

The ring A_{inf}

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Colmez: "One ring to rule them all" (A_{inf} \rightsquigarrow Bars, B_{dR}, \dots)

p prime, E/\mathbb{Q}_p finite, \mathcal{O}_E , $\pi \in \mathcal{O}_E$ unif., $\mathbb{F}_q = \mathcal{O}_E/\pi$

F non-arch, alg. closed/ \mathbb{F}_q , $\mathcal{O}_F = \{x \in F \mid |x| \leq 1\}$ ring of integers
 ↳ complete

w.r.t. a non-trivial norm $| \cdot | : F \rightarrow \mathbb{R}_{\geq 0}$

\mathcal{O}_F perfect \mathbb{F}_q -alg.

Def: $A_{\text{inf}} = A_{\text{inf}, E, F} := W_{\mathcal{O}_E}(\mathcal{O}_F) = \left\{ \sum_{n=0}^{\infty} [x_n] \pi^n \mid x_n \in \mathcal{O}_F \right\} \ni \varphi = F\text{-Witt vector}$

"power series in indeterminant π with coefficients" $\varphi\left(\sum_{n=0}^{\infty} [x_n] \pi^n\right) = \sum_{n=0}^{\infty} [x_n] \pi^n$

Remarks: * ~~$\mathcal{O}_F[[z]]$~~ , $\varphi\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} a_n \pi^n$

Equal characteristic analog:

* $W_{\mathcal{O}_E}(A) =$ ramified Witt vectors (last time), functor $\{\mathcal{O}_E\text{-alg}\} \xrightarrow{W_{\mathcal{O}_E}(G)} \{\mathcal{O}_E\text{-alg}\}$

$W_{\mathcal{O}_E}(A) = A^N \ni (a_0, a_1, \dots) = \sum_{n=0}^{\infty} \pi^n V[a_n]$
 ↳ Teichmüller lift
 assets Verschiebung, $FV = \pi$

~~$\mathcal{O}_E[[z]]$~~ , ~~$\mathcal{O}_E[[z]]$~~

* If R perfect \mathbb{F}_q -alg., $W_{\mathcal{O}_E}(R)$ π -adically complete, π -tors. free

$W_{\mathcal{O}_E}(R)/\pi \simeq R$, $W_{\mathcal{O}_E}(R) \simeq W(R) \otimes_{\mathcal{O}_E} \mathcal{O}_E$, $\mathcal{O}_{E_0} = W(\mathbb{F}_q)$ maximal unramified subcat.
 classical Witt vectors ($E = \mathcal{O}_p, \pi = p$) of \mathcal{O}_E

~~Ring~~

Def: A π -compl. \mathcal{O}_E -alg

$\Rightarrow A^b := \lim_{\leftarrow} A/\pi$ tilt of A
 $x \mapsto x^q$

$$\{(a_0, a_1, \dots) \in \prod_{\mathbb{N}} A/\pi \mid a_{i+1}^q = a_i\}$$

~~W~~ A^b perfect \mathbb{F}_q -algebra : $\text{Frob}_{q, A^b}^{-1}(a_0, a_1, \dots) = (a_0, a_1, \dots)$ ②

Prop: \exists adj: $\{\pi\text{-complete } \mathcal{O}_E\text{-alg}\} \xleftrightarrow[W_{\mathcal{O}_E}(-)]{\text{Frob}_q^{-1}} \{\text{perfect } \mathbb{F}_q\text{-alg.}\}$

Rmk: * The unit $R \rightarrow (W_{\mathcal{O}_E}(R))^b = \varprojlim_{\text{Frob}_q} W_{\mathcal{O}_E}(R)/\pi = \varprojlim_{\text{Fr}} R \rightarrow (r, r^{q}, \dots)$

is an isom

$\Rightarrow W_{\mathcal{O}_E}(-)$ fully faithful, ess. image given by π -compl., π -tors. free \mathcal{O}_E -alg A , s.t. A/π perfect

* The counit $\Theta: W_{\mathcal{O}_E}(A^b) \rightarrow A$ is called Fontaine's map Θ

Proof of Prop: (Only construction of Θ , rest exercise)

Need Key lemma (for everything): B any \mathcal{O}_E -alg, $a, b \in B$, $I \subseteq B$ ideal, $I \subseteq I$

If $a \equiv b \pmod{I}$, then $a^{q^k} \equiv b^{q^k} \pmod{I^{k+1}}$

" p -power map is π -adically contracting"

(was proven last time)

Constr. of Θ : Fix $n \geq 0$

$\Rightarrow w_n: W_{\mathcal{O}_E, n}(A) \rightarrow A/\pi^{n+1}$ morph. of rings (by constr. of $W_{\mathcal{O}_E}(-)$)
 $(a_0, a_1, \dots, a_n) \mapsto \sum_{i=0}^n a_i^{q^{n-i}} \pi^i$

~~W~~ If $a_i \equiv 0 \pmod{\pi}$ $\stackrel{\text{Keyla}}{\Rightarrow} a_i^{q^{n-i}} \equiv 0 \pmod{\pi^{n-i+1}}$

$\Rightarrow \sum_{i=0}^n a_i^{q^{n-i}} \cdot \pi^i \equiv 0 \pmod{\pi^{n+1}}$

(call resulting map $\Theta_n: W_{\mathcal{O}_E, n}(A/\pi) \rightarrow A/\pi^{n+1}$)

Check $W_{O_{E,n+1}}(A/\pi) \xrightarrow{\Theta_{n+1}} A/\pi^{n+2}$ (3)

$$\begin{array}{ccc} \downarrow F & G & \downarrow \text{can} \\ W_{O_{E,n}}(A/\pi) & \xrightarrow{\Theta_n} & A/\pi^{n+1} \end{array}$$

$((a_0, \dots, a_{n+1}) \in W_{O_{E,n+1}}(A/\pi)$ with lift $(\tilde{a}_0, \dots, \tilde{a}_{n+1}) \in W_{O_{E,n+1}}(A)$,

then

$$(a_0, \dots, a_{n+1}) \mapsto \sum_{i=0}^{n+1} (\tilde{a}_i)^q \pi^{n-i+1} \pmod{\pi^{n+2}}$$

$$(a_0^q, \dots, a_n^q) \mapsto \sum_{i=0}^n (\tilde{a}_i^q)^q \pi^{n-i} \pmod{\pi^{n+1}}$$

Passing to \lim_n yields $\Theta: W_O(A^b) \subset \lim_n W_{O_{E,n+1}}(A/\pi) \rightarrow A = \lim_{\pi^n}$ \square

Another application of Key lemma:

Prop: A π -complete O_E -aly, $I \subset A$ ideal, s.t. $\pi \in I$ and A I -adically complete

Then $\lim_{x \mapsto x^q} A \xrightarrow{\sim} (A/I)^b$, $(a_0, a_1, \dots) \mapsto (\bar{a}_0, \bar{a}_1, \dots)$

as mult. monoids ($\text{In part, } \lim_{x \mapsto x^q} A \text{ is naturally a ring}$)

Prf: $x = (x_0, x_1, \dots) \in (A/I)^b$, lift x_i to $\tilde{x}_i \in A$

Then $\{\tilde{x}_i^q\}_{i \geq 0} \subset A$ is Cauchy (for I -adic top.)

(let $j^{2i} = \tilde{x}_j^{q^i} \equiv \tilde{x}_i^{q^i} \pmod{I^{i+1}}$ as $\tilde{x}_j^{q^{j-i}} \equiv \tilde{x}_i^{q^{j-i}} \pmod{I^{j-i}}$)

Set $x^\# := \lim_{i \rightarrow \infty} \tilde{x}_i^q \in A$ (independant of lift by another application of Key Lemma)

Then $(A/I)^b \rightarrow \lim_{x \mapsto x^q} A$, $x \mapsto (x^\#, (\tilde{x}_i^q)^\#, \dots)$ desired inverse \square

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La: $\Theta: W_{\mathcal{O}_E}(A^b) \rightarrow A$ is given by

$$\sum_{i=1}^{\infty} [a_i] \pi^i \mapsto \sum_{i=1}^{\infty} a_i^{\#} \pi^i$$

Prf: Good exercise

Def: 1) A perfect prism over \mathcal{O}_E is a pair $(W_{\mathcal{O}_E}(R), I)$ with
 R perfect \mathbb{F}_q -alg., $I \subseteq W_{\mathcal{O}_E}(R)$ ideal gen. by some $d \in W_{\mathcal{O}_E}(R)$, s.t.
 $\frac{F(d)-d^q}{\pi} \in W_{\mathcal{O}_E}(R)^{\times}$ ("d distinguished") and $W_{\mathcal{O}_E}(R)$ (π, I)-adically complete

2) A perfectoid \mathcal{O}_E -alg if $A \simeq W_{\mathcal{O}_E}(R)/I$ for some perfect prism $(W_{\mathcal{O}_E}(R), I)$ over \mathcal{O}_E

Remarks:

- * $d = \sum_{i=0}^{\infty} [r_i] \pi^i \in W_{\mathcal{O}_E}(R)$ is distinguished iff $r_1 \in R^{\times}$
 $\left(\frac{F(d)-d^q}{\pi} \equiv r_1 \pmod{\pi} \right)$

In this case, $W_{\mathcal{O}_E}(R)$ (p, d) -adically complete iff R (r_1) -adically complete.

* perfect rings are perfectoid (take $d=p$)

* A perfectoid \Rightarrow tilt $A^b \simeq \mathbb{B}(W_{\mathcal{O}_E}(R)/I)^b \simeq (W_{\mathcal{O}_E}(R)/_{(\pi, I)})^b$
 $\simeq W_{\mathcal{O}_E}(R/\bar{I})^b \simeq R$

* Terminology: A is an "untilt" of R , i.e. a pair (A, ι) , A perfectoid ring

* R perfect $\Leftrightarrow \{ \text{untills } (A, \iota) \} / \text{isom} \simeq \{ I \subseteq W_{\mathcal{O}_E}(R) \} \text{ s.t. } (W_{\mathcal{O}_E}(R)/I, \iota: A^b \simeq R)$

Exercise (Tilting equivalence): A perfectoid \mathcal{O}_E -alg. perfect prism $\Rightarrow \{ \text{perfectoid } A\text{-alg} \} \simeq \{ \text{perfectoid } A^b\text{-algebras} \}$

$B \quad \hookrightarrow \quad B^b$

$$W_{\mathcal{O}_E}(S) \otimes_{W_{\mathcal{O}_E}(A^b)} A \quad \hookleftarrow \quad S$$

Hint: $(W_{\mathcal{O}_E}(B))^b / (W_{\mathcal{O}_E}(R) \otimes_{W_{\mathcal{O}_E}(A^b)} I) \rightarrow (W_{\mathcal{O}_E}(B), J)$ is a morph. of perfect prisms

(i.e. $W_{\mathcal{O}_E}(R) \rightarrow W_{\mathcal{O}_E}(S)$ sends I to J), then $J = W_{\mathcal{O}_E}(B) \cdot I$.

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Example: ~~(#)~~ C/E alg. closed non-arch. ext., $C \cap R_{\geq 0} \times \mathbb{R}^{1 \times 1}$ norm

$$\mathcal{O}_C = \{1 \times 1 \leq 1 \mid x \in C\} \Rightarrow \mathcal{O}_C \text{ perfectoid } \mathcal{O}_E\text{-alg.}$$

Pf: (Claim: 1) Let $\pi \in \mathbb{Q}_q$ be a system of q^n -th roots of π ,

$$\pi^b = (\pi, \pi^{\frac{1}{p}}, \dots) \in \varprojlim_{X \rightarrow X^q} O_C^\times \cong O_C^b$$

Then $\Omega_{C/\pi} \simeq \Omega_{C'}^5/\pi^6$

2) $\ker(\Theta: W_{\mathcal{O}_F}(\mathcal{O}_C^b) \rightarrow \mathcal{O}_C)$ is generated by $\pi - [\pi^b] =: \{$

Indeed, ~~a~~ ^{the} Net

for 1) let $\gamma = (\gamma_0, \gamma_1, \dots) \in \varprojlim_{x \mapsto x^q} \mathcal{O}_C^\times \simeq \mathcal{O}_C^\times$ valuation ring
 Then $\pi^b | \gamma \Leftrightarrow \forall n \geq 0 \quad \pi^{\frac{1}{q^n}} | \gamma_n \Leftrightarrow \exists y_0 \forall n \geq 0: |\pi|^{\frac{1}{q^n}} | \gamma_n | y_n$

$$\text{that is } \ker\left(\mathcal{O}_C^b \xrightarrow{\pi^b} \mathcal{O}_{C/\mathbb{F}}\right) = (\pi^b) \quad \Rightarrow 1)$$

$$\textcircled{1} (\pi - [\pi^b]) = \pi - (\pi^b) \cancel{\#} = \pi - \pi = 0$$

$$\text{Let } x \in \ker \Theta \Rightarrow \Theta(x) = \sum_{i=0}^{\infty} x_i \# \pi^i \equiv x_0 \# \pmod{\pi}$$

$$\Rightarrow \Theta = \Theta(x) = \Theta([\pi^b]z_0) = \pi \Theta(z_0)$$

$$\Rightarrow x_1 \in \ker \Theta$$

Inductively, $x = \{ (z_0 + [\pi^b]z_1 + \dots) \in \mathbb{X} \}$

Finally, note that $\Theta: W_E(\mathcal{O}_C^b) \rightarrow \mathcal{O}_C$ is surj ($\mathcal{O}_C^b \xrightarrow{(-)^b} \mathcal{O}_C$)

Next time:

La: A perfectoid, then

$A = \text{ring of integers in non-arch, alg. closed ext. } C/E$

$$\Leftrightarrow A^b \simeq \mathcal{O}_F \quad F/F_q$$

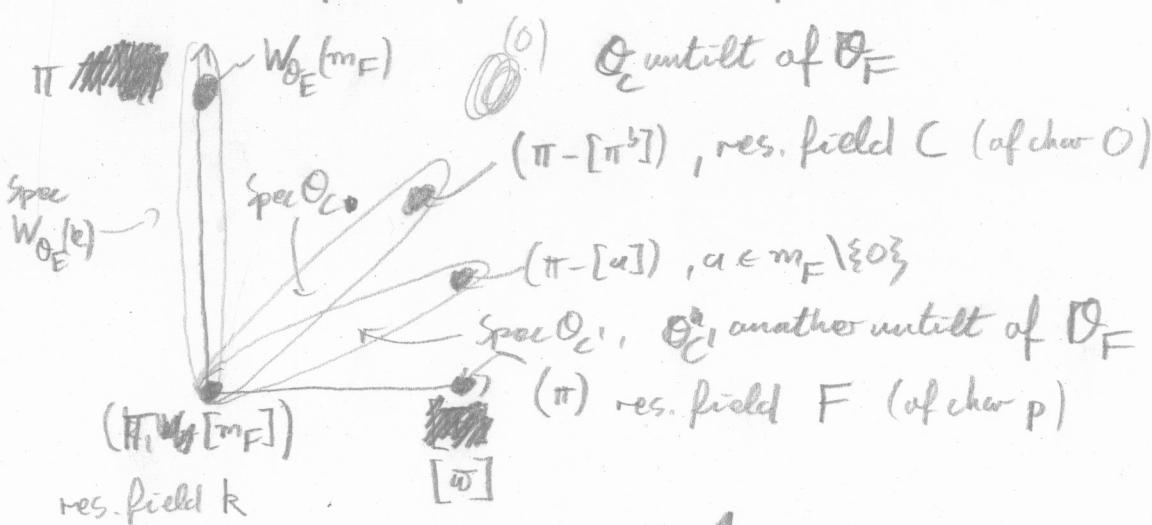
Recall: F/F_q non-arch, alg. closed

$$\Rightarrow A_{\text{inf}} = W_E(\mathcal{O}_F)$$

Remarks: * A_{inf} local integral domain, non-noetherian

* $A_{\text{inf}}, (\pi, [\omega])$ -adically complete for any $\vartheta \in m_F \setminus \{0\}$, $m_F := \{x \in F \mid |x| < 1\}$

Picture of $\text{Spec } A_{\text{inf}}$: $k = \mathcal{O}_F/m_F$ residue field



For $(\mathcal{O}_C, c: \mathcal{O}_C^b \simeq \mathcal{O}_F)$, $\{ \} = \ker (W_E(\mathcal{O}_C^b) \xrightarrow{A_{\text{inf}}} \mathcal{O}_C)$ set \downarrow localisation

$$B_{dR}^+ := A_{\text{inf}} \left[\frac{1}{[\vartheta]} \right] \wedge \quad \text{Next time: } B_{dR}^+, A_{\text{inf}, \{ \}} \text{ DVR}$$

However, $\mathcal{O}_F \not\subseteq \bigcup_{x \in m_F} [x] A_{\text{inf}} \not\subseteq W_E(m_F) \not\subseteq (\pi, W_E(m_F))$ chain of prime ideals

$\Rightarrow \text{Spec } A_{\text{inf}}$ of Krull dim ≥ 3

In fact (Ludwig-Lang): $\text{Spec } A_{\text{inf}}$ has infinite Krull dimension

~~Noemn~~ The "exotic" prime ideals are all contained in $W_E^{(m_F)}$ and non-closed in $(\pi, [\varpi])$ -adic topology

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