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WS 2023/24

Algebraic Geometry I

1. Exercise sheet

Let k be an algebraically closed field.

Exercise 1 (4 points):

1) Classify all closed subsets of $\mathbb{A}^1_k(k)$.
2) Show that the Zariski topology on $\mathbb{A}^2_k(k) = \mathbb{A}^1_k(k) \times \mathbb{A}^1_k(k)$ is not the product topology.

Exercise 2 (4 points):

Let $n \geq 0$. We identify $V := \mathbb{A}^{n^2}_k(k) \cong \mathrm{Mat}_n(k)$ with the set of $n \times n$-matrices with values in k, and we identify $W = \mathbb{A}^n_k(k)$ with the set of monic polynomials in the variable X, which are of degree n and coefficients in k.
1) Show that

$$V \to W,\ A \mapsto \chi_A(X) := \det(X \cdot \mathrm{Id} - A)$$

is a morphism of affine algebraic sets.
2) Let $\mathrm{Disc}(f)$ be the discriminant for $f \in W$. Show that

$$W \to \mathbb{A}^1_k(k),\ f(X) \mapsto \mathrm{Disc}(f)$$

is a morphism of affine algebraic sets.
3) Show that the set of regular semisimple elements, i.e., the set of diagonalizable matrices with pairwise different eigenvalues, is open in V.

Exercise 3 (4 points):

Let $f(x,y) = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6 \in k[x,y]$ be a non-zero polynomial of degree 2 with vanishing locus $V := \{f(x,y) = 0\}$. Show that V is isomorphic to

1. a parabola $\{y - x^2 = 0\}$,
2. the hyperbola $\{xy - 1 = 0\}$,
3. the union of the coordinate axis $\{xy = 0\}$,
4. the disjoint union of two lines $\{x(x-1) = 0\}$,
5. or a single line $\{x = 0\} = \{x^2 = 0\}$.

Hint: Using suitable coordinate transformations of x, y show first that wlog $a_1x^2 + a_2xy + a_3y^2 \in \{x^2, xy\}$.

Exercise 4 (4 points):

Determine which of the 5 options in exercise 3 are isomorphic and which not.
Hint: Determine the respective coordinate rings.

To be handed in on: Thursday, 19.10.2023 (during the lecture, or via eCampus).