

①

The ring B

P prime, E/\mathbb{Q}_p finite, $\pi \in \mathcal{O}_F \rightarrow \mathbb{F}_q$, F/\mathbb{F}_q non-arch, alg. clcl, $v : F \rightarrow \mathbb{R} \cup \{\infty\}$

Thm (Fargues-Fontaine) $f \in A_{\text{inf}} = W_{\mathcal{O}_E}(\mathcal{O}_F)$, $\pi \neq 0$ slope of $\text{Newt}(f)$

$\Rightarrow a \in \mathcal{O}_F$, s.t. $v(a) = -\pi$ and $f = (\pi - [a])g$, $g \in A_{\text{inf}}$

Equiv, ex. $y \in |\mathbb{Y}_r|$ with $r = -\pi$, $f(y) = 0$, where $|\mathbb{Y}| = \text{Prim}_{\mathcal{O}_F}$



End of proof! Can assume f prim. of degree d , $f = \sum_{n=0}^{\infty} [x_n] \pi^n$, $x_d \in \mathcal{O}_F^\times$

Aim: Construct sequence $y_n \in |\mathbb{Y}|$, s.t.

$$a) v(f(y_n)) \geq -(d+n)\pi$$

$$b) d(y_n, y_{n+1}) \geq -\frac{d+n}{d}\pi$$

$$c) v(\pi(y_n)) = -\pi$$

$\left(\begin{array}{l} \text{=} \\ \{y_n\}_n \text{ converge to searched } y \in |\mathbb{Y}| \\ \text{by completeness of } |\mathbb{Y}| \end{array} \right)$

Last time $y_1 = (\pi - [z])$ with $z \in \mathcal{O}_F$ zero of $\sum_{i=0}^d x_i T^i \in \mathcal{O}_F[T]$

Assume y_n constructed. Write $f = \sum_{i=0}^{\infty} [a_i] \xi_{y_n}^i$, $a_i \in \mathcal{O}_F$

Let $z \in F$ be a zero of $\sum_{i=0}^d a_i T^i$ of maximal valuation

Then $z \in \mathcal{O}_F$ (check $a_d \in \mathcal{O}_F^\times$ using $A_{\text{inf}} \rightarrow W_{\mathcal{O}_E}(k)$, $k = \mathcal{O}_F/m_F$)

Check $y_{n+1} = (\xi_{y_n} - [z])$ works

Rmk: * $m_F \setminus \{0\} \rightarrow |\mathbb{Y}|$, $a \mapsto (\pi - [a])$ surj. But can give better description if $E = \mathbb{Q}_p$ (for general E use Lubin-Tate formal groups)

Exercise: 1) $\varepsilon \in 1 + m_F \setminus \{1\}$, $u_\varepsilon := \frac{[\varepsilon]-1}{[\varepsilon p]-1}$ (2)

Show u_ε primitive of degree 1

2) Show $1 + m_F \setminus \{1\} \rightarrow |\mathbb{Y}|$, $\varepsilon \mapsto (u_\varepsilon)$ is surjective

(Hint: If ζ_{q_p} non-alg. over alg. cl., $\varepsilon = (1, \zeta_p, \dots) \in \mathcal{O}_C^\times \cong \mathcal{O}_F$,

show $\ker \Theta: A_{\text{inf}} \rightarrow \mathcal{O}_C$ gen. by $u_\varepsilon^{\frac{1}{p}}$ using ~~that~~:

$a, b \in A_{\text{inf}}$ primitive of degree 1, $a \in (b) \Leftrightarrow (a) = (b)$.]

3) $|\mathbb{Y}| \cong (1 + m_F \setminus \{1\}) / \mathbb{Z}_p^\times$ where \mathbb{Z}_p^\times acts on $1 + m_F \setminus \{1\}$ via

$$a * \varepsilon = \varepsilon^a := \sum_{i=0}^{\infty} \binom{a}{i} (\varepsilon^{-1})^i, \text{ i.e. } (u_\varepsilon) = (u_{\varepsilon^a}) \Rightarrow \varepsilon^a = \varepsilon^{a \cdot p} \text{ for some } a \in \mathbb{Z}_p^\times$$

(Hint: $C := A_{\text{inf}} / (u_\varepsilon) [\frac{1}{p}] \Rightarrow \varepsilon \in \mathcal{O}_F = \mathcal{O}_C^\times$ generator of

$$T_p C^\times = \left\{ (1, a_1, \dots) \mid a_{i+1}^p = a_i \right\} \subseteq \mathcal{O}_C^\times$$

If $(u_\varepsilon) = (u_{\varepsilon'})$, show $\varepsilon' \in T_p C^\times$ and use that $T_p C^\times \cong \mathbb{Z}_p \cdot \varepsilon$)

Recall: $r \geq 0 \rightsquigarrow v_r: A_{\text{inf}} \rightarrow \mathbb{R} \cup \{\infty\}$, $f = \sum_{i=0}^{\infty} [x_i] \pi^i \mapsto \inf_{i \in \mathbb{Z}} \{v(x_i) + ir\}$
valuation

$$\mathbb{I}(\text{Newt}(f)) = \begin{cases} v_r(f), & r \geq 0 \\ -\infty, & r < 0 \end{cases}$$

Can extend $v_r, \text{Newt}(f)$ to

$$\mathbb{B}^b := A_{\text{inf}} \left[\frac{1}{\pi}, \frac{1}{[\infty]} \right], \infty \in m_F \setminus \{0\}$$

$$\left\{ \sum_{i=-\infty}^{\infty} [x_i] \pi^i \mid x_i = 0, i < 0, \inf_{i \in \mathbb{Z}} \{v(x_i)\} > -\infty \right\}$$

(3)

~~Defn~~ $I \subseteq (0, \infty)$ interval

Def: $B_I :=$ completion of B^b w.r.t. to the family of valuations $(v_r)_{r>0}$

Intuition: " $B_I = \mathcal{O}(I\gamma_I)$ " where $|Y_I| = \{y \in Y \mid d(y, 0) = v(\pi(y))\} \cap I\}$

Rank: * R topological ring, s.t. 0 has fund. system of nbhds which are subgroups
(i.e. closed under addition), then

$\hat{R} = \lim_{\leftarrow} R/\mathcal{U}_U$ is the completion of R

and by continuity \hat{R} is again a ring
of multiplication

For B_I take $\mathcal{F} = \{\hat{\mathcal{A}}\tilde{J}_r^{-1}([m\infty]) \mid n, m \in \mathbb{N}, r \in I\}$

* $B_I = \lim_{\leftarrow} B_J$
 $J \subseteq I$
compact

Most important case: $I = (0, \infty)$

Def: $B := B_{(0, \infty)}$ (" = $\mathcal{O}(Y_I)$ ")

Note φ^{B^b} extends by continuity to \hat{B}
an autom.

(More gen. p: $B_I \cong B_{qI}$ for $I \subseteq (0, \infty)$)

Def: $X := X_{FF} = \text{Proj}(\bigoplus_{d \geq 0} B^{\varphi = \bar{\pi}^d})$ " $\mathbb{Y}_{\mathbb{Z}}$ "
(More prec., $X_{E,F}$)

"schematic Fargues-Fontaine curve"

$$\text{Rmk: } * \quad Y = \text{Spf}(A_{\text{inf}}) V(\pi, [\varnothing])$$

$X = Y/\varphi^{\mathbb{Z}}$ adic Fargues-Fontaine
curve

$$\pi! \cdot \varphi: \varphi^* \mathcal{O}_Y \xrightarrow{\sim} \mathcal{O}_Y$$

defines the descent datum

for a line bdl. $\mathcal{O}(1)$ on X

Moreover, $H^0(X, \mathcal{O}(1)^{\otimes d}) \simeq B^{(\pi^{-1}\varphi)^d} = B^{\varphi = \pi^d}$ and
($\mathcal{O}(1)$ in some sense "ample")

La:

$$(B^b)^{\varphi = \pi^d} = \begin{cases} E, & d=0 \\ 0, & d \neq 0 \end{cases}$$

Prf: $d=0$: Let $f \in (B^b)^{\varphi=1}$ ~~then $f = \sum_{i=-\infty}^{\infty} [x_i] \pi^i$~~ , $f = \sum_{i=-\infty}^{\infty} [x_i] \pi^i$

~~that $x_i \in \mathcal{O}_F$ for all i~~

$$= 1 \forall i \in \mathbb{Z} \quad \varphi(x_i) = x_i, \text{ i.e. } x_i \in \mathcal{O}_F \subseteq \mathcal{O}_F \Rightarrow f \in E$$

$d \neq 0$:

$$\text{Neut}(\varphi(f)) \stackrel{(x)}{=} \text{Neut}(\varphi)(x) = \text{Neut}(f)(x+nd)$$

$$= 1 \stackrel{q \text{ Neut}(f)(x)}{\text{Neut}}(\varphi^n(f))(x) = \text{Neut}(f)(x+nd)$$

If $d \neq 0 \Rightarrow \text{Neut}(f)(x+nd) = +\infty$ for $n \gg 0$

$$\Rightarrow q^n \text{Neut}(f)(x) = +\infty \quad \forall x \Rightarrow f = 0$$

If $d \neq 0$, pick $x_0 < 0$ with $\text{Neut}(f)(x_0) = +\infty$

For $\exists x \in \mathbb{C}^n, s.t. \text{Neut}(f)(x) \geq \text{Neut}(f)(x_0 + nd) = q^n \text{Neut}(f)(x_0) = +\infty$

$$\Rightarrow f = 0$$



(4)

(5)

~~La~~: Let $(x_n)_{n \in \mathbb{Z}}$ be a sequence of elements in F , s.t.

$$\lim_{|n| \rightarrow \infty} v(x_n) + nr = \infty, \text{ for all } r \in (0, \infty)$$

Then $\sum_{n \in \mathbb{Z}} [x_n] \cdot \pi^n$ converges in B

Prf: \downarrow Sufficient to show $v([x_n] \cdot \pi^n) \rightarrow 0$, $|n| \rightarrow \infty$ for all $r \in (0, \infty)$

$$v(x_n) + rn$$

but this is the assumption \square

Rmk: * $a \in m_F \Rightarrow f_a := \sum_{i \in \mathbb{Z}} [aq^i] \cdot \pi^i$ converges in B and $f_a \in B^{\Phi=\pi}$.

$$(v(aq^i) + ir = q^i \cdot v(a) + ir \rightarrow \infty, |i| \rightarrow \infty, \text{ as } v(a) > 0)$$

$$\varphi(f_a) = \sum_{i \in \mathbb{Z}} [\varphi(aq^i)] \pi^i = \sum_{i \in \mathbb{Z}} [a q^{(i+1)}] \pi^{i+1} \cdot \pi = \pi \cdot f_a$$

In fact,
~~Later~~ $m_F \simeq B^{\Phi=\pi}$ (later)

* In general, it is unknown whether elements in B can be written as $\sum_{n \in \mathbb{Z}} [x_n] \pi^n$, $x_n \in F$.

Thm (Fargues-Fontaine): $X = \text{Proj}_{\mathbb{Z}}((\bigoplus_{d \geq 0} B^{\Phi=\pi^d})^\vee)$ is a Dedekind scheme,

more precisely for $t \in B^{\Phi=\pi}$, $D_t(t) \simeq \text{Spec}(B[\frac{1}{t}]^{\Phi=1})$ is the spectrum of a principal ideal domain

To prove this we need Newton polygons for elements in B_I , $I \subseteq (0, \infty)$ ideal

For $r \in I$, $v_r: B^b \rightarrow \mathbb{R} \cup \{\infty\}$ extends to $v_r: B_I \rightarrow \mathbb{R} \cup \{\infty\}$ by continuity

Def: Assume $I \subseteq (0, \infty)$ open, $f \in B$

Set Nent_I^0 as the decreasing convex fct. whose legendre transform is

(6)

$$r \mapsto \begin{cases} v_r(f), & r \in I \\ -\infty, & r \notin I \end{cases}$$

and $\text{Went}_I^o(f) \subseteq \mathbb{R}^2$ as the subset of the graph of $\text{Went}_I(f)$ with slopes contained in $-I$

Rmk: * If $K \subseteq I$ is compact, $f_n \in B^b$ converging to $f, f \neq 0$, then

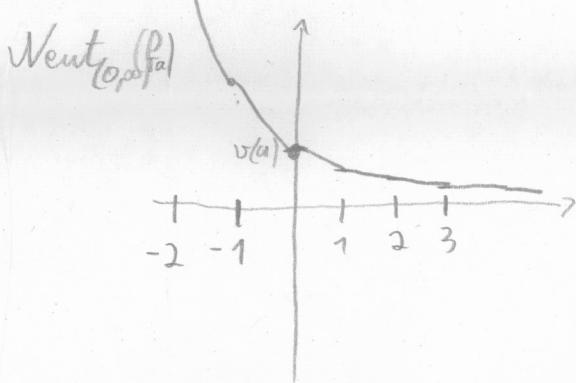
ex. N , s.t. for all $n \geq N$, $v_r(f_n) = v_r(f)$ for all $r \in K$

($\Rightarrow L(\text{Went}_I^o(f))$ is a concave fct with integral slopes)

$\Rightarrow \text{Went}_I^o(x)$ is a decreasing convex polygon with integral breakpoints)

Rmk

~~Ex:~~: * $a \in m_F$, $f_a = \sum_{i \in \mathbb{Z}} [a q^{-i}] \pi^i$



(~~maybe this~~ polygon version of $\# v(a) \cdot q^{-i}$,
note there exists no x with

$$\text{Went}_{(0,0)}(f_a)(x) = +\infty$$

$$1) \pi_i < 0$$

* ~~maximize~~ $f \in B$, π_i slope of $\text{Went}_{(0,0)}(f)$ on $[i, i+1]$, then $\lim_{i \rightarrow \infty} \pi_i = 0$
Need different definition for $I \subseteq (0, \infty)$ compact $\lim_{i \rightarrow -\infty} \pi_i = -\infty$

Def: $I = [a, b]$ compact, $f \in B_I$, $f \neq 0$

Set $\text{Went}_I^o(f)$ as the decreasing convex fct whose Legendre transform is

$$r \mapsto \begin{cases} v_r(x), & r \in I \\ v_a(x) + (r-a)\partial_g v_a(x), & r < a \\ v_b(x) + (r-b)\partial_d v_b(x), & r \geq b \end{cases}$$

and $\text{Went}_I(x) \subseteq \mathbb{R}^2$ as the ~~slope~~ subset of the graph of $\text{Went}_I^o(x)$ with slopes in $-I$

Rmk: * If $f_n \rightarrow f$, then $\partial_g v_r(f) = \lim_{n \rightarrow \infty} \partial_g^{\text{left}} v_r(f_n)$ left derivative
 \uparrow
 B^b at r
 $\partial_d v_r(f) = \dots \partial_d^{\text{right}} v_r(f_n)$ right derivative

(independent of seq. converging to f)

* For $f \in B^b$, π a slope of $\text{Neut}(f)$, then

$\partial_g v_{-\pi}(f) - \partial_d v_{-\pi}(f)$ is the multiplicity of π in $\text{Neut}(f)$

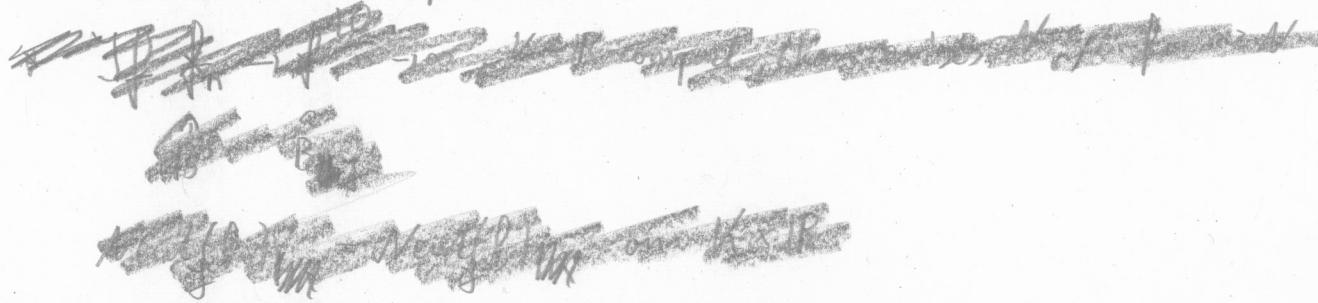
(\Rightarrow have to ~~use~~ use modified definition if I compact)

* ~~Neut~~ ^{intervall}

* $J \subseteq (0, \infty)$ general, $f \in B_J$

$$\text{Neut}_J(f) = \bigcup_{\substack{I \subseteq J \\ \text{compact}}} \text{Neut}_I(f) \quad (\text{Neut}_J(0) := \emptyset)$$

* $\text{Neut}_J(f)$ is a decreasing convex polygon and all slopes have finite multiplicities



$$* \text{Neut}_J^\circ(f \cdot g) = \text{Neut}_J^\circ(f) * \text{Neut}_J^\circ(g)$$