

The classifications of vector bundles

(1)

$X = X_{E,F}$ FF-curve associated to E,F

$$\check{E} = \text{W}_E(\bar{\mathbb{F}}_q)[\frac{1}{\pi}] \subseteq B$$

$$\mathcal{E}(-) : \varphi\text{-Mod}_{\check{E}} \rightarrow \text{Bun}_X, (D, \varphi) \mapsto \bigoplus_{d \geq 0} (\mathcal{B}_{\check{E}} \otimes D)^{\overset{\varphi \otimes \varphi_D = \pi^d}{\sim}}$$

Theorem $\mathcal{E}(-)$ essentially surjective

(Forques-Fontaine)

By pullback to $X_h := X_E^{\otimes E_h}$, E_h/E unram. of degree d , descent,

reduced to (necessary)

Theorem ~~Assume~~ $n \geq 0$

1) If $0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X(\frac{1}{n}) \rightarrow F \rightarrow 0$ exact

with F torsion sheaf of degree 1, then $\mathcal{E} \simeq \mathcal{O}_X^n$

2) If $0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X^n \rightarrow F \rightarrow 0$ exact with F torsion sheaf of degree 1, then ex. $m \in \{1, \dots, n\}$ s.t.

$$\mathcal{E} \simeq \mathcal{O}_X^m \oplus \mathcal{O}_X(\frac{1}{m})$$

Fix $x \in X$ closed, $C := k(x)$ "tilt of F ", $i : \text{Spec } k(x) \rightarrow X$ inclusion
~~Let~~ $\mathcal{E}' \in \text{Bun}_X$. Consider

$$M_{\mathcal{E}'} = \left\{ \mathcal{E} \subseteq \mathcal{E}' \mid \mathcal{E}/\mathcal{E} \simeq i^*(C) \right\} \simeq \text{IP}(\mathcal{E}'(x))(C)$$

$$(\mathcal{E} \subseteq \mathcal{E}') \mapsto (\mathcal{E}' \otimes_C \mathcal{E}/\mathcal{E})$$

Get decomposition

$$\text{IP}(\mathcal{E}'(x))(C) = \prod_{[\mathbb{F}] \in \text{Ban}_{X/\text{isom}}} \text{IP}(\mathcal{E}'(x))(C)_{[\mathbb{F}]} \quad \text{locus where } \mathcal{M} \mathcal{E} \simeq \mathcal{F}$$

For simplicity $E = \mathbb{Q}_p$ (in general replace p -div. groups by π -div. \mathbb{Q}_p -module)
 (not sufficient: for classification of Bun_X need Theorem also
 for E_h/E unramified)

Last time:

$$\begin{aligned} \{ p\text{-div. grp}/\mathcal{O}_C \} &\rightarrow \{ \text{modifications} \} & \{ \text{modifications} \} \\ \text{admissible} & & \text{covariant Dieudonné module} \\ G &\mapsto (0 \rightarrow \mathcal{O}_X \otimes_{\mathbb{Z}_p} T_p G \rightarrow \mathcal{E}(M(G_s)) \rightarrow \text{Lie } G \rightarrow 0) \end{aligned}$$

Fix $H_{/\mathbb{F}_p}$ p -div. grp of dimension 1, ht n (unique up to isom.)

$$M_{H,\eta}^{\text{ad}}(C) := \{ (G, \alpha) \mid G \text{ p-div. grp over } \mathcal{O}_C, \alpha: G \otimes_{\mathcal{O}_C} H \simeq H \otimes_{\mathbb{Z}_p} \mathcal{O}_{C/p} \}_{\text{isom.}}$$

" \mathcal{O}_C -points of adic generic fiber of Lubin-Tate space"
 $(\simeq \mathbb{M}_C^{n-1})$

or Gross-Hopkins

Get "de Rham" period morphism

$$\pi_{\text{dR}}: M_{H,\eta}^{\text{ad}}(C) \rightarrow \text{IP}^{n-1}(C) \stackrel{\cong}{=} \text{IP}(M(H_s) \otimes C)(C), (G, \alpha) \mapsto (M(H) \otimes C \simeq M(G_s) \otimes C) \downarrow \text{Lie } G$$

Theorem (Gross-Hopkins): π_{dR} is surjective.

(3)

This implies cond. 1):

~~$\mathcal{E} \simeq \mathcal{O}_X(1)$~~ and π_{dR} factors as

$$M_{H, \eta}^{\text{ad}}(C) \rightarrow M_{H, \eta, \mathbb{F}_p} \simeq \mathbb{P}(\mathcal{E}'(x))(C) \simeq \mathbb{P}^{n-1}(C)$$

$M(H) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$

Conversely ~~we want to~~ the classification of vector bundles implies π_{dR} surjectivity. (exists proof independent of p-div. gops)

By Scholze/Weinstein subvectorpace

$$\{ \text{p-div. gops}/\mathcal{O}_C \} \simeq \{ (T, W) \mid T \text{ finite free } \mathbb{Z}_p\text{-lattice}, W \in T \otimes_{\mathbb{Z}_p} (1^{-1}) \}$$

needs classification

$\Rightarrow \text{Im}(\pi_{\text{dR}}) \subseteq \mathbb{P}(\mathcal{E}'(x))(C)$ is the "admissible locus", i.e., where the corresponding ~~new~~ modification $\mathcal{E}' \simeq \mathcal{E}$ of \mathcal{E} is semistable (of degree 0)

But if $0 \rightarrow \mathcal{E} \rightarrow \mathcal{E}' \rightarrow \mathcal{O} \rightarrow 0$ is exact ~~then~~ and \mathcal{E} is not semistable, then ex. $\mathcal{O}_X(n) \subseteq \mathcal{E}$, $n > 0$, $n = \frac{d}{r}$, $r < n$. and nonzero $\mathcal{O}_X(n) \not\cong \mathcal{O}_X(1)$

$\Rightarrow \pi_{\text{dR}}$ surj (assuming classification of v.b.)

For condition 2)

Let us first describe the decomp.

$$\mathbb{P}(\mathcal{E}(x))(C) = \coprod_{[F] \in \text{Bun}_{X, \text{tors}}} \mathbb{P}(\mathcal{E}'(x))(C)$$

for $\mathcal{E}' = \mathcal{O}_X^n$

(assuming the classification of v.b.)

$$|P(\mathcal{E}(x))(C)|_{[F]} \neq \emptyset \quad \text{if and only if } F = \mathcal{O}_X^m \oplus \mathcal{O}_X(-\frac{1}{m})^{n-m} \quad (4)$$

(~~all~~ all v.b. of deg 1, when admitting an injection to $\mathcal{O}_x^{\oplus n}$
are of this form)

Moreover, $\mathbb{E} \subset \mathcal{O}_X(\frac{1}{n}) \iff H^0(X, \mathbb{E}) = 0$

$$\Rightarrow H^0(X, \mathcal{E}') \hookrightarrow C \text{ injective}$$

$$\mathbb{Q}_p^{n^r} \cong \mathbb{P}^{n-1}(C)$$

\Leftrightarrow the point of $IP(E)(C)$ defined by E
 lies on $\mathcal{P}(C) \subseteq IP(C)$

complement of all \mathbb{Q}_p -rational hyperplanes hyperplanes "Drinfeld's upper halfplane"

(n = 2 => $\mathbb{R}^1 = (\mathbb{P}_C^1 \setminus \mathbb{P}^1(\mathbb{Q}_p))$ open subset \mathfrak{X} , analogous to $(\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}))$
 \mathfrak{X} profinite

Thus

$$\mathbb{P}^{\mathbb{Z}}(\mathcal{E}'(x))(C) = \mathcal{R}_{\mathbb{Z}}^{n-1}(C) \amalg \mathcal{R}_{\mathbb{Z}}^{n-2}(C) \amalg \mathcal{R}_{\mathbb{Z}}^{n-3} \amalg \dots$$

$\mathbb{Z} \subseteq \mathbb{P}^{n-1}$
 \mathbb{P}^{n-2} \mathbb{Q}_p -rational
 \mathbb{P}^{n-3} linear subspace

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 subspace

of dimension m

Given a linear subspace $V \subseteq H^0(X, \mathcal{E})$ we can look at

$$\{ \mathcal{E} \subset \mathcal{E}' \mid \exists x \in \mathcal{E} \text{ such that } H^0(x, \mathcal{E}) = V \}$$

This will map to $\mathbb{R}^{n-m-1}(c) \subseteq \text{IP}(\mathbb{R}^m(c))$

\Rightarrow It is sufficient to show that

(for obtaining and 2))

$\mathcal{R}^{n-1}(C)$ is given by the locus where the associated modification E of

\mathcal{E}' is then isomorphic to $\mathcal{O}_X(-\frac{1}{n})$

Drinfeld constructed a moduli problem $M_{\text{Dr}, \eta}^{\text{ad}}$ for p -div groups (+ extra structure) together with a period morph. (5)

$$\pi_{\text{dR}}: M_{\text{Dr}, \eta}^{\text{ad}}(C) \rightarrow \mathbb{P}^{n-1}(C)$$

with image $S^{n-1}(C)$

$$\pi_{\text{dR}} \text{ factors as } \pi_{\text{dR}}: M_{\text{Dr}, \eta}^{\text{ad}}(C) \rightarrow \left\{ \varepsilon \in \varepsilon' \mid \varepsilon'/\varepsilon \cong \text{det}_{\mathbb{Q}_p}(C), \varepsilon \in \mathcal{O}_X(-\frac{1}{n}) \right\} \xrightarrow{\sim} \mathbb{P}^{n-1}(C)$$

This implies condition 2).

The adic space $M_{\text{Dr}, \eta}^{\text{ad}}, M_{\text{LT}, \eta}^{\text{ad}}$ are interesting by their

relation with the local Langlands correspondence.

There are the " ∞ -level LTI Drinfeld spaces"

$$\begin{array}{ccc}
 M_{\text{LT}, \eta, \infty}^{\text{ad}} & \xrightarrow{\sim} & M_{\text{Dr}, \eta, \infty}^{\text{ad}} \\
 \downarrow \text{GL}_n(\mathbb{Q}_p)/\text{tors} & & \downarrow D^\times\text{-torsor} \\
 \{ M_{\text{LT}, \eta}^{\text{ad}} \} & \text{Thm (Fultings) Fargues, } & M_{\text{Dr}, \eta}^{\text{ad}} \\
 \text{trivialize Tate-module} & & \downarrow \text{GL}_n(\mathbb{Q}_p) \\
 & \text{D}^\times &
 \end{array}$$

$(D/\mathbb{Q}_p \text{ central div. algebra of invariant } \frac{1}{n})$

and (roughly)

$$\begin{array}{c}
 W_{\mathbb{Q}_p} - \text{repr.} \xrightarrow{\sim} \text{ass. to } \pi(\text{LLC}) \\
 \downarrow \text{ass. to } \pi(\text{FL}) \\
 H_c^{n-1}(M_{\text{LT}, \eta, \infty, \overline{\mathbb{Q}_p}}^{\text{ad}}, \overline{\mathbb{Q}_e}) = \oplus \\
 \text{Supercuspidal} \\
 \text{Supercuspidal}
 \end{array}$$

$$W_{\mathbb{Q}_p} \times \text{GL}_n(\mathbb{Q}_p) \times D^\times$$

The space $M_{\text{LT}, \eta, \infty}^{\text{ad}}$ can be defined via the Fargues-Fontaine curve

$$M_{\text{LT}, \eta, \infty}^{\text{ad}} = \left\{ \mathcal{O}_X^n \hookrightarrow \mathcal{O}_X\left(\frac{1}{n}\right) \mid \text{corencl supported at } \infty \right\}$$

(~ to make this precise one needs ~~perfectoids~~, aka perfectoid spaces, diamonds, etc.)

The switch from p-div. grps to vectorbundles on the Fargues-Fontaine curve leads to another important generalization (6)

Let G/\mathbb{Q}_p be a reductive group (e.g. $G = \mathrm{GL}_n, \mathrm{GSp}_{2n}, \dots$)

Def: $B(G) := G(\mathbb{Q}_p)/_{\mathbb{Q}_p\text{-conj.}}$

Thm (Fargues): $B(G) \simeq H^1_{\text{ét}}(X, G) (\simeq \{G\text{-torsors on } X\}_{\text{geom}})$
 $\simeq \{\text{exact } \otimes\text{-formalism vectors}$
 \uparrow
 $\text{Tannaka} \quad \text{Rep}_{\mathbb{Q}_p}^{G \rightarrow \mathrm{Bun}_X}\}$

$$b \mapsto ((V, \varphi) \mapsto \mathcal{E}(Q_p \otimes V, \varphi_b)(\varphi \otimes 1))$$

$$\mathcal{E}_b =$$

Def: A local Shimura datum is a triple $(G, [b], \{\mu\})$ consisting of

- G/\mathbb{Q}_p reductive group

- $[b] \in B(G)$

- $\{\mu\}$ conj. class of minuscule geom. cocharacters $\mu: G_m, \bar{\mathbb{Q}}_p \xrightarrow{\sim} G(\bar{\mathbb{Q}}_p)$ $G(B_{dR})_{U^1}$

Def: $(G, [b], \{\mu\})$ local Shimura datum lies in $G(B_{dR}^+ \cdot t^\mu \mathcal{O}_p^\times \cdot G(B_{dR}^+)^{-1})$

$\rightsquigarrow \mathrm{Sh}_{(G, [b], \mu), \infty} := \left\{ \mathcal{E}_b \mid \begin{array}{l} \text{relative pos. of } \mathcal{E} \text{ bdd by } \mu \\ \mathcal{E} \text{ has minuscule weights} \end{array} \right\}$

$G(\mathbb{Q}_p) \times \mathcal{J}_b(\mathbb{Q}_p), \quad \mathcal{J}_b(\mathbb{Q}_p) = \mathrm{Aut}(\mathcal{E}_b)$ depends in terms of B_{dR}^+ affine
~~weights~~

Conj. (Kottwitz): $H_c^*(\mathrm{Sh}_{(G, [b], \mu), \infty}, \bar{\mathbb{Q}}_p)$ realizes local Langlands corr. for G
 (roughly)

$$G(\mathbb{Q}_p) \times \mathcal{J}_b(\mathbb{Q}_p) \times W_{E(G, \{\mu\})}$$

~~field of definition of $\{\mu\}$~~