

The theorem "weakly admissible implies admissible"

(1)

K discr. valued non-archimedean ext. of \mathbb{Q}_p , perfect res. field k

1)

$K_0 = W(k)[\frac{1}{p}]$ maximal unramified subextension

\bar{K} analog closure, $C := \hat{\bar{K}}$, $G_K := \text{Gal}(\bar{K}/K) = \text{Aut}_{cts}(C/K)$

~~Forrestalization~~

~~for~~

Today:

~~Witt~~

Thm (Colmez - Fontaine)

$\{ \text{cryst. } G_{\mathbb{Q}_p} \text{-repr.} \} \simeq \{ \text{weakly adm. filtered } \varphi \text{-modules} \}$

Idea of proof: $F := C^G = \lim_{x \mapsto x^p} C \rtimes G_K$

$X = X_{\mathbb{Q}_p, F}$ associated FF-curve

G
 G_K

\curvearrowright equivariant v. b.

Embed $\text{Rep}_{\mathbb{Q}_p} G \hookrightarrow \text{Bun}_X^{G_K}$ and intersect.

cts rep. of G_K on f.d. \mathbb{Q}_p -v.s.

\uparrow

\downarrow

desp φ -Fil Mod $_{K/K_0}$

Notations: $B_{\text{crys}} = B_{\text{crys}}^+ [\frac{1}{\epsilon}], B_e = B_{\text{crys}}^{\varphi=1}$

$\infty \in X$ closed pt. corresponding to $V^+(t)$, $t = \log[\epsilon] \in \mathbb{B}^{\varphi=p}$

$\epsilon = (1, \gamma_p, \dots) \in F$

$B_{dR}^+ := \widehat{\mathcal{O}}_{X, \infty}, B_{dR} = \text{Frac}(B_{dR}^+)$

G_K acts on all of these, on \mathbb{Q}_p^\times via cyclotomic char $\chi_{\text{cycl}}: G_K \rightarrow \mathbb{Z}_p^\times$

We need

Theorem (Tate) : $\chi : G_K \rightarrow \mathbb{Z}_p^\times$ cont. character, $I_K \subseteq G_K$ ramification subgroup. Then

$$H^i_{cts}(G_K, (\chi)) = \begin{cases} 0, & \text{if } i \geq 2 \text{ or } i \text{ arbitrary and } \chi(I_K) \text{ is infinite} \\ \cong K, & \text{otherwise} \end{cases}$$

Proposition :

$$1) B_{dR}^{GK} = K$$

$$2) K \otimes_{K_0} B_{\text{crys}} \rightarrow B_{dR}, \text{ and in particular } B_{\text{crys}}^{GK} \cong K_0$$

$$3) B_e^{GK} = K_0$$

Proof: $\text{Fil}^i B_{dR} = t^i B_{dR}^+, i \in \mathbb{Z}$. Then $\text{gr}^i B_{dR} \cong \frac{\text{Fil}^i}{\text{Fil}^{i+1}} \cong C(x_{\text{cycl}}^i)$
 t uniformizer in B_{dR}^+ G_K -equiv.

$$\Rightarrow (\text{gr}_d^i B_{dR})^{GK} = \begin{cases} K, & i=0 \\ 0, & i \neq 0 \text{ Exercise} \end{cases}$$

$$2) \text{ Injectivity } K \otimes_{K_0} B_{\text{crys}} \rightarrow B_{dR}, \text{ cf. [FF, Corollaire 10.2.8]}$$

$$\Rightarrow B_{\text{crys}}^{GK} = K_0 \text{ by 1)}$$

$$3) \text{ apply 1) and inj. of } K \otimes_{K_0} B_e \hookrightarrow B_{dR} \quad \square$$

Recall: can. morph.

$V \in \text{Rep}_{\mathbb{Q}_p} G_K$ crystalline if $(V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{GK} \otimes_{K_0} B_{\text{crys}} \xrightarrow{\sim} V \otimes_{\mathbb{Q}_p} B_{\text{crys}}$ is an isomorphism (in part. $V \otimes_{\mathbb{Q}_p} B_{\text{crys}} \cong B_{\text{crys}}^{\dim V}$ as G_K -modules)

E.g.: G p-div. grp/ \mathcal{O}_K $\Rightarrow V := T_p G(C) \otimes_{\mathbb{Z}_p[\frac{1}{p}]} \mathbb{Z}_p[G_K]$ crystalline

Now pass to G_K -equivariant vector bundles on X

(3)

Def: $\mathcal{E} \in \text{Bun}_X$. A G_K -action on \mathcal{E} is a collection of
isom. $c_\sigma: \sigma^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}$ s.t. $c_{\sigma\tau} = \sigma c_\tau \circ \tau^*(c_\sigma)$ for all $\sigma, \tau \in G_K$.

E.g. V any \mathbb{Q}_p -f.d. \mathbb{Q}_p -v.s. with action of G_K (not nec. cont.)

$$= 1 \quad \mathcal{E} := V \otimes_{\mathbb{Q}_p} \mathcal{O}_X \quad \text{with } c_\sigma: \sigma^* \mathcal{E} \xrightarrow{\sim} \mathcal{E} \xrightarrow{\sigma \otimes \text{Id}} \mathcal{E}$$

can. isom.

Have to put condition of continuity

Def. Note $(\mathcal{E}, (c_\sigma)_{\sigma \in G_K})$ v.b. + G_K -action

$$= 1 \quad \mathcal{E}'_\infty := \mathcal{E} \otimes_{\mathbb{Q}_p} B_{dR}^+ \quad \text{has semilinear } G_K\text{-action}$$

Lemma: V f.d. \mathbb{Q}_p -v.s. with G_K -action. Then

$V \in \text{Rep}_{\mathbb{Q}_p}^{G_K}$ (i.e. $G_K \rightarrow GL(V)$ cont for can. topology on V)

if and only if on $V \otimes_{\mathbb{Q}_p} B_{dR}^+$ continuous for can. top.

$(B_{dR}^+ = \varprojlim \text{Aff}(\mathbb{Z}/p)[\frac{1}{p}]$ has canonical top. where $A/(\mathfrak{p}) \not\simeq A/\mathfrak{p}[\frac{1}{p}]$
is open with p -adic topology)

Moreover, each finite free B_{dR}^+ -module has canonical topology

Proof: " $=$ " $GL(V) \rightarrow GL(V \otimes_{\mathbb{Q}_p} B_{dR}^+)$

" \leq " $V \subseteq V \otimes_{\mathbb{Q}_p} B_{dR}^+$ has subspace topology

Definition A G_K -equivariant v.b. is a v.b. \mathcal{E} with G_K -action
s.t. the associated action $G_K \times \mathcal{E}'_\infty \xrightarrow{\text{on } X} \mathcal{E}'_\infty$ is cont.

$\text{Bun}_X^{G_K} = \text{cat. of } G_K\text{-equiv. v.b.}$

(Corollary (of classification of v.b.))

The functor $\text{Rep}_{\mathbb{Q}_p}^{G_K} \rightarrow \text{Bun}_X^{G_K}$, $V \mapsto V \otimes_{\mathbb{Q}_p} \mathcal{O}_X$
is fully faithful with essential image

all G_n -equiv. v.b. \mathcal{E} whose underlying v.b. is semistable of slope $\frac{1}{n}$, i.e., trivial

Now, filtered φ -modules

Def: A filtered φ -module over K is a triple $(D, \varphi_D, \text{Fil}^\bullet)$

with $(D, \varphi_D) \in \varphi\text{-Mod}_{K_0}$ and Fil^\bullet a decreasing, exhaustive, separated filtration on $D_K := D \otimes_{K_0} K$

$\varphi\text{-FilMod}_{K_0} = \text{cat. of fields such}$

E.g. $V \in \text{Rep}_{\mathbb{Q}_p} G_n$ crystalline

$$= D := D_{\text{crys}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{\varphi_D^{G_K}} \quad \varphi\text{-module (of } \dim = \dim_{\mathbb{Q}_p} V\text{)}$$

$\begin{matrix} G \\ \varphi_D \end{matrix}$

$$K \otimes_{K_0} D_{\text{crys}} \simeq D_{\text{dR}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\text{dR}})^{G_K} \supseteq \text{Fil}^\bullet(D_{\text{dR}}(V)) = (V \otimes_{\mathbb{Q}_p} t^i B_{\text{dR}}^+)^{G_K}$$

$$\Rightarrow (D_{\text{crys}}, \varphi_D, \text{Fil}^\bullet) \in \varphi\text{-FilMod}_{K_0}$$

Aim this functor $\text{Rep}_{\mathbb{Q}_p}^{\text{crys}} \rightarrow \varphi\text{-FilMod}_{K_0}$ fully faithful,
essential image defined by "weakly admissibility"

For fully faithfulness: $\downarrow \varphi\text{-equiv, comp. with filtration after } \begin{smallmatrix} \otimes \\ B_{\text{crys}} \end{smallmatrix}$

$$V \in \text{Rep}_{\mathbb{Q}_p} G_n \xrightarrow{\sim} B_{\text{crys}} \otimes_{\mathbb{Q}_p} V \simeq D_{\text{crys}}(V) \otimes_{K_0} B_{\text{crys}}$$

$$\Rightarrow V \xrightarrow{\sim} \text{Fil}^\bullet((B_{\text{crys}} \otimes_{\mathbb{Q}_p} V)^{\varphi^{-1}}) \simeq \text{Fil}^\bullet(D_{\text{crys}}(V) \otimes_{K_0} B_{\text{crys}})$$

$$\text{fund. ex. seq. } 0 \rightarrow \mathbb{Q}_p \rightarrow B_{\text{crys}} \xrightarrow{\sim} B_{\text{dR}}^+ \rightarrow 0$$

$\mathcal{E} = \varphi\text{-FilMod}_{K/K_0}$ has HN-formalism:

(exact (if seq. on ass. graded exact))

$$\deg(D, \varphi_D, \text{Fil}^\circ) := \sum_{i \in \mathbb{Z}} i \dim_K \text{gr}^i(D_K) - \deg(D, \varphi_D)$$

$$\text{rk}(\dots) := \dim_{K_0} D$$

$$F: \mathcal{E} \rightarrow \text{Vect}_K, (D, \varphi_D, \text{Fil}^\circ) \mapsto D_K$$

Definition: $(D, \varphi_D, \text{Fil}^\circ) \in \varphi\text{-FilMod}_{K/K_0}$ weakly admissible
if semistable of slope 0

~~Proposition~~ \mathcal{V} : $\varphi\text{-Mod}_{K_0} \rightarrow \text{Rep}_{B_e} G_K$, $(D, \varphi_D) \mapsto (D \otimes_{K_0} B_{\text{crys}})^{\varphi=1}$
 $D \otimes: \text{Rep}_{B_e} G_K \rightarrow \varphi\text{-Mod}_{K_0}$, $W \mapsto (W \otimes_{B_e} B_{\text{crys}})^{G_K}$

adjoint, \mathcal{V} fully faithful, $W \in \text{Rep}_{B_e} G_K$ in ess. image iff $W \cong (W \otimes_{B_e} B_{\text{crys}})^{G_K} \otimes_{K_0} B_{\text{crys}}$

Proof: Only $\mathcal{D} \circ \mathcal{V}(D, \varphi_D) \simeq (D, \varphi_D)$

$$\begin{aligned} & \text{wlog } k = \bar{k} \\ & \Rightarrow \text{wlog } D = (K_0^\Gamma, \varphi_D = \begin{pmatrix} 0 & p^s \\ 1 & 0 \end{pmatrix}, \pi = \frac{s}{r}) \quad (s, r) \text{ coprime} \\ & \qquad \qquad \qquad \xrightarrow{\text{pd. } t^d \cdot t^{-d} \hookrightarrow \text{pd. } 1} \end{aligned}$$

Dieudonné-Manin

Reksza ~~DKDy~~ For simplicity $r=1$

$$\mathcal{D} \circ \mathcal{V}(D, p^d \cdot \varphi) = \mathcal{D}(B_e \cdot t^{-d}) = (B_{\text{crys}} \cdot t^{-d})^{G_K} = K_0$$

Arg shows ~~(D, p^d)~~

$\varphi \otimes p^d \varphi$

Descr. of ess. image follows by adjunction.

Refined arg. shows $(D \otimes_{K_0} B_{\text{crys}})^{\varphi=1} \otimes_{B_e} B_{\text{crys}} \simeq D \otimes_{K_0} B_{\text{crys}}$

(6)

Lemma Let D' be a finite dim. K -v.s., then there exists a bijection

$$\{ \text{filtrations on } D' \} \xrightarrow{1:1} \{ G_K\text{-stable } B_{dR}^+ \text{-lattices } \underline{\Xi} \subseteq B_{dR} \otimes_K D' \}$$

$$\text{Fil}^\bullet \mapsto \text{Fil}^\bullet(V \otimes_K B_{dR})$$

\Downarrow
tensor product filtration

$$(t^i \underline{\Xi})_{i \in \mathbb{Z}}^{G_K} \hookrightarrow \underline{\Xi}$$

In other words (B.-L.), there exists a cartesian diagram

$$\underline{\Xi} \mapsto H^0(X/\mathbb{Q}_p, \underline{\Xi})$$

$$\begin{array}{ccc} \text{modify } \underline{\Xi}(D, \varphi_0) & \text{via } \text{Bun}_X^{G_K} \rightarrow \text{Rep}_{B_e}^{G_K} \\ \text{according to lattice} & \text{associated to } \text{Fil}^\bullet & \underline{\Xi}(D, \varphi_0, \text{Fil}^\bullet) \\ \downarrow & \uparrow \text{Gal}(K_{\varphi_0}/K_0) & \uparrow \mathcal{V} \\ (D, \varphi_0, \text{Fil}^\bullet) \in \varphi\text{-Fil Mod}_{K_{\varphi_0}} & \xrightarrow{\quad} & \varphi\text{-Mod}_{K_0} \end{array}$$

$$(\text{using } \mathcal{V}(D, \varphi_0) \otimes_{B_e} B_{dR} \simeq D_K \otimes_K B_{dR} \text{ resp. } \underline{\Xi}(D, \varphi_0) \simeq \mathcal{V}(D, \varphi_0))$$

Lemma: ~~The functor $\text{Fil}(D, \varphi)$ respects Fil^\bullet and $H^0(X/\mathbb{Q}_p, \underline{\Xi})$~~

$$\text{exact iff } \underline{\Xi}(-) : \varphi\text{-Fil Mod}_{K_{\varphi_0}} \rightarrow \text{Bun}_X^{G_K}$$

preserves degrees and Harder-Narasimhan filtrations
 $(= \text{HN-filtration of underlying bundle})$

Proof For degree: Exercise

(Sketch) For HN-filtration: Use Galois equivariance of HN-filtration and that G_K -inv. B_e -submodules of $\mathcal{V}(D, \varphi_0)$ lie again in the image of \mathcal{V} □

Proof of "wa = da":

(7)

Consider

$$\begin{array}{ccccc} \text{Rep}_{\mathbb{Q}_p} G_N & \hookrightarrow & \text{FilBun}_X^{G_N} & \rightarrow & \text{Rep}_{B_e} G_N \\ \uparrow & & \downarrow & & \uparrow \\ \mathcal{C} & \xrightarrow{\subseteq} & \varphi\text{-Mod}_{K/K_0}^{\text{Filt}} & \xrightarrow{\subseteq} & \varphi\text{-Mod}_{K_0} \end{array}$$

By previous lemma + $\text{Rep}_{\mathbb{Q}_p} G_N \simeq \text{Bun}_X^{G_N, \text{ess}, \mu=0}$

$$\Rightarrow \mathcal{C} \simeq \varphi\text{-FilMod}_{K/K_0}^{wa}$$

But $\mathcal{C} \simeq \text{Rep}_{\mathbb{Q}_p} G_N \times_{\text{Rep}_{B_e} G} \varphi\text{-Mod}_{K_0}$

and for $V \in \text{Rep}_{\mathbb{Q}_p} G_N$

$V \otimes_{B_e} \text{Rep}_{B_e} G_N$ is in the ess. image of \mathcal{V}

$$(\Rightarrow) (V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{G_K} \otimes_{B_{\mathbb{Q}_p} K_0} B_{\text{crys}} \simeq V \otimes_{\mathbb{Q}_p} B_{\text{crys}}$$

$(\Rightarrow) V$ crystalline

$\Rightarrow \mathcal{C} \simeq \text{Rep}_{\mathbb{Q}_p}^{crys} G_N$ and the equivalence $\text{Rep}_{\mathbb{Q}_p}^{crys} G_N \simeq \mathcal{C} \simeq \varphi\text{-FilMod}_{K/K_0}^{wa}$
 is given by $V \mapsto (\text{D}_{\text{crys}}(V), \varphi, \text{Fil}^\bullet)$ □