of the normalized intertwining operators and its compatibility with Aubert–Zelevinski duality.

Unfortunately, neither (8.3.13) nor (8.3.14) appears to be directly approachable by the methods developed so far. Arthur relegates their proof to one of the unwritten articles, namely [A25]. More precisely, Arthur formally defines the subset $\Pi_\psi^G \subset \Pi_\psi(G)$ to consist of those elements $\widehat{\pi}$ for which (8.3.13) holds, and analogously the subset $\Pi_{\psi_M}^G \subset \Pi_{\psi_M}(M)$ the subset of those $\widehat{\pi}_M$ for which (8.3.14) holds, and asserts in [Art13, Lemma 7.1.2] that these sets can be proven to be large enough so that the resulting local information is sufficient to be used to control the global argument, at least for sufficiently many co-tempered parameters.

Both identities were recently obtained in [AGI+24] in the setting of all co-tempered parameters, thus proving a generalization of [Art13, Lemma 7.1.2]. Identity (8.3.13) is proved in [AGI+24, Proposition 5.1.2], from which the desired endoscopic character identities follow somewhat formally in [AGI+24, Theorem 5.4.1], but not before the correct pairing between S_ψ and Π_ψ has been defined. Once these have been established, one can reinterpret (see [AGI+24, Lemma 1.10.2]) identity (8.3.14) as the simpler looking identity

$$R_P(\psi, \pi_M, w)|_\pi = \langle u, \pi_M\rangle^{-1}\langle u, \pi\rangle \tag{8.3.15}$$

which describes the scalar by which the normalized intertwining operator acts on any π appearing in the parabolic induction $\mathcal{I}_P(\pi_M)$ as the scalar $\langle u, \pi\rangle$ expressed in terms of the pairing between $\mathcal{S}_\psi$ and Π_ψ defined above, and its analog for the Levi subgroup M. This simplification is one of technical, but not substantive, nature, because one still needs to understand the mysterious action of the normalized intertwining operator. This identity is proved in [AGI+24, §6,§7] by a sequence of reductions which crucially involve the theory of derivatives due to Atobe and Minguez.

8.3.6 The local intertwining relation for non-generic parameters

The proof of the local intertwining relation for non-generic parameters follows the same outline as that for generic parameters.

The first step is the application of a globalization result [Art13, Proposition 7.2.1]. Its statement is almost identical to that of [Art13, Proposition 6.3.1], which we have recalled here as Proposition 8.3.5, with the main difference that the set of real places in parts 5., 6., and 7, has been replaced with a finite set V of non-archimedean places for which $\dot\psi_v$ is co-tempered and for which the subset of $\Pi_{\dot\psi_M,V}$ consisting of those elements for which (8.3.13) and (8.3.14) hold, maps surjectively onto $\widehat{\mathcal{S}}_{\psi_M}$ (in light of [AGI$^+$24] the requirement is simply that $\Pi_{\dot\psi_M,V}$ maps surjectively onto $\widehat{\mathcal{S}}_{\psi_M}$). The proof of this globalization result is also very close to that of Proposition 8.3.5, and we will not review it.

The second step is the derivation of the weak global intertwining relation for the globalized parameter $\dot\psi$ that is produced by the globalization step. This is [Art13, Lemma 7.3.1], and is the direct analog of Proposition 8.3.6. Its proof is also quite similar. The first part of the proof, up to (8.3.4), is the same, as it relies on Corollaries 8.2.4 and 8.2.6 which are available in the necessary generality of not necessarily generic parameters. The second part of the proof relied on Equation (8.3.4), which asserted that a weaker form of the local intertwining relation is known at archimedean places. In the current setting, the set of archimedean places is replaced by the set V of non-archimedean places at which $\dot\psi$ has co-tempered localization. The analogous weaker version of the local intertwining relation is extracted from the work of D. Ban [Ban02], which identifies the summands in the definition of $f_G(\psi,n)$ with those of $(Df_G)(\phi,s_\psi n)$ up to a sign, and the result of Hiraga [Hir04] discussed above, which identifies $f'_G(\psi,n)$ with $(Df_G)'(\phi,n)$ up to a sign. The rest of the argument proceeds analogously to its generic counterpart.

The third step is to supply the validity of the local intertwining relation at all unramified places, as well as at all places in the control set V, at least for a sufficient supply of functions. This is [Art13, Lemma 7.3.3, 7.3.4]. The first of these states that, for a place v for which $\dot\psi_v$ is co-tempered, if the test function is chosen so as to kill all representations in the local co-tempered packet $\Pi_{\dot\psi_v}(G)$ which fail the identity (8.3.14), then the local intertwining relation holds for that test function. This is essentially by definition, and the statement of this lemma was made necessary due to the particular form in which Lemma 7.1.2 was stated.

The second lemma states that for any unramified place v, the localization $\dot\psi_v$ is co-tempered, and the characteristic function of the hyperspecial maximal compact satisfies the local intertwining relation. This essentially follows from the fundamental lemma. With this, the argument proceeds roughly as before and produces the local intertwining relation at the place u of interest. A small deviation is that, instead of a single place u where the desired local statement is unknown, a-priori one now has to deal with the finite set of places U consisting of u and all archimedean places, for the archimedean places are not as easy to control in the non-generic setting. But the required modification of the argument is not difficult, and we refer to the final page of [Art13, §7.3] for discussion.

The results of [AGI$^+$24], which were obtained much after [Art13] was written, simplify this proof, because they directly provide the local intertwining relation at all places v where $\dot{\psi}_v$ is co-tempered.

8.3.7 The local classification theorem for non-generic parameters

We can now complete the proof of the local classification theorem (Theorem 8.1.3) for arbitrary $\psi \in \Psi(G)$. If $\psi \notin \Psi_2(G)$, there exists a proper Levi subgroup $M \subset G$ and $\psi_M \in \Psi_2(M)$ such that $\psi_M \mapsto \psi$. In the previous section we established the local intertwining relation for this setting. Then Proposition 8.3.2 establishes Theorem 8.1.3 for ψ.

We may thus assume $\psi \in \Psi_2(G)$. We apply the same globalization result ([Art13, Proposition 7.2.1]) that was used in the proof of the local intertwining relation in §8.3.6, and that is the non-generic analog of Proposition 8.3.5, to obtain a pair $(\dot{G}, \dot{\psi})$ over a global field $\dot{F}$ which at a place u of $\dot{F}$ localizes to the given local pair (G, ψ). Note however that, while Proposition 8.3.5 requires the generic parameter to not be simple (so that all theorems are available for its simple components by induction), its non-generic analog does not have that requirement, because the inductive hypothesis is used on the generic pieces of its simple components, and even if ψ itself is simple, its generic component will be smaller due to the non-trivial SU_2-part. Therefore, we can treat both the simple and non-simple non-generic $\psi \in \Psi_2(G)$ uniformly.

We apply Proposition 8.2.2 and obtain

$$I^{\dot{G}}_{\mathrm{disc},\dot{\psi}}(\dot{f}) = \sum_{\dot{G}'} \iota(\dot{G}, \dot{G}') S^{\dot{G}'}_{\mathrm{disc},\dot{\psi}}(\dot{f}').$$

Next we check that the stable multiplicity formula (Conjecture 7.9.1) holds for $(\dot{G}', \dot{\psi})$. This is [Art13, Lemma 7.3.2], which is proved just the same as the argument given in the proof of Proposition 8.3.12(2) except in the slightly more difficult case when N is even and η_ψ is trivial.

On the other hand, since $\dot{\psi}$ is discrete for $\dot{G}$, it doesn't factor through a proper Levi subgroup of $\dot{G}$, and one concludes that

$$I^{\dot{G}}_{\mathrm{disc},\dot{\psi}}(\dot{f}) = \mathrm{tr}\big(R^{\dot{G}}_{\mathrm{disc},\dot{\psi}}(\dot{f})\big) = \sum_{\dot{\pi}_G} n_{\dot{\psi}}(\dot{\pi}_G) \dot{f}_{\dot{G}}(\dot{\pi}_G),$$

where the sum runs over discrete automorphic representations $\dot{\pi}_G$ of $\dot{G}(\mathbb{A})$ and $n_{\dot{\psi}}(\dot{\pi}_G)$ are non-negative integers.

Plugging these into the above identity leads to

$$\sum_{\dot{\pi}_G} n_{\dot{\psi}}(\dot{\pi}_G) \dot{f}_{\dot{G}}(\dot{\pi}_G) = |\mathcal{S}_{\dot{\psi}}|^{-1} \sum_{\dot{x} \in \mathcal{S}_{\dot{\psi}}} \epsilon'(\dot{\psi}') \dot{f}'(\dot{\psi}'), \tag{8.3.16}$$

where $(\dot{G}', \dot{\psi}')$ corresponds to $(\dot{\psi}, x)$ under (6.4.5). This is the non-generic analog of Lemma 8.3.15.

Consider the complex number $\dot{f}'(\dot{\psi}')$. It depends on the test function $\dot{f}$ (through its endoscopic transfer $\dot{f}'$), and is thus a distribution on $\dot{G}(\mathbb{A})$. But it also depends on the element $\dot{x}$, because $(\dot{G}', \dot{\psi}')$ arise from $(\dot{\psi}, \dot{x})$ via (6.4.5).

In the generic case in Proposition 8.3.16, we proceeded to decompose $\dot{f}'(\dot{\psi}')$ in terms of the variable $\dot{f}_u$, thus into irreducible characters of $G(F)$ with unknown coefficients $c_\psi(-, \pi)$, and then to identify these coefficients as characters of $\mathcal{S}_\psi$ using the orthogonality relations for elliptic tempered characters. In the present (non-tempered) case we will decompose $\dot{f}'(\dot{\psi}')$ in terms of the variable x, thus into characters of $\mathcal{S}_\psi$ with unknown coefficients $f_G(\sigma)$, and will identify these coefficients (which depend on f and are thus distributions) with (the characters of) irreducible representations of $G(F)$.

Thus, we write

$$\dot{f}'(\dot{\psi}') = \sum_{\dot{\sigma}} \langle s_{\dot{\psi}} \cdot \dot{x}, \dot{\sigma} \rangle \dot{f}_{\dot{G}}(\dot{\sigma}),$$

where $\dot{\sigma}$ runs over the characters of $\mathcal{S}_{\dot{\psi}}$, and we have denoted the evaluation of this character at $s_{\dot{\psi}} \cdot \dot{x}$ by $\langle s_{\dot{\psi}} \cdot \dot{x}, \dot{\sigma} \rangle$, and $\dot{f}_{\dot{G}}(\dot{\sigma})$ is the coefficient with which this character enters the decomposition of the function $\dot{x} \mapsto \dot{f}'(\dot{\psi}')$.

In fact, this decomposition can be done at each place v, and doing this shows that $\dot{\sigma}$ is a tensor product $\otimes_v \dot{\sigma}_v$ with $\dot{\sigma}_v$ a character of $\mathcal{S}_{\dot{\psi}_v}$. At the same time, for all places v away from U and V, the local classification theorem is known. There the distribution $\dot{f}_{v,\dot{G}_v}(\dot{\sigma}_v)$ is known to be an irreducible character of $\dot{G}(\dot{F}_v)$. Furthermore, we can choose an irreducible representation $\dot{\pi}_v \in \Pi_{\dot{\psi}_v}$ in such a way that $\langle -, \dot{\pi}_v \rangle = 1$. Set $\dot{\pi}^{U,V} = \otimes_{v \notin U \cup V} \dot{\pi}_v$.

Now consider a fixed character ξ of $\mathcal{S}_\psi = \mathcal{S}_{\dot{\psi}}$. While the local classification theorem is not fully known at the set V of control places, we do have available the local packet $\Pi_{\dot{\psi}_v}$ there, as well as a subset $\Pi^G_{\dot{\psi}_v}$ for which the sign identity (8.3.13) is known, and which is rich enough (according to the unproven [Art13, Lemma 7.1.2]). Note that these local packets were defined "by hand", and we do not yet know that their constituents are unitary, but this is not relevant at this point, and we will be able to conclude the unitarity at the end of the argument, see the final paragraph of this section. Regardless, this allows to choose a member $\dot{\pi}_v \in \Pi^G_{\dot{\psi}_v}$ for each $v \in V$ so that the product $\dot{\pi}_V(\xi) = \otimes_{v \in V} \dot{\pi}_v$ has the property $\langle -, \dot{\pi}_V(\xi) \rangle = \epsilon_{\dot{\psi}}^{-1} \xi^{-1}$.

We now choose the test function $\dot{f}$ away from the places in U so that $\dot{\sigma}_V \mapsto \dot{f}_V(\dot{\sigma}_V)$ is the indicator function of $\dot{\pi}_V(\xi)$, and $\dot{\sigma}^{U,V} \mapsto \dot{f}^{U,V}(\dot{\sigma}^{U,V})$ is the indicator function of $\dot{\pi}^{U,V}$. Then (8.3.16) becomes

$$\sum_{\dot{\pi}_U} n_\psi(\dot{\pi}_U \otimes \dot{\pi}_V(\xi) \otimes \dot{\pi}^{U,V}) \dot{f}_{U,\dot{G}}(\dot{\pi}_U) = \sum_{\dot{\sigma}_U} |\mathcal{S}_{\dot{\psi}}|^{-1} \sum_{x \in \mathcal{S}_\psi} \langle \dot{x}, \dot{\sigma}_U \rangle \xi(x)^{-1} \dot{f}_{U,\dot{G}}(\dot{\sigma}_U). \tag{8.3.17}$$

Here $\dot{\pi}_U$ runs over the set of irreducible admissible unitary representations of $\prod_{v \in U} \dot{G}(\dot{F}_v)$, while $\dot{\sigma}_U$ runs over the characters of $\prod_{v \in U} \mathcal{S}_{\dot{\psi}_v}$. Fourier inversion on the abelian group $\mathcal{S}_\psi$ reduces the above identity to

$$\sum_{\dot\pi_U} n_\psi(\dot\pi_U \otimes \dot\pi_V(\xi) \otimes \dot\pi^{U,V}) \dot f_{U,\dot G}(\dot\pi_U) = \sum_{\dot\sigma_U} \dot f_{U,\dot G}(\dot\sigma_U), \tag{8.3.18}$$

where now $\dot\sigma_U$ runs over those characters of $\prod_{v\in U} \mathcal{S}_{\dot\psi_v}$ whose restriction to the diagonal $\mathcal{S}_{\dot\psi}$ equals ξ. If U consists of the single place u of interest, then the sum on the right hand side has a single term $f_G(\sigma)$, which makes it clear that this σ is a (a-priori not irreducible) unitary representation of $G(F) = \dot G(\dot F_u)$, namely the sum of the terms $\dot\pi_u$ on the right with the non-negative integral multiplicities $n_\psi(\dots)$. One thus defines $\Pi_\psi(G)$ to be the set of these (possibly reducible) representations of $G(F)$, or rather the multiset of their irreducible pieces, and the mapping $\Pi_\psi(G) \to (\mathcal{S}_\psi)^*$ maps each of the representations occurring in the left side of (8.3.18) to ξ. The character identity of the local classification theorem (Theorem 8.1.3) now holds by construction, and this completes the proof of that theorem. In general one needs to separate the various places in U, which is not particularly difficult, and takes places in the page before the statement of [Art13, Proposition 7.4.3].

We come back to the question of unitarity of the constituents of the local packets constructed here. The construction of the local packet $\Pi_\psi(G)$ given here proceeds by global means, in the sense that the constituents of $\Pi_\psi(G)$ are local components of automorphic representations, hence unitary. Thus, a by-product of the construction is the unitarity of the members of $\Pi_\psi(G)$. But in this global construction we used, at the control places in the set V, the local packets for co-tempered parameters that were provided to us by [Art13, Lemma 7.1.2], thus in some sense constructed "by hand", whose unitarity a-priori we do not know. However, what we do know is that these packets satisfy the endoscopic character identities, both ordinary and twisted. Since the local packets $\Pi_\psi(G)$ produces by the above global construction also satisfy these identities, we now have, for a fixed co-tempered local parameter ψ, two separate, hence competing, constructions of the local packet $\Pi_\psi(G)$ – one "by hand", and one via global means. The second one has the property that the result is unitary. Both have the property that they satisfy the endoscopic character identities. But these identities, together with the linear independence of characters, imply that the two constructions produce the same representations. Thus, in hindsight, the two constructions agree.

8.4 Completion of the proof of the global theorems

With all the local theorems at this point established, Arthur turns to the completion of the proof of the global theorems. These are the global classification theorem 8.1.4, the seed theorems 8.1.1, 8.1.2, 8.1.5, and the stable multiplicity formula Conjecture 7.9.1. Recall that the latter is equivalent to the vanishing of the distribution ${}^0S^G_{\text{disc},\psi}$ that was defined to be the discrepancy between $S^G_{\text{disc},\psi}$ and its conjectural expression.

We take F to be a global field. Before we can make sense of the stable multiplicity formula, we need to define the key stable distribution that is part of its statement, namely $f^G(\psi)$.

Lemma 8.4.1. *Let G be an elliptic twisted endoscopic group of GL_N. Given $\psi \in \Psi(G)$ there exists a unique stable linear form $f^G(\psi)$ whose twisted transfer to GL_N is the stable linear form $\tilde{f}^N(\psi)$.*

Proof. We can reduce immediately to the case that G is simple, because a non-simple G is a product of a symplectic and an orthogonal group, each of smaller dimension than G, and one can apply the inductive hypothesis on the dimension. Thus assume from now on that G is simple.

Consider the localization ψ_v at a place v. If we knew $\psi_v \in \Psi_v(G)$, then the local classification theorem asserts that the local analog of the desired statement is true, and taking the product of all local stable linear forms $f^G(\psi_v)$ over all places one obtains the desired global form.

The assertion $\psi_v \in \Psi_v(G)$ is the content of the first seed theorem, Theorem 8.1.1. By induction this theorem is known when ψ is composite (apply it to its irreducible pieces), or simple but non-generic (apply it to the generic factor).

Thus one can assume that $\psi = \phi$ is simple generic. Then Theorem 8.1.1 is not yet available, and we proceed globally. The stabilization identity of Proposition 8.2.2 applied to twisted GL_N implies

$$\tilde{f}^N(\phi) = \mathrm{tr}\big(\tilde{R}^N_{\mathrm{disc},\phi}(\tilde{f})\big) = \tilde{I}^N_{\mathrm{disc},\phi}(\tilde{f}) = \sum_{G^* \in \tilde{\mathcal{E}}_{\mathrm{sim}}(N)} \tilde{\iota}(N, G^*) S^{G^*}_{\mathrm{disc},\phi}(\tilde{f}^{G^*}).$$

Ignoring the case "N even and η_ϕ=1", Remark 8.3.11 shows that there is exactly one summand with non-zero contribution, namely the summand for $G^* = G$, and we conclude

$$\tilde{f}^N(\phi) = \tilde{\iota}(N, G) S^G_{\mathrm{disc},\phi}(\tilde{f}^G).$$

We then set $f^G(\phi) := \tilde{\iota}(N, G) S^G_{\mathrm{disc},\phi}(f^G)$. □

We now turn to the proof of the global theorems. Let $\psi \in \tilde{\Psi}(N)$. Theorem 8.1.4 and Conjecture 7.9.1 have already been established when ψ does not lie in $\tilde{\Psi}_{\mathrm{ell}}(N)$ or in $\Psi_{\mathrm{ell}}(G)$ for any simple twisted endoscopic group G of GL_N: this is Proposition 8.2.9, which was based on the comparison of trace formulas dubbed the "standard model". This proposition requires as an input the global intertwining relation, which, being the product of the local intertwining relations, has now been verified[8].

As a next case Arthur considers $\psi \in \tilde{\Psi}(N)$ which does not lie in $\tilde{\Psi}_{\mathrm{ell}}(N)$, but does lie in $\Psi_{\mathrm{ell}}(G)$ for some simple twisted endoscopic group G of GL_N. Then G is uniquely determined by ψ, due to the shape of S_ψ. By assumption $\psi \notin \Psi_2(G) \subset \tilde{\Psi}_{\mathrm{ell}}(N)$. Therefore, the statement of Theorem 8.1.4 is that $R_{\mathrm{disc},\psi} = 0$.

[8]Actually, as we have recorded in Proposition 8.2.10, the global intertwining relation can be verified directly for such ψ, except when ψ is an exceptional parameter, i.e. lies in one of the families (8.2.6), (8.2.7). The validity of Theorem 8.1.4 and Conjecture 7.9.1 in the exceptional cases can also be verified directly, without appeal to the global intertwining relation, as is done in the second half of the proof of [Art13, Propostion 4.5.1].

The argument used in the proof of Proposition 8.3.6 applies here as well (it is the argument of [Art13, Lemma 5.2.1]) and established the identity (8.3.3), which we recall is

$$\sum_{x \in \mathcal{S}_{\psi,\text{ell}}} \Big(f_G(\phi, x) - f'_G(\phi, x) \Big) = C \cdot \text{tr}(R_{\text{disc},\psi})(f),$$

for some positive constant C. Since the global intertwining relation is known, the left hand side of this identity is zero, and we obtain $\text{tr}(R_{\text{disc},\psi})(f) = 0$, thus $R_{\text{disc},\psi} = 0$, as required. Conjecture 7.9.1 also follows from the argument given in the proof of Proposition 8.3.6, which showed that

$$0 = \text{tr}(R^N_{\text{disc},\psi})(\tilde{f}) = \sum_{G^* \in \tilde{\mathcal{E}}_{\text{sim}}(N)} \tilde{\iota}(N, G^*)^0 S^{G^*}_{\text{disc},\psi}(\tilde{f}^{G^*}),$$

where the first identity comes from $\psi \notin \tilde{\Psi}_2(N)$ and the second comes from the stabilization of the twisted trace formula for GL_N. The summands for $G^* \neq G$ vanish because of Remark 8.3.11. This means that ${}^0S^G_{\text{disc},\psi}(\tilde{f}^G)$ also vanishes, and this is Conjecture 7.9.1.

The final case is $\psi \in \Psi_2(G)$ for a (as above uniquely determined) simple twisted endoscopic group G of GL_N. The argument that we just reviewed for the proof of the stable multiplicity formula is still valid in this case, as it relied only on Remark 8.3.11 and ${}^0\tilde{r}^N_{\text{disc},\psi}(\tilde{f}) = 0$. The argument for the proof of Theorem 8.1.4 is quite different, as it has to be, because this theorem now asserts an actual formula for $R^G_{\text{disc},\psi}$, rather than just a vanishing statement. This argument is given in [Art13, §4.7], and is basically the reverse of the argument presented in §6.4. It goes roughly as follows.

Since $\psi \in \Psi_2(G)$, this parameter does not contribute to any of the summands with $M \neq G$ in the expansion of I^G_{disc} given in (8.2.2). This of course depends on the validity of the global theorem (Theorem 8.1.4) for all proper Levi subgroups of G, which is assumed by induction. Thus

$$I^G_{\text{disc},\psi}(f) = \text{tr}\big(R^G_{\text{disc},\psi}(f)\big).$$

On the other hand, the stabilization identity of Proposition 8.2.2 together with the stable multiplicity formula lead to

$$I^G_{\text{disc},\psi}(f) = \sum_{(G',\psi')} \iota(G, G') |\mathcal{S}_{\psi'}|^{-1} \epsilon'(\psi') f^{G'}(\psi'),$$

where the sum runs over the pairs (G', ψ') consisting of an elliptic endoscopic datum G' for G and parameter ψ' for G' that maps to ψ; note that $\sigma(\bar{S}^0_{\psi'})$ is trivial because the group $\bar{S}^0_{\psi'}$ is trivial, due to the discreteness of ψ. We now use the bijection (6.4.5) to switch the set (G', ψ') to the set $\mathcal{S}_\psi$. The sign lemma [Art13, Lemma 4.4.1] shows $\epsilon'(\psi') = \epsilon_\psi(s_\psi \cdot s)$, while an explicit computation shows $\iota(G', G)|\Psi(G', \psi)||\mathcal{S}_{\psi'}|^{-1} = |\mathcal{S}_\psi|^{-1}$, where $\Psi(G', \psi)$ is the set of parameters ψ'

for G' that map to ψ. Finally, the local character identity of the local classification theorem (Theorem 8.1.3), taken over all places, provides the global identity $f'(\psi') = \sum_{\pi \in \Pi_\psi(G)} \langle s_\psi \cdot s, \pi \rangle f_G(\pi)$. Putting these together we obtain

$$\mathrm{tr}\big(R^G_{\mathrm{disc},\psi}(f)\big)(f) = |\mathcal{S}_\psi|^{-1} \sum_{s \in \mathcal{S}_\psi} \sum_{\pi \in \Pi_\psi(G)} \epsilon_\psi(s_\psi \cdot s) \langle s_\psi \cdot s, \pi \rangle f_G(\pi).$$

Substituting $s_\psi \cdot s$ for s in the sum over $\mathcal{S}_\psi$ leads to the identity claimed in Theorem 8.1.4.

This completes the review of the proofs of Theorems 8.1.4 and Conjecture 7.9.1. We remind the reader that we have been omitting the case "N even and $\eta_\psi = 1$". This case is more difficult and requires the arguments of [Art13, §8.2].

Continuing to omit the case "N even and $\eta_\psi = 1$", we can now review the proof of the seed theorems, Theorem 8.1.1 and 8.1.2. This is essentially [Art13, Corollary 5.4.7].

Corollary 8.4.2. *For given $\phi \in \tilde{\Phi}_{sim}(N)$ and $G \in \tilde{\mathcal{E}}_{sim}(N)$ the following statements are equivalent.*

1. *The linear form $S^G_{disc,\phi}$ doesn't vanish.*
2. *Theorem 8.1.1 holds and the unique group asserted by its statement is G.*

Moreover, Theorem 8.1.2 holds for ϕ.

Proof. Let $G^* \in \tilde{\mathcal{E}}_{\mathrm{sim}}(N)$. Since ϕ is simple, it does not contribute to any proper Levi subgroup of G, or any proper endoscopic subgroup of G, according to the global theorem applied to each of these groups. Therefore, the combination of Proposition 8.2.2 and (8.2.2) provide

$$\mathrm{tr}\big(R^{G^*}_{\mathrm{disc},\phi}(f^*)\big) = S^{G^*}_{\mathrm{disc},\phi}(f^*).$$

Therefore $S^{G^*}_{\mathrm{disc},\phi}(f^*) \neq 0$ is equivalent to $\mathrm{tr}\big(R^{G^*}_{\mathrm{disc},\phi}(f^*)\big) \neq 0$. This shows that (1) and (2) are equivalent, noting that we have already argued that there is at most one $G^* \in \tilde{\mathcal{E}}_{\mathrm{sim}}(N)$ with $S^{G^*}_{\mathrm{disc},\phi} \neq 0$.

Regarding Theorem 8.1.2, we couple the non-vanishing of $S^G_{\mathrm{disc},\phi}$ with the stable multiplicity formula to obtain the non-vanishing of $f^G(\phi)$. This stable linear form is the product over all places of the local stable linear forms $f_v^{G_v}(\phi_v)$. The form $f_v^{G_v}(\phi_v)$ transfers to the form $\tilde{f}_v^N(\phi_v)$, which implies that $\phi_v \in \Phi(G_v)$. □

Appendices

A Eisenstein series and the spectral decomposition of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$

In this appendix we will give a brief and superficial review of the results of Langlands' work on Eisenstein series and the spectral decomposition of the space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$, in particular of its continuous spectrum. We have drawn on various sources, including [Art05, §7,§12], [Gel75, §8], and [Bor97, §10]. For a more detailed discussion we refer to Lapid's article [Lap22] in these proceedings.

Let F be a global field of characteristic zero with ring of adeles $\mathbb{A}$, G a connected reductive F-group, and $P = MN \subset G$ a parabolic subgroup. Let $K \subset G(\mathbb{A})$ be a maximal compact subgroup such that the Iwasawa decomposition $G(\mathbb{A}) = P(\mathbb{A}) \cdot K$ holds.

A.1 Different presentations of parabolic induction

In the literature one often uses without comment multiple equivalent presentations of parabolically induced representations. We review some of them here, and refer the reader to [Kna86, §VII.1] for further discussion in the context of real groups.

Given a unitary representation (π, V_π) of $M(\mathbb{A})$ we have the parabolically induced representation $\mathcal{I}_P(\pi)$ with underlying vector space $\mathcal{H}_P(\pi)$ consisting of measurable functions $f\colon G(\mathbb{A}) \to V_\pi$ that satisfy the properties

1. $f(nmg) = \delta_P(m)^{1/2}\pi(m)f(g)$,
2. $\|f\|^2_{\mathcal{I}_P(\pi)} = \int_K \|f(k)\|^2_\pi dk < \infty$,

and where $\delta_P\colon M(\mathbb{A}) \to \mathbb{R}_{>0}$ is the (restriction to $M(\mathbb{A})$ of the) modulus character of the locally compact group $P(\mathbb{A})$. Slighly more generally, we can apply this construction to the case where the restriction of π to $M(\mathbb{A})^1$ is unitary. A key example for us will be the representation $L^2_{\text{disc}}(M(F)\backslash M(\mathbb{A})^1) \otimes e^{\langle H_M(-),\lambda\rangle}$ of $M(\mathbb{A}) = M(\mathbb{A})^1 \times A_M^+$ for some $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$.

The action of $G(\mathbb{A})$ on this space is via the right regular action, that is, $(\mathcal{I}_P(\pi, x)f)(g) = f(gx)$. We recall that $\delta_P\colon M(\mathbb{A}) \to \mathbb{R}_{>0}$ is the unique character whose restriction to $A_M(\mathbb{A}_F)$ is given by $\delta_P(x) = \prod_{\alpha\in R(A_M,N)} |\alpha(x)|^{\dim(\mathfrak{n}_\alpha)}$, where $A_M \subset M$ is the maximal split central torus, $R(A_M, N)$ is the set of weights for the adjoint action of A_M on $\mathfrak{n} := \text{Lie}(N)$, and $\mathfrak{n}_\alpha$ is the eigenspace for the weight α. If $M \to A'_M$ is the maximal split quotient torus, the natural map $A_M \to A'_M$ is an isogeny and the character on $A_M(\mathbb{A})$ defined by the above formula extends uniquely to $A'_M(\mathbb{A}_F)$, and can then be pulled back to $M(\mathbb{A}_F)$. In terms of the function H_M, we have

$$\delta_P(m) = e^{\langle H_M(m), 2\rho_P\rangle}, \qquad \rho_P = \frac{1}{2}\sum_{\alpha\in R(A_M,N)} \dim(\mathfrak{n}_\alpha)\alpha \in \mathfrak{a}^*_M.$$

For many applications one embeds the representation π into a continuous family $\pi_\lambda = \pi \otimes e^{\langle H_M(-),\lambda\rangle}$ indexed by $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$, or some subspace thereof. The vector space $\mathcal{H}_P(\pi_\lambda)$ then depends on λ, and this is inconvenient when one wants to study analytic questions in the variable λ. To avoid this problem one can consider the vector space

$$\mathring{\mathcal{H}}_P(\pi) := \left\{ \mathring{f}\colon G(\mathbb{A}) \to V_\pi \,\middle|\, \mathring{f}(nmg) = \pi(m)\mathring{f}(g) \right\}$$

subject to the same requirement $\|\mathring{f}\|^2_{\mathcal{I}_P(\pi)} < \infty$ as above, but now the action of $x \in G(\mathbb{A})$ on this space is given by

$$(\mathring{\mathcal{I}}_P(\pi,\lambda,x)\mathring{f})(g) = e^{\langle H_P(gx)-H_P(g),\lambda+\rho_P\rangle}\mathring{f}(gx),$$

where we recall from (7.3.7) that $H_P(nmk) = H_M(m)$ with $n \in N(\mathbb{A})$, $m \in M(\mathbb{A})$, $k \in K$. We have thus recorded the parameter λ in the action, rather than in the vector space. The isomorphism $\mathcal{H}_P(\pi_\lambda) \to \mathring{\mathcal{H}}_P(\pi)$ sending f to $\mathring{f}(g) = e^{-\langle H_P(g),\lambda+\rho_P\rangle}f(g)$ intertwines the representations $\mathcal{I}_P(\pi_\lambda)$ and $\mathring{\mathcal{I}}_P(\pi,\lambda)$.

Given a second parabolic subgroup P' with Levi factor M one has the standard intertwining operator $J_{P'|P}(\pi)\colon \mathcal{H}_P(\pi) \to \mathcal{H}_{P'}(\pi)$ defined by

$$J_{P'|P}(\pi)(f)(g) = \int_{(N(\mathbb{A})\cap N'(\mathbb{A}))\backslash N'(\mathbb{A})} f(n'g)dn'.$$

One then formally sees that $J_{P'|P}$ intertwines the representations $\mathcal{I}_P(\pi)$ and $\mathcal{I}_{P'}(\pi)$. For this, it is useful to note that

$$\int_{(N(\mathbb{A})\cap N'(\mathbb{A}))\backslash N'(\mathbb{A})} f(mn'm^{-1})dn' = \delta_P^{1/2}(m)\delta_{P'}^{-1/2}(m)\int_{(N(\mathbb{A})\cap N'(\mathbb{A}))\backslash N'(\mathbb{A})} f(n')dn'.$$

The integral defining $J_{P'|P}(\pi)(f)$ doesn't always converge, but if one replaces π with π_λ for sufficiently positive λ then it does. For this it is useful to pass to $\mathring{\mathcal{H}}(\pi)$, which is unaffected by this twist. Under the isomorphism $\mathcal{I}_P(\pi_\lambda) \to \mathring{\mathcal{I}}_P(\pi,\lambda)$ described above the operator $J_{P'|P}(\pi_\lambda)$ is translated to the operator $\mathring{J}_{P'|P}(\pi,\lambda)\colon \mathring{\mathcal{H}}_P(\pi) \to \mathring{\mathcal{H}}_{P'}(\pi)$ defined by

$$\mathring{J}_{P'|P}(\pi,\lambda)(\mathring{f})(g) = e^{-\langle H_{P'}(g),\lambda+\rho_{P'}\rangle}\int_{(N(\mathbb{A})\cap N'(\mathbb{A}))\backslash N'(\mathbb{A})} e^{\langle H_P(n'g),\lambda+\rho_P\rangle}\mathring{f}(n'g)dn'.$$

A.2 Eisenstein series

In Appendix A.1 we discussed various presentations of the parabolically induced representation $\mathcal{I}_P(\pi)$, where π is a representation of $M(\mathbb{A})$ with unitary restriction to $M(\mathbb{A})^1$. Of central importance is $\pi = L^2_{\text{disc}}(M(F)A_M^+\backslash M(\mathbb{A}))$. The space $\mathcal{H}_P(\pi)$ now consists of functions $f\colon G(\mathbb{A}) \times (M(F)A_M^+\backslash M(\mathbb{A})) \to \mathbb{C}$ such that $f(nmg,m') = \delta_P(m)^{\frac{1}{2}}f(g,m'm)$ for $n \in N(\mathbb{A})$, $m,m' \in M(\mathbb{A})$, $g \in G(\mathbb{A})$. Therefore we can forget the second variable and interpret this as a function $\phi\colon N(\mathbb{A})M(F)\backslash G(\mathbb{A}) \to \mathbb{C}$, namely via $\phi(g) = f(g,1)$ and $f(g,m) = \delta_P(m)^{-\frac{1}{2}}\phi(mg)$. The function ϕ has to satisfy the properties that for all $x \in G(\mathbb{A})$, $\phi_x \in L^2_{\text{disc}}(M(F)\backslash M(\mathbb{A})^1)$, where $\phi_x(m) = \phi(mx)$, and moreover that

$$\|\phi\|_2 = \int_K \int_{M(F)\backslash M(\mathbb{A})^1} |\phi(mk)|^2 < \infty,$$

where we have taken arbitrary Haar measures on all groups (the choice does not influence the condition).

The action of $G(\mathbb{A})$ on this space is given again by the right regular action, i.e.

$$(\mathcal{I}_P(x)\phi)(g) = \phi(gx).$$

We can do the same also with $\pi_\lambda = L^2_{\text{disc},\lambda}(M(F)\backslash M(\mathbb{A}))$ for some $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, in which case one would be looking at functions $\phi\colon N(\mathbb{A})M(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ which transform on the left under $z \in A_M^+$ via $e^{\langle H_M(z),\lambda\rangle}$. The resulting induced representation shall be denoted by $\mathcal{I}_P(\lambda)$.

The function ϕ on $G(\mathbb{A})$ is left-invariant under $P(F)$, but not under $G(F)$, i.e. it is not itself an automorphic form. The Eisenstein series construction

$$E(x,\phi) = \sum_{p\in P(F)\backslash G(F)} \phi(px) \tag{A.2.1}$$

produces a function in $x \in G(\mathbb{A})$ that is left-invariant under $G(F)$, provided the sum converges. Moreover, formally, the map

$$\phi \mapsto E(-,\phi)$$

intertwines the representation $\mathcal{I}_P(\pi_\lambda)$ on $\mathcal{H}_P(L^2_{\text{disc},\lambda}(M(F)\backslash M(\mathbb{A})))$ with the right regular representation of $G(\mathbb{A})$ on a subspace of the space of automorphic forms with central character λ, where we treat λ as an element of $\mathfrak{a}^*_{G,\mathbb{C}}$ via the projection $\mathfrak{a}^*_M \to \mathfrak{a}^*_G$. However, these automorphic forms need not be square-integrable, and one needs a further Fourier transform to obtain an element of $L^2_\lambda(G(F)\backslash G(\mathbb{A}))$, cf. Appendix A.3.

Consider now a second parabolic subgroup $P' = M'N'$ with the same Levi factor $M' = M$. As in Appendix A.1 we have the intertwining operator

$$J_{P'|P}\colon \mathcal{H}_P(L^2_{\text{disc},\lambda}(M(F)\backslash M(\mathbb{A}))) \to \mathcal{H}_{P'}(L^2_{\text{disc},\lambda}(M(F)\backslash M(\mathbb{A})))$$

defined by

$$(J_{P'|P}\phi)(g) = \int_{(N(\mathbb{A})\cap N'(\mathbb{A}))\backslash N'(\mathbb{A})} \phi(n'g)dn'.$$

It is useful to rigidify the setting by fixing a minimal parabolic subgroup $P_0 = M_0N_0$. Every $G(F)$-conjugacy class of parabolic subgroups has a unique representative P (called standard) that contains P_0, and P has a unique Levi factor M (again called standard) that contains M_0. Another standard parabolic subgroup P' is called *associate* to P if the standard Levi factors M and M' of P and P' are $G(F)$-conjugate, equivalently $N_G(M_0)(F)$-conjugate. In that case, let $s \in W^G(M_0)(F) = N_G(M_0)(F)/M_0(F)$ be an element such that $sMs^{-1} = M'$. We can form the operator

$$M_{P'|P}(s,\lambda)\colon \mathcal{H}_P(L^2_{\text{disc},\lambda}(M(F)\backslash M(\mathbb{A}))) \to \mathcal{H}_{P'}(L^2_{\text{disc},s\lambda}(M'(F)\backslash M'(\mathbb{A})))$$

defined as the composition $M_{P'|P}(s,\lambda)(s) = J_{P'|sPs^{-1}} \circ \ell_s$, where

$$\ell_s\colon \mathcal{H}_P(L^2_{\text{disc},\lambda}(M(F)\backslash M(\mathbb{A}))) \to \mathcal{H}_{sPs^{-1}}(L^2_{\text{disc},s\lambda}(sMs^{-1}(F)\backslash sMs^{-1}(\mathbb{A})))$$

is defined by $(\ell_s\phi)(g) = \phi(\dot{s}^{-1}g)$ for an arbitrary lift $\dot{s} \in N_G(M_0)(F)$ of s; the $M(F)$-equivariance of ϕ implies that the choice of lift is irrelevant. In the literature this operator is oftentimes abbreviated to $M(s,\lambda)$ or $M(s)$, when the remaining parameters are clear from the context. In the special case of $s = 1$ the operator $M_{P'|P}(s)$ recovers the operator $J_{P'|P}$, and is sometimes denoted by $M_{P'|P}$. We also have the version $M_{P'|P}(s,\sigma_\lambda) : \mathcal{H}_P(\sigma_\lambda) \to \mathcal{H}_{P'}(\sigma_\lambda)$, given by the same definition, but where now σ is any representation of $M(\mathbb{A})$ with unitary restriction to $M(\mathbb{A})^1$.

In summary, the intertwining operator $M(s)$ shows that the unitary $G(\mathbb{A})$-representation $\mathcal{H}_P(L^2_{\text{disc},\lambda}(M(F)\backslash M(\mathbb{A})))$ depends only on the conjugacy class of M; we remind ourselves that λ is imaginary and that, in the global context that we are discussing, the intertwining operator $M(s)$ is holomorphic at imaginary λ.

The definitions of the Eisenstein series and of the intertwining operator are given by a sum and an integral over non-compact spaces, and in general will not converge, and the serious difficulties in the theory of Eisenstein series arise from this fact. They converge when $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ has sufficiently positive real part, and Langlands has shown that as functions in λ they can be analytically continued to those λ whose real part is zero, which is the case of interest for us.

For this reason one usually works in the presentation $\mathring{\mathcal{H}}_P$ that is independent of λ, as discussed in Appendix A.1. Setting $\mathring{\phi}(g) = \mathring{f}(g,1)$ we obtain the space of functions $\mathring{\phi}\colon N(\mathbb{A})M(F)A_M^+\backslash G(\mathbb{A}) \to \mathbb{C}$ satisfying the properties that $\mathring{\phi}_x \in L^2_{\text{disc}}(M(F)A_M^+\backslash M(\mathbb{A}))$ and $\|\mathring{\phi}\|_2 < \infty$. The dependence of λ is recorded in the action of $G(\mathbb{A})$, which is by

$$(\mathring{\mathcal{I}}_P(\lambda,x)\mathring{\phi})(g) = e^{\langle H_P(gx)-H_P(g),\lambda+\rho_P\rangle}\mathring{\phi}(gx).$$

The passage between ϕ and $\mathring{\phi}$ is again via $\mathring{\phi}(g) = e^{-\langle H_P(g),\lambda+\rho_P\rangle}\phi(g)$. In these coordinates the Eisenstein series construction becomes

$$E(x,\mathring{\phi},\lambda) = \sum_{p\in P(F)\backslash G(F)} \mathring{\phi}(px)e^{\langle H_P(px),\lambda+\rho_P\rangle},$$

and the intertwining operator becomes

$$M(s,\lambda)(\mathring{\phi})(g) = e^{-\langle H_{P'}(g),s\lambda+\rho_{P'}\rangle}\int_{(sNs^{-1}(\mathbb{A})\cap N'(\mathbb{A}))\backslash N'(\mathbb{A})} e^{\langle H_P(\dot{s}^{-1}n'g),\lambda+\rho_P\rangle}\mathring{\phi}(\dot{s}^{-1}n'g)dn'.$$

Example A.2.1. We give an example connecting the general definition of Eisenstein series (A.2.1) to the classical Eisenstein series, cf. [Bor97, §10.8], [Gel75, Remark 8.5].

Let $k \geq 2$ be an integer. Recall that the classical *holomorphic Eisenstein series* is defined as

$$G_{2k}(z) = \sum_{(0,0)\neq(c,d)\in\mathbb{Z}^2} \frac{1}{(cz+d)^{2k}}.$$

This is a function $\mathcal{H} \to \mathbb{C}$, where $\mathcal{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ is the upper half plane of the complex numbers, and is a modular form of weight $2k$ for the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. In particular $G_{2k}(\gamma z) = \mu(\gamma, z)^{2k} G_{2k}(z)$, where

$$\gamma z = \frac{az+b}{cz+d}, \qquad \mu(\gamma, z) = cz + d, \qquad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix};$$

the term $\mu(\gamma, z)$ is called the factor of automorphy. If we define

$$e_{2k}(z) = \frac{1}{2} \sum_{\substack{(0,0) \neq (c,d) \in \mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{1}{(cz+d)^{2k}}$$

then a simple calculation shows $G_{2k}(z) = 2\zeta(2k) \cdot e_{2k}(z)$. The function e_{2k} is thus still a modular form of weight $2k$ for Γ, called a *holomorphic Eisenstein series*.

A related object is the *real analytic Eisenstein series*, defined for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ as

$$e_s(z) = \frac{1}{2} \sum_{\substack{(0,0) \neq (c,d) \in \mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{\mathrm{Im}(z)^s}{|cz+d|^{2s}},$$

and via analytic continuation for all $s \in \mathbb{C} \setminus \{1\}$. This is again a function $\mathcal{H} \to \mathbb{C}$ and behaves like a modular form of weight 0, i.e. it is invariant on the left under Γ, and is real analytic, but in general not holomorphic.

We can reinterpret the summation index in the definitions of e_{2k} and e_s using the group $G = \mathrm{SL}_2/\mathbb{Q}$. Let $B \subset G$ be the standard Borel subgroup, consisting of upper triangular matrices. Write $B = TU$, with T the standard maximal torus, consisting of diagonal matrices, and U the subgroup consisting of unipotent upper triangular matrices.

Consider the (right) action of $\mathrm{GL}_2(\mathbb{Q})$ on $\mathbb{Q}^2$ given by interpreting $\mathbb{Q}^2$ as the set of row vectors (a, b) and multiplying them on the right by the corresponding matrix. This action commutes with the scaling action of $\mathbb{Q}^\times$ and induces a transitive action of $\mathrm{GL}_2(\mathbb{Q})$ on $\mathbb{P}^1(\mathbb{Q}) = (\mathbb{Q}^2 \setminus \{(0,0)\})/\mathbb{Q}^\times$. The stabilizer of $[0:1] \in \mathbb{P}^1(\mathbb{Q})$ is the group of upper triangular matrices in $\mathrm{GL}_2(\mathbb{Q})$. In this way we obtain a bijection

$$B(\mathbb{Q}) \backslash G(\mathbb{Q}) \to \mathbb{P}^1(\mathbb{Q}),$$

which explicitly extracts the bottom row of the matrix and interprets it as a row vector of homogenous coordinates of an element of $\mathbb{P}^1(\mathbb{Q})$. On the other hand, it is immediate that the elements of $\mathbb{P}^1(\mathbb{Q})$ are in 1-1 correspondence with tuples (c, d) of coprime integers with $(c, d) \neq (0, 0)$, taken up to the equivalence relation $(c, d) \sim (-c, -d)$. By the Euclidean algorithm we can find $a, b \in \mathbb{Z}$ such that $ad - bc = 1$. We obtain the bijections

$$B(\mathbb{Z}) \backslash G(\mathbb{Z}) \to B(\mathbb{Q}) \backslash G(\mathbb{Q}) \to \{(0,0) \neq (c,d) \in \mathbb{Z}^2 | \gcd(c,d) = 1\} / \sim .$$

Using this bijection, and the computation

$$\operatorname{Im}(\gamma z) = \operatorname{Im}\left(\frac{(az+b)(\overline{cz+d})}{|cz+d|^2}\right) = \frac{\operatorname{Im}(z)}{|cz+d|^2},$$

we can rewrite the definitions of e_{2k} and e_s as

$$e_{2k}(z) = \sum_{\gamma \in G(\mathbb{Z})\backslash B(\mathbb{Z})} \mu(\gamma, z)^{-2k}, \qquad e_s(z) = \sum_{\gamma \in G(\mathbb{Z})\backslash B(\mathbb{Z})} \operatorname{Im}(\gamma z)^s.$$

We now recall the standard passage between modular forms $\mathcal{H} \to \mathbb{C}$ and automorphic forms $G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$, cf. [Gel75, (3.4)]. It is a two-step process. The first step associates to a modular form $f\colon \mathcal{H} \to \mathbb{C}$ of weight $2k$ the function $\phi\colon G(\mathbb{R}) \to \mathbb{C}$ defined by $\phi(g) = f(g \cdot i)\mu(g, i)^{-2k}$. This function is now left-invariant under $G(\mathbb{Z})$. The second step uses the fact that the inclusion $G(\mathbb{R}) \to G(\mathbb{A})$ induces a bijection $G(\mathbb{Q})\backslash G(\mathbb{A})/K_f = G(\mathbb{Z})\backslash G(\mathbb{R})$, where $K_f = G(\widehat{\mathbb{Z}}) \subset G(\widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q})$, in order to interpret ϕ as a function on $G(\mathbb{Q})\backslash G(\mathbb{A})$. Applying the first step to the holomorphic Eisenstein series e_{2k} and using the cocycle identity $\mu(g_1 g_2, z) = \mu(g_1, g_2 z)\mu(g_2, z)$ we obtain the function

$$E_{2k}\colon G(\mathbb{Z})\backslash G(\mathbb{R}) \to \mathbb{C}, \qquad E_{2k}(g) = \sum_{\gamma \in G(\mathbb{Z})\backslash B(\mathbb{Z})} \mu(\gamma g, i)^{-2k}.$$

Applying the same procedure to the real analytic Eisenstein series e_s we obtain

$$E_s\colon G(\mathbb{Z})\backslash G(\mathbb{R}) \to \mathbb{C}, \qquad E_s(g) = \sum_{\gamma \in G(\mathbb{Z})\backslash B(\mathbb{Z})} \operatorname{Im}(\gamma g i)^s.$$

Recall the Iwasawa decomposition $G(\mathbb{R}) = U(\mathbb{R})T(\mathbb{R})K_\infty$, where $K_\infty = \mathrm{SO}(2) \subset G(\mathbb{R})$. Let us parameterize the three groups $U(\mathbb{R})$, $T(\mathbb{R})$, and K_∞ by $n \in \mathbb{R}$, $a \in \mathbb{R}^\times$, and $\theta \in [0, 2\pi)$, using

$$u(n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}, \quad t(a) = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, k(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix},$$

A direct computation shows $\operatorname{Im}(u(n)t(a)k(\theta)i) = a^2 = |a|^2$, while the cocycle identity for μ implies $\mu(u(n)t(a)k(\theta), i) = a^{-1}e^{-i\theta}$. In fact, we can use the same formulas to parameterize also $U(\mathbb{Q}_p)$ and $T(\mathbb{Q}_p)$ by $n \in \mathbb{Q}_p$ and $a \in \mathbb{Q}_p^\times$, as well as the adelic groups $U(\mathbb{A})$ and $T(\mathbb{A})$ by $n \in \mathbb{A}$ and $a \in \mathbb{A}^\times$. Then we can define the functions $\varphi_{2k}, \varphi_s\colon T(\mathbb{Q})\backslash T(\mathbb{A}) \cdot K_\infty K_f \to \mathbb{C}$ by

$$\varphi_{2k}(t(a) \cdot k(\theta) \cdot k_f) = |a|^{2k} e^{2ik\theta}, \quad \varphi_s(t(a) \cdot k(\theta) \cdot k_f) = |a|^s,$$

where $a \in \mathbb{A}^\times$, $\theta \in [0, 2\pi)$, and $k_f \in K_f$.

These functions are elements of $\mathcal{I}_B^G(L^2_\chi(T(\mathbb{Q})\backslash T(\mathbb{A}))$ for the characters $\chi(t(a)) = |a|^{2k-1}$ and $\chi(t(a)) = |a|^{s-1}$, respectively, where we note that the modulus character $\delta_B\colon T(\mathbb{A}) \to \mathbb{R}_{>0}$ is given by $\delta_B(a) = |a|^2$. This reveals the functions $E_{2k}, E_s\colon G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ as coming from the general Eisenstein series construction (A.2.1), namely

$$E_{2k}(g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \varphi_{2k}(\gamma g), \qquad E_s(g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \varphi_{2s}(\gamma g).$$

A.3 The spectral decomposition

In Appendix A.2 we reviewed some basic constructions in the theory of Eisenstein series. We will now give a very brief summary of how they are used to obtain a description of the space $L^2(G(F)\backslash G(\mathbb{A}))$ of square-integrable automorphic forms. This description is called the *spectral decomposition*, and is the subject of Lapid's article [Lap22] in these proceedings, to which we refer for a detailed discussion. A summary can also be found in [Art05, §7].

The spectral decomposition theorem of Langlands can be stated in a very rough form as follows.

$$L^2(G(F)\backslash G(\mathbb{A})) = \bigoplus_M \mathrm{Ind}_P^G(L^2_{\text{disc}}(M(F)\backslash M(\mathbb{A})))^{W^G(M)}, \tag{A.3.1}$$

where the index runs over the set of $G(F)$-conjugacy classes of Levi subgroups $M \subset G$, and Ind_P^G denotes normalized parabolic induction with respect to an arbitrary parabolic subgroup P with Levi factor M. The special summand for $M = G$ is the discrete spectrum $L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A}))$. The remaining summands constitute together the continuous spectrum $L^2_{\text{cts}}(G(F)\backslash G(\mathbb{A}))$. The continuous quality comes from the fact that each space $L^2_{\text{disc}}(M(F)\backslash M(\mathbb{A}))$ decomposes as a direct integral

$$\int_{i\mathfrak{a}_M^*} L^2_{\text{disc},\lambda}(M(F)\backslash M(\mathbb{A}_F))d\lambda \tag{A.3.2}$$

as described in (4.2.2).

The identity (A.3.1) is not just a G-equivariant isomorphism of vector spaces, but an isometry of Hilbert spaces. It is obtained by putting together various G-equivariant isometries

$$\mathrm{Ind}_P^G(L^2_{\text{disc}}(M(F)\backslash M(\mathbb{A})))^{W^G(M)} \to L^2(G(F)\backslash G(\mathbb{A})).$$

These isometries are given by Eisenstein series.

Slightly more precisely, recall from Appendix A.2 that an element of the left hand side can be thought of as a function $\mathring{\phi}\colon N(\mathbb{A})M(F)A_M^+\backslash G(\mathbb{A}) \to \mathbb{C}$ satisfying certain properties, in particular $\mathring{\phi}_x \in L^2_{\text{disc}}(M(F)\backslash M(\mathbb{A})^1)$ for all $x \in G(\mathbb{A})$, and to each such function one can try to form the Eisenstein series $E(x, \mathring{\phi}, \lambda)$. Langlands has proved that if $\mathring{\phi}$ is K-finite and λ has sufficiently positive real part, than both the Eisenstein series $E(x, \mathring{\phi}, \lambda)$ and the intertwining operator $M(s, \lambda)(\mathring{\phi})(x)$ converge. The function $x \mapsto E(x, \mathring{\phi}, \lambda)$ is automorphic. However, it is usually not square-integrable. In order to obtain a square-integrable automorphic form, thus an element of $L^2(G(F)\backslash G(\mathbb{A}))$, one applies a Fourier transform.

The first kind of Fourier transform, which avoids the convergence problems of Eisenstein series, is the following. Let $\Psi\colon \mathfrak{a}^*_{M,\mathbb{C}} \times N(\mathbb{A})M(F)A_M^+\backslash G(\mathbb{A}) \to \mathbb{C}$ be a function that is entire in the factor $\mathfrak{a}^*_{M,\mathbb{C}}$, and for each $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ the function $\Psi(\lambda, -)$ is K-finite and for each $x \in G(\mathbb{A})$ the function $m \mapsto \Psi(\lambda, mx)$

lies not just in $L^2_{\text{disc}}(M(F)\backslash M(\mathbb{A})^1)$, but in fact in a fixed isotypic summand of $L^2_{\text{cusp}}(M(F)\backslash M(\mathbb{A})^1)$. Then for any $\Lambda \in \mathfrak{a}_P^*$ the function

$$\psi(x) = \int_{\Lambda + i\mathfrak{a}_M^*} e^{\langle H_P(x), \lambda + \rho_P \rangle} \Psi(\lambda, x) d\lambda$$

is compactly supported in $H_P(x)$ and the Eisenstein series

$$E\psi(x) = \sum_{p \in P(F)\backslash G(F)} \psi(px)$$

converges and lies in $L^2(G(F)\backslash G(\mathbb{A}))$. This is a result of Langlands and leads to the following preliminary decomposition of the space $L^2(G(F)\backslash G(\mathbb{A}))$. Let $\mathfrak{X}$ denote the set of pairs (P, σ) consisting of a (standard) parabolic subgroup $P = M_P N_P \subset G$ and a cuspidal automorphic representation σ of $M_P(\mathbb{A})^1$, taken up to conjugacy. For such a pair $\chi = (P, \sigma)$ let $L^2_\chi(G(F)\backslash G(\mathbb{A}))$ denote the subspace of $L^2(G(F)\backslash G(\mathbb{A}))$ generated by the functions $E\psi$, for all Ψ as above coming from the σ-isotypic summand of $L^2_{\text{cusp}}(M(F)\backslash M(\mathbb{A})^1)$. Then one has the orthogonal decomposition

$$L^2(G(F)\backslash G(\mathbb{A})) = \bigoplus_{\chi \in \mathfrak{X}} L^2_\chi(G(F)\backslash G(\mathbb{A})).$$

This decomposition is only preliminary. For example, the non-cuspidal parts of the discrete spectrum of G are scattered among the various $L^2_\chi(G(F)\backslash G(\mathbb{A}))$ for χ coming from proper parabolic subgroups. Moreover, the description of the scalar product on $L^2(G(F)\backslash G(\mathbb{A}))$ involves integration along a vertical axis with positive real part (cf. [Art05, Lemma 12.3]), and does not come from the scalar product of a unitary induced representation.

In order to obtain the desired decomposition (A.3.1) from this one, one must shift the contour of integration from a vertical line with positive real part to the imaginary axis, thereby lining it up with (A.3.2). This requires not only the analytic continuation of Eisenstein series and intertwining operators, but also a control of their poles. These poles account for the non-cuspidal discrete spectrum of G. The result is summarized in [Art05, Theorem 7.2].

We will now discuss briefly the example of $G = \mathrm{SL}_2/\mathbb{Q}$. If τ is an irreducible representation of the maximal compact subgroup $K = \mathrm{SO}(2)\mathrm{SL}_2(\widehat{\mathbb{Z}})$ of $G(\mathbb{A})$, and $\psi\colon \mathbb{C} \to \tau$ is an entire function that is the Fourier transform of a compactly supported smooth function on $i\mathbb{R}$, then $E\psi(x) = \frac{1}{2\pi}\int_{\mathrm{Re}(z)=z_0} E(x, \psi(z), z) dz$ lies in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$, where $z_0 > 1$ is fixed. The functions $E\psi$, as τ varies, generate a subspace whose closure is a complement to the cuspidal spectrum $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. The inner product of two such functions is given by

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} E\psi_1(x) E\psi_2(x) dx = \frac{1}{2\pi} \int_{\mathrm{Re}(z)=z_0} (\langle \psi_1(z), \psi_2(-\bar{z}) \rangle + \langle M(z)\psi_1(z), \psi_2(\bar{z}) \rangle) dz.$$

Here $M(z)$ is the standard intertwining operator for the standard Borel and its opposite. If we move the contour of integration from $\mathrm{Re}(z) = z_0$ to $\mathrm{Re}(z) = 0$

we encounter the poles of Eisenstein series and intertwining operators. In the current example there is only a single pole, at $z = 1$, when τ is the trivial representation. The result becomes

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} E\psi_1(x)E\psi_2(x)dx = \frac{1}{2\pi}(\mu\psi_1^0, \psi_2^0) + \frac{1}{4\pi}\int_{i\mathbb{R}} (\psi_1^1(z), \psi_2^1(z))dz,$$

where $\psi^0 = \psi(1)$, $\psi^1(z) = \frac{1}{2}(\psi(z) + M(-z)\psi(-z))$, and μ is the residue of $2\pi i M(z)$ at $z = 1$. The first summand on the right accounts for the contribution of the trivial representation, which is discrete but not cuspidal. The second summand accounts for the purely continuous spectrum. The sum $\psi^1(z) = \frac{1}{2}(\psi(z)+M(-z)\psi(-z))$ is to be seen as coming from the collection of two functions, one associated to the standard Borel, and one to its opposite, satisfying the coherence requirement in [Art05, Theorem 7.2(b)].

B THE TAMAGAWA MEASURE

Let F be a global field of characteristic zero and G a connected linear algebraic F-group. We continue to use the basic notation established in §4.1, and in addition write O for the ring of integers of F. We will recall the Tamagawa measures dg on $G(\mathbb{A})$ and dg^1 on $G(\mathbb{A})^1$, as well as the Tamagawa number $\tau(G)$ and its computation by Langlands, Lai, Kottwitz, and Sansuc, following [Wei82, Appendix 2], [Ono66], [Kot84], and [Kot88].

As in the reductive case, we let $X^*(G)_F$ be the group of F-rational algebraic characters of G, and define $\mathfrak{a}_G = \mathrm{Hom}(X^*(G)_F, \mathbb{R})$,

$$H_G \colon G(\mathbb{A}) \to \mathfrak{a}_G, \qquad \langle H(g), \chi\rangle = \log|\chi(g)|_{\mathbb{A}}, \quad \forall \chi \in X^*(G)_F,$$

and $G(\mathbb{A})^1 = \ker(H_G)$.

We recall first the following fundamental result from reduction theory.

Theorem B.0.1 ([PR94, §5.3, Theorem 5.5])**.** *The volume of $G(\mathbb{A})^1/G(F)$ is finite (with respect to any Haar measure on $G(\mathbb{A})^1$).*

The Tamagawa measure dg is a canonical measure on $G(\mathbb{A})$, whose definition we will recall below. For now assume that it has been defined and proceed to define the Tamagawa number in terms of it.

The real vector space $\mathfrak{a}_G$ has an integral structure, namely $\mathrm{Hom}(X^*(G)_F, \mathbb{Z})$. Therefore, it has a canonical Euclidean measure dx, obtained by choosing a basis for $\mathrm{Hom}(X^*(G)_F, \mathbb{Z})$, identifying $\mathfrak{a}_G$ with $\mathbb{R}^d$ via this basis, and transporting the Lebesgue measure under this identification. Since the basis is ambiguous up to the action of $\mathrm{GL}_d(\mathbb{Z})$, the result is independent of the choice of basis. Define the Haar measure dg^1 on $G(\mathbb{A})^1$ to be the unique measure such that $dx = dg/dg^1$.

Definition B.0.2. The *Tamagawa number* $\tau(G)$ is the measure of $G(\mathbb{A})^1/G(F)$ with respect to the quotient of the measure dg^1 of $G(\mathbb{A})^1$ and the counting measure on the discrete group $G(F)$.

The key result about $\tau(G)$ is the following conjecture of Weil, which has been proved by Weil and Mars for all classical grous, by Langlands for split groups, by Lai for quasi-split groups cf. [Wei82, Appendix 2, §5], and by Kottwitz in general, cf. [Kot88].

Theorem B.0.3. *Assume that G is semi-simple and simply connected, then $\tau(G) = 1$.*

Assuming this result, a formula of Sansuc [San81, Theorem 10.1] gives the expression $\tau(G) = |\mathrm{Pic}(G)| \cdot |\ker^1(F,G)|^{-1}$ for a general connected reductive group G. Kottwitz reinterpreted $|\mathrm{Pic}(G)|$ in terms of Borovoi's fundamental group in [Kot84, (2.4.1)] and derived the following result.

Theorem B.0.4. *For any connected reductive group G*

$$\tau(G) = |\pi_1(G)_{\Gamma,tor}| \cdot |ker^1(F,G)|^{-1}.$$

We recall here from §6.1 that $\pi_1(G)$, the Borovoi fundamental group, is a finitely generated abelian group with Γ-action, functorial in G, and has the property that $X^*(Z(\widehat{G})) = \pi_1(G)$ (although note that the left-hand side is not obviously functorial in G). We have written $\pi_1(G)_{\Gamma,\mathrm{tor}}$ for the torsion subgroup of the Γ-coinvariants in $\pi_1(G)$. Furthermore, $\ker^1(F,G)$ is the kernel of the total localization map $H^1(F,G) \to \prod_v H^1(F_v,G)$, and is known to be a finite abelian group.

Since both $\pi_1(G)$ and $\ker^1(F,G)$ are invariant under inner twisting, the above theorem implies the following result.

Corollary B.0.5. *If G_1 and G_2 are connected reductive groups F-groups that are inner forms of each other, then $\tau(G_1) = \tau(G_2)$.*

In fact, this corollary was the main theorem of [Kot88], which together with the work of Langlands and Lai led to the proof of Theorem B.0.3. Thus, the historical development follows a different path than our exposition here.

The above results were initially stated under the assumption that G have no factors of type E_8, but the necessity of this assumption has since been removed by the work of Chernousov [Che89].

We now come to the definition of the Tamagawa measure dg on $G(\mathbb{A})$, following [Ono66]. Let ω be a non-zero alternating form of top degree on $\mathrm{Lie}(G)(F)$ and let Λ be a non-trivial complex-valued continuous character of $\mathbb{A}/F$. Composing Λ with the natural map $F_v \to \mathbb{A}/F$ for a place v of F we obtain a non-trivial character $\Lambda_v \colon F_v \to \mathbb{C}^\times$. Let $(dx)_v$ be the Haar measure on F_v that is self-dual with respect to Λ_v, i.e. the one for which the Fourier inversion formula

$$\widehat{\widehat{f}}\,(x) = f(-x)$$

holds for any smooth compactly supported function f on F_v, where

$$\widehat{f}(y) = \int_{F_v} f(x)\Lambda_v(xy)(dx)_v.$$

The measure $(dx)_v$ and the form ω induce a Haar measure $(dg)_v$ on $G(F_v)$ in the usual way: the form ω can be identified with a nowhere vanishing top differential form on $G(F_v)$ via left translations; given a function $f \in \mathcal{C}_c^\infty(G(F_v))$ one defines its integral as the integral of the differential form $f \cdot \omega$ on the (real or p-adic) manifold $G(F_v)$; the latter can be reduced by a partition of unity to the case when the support of f is contained in the domain of definition of a chart, after which $f \cdot \omega$ can be transported via this chart to a top form on $F_v^{\dim(G)}$ and then integrated with respect to $(dx)_v$.

We have thus obtained a collection $((dg)_v)_v$ of measures and would like to define a measure on $G(\mathbb{A})$ as the product $\prod_v (dg)_v$. The question is whether this product converges. Before we discuss this question, let us argue that this product is independent of the choices of ω and Λ. Indeed, each of these is unique up to multiplication by an element of $F^\times$, which means that changing either of them changes the collection of local measures $((dx)_v)_v$ to a collection $(|y|_v (dx)_v)_v$ for some $y \in F^\times$. The product formula $\prod_v |y|_v = 1$ implies that the product $\prod_v (dg)_v$ is unchanged.

Coming to the convergence question, we would like to know that for any open subset of $G(\mathbb{A})$ of the form $\prod_v U_v$ with $U_v \subset G(F_v)$ open and relatively compact, the product $\prod_v \mathrm{vol}(U_v; (dg)_v)$ converges to a non-zero quantity. We may omit finitely many factors from this product, since each individual factor $\mathrm{vol}(U_v; (dg)_v)$ is always finite and non-zero. Consider a finite set of places S of F containing all archimedean places, such that G has a smooth model over the ring of S-integers of F. It is enough to check that $\prod_{v \notin S} \mathrm{vol}(G(O_v); (dg)_v)$ converges to a non-zero quantity. To investigate the convergence of this product we use the following result, cf. [Wei82, Theorem 2.2.5].

Theorem B.0.6. *Fix a finite set of places S of F containing all archimedean places, such that G has a smooth model over the ring of S-integers of F, Λ_v is unramified for $v \notin S$, and ω is integral at all $v \notin S$ (i.e. it lies in $\bigwedge^{top} Lie(G)(O_v)$ and has non-zero image in $\bigwedge^{top} Lie(G)(k_v)$). Then for all $v \notin S$ we have*

$$vol(G(O_v); (dg)_v) = q_v^{-\dim(G)} \# G(k_v),$$

where q_v is the size of the residue field of F_v.

Let us compute the right hand side of the above theorem for an algebraic group G defined over the finite field $k = k_v$ of size $q = q_v$ in the following basic cases.

- G is unipotent. Then G is isomorphic as a variety to affine space, hence $q^{-\dim(G)} \# G(k) = 1$.

- G is semi-simple. Then [Ste68b, Theorem 25] implies that

$$\prod_{i=1}^{l} (1 - q^{-a_i}) \leq q^{-\dim(G)} \# G(k) \leq \prod_{i=1}^{l} (1 + q^{-a_i}),$$

  where l is the rank of G and $a_1, \ldots, a_l$ are the invariant degrees. As discussed in [Ste68b, Theorem 24], we have $a_i \geq 2$.

- $G = T$ is a torus. Choose an isomorphism of groups $\bar{k}^\times \to (\mathbb{Q}/\mathbb{Z})^{(p)}$, where the superscript $^{(p)}$ indicates that we are taking away the p-torsion subgroup. This isomorphism gives the isomorphism $T(\bar{k}) = X_*(T) \otimes_{\mathbb{Z}} \bar{k}^\times \to X_*(T) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})^{(p)}$. Thus $T(k) = T(\bar{k})^\phi = (X_*(T) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})^{(p)})^{\phi\otimes q}$, where ϕ denotes the Frobenius automorphism[9] of $T(\bar{k})$ as well as of $X_*(T)$, and q denote multiplication by q on the abelian group $(\mathbb{Q}/\mathbb{Z})^{(p)}$. Since the endomorphism $\phi \otimes q$ is divisible by p, we also have $T(k) = (X_*(T) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}))^{\phi\otimes q}$, giving the exact sequence

  $$1 \to T(k) \to X_*(T) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) \overset{\phi\otimes q-1}{\longrightarrow} X_*(T) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) \to 0.$$

  Taking $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ we obtain the exact sequence

  $$1 \to X^*(T) \overset{\phi^{-1}\otimes q-1}{\longrightarrow} X^*(T) \to \mathrm{Hom}(T(k), \mathbb{Q}/\mathbb{Z}) \to 0,$$

  from which we conclude

  $$|T(k)| = |\mathrm{Hom}(T(k), \mathbb{Q}/\mathbb{Z}))| = |\det(q\phi^{-1} - 1)| = |\det(q - \phi)|.$$

  Since ϕ is an automorphism of a lattice and has finite order, its eigenvalues are roots of unity, and all non-real such come in conjugate pairs. Therefore $\det(q - \phi)$ is positive, and we see

  $$q^{-\dim(T)}|T(k)| = \det(1 - q^{-1}\phi).$$

These computations allow us to understand the convergence of

$$\prod_{v \notin S} \mathrm{vol}(G(O_v); (dg)_v).$$

- When G is unipotent, all factors equal 1.

- When G is semi-simple, the product $\prod_v \prod_i (1 + q_v^{-a_i})$ can be reduced to considering $\prod_v (1 + q_v^{-a_i})$, which is clearly non-zero if it is finite, and is dominated by the exponential of $\sum_v q_v^{-a_i}$. Since $a_i \geq 2$, the latter sum is finite. On the other hand, consideration of the product $\prod_v \prod_i (1 - q_v^{-a_i})$ reduces to $\prod_v (1 - q_v^{-a_i})$, which again is clearly non-zero if it is finite. By definition, it is the inverse value of the partial Dedekind zeta function at a_i. Again since $a_i \geq 2$ it is known that this value is finite and non-zero, cf. [Neu13, Chapter VII, Proposition 5.2].

[9] We note that our convention is that $\phi\lambda = \phi \circ \lambda \circ \phi^{-1}$ for $\lambda \in X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$, so that ϕ is a finite order automorphism of $X_*(T)$. This convention is different from the convention typically used in treatments of finite groups of Lie type, such as [DL76] or [Car93], which use $F\lambda = \phi \circ \lambda$, so that $F = q \cdot \phi$.

- When $G = T$ is a torus the complex vector space $V = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ is a representation of the Galois group of F and $\prod_{v \notin S} \mathrm{vol}(G(O_v); (dg)_v) = L^S(1, V)^{-1}$. Let us decompose $V = V_0 \oplus V_1$, where $V_0 = V^{\mathrm{Gal}(\bar{F}/F)}$ is the isotypic component for the trivial representation, and V_1 is the sum of all other isotypic components. Then $L^S(s, V) = L^S(s, V_0) \cdot L^S(s, V_1)$. It is known, cf. [IK04, §5.13, Corollary 5.47], that $L^S(s, V_1)$ is regular at $s = 1$ with non-zero value. On the other hand, V_0 is direct sum of copies of the trivial representation and hence $L^S(s, V_0)$ has a pole at $s = 1$ of order $\dim(V_0)$, cf. [Neu13, Chapter VII, Corollary 5.11]. Therefore, the product $\prod_{v \notin S} \mathrm{vol}(G(O_v); (dg)_v)$ converges to a non-zero value if and only if $V_0 = \{0\}$, equivalently T is anisotropic.

The conclusion of this analysis is that $\prod_v (dg)_v$ will generally not converge to a Haar measure on $G(\mathbb{A})$. We need to introduce convergence factors $\lambda_v \in \mathbb{R}_{>0}$ and take $\prod_v \lambda_v (dg)_v$. Furthermore, this analysis offers a natural definition of such convergence factors. In the case of a torus, we can take $\lambda_v = L_v(1, V)$ when v is non-archimedean, and $\lambda_v = 1$ when v is archimedean.

In fact, we can take this definition of convergence factors in the general case of an arbitrary connected affine algebraic F-group G, where we take $V = X^*(G) \otimes_{\mathbb{Z}} \mathbb{C}$ as a representation of the Galois group. Of course $X^*(G) = X^*(T)$, where T is the maximal torus quotient of G. This leads to the following definition of the Tamagawa measure dg.

Definition B.0.7. The Tamagawa measure on $G(\mathbb{A})$ is the Haar measure

$$dg = |\Delta_F|^{-\dim(G)/2} \rho_G^{-1} \prod_{v|\infty} (dg)_v \prod_{v \nmid \infty} L_v(1, V)(dg)_v,$$

where $\rho_G = \lim_{s \to 1} (s-1)^r L(s, V)$, r is the dimension of the maximal split torus quotient (equivalently maximal split central torus) of G, and Δ_F is the discriminant of F.

The factors, $|\Delta_F|^{-\dim(G)/2} \rho_G^{-1}$ are included to normalize the measure so as to ensure that $\tau(\mathbb{G}_m) = 1$ and $\tau(\mathrm{Res}_{E/F} G) = \tau(G)$.

Example B.0.8. Consider $G = \mathbb{G}_m$ over the number field F. Verify explicitly that $\tau(G) = 1$. This computation will highlight the number-theoretic nature of the Tamagawa number.

By definition, $\tau(G) = \mathrm{vol}(\mathbb{A}^1/F^\times)$ taken with respect to the measure on this quotient determined by the counting measure on $F^\times$ and the measure on $\mathbb{A}^1$ that, together with the Haar measure $\prod_{v|\infty} (dg)_v \prod_{v \nmid \infty} L_v(1, V)(dg)_v$ on $\mathbb{A}^\times$, induces the Lebesgue measure on $\mathbb{R}$ via the exact sequence $1 \to \mathbb{A}^1 \to \mathbb{A}^\times \to \mathbb{R} \to 0$ given by $\log|-|_{\mathbb{A}}$. Here V is the trivial one-dimensional Galois representation, so $L_v(1, V) = (1 - q_v^{-1})^{-1}$.

We have the canonical identification $\mathrm{Lie}(G)(F) = F$ and we take the standard differential form ω corresponding to $1 \in F$ under the tautological identification $\bigwedge^1 F = F$. We take the character $\Lambda_{\mathbb{Q}} \colon \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \to \mathbb{C}^\times$ that is the product of the character $e^{-2\pi i x}$ on $\mathbb{R}$ and the character $e^{2\pi i x}$ on $\mathbb{Q}/\mathbb{Z} = \prod_p' \mathbb{Q}_p/\mathbb{Z}_p$. We take on $\mathbb{A}_F/F$ the character $\Lambda = \Lambda_{\mathbb{Q}} \circ \mathrm{tr}_{F/\mathbb{Q}}$. For a real place v we then have $\Lambda_v(x) = e^{-2\pi i x}$ and the self-dual measure on $F_v = \mathbb{R}$ is the Lebesgue measure. For a complex place v we have $\Lambda_v(x) = e^{-2\pi i x \bar{x}}$ and the self-dual measure on $F_v = \mathbb{C}$ is *twice* the Lebesgue measure. For $v \nmid \infty$ the self-dual measure on F_v assigns volume 1 to O_v. Therefore the measure $L_v(1, V)(dx)_v = \frac{dx_v}{1-1/q_v}$ assigns volume 1 to $O_v^\times$.

We now compute $\mathrm{vol}(\mathbb{A}^1/F^\times)$. Recall the section $\mathbb{R}_{>0} \to \prod_{v|\infty} F_v^\times \subset \mathbb{A}^\times$ of the idele norm introduced in §4.1. It provides the direct product decomposition $\mathbb{A}^\times = \mathbb{A}^1 \times \mathbb{R}_{>0}$. We are thus computing $\mathrm{vol}(\mathbb{A}^\times/\mathbb{R}_{>0}F^\times)$. Let $\mathbb{A}(\infty) = \prod_{v|\infty} F_v^\times \times \prod_{v\nmid\infty} O_v^\times$. Then $\mathbb{A}^\times/\mathbb{A}(\infty)F^\times$ is a finite group whose order equals the class number h of F, cf. [Neu13, Chap. VI, Prop 1.3]. Since $F^\times \cap \mathbb{A}(\infty) = O^\times$ we obtain $\mathrm{vol}(\mathbb{A}^\times/\mathbb{R}_{>0}F^\times) = h \cdot \mathrm{vol}(\mathbb{A}(\infty)/\mathbb{R}_{>0}O^\times)$. Now $O^\times$ is embedded diagonally in $\mathbb{A}(\infty)$, but replacing this by the diagonal embedding into $F_\infty^\times = \prod_{v|\infty} F_v^\times \subset \mathbb{A}(\infty)$ does not change the quotient measure. With this and the identity $\mathrm{vol}(O_v^\times) = 1$ we see $\mathrm{vol}(\mathbb{A}^\times/\mathbb{R}_{>0}F^\times) = h \cdot \mathrm{vol}(F_\infty^\times/\mathbb{R}_{>0}O^\times)$. Consider now the homomorphism $F_\infty^\times \to \mathbb{R}_{>0}$ obtained by restricting the idele norm. Its kernel F_∞^1 contains $O^\times$ and $\mathbb{R}_{>0}$ is a section of this homomorphism. Thus $\mathrm{vol}(\mathbb{A}^1/F^\times) = h \cdot \mathrm{vol}(F_\infty^1/O^\times)$.

To compute $\mathrm{vol}(F_\infty^1/O^\times)$ we consider the following diagram of locally compact abelian groups with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \prod_{v:F_v=\mathbb{R}} \{\pm 1\} \times \prod_{v:F_v=\mathbb{C}} \mathbb{S}^1 & \longrightarrow & F_\infty^1 & \longrightarrow & H & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \prod_{v:F_v=\mathbb{R}} \{\pm 1\} \times \prod_{v:F_v=\mathbb{C}} \mathbb{S}^1 & \longrightarrow & F_\infty^\times & \xrightarrow{f} & \bigoplus_{v|\infty} \mathbb{R} & \longrightarrow & 0 \\
 & & & & \downarrow g & & \downarrow h & & \\
 & & & & \mathbb{R} & = & \mathbb{R} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where we have $f((x_v)_v) = (\log |x_v|_v)_v$, $h((r_v)_v) = \sum_v r_v$, and $g((x_v)_v) = \log \prod_v |x_v|_v = \sum_v \log |x_v|_v$.

We recall that the measure on

$$F_\infty^\times = \prod_{v:F_v=\mathbb{R}} \mathbb{R}^\times \times \prod_{v:F_v=\mathbb{C}} \mathbb{C}^\times$$

is the product of the Haar mesure on each copy of $\mathbb{R}$ and twice the Haar measure on each copy of $\mathbb{C}$, the measure on the target of g is the Lebesgue measure, the measure on F^1_∞ is the measure induced by these two, and the measure on $O^\times$ is the counting measure.

We further endow $\mathbb{R}$ on the bottom right corner with the Lebesgue measure and $\bigoplus_{v|\infty}\mathbb{R}$ with the product of the Lebesgue measures. This fixes measures on H as well as on the group

$$C := \prod_{v:F_v=\mathbb{R}}\{\pm 1\} \times \prod_{v:F_v=\mathbb{C}}\mathbb{S}^1.$$

One checks that the volume of C is $2^r(2\pi)^s$, where r is the number of real places and s is the number of complex places. On the other hand, $C \cap O^\times = \mu(F)$ is the group of roots of unity of F, a finite group whose size we denote by w. Therefore

$$\mathrm{vol}(\mathbb{A}^1/F^\times) = h \cdot \mathrm{vol}(F^1_\infty/O^\times) = \frac{h2^r(2\pi)^s}{w}\mathrm{vol}(H/\ell(O^\times)).$$

It remains to compute $\mathrm{vol}(H/\ell(O^\times))$. This was done in [Neu13, Chap. I, Proposition 7.5] and the result is

$$\overline{\mathrm{vol}}(H/\ell(O^\times)) = \sqrt{r+s}\cdot R,$$

where R is the *regulator* of F. We have placed an overline on the volume to indicate that this result uses a different measure on H than what we use here, so we have to compare the measures. The measure used in [Neu13] is obtained by taking the isomorphism

$$H \times \mathbb{R} \to \mathbb{R}^{r+s}, \quad (x,a) \mapsto x + a\lambda_0, \quad \lambda_0 = (\sqrt{r+s})^{-1}(1,1,\dots,1),$$

and endowing H with the measure whose product with the Lebesegue measure on $\mathbb{R}$ will match the Lebesgue measure on $\mathbb{R}^{r+s}$ under this isomorphism. Comparing with our choice of measure we conclude

$$\mathrm{vol}(H/\ell(O^\times)) = R,$$

and hence

$$\mathrm{vol}(\mathbb{A}^1/F^\times) = \frac{h2^r(2\pi)^s}{w}R.$$

On the other hand, the analytic class number formula [Neu13, Chap. VII, Corollary 5.11] implies that the same quantity is equal to $\rho_G|\Delta_F|^{1/2}$. Thus $\tau(G) = 1$.

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