

More on  $A_{inf}$

$p$  prime,  $E/\mathbb{Q}_p$  finite,  $\mathcal{O}_E, \varpi \in \mathcal{O}_E$  unif.,  $\mathbb{F}_q = \mathcal{O}_E/\varpi$

$F/\mathbb{F}_q$  non-arch, alg. cld,  $\vartheta \in m_F \setminus \{0\}$  ("  $\vartheta$  a pseudo-uniformizer ")

$$m_F \subseteq \mathcal{O}_F \subseteq F, \quad k := \mathcal{O}_F/m_F$$

$\uparrow$   
non-noeth. val. ring,  $m_F^2 = m_F$ ,  $\varpi$ -adically complete

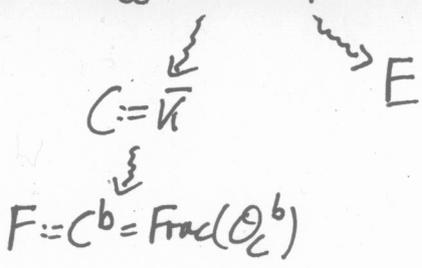
Last time:

$$A_{inf} = A_{inf, E, F} = W_{\mathcal{O}_E}(\mathcal{O}_F) = \left\{ \sum_{i=0}^{\infty} [x_i] \pi^i \mid x \in \mathcal{O}_F \right\} \supseteq \varphi$$

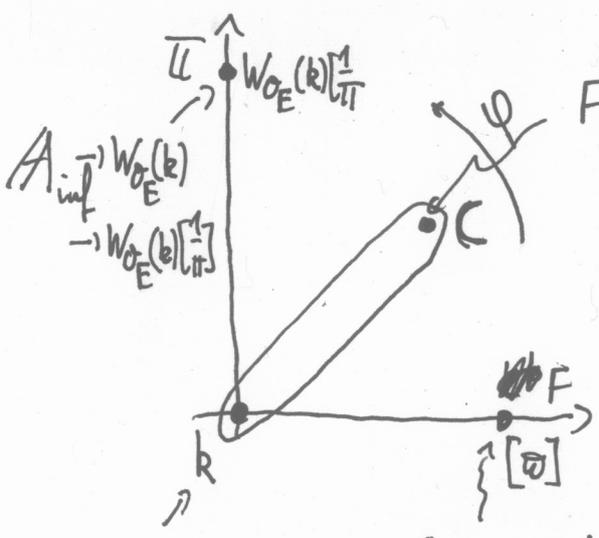
[Side remark: Why this setup?

$W_{\mathbb{Q}_p}$  disc. val, non-arch, perfect residue field,  $X \rightarrow \text{Spec } K$  proper + smooth

Of interest in  $p$ -adic Hodge theory:  $H_{\text{et}}^*(X_{\bar{k}}, \mathbb{Q}_p)$



Picture of  $\text{Spec } A_{inf}$ :



For  $C/E$  alg. cld,  $i: \mathcal{O}_C^b \cong \mathcal{O}_F$

$$\ker(A_{inf} \cong W_{\mathcal{O}_E}(\mathcal{O}_C^b) \xrightarrow{i} \mathcal{O}_C) = (\pi - [z^b])$$

Today:  $B_{dR}^+ = A_{inf} \left[ \frac{1}{[\vartheta]} \right]^{\wedge}, A_{inf, (\vartheta)}$   
DVR

$A_{inf}/\text{max. ideal} \rightarrow W_{\mathcal{O}_F}(F) \rightarrow F$

Def:  $x = \sum_{i=0}^{\infty} [x_i] \pi^i \in A$  primitive if  $x_0 \neq 0$  and ex.  $d \geq 0$ , s.t.  $x_d \in \mathcal{O}_F^*$

degree of  $x := \min \{d \mid x_d \in \mathcal{O}_F^*\}$

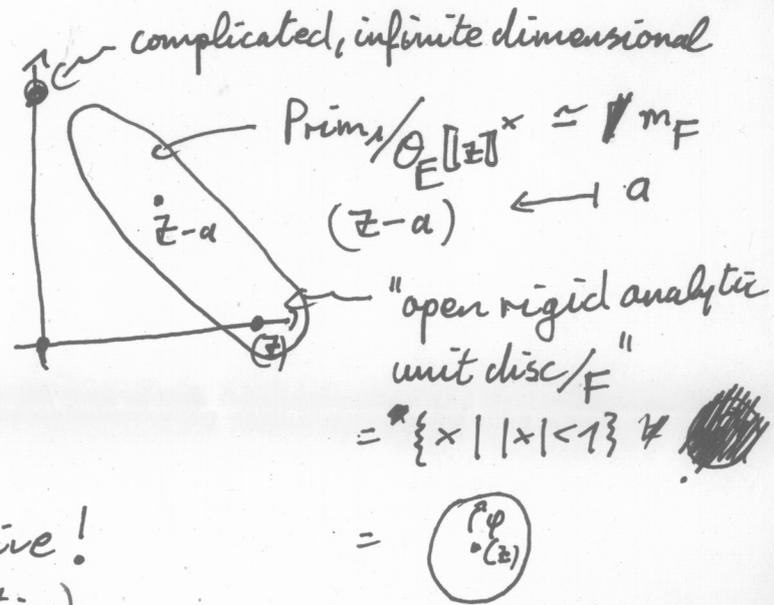
$\text{Prim}_d :=$  set of degree primitive elements.

Ex:  $\text{Prim}_0 = A_{\text{inf}}^*$ ,  $x \in \text{Prim}_{\neq 0} \Rightarrow x$  distinguished  
 $\Leftrightarrow$   
 if  $x \neq 0$

~~Next~~ ~~time~~ ~~Next~~  $a \in m_F \setminus \{0\} \Rightarrow (\pi - [a]) \in A_{\text{inf}}$  prime,  $A_{\text{inf}} / (\pi - [a]) \cong \mathcal{O}_C$ ,  $C/E$  non-arch alg. clsd

Next

For  $\mathcal{O}_F[[z]]$  similar picture:



BUT for  $A_{\text{inf}}$

$m_F \rightarrow \text{Prim}_{A_{\text{inf}}} / A_{\text{inf}}^*$  not bijective!  
 (still surjective)  
 $a \mapsto (\pi - [a])$

(e.g.  $\ker(A_{\text{inf}} \rightarrow \mathcal{O}_C) = (\pi - [\pi^b])$ ,  $\pi^b = (\pi, \pi^{\frac{1}{q}}, \dots) \in \mathcal{O}_C^b$ )  
 $\nwarrow$  not unique

Back to proofs:

Let  $C/E$  non-arch, alg. clsd,  $|\cdot|: C \rightarrow \mathbb{R}_{>0}$  norm.

Recall:  $\mathcal{O}_C^b = \varprojlim_{\text{Frob}} \mathcal{O}_C / \pi \cong \varprojlim_{x \mapsto x^q} \mathcal{O}_C$ ,  $(x^\#, (x^{\frac{1}{q}})^\#, \dots)$

La: 1)  $|\cdot|^b: \mathcal{O}_C^b \rightarrow \mathbb{R}_{>0}$ ,  $x \mapsto |x^\#|$  valuation  $\mathcal{O}_C^b$ ,  $\mathcal{O}_C^b$  complete for top. induced by  $|\cdot|^b$

2)  $C^b := \text{Frac}(\mathcal{O}_C^b)$  non-arch, alg. clsd  
 $(\uparrow$  via  $|\cdot|^b)$

Prof: 1) Clear,  $|x \cdot y|^b = |x|^b \cdot |y|^b$ ,  $|1| = 1$   
 $|x| = 0 \Leftrightarrow x = 0$

Moreover,

$$|x+y|^b = |(x+y)^\#| = \left| \lim_{n \rightarrow \infty} \left( (x^{\frac{1}{q^n}})^\# + (y^{\frac{1}{q^n}})^\# \right)^{q^n} \right| = \lim_{n \rightarrow \infty} \left| (x^{\frac{1}{q^n}})^\# + (y^{\frac{1}{q^n}})^\# \right|^{q^n}$$

$$= \lim_{n \rightarrow \infty} \max \left( \left| (x^{\frac{1}{q^n}})^\# \right|^{q^n}, \left| (y^{\frac{1}{q^n}})^\# \right|^{q^n} \right) = \max(|x|^b, |y|^b)$$

~~Claim~~ Claim: norm top. for  $|-|$  = inverse limit top on  $\mathcal{O}_C^b \cong \varprojlim \mathcal{O}_C$   
 $(\Rightarrow \mathcal{O}_C^b$  complete, as  $\mathcal{O}_C$  complete)

basis of  $\mathcal{O}$  for norm topology:  $\{ |x|^b < \varepsilon \}$ ,  $\varepsilon > 0$

↳ " inverse limit top:  $\{ \exists x \in \mathcal{O}_C^b, \underbrace{\left| (x^{\frac{1}{q^n}})^\# \right|}_{= (|x|^b)^{\frac{1}{q^n}}} < \delta \}$ ,  $\delta > 0$

$\Rightarrow$  Claim

2) Let  $f(T) \in \mathcal{O}_C^b[T]$ ,  $f(T) = a_d T^d + a_{d-1} T^{d-1} + \dots + a_0$ ,  $a_d = 1$

Set  $f_n(T) := (a_d^{\frac{1}{q^n}})^\# T^d + (a_{d-1}^{\frac{1}{q^n}})^\# T^{d-1} + \dots + (a_0^{\frac{1}{q^n}})^\# \in \mathcal{O}_C[T]$

Then  $f_{n+1}(T)^q \equiv f_n(T)^q \pmod{\pi}$

Fix  $n$  and let  $x \in \mathcal{O}_C$  be a zero of  $f_n$ . Choose  $y \in \mathcal{O}_C$ , s.t.  $y^q = x$

$\Rightarrow |f_{n+1}(y)| \leq |\pi|^{\frac{1}{q}}$

Let  $z_1, \dots, z_d \in \mathcal{O}_C$  be the zeros of  $f_{n+1}$

$\Rightarrow |f_{n+1}(y)| = \prod_{i=1}^d |y - z_i| \leq |\pi|^{\frac{1}{q}}$

$\Rightarrow \exists i$ , s.t.  $|y - z_i| \leq |\pi|^{\frac{1}{dq}}$

$\Rightarrow |y - z_i|^q \leq |\pi|^{\frac{1}{d}} \Rightarrow |x - z_i^q| \leq |\pi|^{\frac{1}{d}}$

(If  $|x - z_i^q| > |\pi|^{\frac{1}{d}}$ , then  $|y - z_i|^q - z_i^q + x > |\pi|^{\frac{1}{d}}$   
 $\frac{|y - z_i|^q - z_i^q + x}{\text{div. by } q} > |\pi|^{\frac{1}{d}}$ )

By induction obtain sequence  $(x_n)_{n \geq 0}$ , s.t.  $x_n \in \mathcal{O}_C$ ,

$P_n(x) = 0, |x_{n+1}^q - x_n| \leq |\pi|^{1/q}$ . Set  $\mathcal{O}_C := \{y \in \mathcal{O}_C \mid |\pi|^{1/q} \geq |y|\}$

$\Rightarrow x := (x_n) \in \varprojlim_{\text{Frob}} \mathcal{O}_C/\mathcal{O}_C \xrightarrow{\text{Key La}} \varprojlim \mathcal{O}_C/\pi \cong \mathcal{O}_C^b$

Clear:  $P(x) = 0$  as <sup>each</sup> proj. to  $\mathcal{O}_C/\mathcal{O}_C$  of  $P(x)$  is 0. □

Recall:  $\mathcal{O}_C \cong A_{\text{inf}}(\xi), \xi = \pi - [\pi^b]$

Def:  $B_{dR}^+ := B_{dR}^+(C) := A_{\text{inf}}[\frac{1}{\pi}]^{\wedge} = \varprojlim_n A_{\text{inf}}/\xi^n[\frac{1}{\pi}], B_{dR}^+(\xi) \cong C$

Stacks project, Tag 05GG:

$R$  ring,  $I \subseteq R$  fin. gen.,  $M$   $R$ -module

- $\Rightarrow$  1)  $\hat{M} := \varprojlim_n M/I^n M$  ~~is naturally complete~~  $I$ -adic completion of  $M$   
(in part.  $\hat{M}$   $I$ -adically complete)
- 2)  $M/I^n M \cong \hat{M}/I^n \hat{M}$

La: 1)  $A_{\text{inf}} \hookrightarrow B_{dR}^+$

2)  $B_{dR}^+, A_{\text{inf}}(\xi)$  discr. valuation rings, res. field  $C$

Prof: 1)  $\xi \in A_{\text{inf}}$  non-zero div +  $\mathcal{O}_C$   $\pi$ -tors. free  $\Rightarrow A_{\text{inf}}/\xi^n$   $\pi$ -tors. for all  $n \geq 0$   
 $\Rightarrow A_{\text{inf}}/\xi^n \hookrightarrow A_{\text{inf}}/\xi^n[\frac{1}{\pi}]$  for all  $n \geq 0 \Rightarrow \varprojlim_{\text{left exact}} A_{\text{inf}} \cong \varprojlim_n A_{\text{inf}}/\xi^n \hookrightarrow B_{dR}^+$

Used  $A_{\text{inf}}(\pi, [\pi^b])$ -adically complete and

Stack SP, Tag 03OT:  $R$  ring,  $I \subseteq J \subseteq R$  ideals,  $I$  fin. gen.,  $M$   $J$ -adically complete  $R$ -module. Then  $M$   $I$ -adically complete

( $\hookrightarrow A_{\text{inf}}(\xi)$ -adically complete)

2) SP, Tag 05G4: Ring,  $I \subseteq R$  fin. gen. Assume  $R/I$  noetherian

(5)

$$\Rightarrow R_I^1 = \varinjlim_n R/I^n \text{ noetherian}$$

$\Rightarrow B_{dR}^+$  noetherian

1)

Moreover,  $B_{dR}^+$  local, maximal ideal gen. by one element,  $\dim \text{Spec } B_{dR}^+ = 1$

$\Rightarrow B_{dR}^+(\mathbb{C})$  DVR

For  $A_{inf,(\xi)}$ : ~~Let~~  $\mathfrak{p} \subseteq A_{inf}$  prime ideal, s.t.  $\xi \in \mathfrak{p}$

$$\text{If } a \in \mathfrak{p} \Rightarrow a = \underbrace{b}_{\in A_{inf,(\xi)}} \cdot \underbrace{\xi}_{\in \mathfrak{p}} \Rightarrow b = \frac{a}{\xi} \in \mathfrak{p}, \text{ i.e. } \xi \cdot \mathfrak{p} = \mathfrak{p}$$

$$\text{Set } \mathfrak{a}_{\mathfrak{p}} = \mathfrak{p} \cdot B_{dR}^+ \Rightarrow \xi \mathfrak{a}_{\mathfrak{p}} = \mathfrak{a}_{\mathfrak{p}} \stackrel{B_{dR}^+ \text{ DVR}}{\Rightarrow} \mathfrak{a}_{\mathfrak{p}} = 0 \stackrel{A_{inf,(\xi)} \hookrightarrow B_{dR}^+}{\Rightarrow} \mathfrak{p} = 0$$

$\Rightarrow \text{Spec } A_{inf,(\xi)} = \{(\mathfrak{p}), (\xi)\} \Rightarrow A_{inf,(\xi)}$  noetherian + DVR  
 fin. gen. □

Up to now:  $\{C/E \text{ alg. fld, } \iota: \mathcal{O}_C \xrightarrow{\sim} \mathcal{O}_F\} \xleftrightarrow{\sim} \text{Prim}_{A_{inf}^x} = \{I \subseteq A_{inf} \text{ gen. by prim. elem. of degree } 1\}$

Next aim: Even bijection

~~Under suitable conditions~~

(needs theory of Newton polygons for elements in  $A_{inf}$ )

Sidequestion: Why  $A_{inf}$ ?

Def: ~~Let~~  $R$   $\pi$ -complete  $\mathcal{O}_E$ -alg. A surj.  $D \twoheadrightarrow R$  of  $\mathcal{O}_E$ -alg. with kernel  $I$ , s.t.  $D/I + (\pi)$ -adically complete is called a ~~regular~~  $\pi$ -adic

pro-infinitesimal thickening of  $R$

E.g.:  $R \in \{\mathcal{O}_F, \mathcal{O}_E\} \Rightarrow A_{inf} \twoheadrightarrow R$   $\pi$ -adic pro-inf thickening

La:  $R \in \{\mathcal{O}_C/p, \mathcal{O}_C\}$ ; ~~regular~~

$D \twoheadrightarrow R$   $\pi$ -adic pro-inf. thic.

$\Rightarrow \exists!$  morph.  $A_{inf} \rightarrow D$  over  $R$

Prof: Key La (last time)  $\Rightarrow D^b \approx R^b \Rightarrow \exists! A_{inf} \rightarrow D$  over  $R$  (6)  
 + corollary adj.  $W_{O_E}(-) = (-)^b$

Now,  $E = \mathbb{Q}_p$

Def: A  $p$ -adic PD-thickening of  $R$  is a triple  $(D, D \rightarrow R, (\mathcal{J}_n))$ ,  $D$   $p$ -complete  $\mathbb{Z}_p$ -algebra  
 $D \rightarrow R$  ~~is a~~  $(\mathcal{J}_n)_{n \geq 0}$  is a divided power str. on  $\mathcal{J} := \ker(D \rightarrow R)$

~~Why~~ Rmks: \* ~~...~~  $\mathcal{J}_n(x) = \frac{x^n}{n!}$  comp. wt div. powers on  $(p)$

\* ~~...~~ Normalise  $|\cdot|: \mathbb{C} \rightarrow \mathbb{R}_{\geq 0}$ , s.t.  $|p| = \frac{1}{p}$ .

Then  $|\frac{x^n}{n!}| \leq 1 \forall n \geq 0 \Leftrightarrow |x| \leq p^{-\frac{1}{p-1}}$

$\Rightarrow \{x \mid |x| \leq p^{-\frac{1}{p-1}}\}$  largest ideal in  $\mathcal{O}_{\mathbb{C}}$  admitting divided powers

Def:  $A_{cris}$  := universal  $p$ -adic PD-thickening of  $\mathcal{O}_{\mathbb{C}}$  ( $\Leftrightarrow$  of  $\mathcal{O}_{\mathbb{C}}/p$ )

$$= H^0_{cris}(\mathcal{O}_{\mathbb{C}}/p) = H^0_{cris}(\mathcal{O}_{\mathbb{C}}/p/\mathbb{Z}_p)$$

Concretely,

$A_{cris} =$  ~~...~~ divided power envelope of  $\ker \theta \subseteq A_{inf}$

$$\stackrel{(\S)}{=} A_{inf} \left[ \frac{\xi^n}{n!} \right]_p \simeq A_{inf} \hat{\otimes}_{\mathbb{Z}_p[x]} D_{\mathbb{Z}_p[x]}((x)) \text{ with } D_{\mathbb{Z}_p[x]}((x))$$

$\xi \in A_{inf}$  non-zero divisor

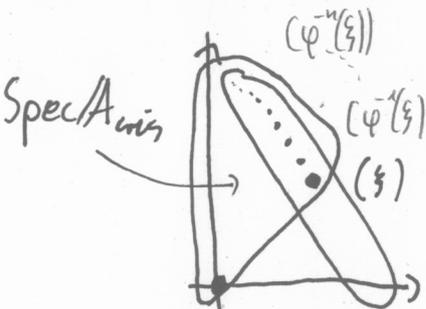
$$= \hat{\bigoplus}_{n \geq 0} \mathbb{Z}_p \cdot \frac{x^n}{n!}$$

$$\mathcal{O}_{\mathbb{C}}/p \simeq \mathcal{O}_{\mathbb{C}}/p$$

$$\simeq (\mathbb{Z}_p[y_0, y_1, \dots]) / (y_0 = x, y_1^p = p y_1, \dots)$$

In part, ~~...~~  $A_{cris}/p \simeq A_{inf}/(p) \hat{\otimes}_{\mathbb{F}_p} \mathbb{F}_p[y_1, y_2, \dots]_{y_i}$

Intuition: The image of  $A_{cris}$  in  $\text{Spec } A_{inf}$  is



Concretely (Exercise)

~~...~~  $a \in m_{\mathbb{F}}$ . If  $|a| \leq |p| = |p|$ , then  $(p-[a]) \cdot A_{cris} = (p)$

If ~~...~~  $a = (p^b)$ , then  $A_{inf} \rightarrow A_{cris}$

$$\mathcal{O}_n = \mathcal{O} \circ \varphi^n$$

$\Rightarrow A_{inf} \xrightarrow{\varphi^n} \mathcal{O}_n$  factors over  $A_{cris}$