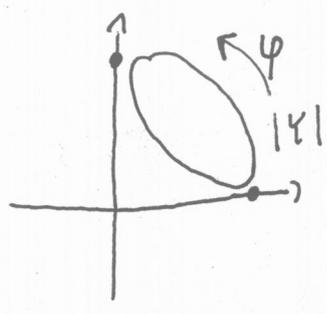


Crucial: Interpret A_{inf} as "functions on punctured open unit disc" (2)

Def: $|Y| := \{ \mathfrak{p} \in A_{inf} \mid \mathfrak{p} \text{ gen. by a primitive elt. of degree 1} \}$

$\varphi: C \cong \text{Prim}_1 / A_{inf}^{\times} = \{ C/E \text{ non-arch, alg. cl'd, } \mathcal{O}_C^b \cong \mathcal{O}_F \}$



~~Notation~~

Notation: * $\gamma \in |Y| \Rightarrow \mathfrak{p}_{\gamma} \subseteq A_{inf}$ corresp. prime ideal, res. field $C_{\gamma}, v_{\gamma}: C_{\gamma} \rightarrow (\mathbb{R} \cup \{\infty\})$
 $\mathcal{O}_{\gamma}: A_{inf} \rightarrow C_{\gamma}$ ass. val., $v_{\gamma}(\mathcal{O}_{\gamma}([x])) = v(x)$
non-arch., alg. closed

* $f \in A_{inf} \Rightarrow f(\gamma) \equiv$ class of $f \in C_{\gamma}$ (\sim "f in $\mathcal{O}(|Y|)$ ")
 $v(f(\gamma)) := v_{\gamma}(f(\gamma))$

* $d(\gamma_1, \gamma_2) = v_{\gamma_1}(\mathcal{O}_{\gamma_1}(\xi_{\gamma_2}))$, $\xi_{\gamma_2} \in \mathfrak{p}_{\gamma_2}$ gen.

~~Later~~ ("distance from γ_1 to γ_2 ")

Later: $d(\gamma_1, \gamma_2)$ defines metric $|Y| \times |Y| \rightarrow (\mathbb{R} \cup \{\infty\})$

$d(\gamma, 0) := v(\pi(\gamma))$ "distance to origin"

Rmk: * $Y = \text{Spf } A_{inf} \setminus V(\pi \cdot [\pi])$ adic space ~~properly disc~~

$|Y| \subseteq Y$ as "classical points", $\mathcal{O}_{A_{inf}} \subset \mathcal{O}(Y)$

* $\varphi @ Y$ properly discontinuous ($d(\varphi(\gamma), 0) = \frac{1}{q} (d(\gamma, 0))$)

$\Rightarrow X^{rad} := Y / \varphi^{\mathbb{Z}}$ "adic Fargues-Fontaine curve"

* Thm of Fargues-Fontaine has the reformulation:

$f \in A_{inf}^{\times}, r \neq 0$ slope of $\text{Newt}(f) \Rightarrow \exists \gamma \in |Y|, \text{ s.t. } v(\pi(\gamma)) = -r \text{ and } f(\gamma) = 0$

La: Set $\alpha_r = \{x \in A_{\text{inf}} \mid v_0(x) = \inf_{i \in \mathbb{Z}} \{v(x_i)\} \geq r\}$
 if $x = \sum_{i=0}^{\infty} [x_i] \pi^i$

$\Rightarrow d(\gamma_1, \gamma_2) = \sup_{r \geq 0} \{ \rho_{\gamma_1} + \alpha_r = \rho_{\gamma_2} + \alpha_r \}$

in part, $d(-, -)$ is an ultrametric space, i.e.,

- 1) $d(\gamma_1, \gamma_2) = d(\gamma_2, \gamma_1)$
- 2) $d(\gamma_1, \gamma_3) \geq \inf\{d(\gamma_1, \gamma_2), d(\gamma_2, \gamma_3)\}$
- 3) $d(\gamma_1, \gamma_2) = \infty \Leftrightarrow \gamma_1 = \gamma_2$

Prof: Let $\rho_{\gamma_i} = (\xi_{\gamma_i})$, write $\xi_{\gamma_1} = \sum_{n=0}^{\infty} [x_n] \xi_{\gamma_2}^n$ (use $\Theta_F \rightarrow \Theta_{C_{\gamma_2}}$)

$\Rightarrow d(\gamma_2, \gamma_1) = v_{\gamma_2}(\Theta_{\gamma_2}(\xi_{\gamma_1})) = v(x_0)$

Apply $\Theta_{\gamma_1} \Rightarrow 0 = \sum_{n=0}^{\infty} \Theta_{\gamma_1}([x_n]) \cdot \Theta_{\gamma_1}(\xi_{\gamma_2})^n$, i.e. $\Theta_{\gamma_1}([x_0]) = \Theta_{\gamma_1}(\xi_{\gamma_2})$
 $(\sum_{n=1}^{\infty} \Theta_{\gamma_1}([x_n]) \cdot \Theta_{\gamma_1}(\xi_{\gamma_2})^n)$

$\Rightarrow d(\gamma_2, \gamma_1) = v(x_0) = v_{\gamma_1}(\Theta_{\gamma_1}([x_0])) \geq v_{\gamma_1}(\Theta_{\gamma_1}(\xi_{\gamma_2})) = d(\gamma_1, \gamma_2)$
 with equality iff $x_1 \in \mathcal{O}_F^*$ (as $v_{\gamma_1}(\Theta_{\gamma_1}(\xi_{\gamma_2})) > 0$)

By symmetry, $d(\gamma_2, \gamma_1) = d(\gamma_1, \gamma_2)$. Moreover, $\rho_{\gamma_1} + \alpha_{v(x_0)} = \rho_{\gamma_2} + \alpha_{v(x_0)}$.

~~scribbles~~ If $r \geq 0$, s.t.
 $\rho_{\gamma_1} + \alpha_r = \rho_{\gamma_2} + \alpha_r$
 $\Rightarrow \Theta_{\gamma_1}(\xi_{\gamma_2}) \in \{x \in \mathcal{O}_{C_{\gamma_1}} \mid v_{\gamma_1}(x) \geq r\}$
 $\Theta_{\gamma_1}([x_0]) \Rightarrow d(\gamma_1, \gamma_2) \geq v(x_0) \geq r$

1), 2) are clear.

To see 3) use $A_{inf} = \varprojlim_{r \geq 0} A_{inf}/\alpha_r$ and that \mathcal{P}_{γ_i} are closed. \square

~~Prop~~

Def: $r > 0, |Y_r| := \{ \gamma \in |Y| \mid d(\gamma, 0) = r \}$

Δ Prop: $r > 0 \Rightarrow \{ |Y_r| \}$ is complete for metric d

Prof: Let $\{ \gamma_n \}_{n \geq 0} \subset |Y_r|$ be a Cauchy sequence

Claim: $\forall r' > 0$ the sequence $\{ \mathcal{P}_{\gamma_n} + \alpha_{r'} \}_{n \geq 0}$ of ideals is const. for $n \gg 0$

Indeed, ex. n_0 , s.t. $d(\gamma_n, \gamma_m) \geq r' \forall n, m \geq n_0$

then $\mathcal{P}_{\gamma_n} + \alpha_{r'} = \mathcal{P}_{\gamma_m} + \alpha_{r'}$
(last lemma)

Set $I_{r'} := \mathcal{P}_{\gamma_n} + \alpha_{r'} / \alpha_{r'} \subseteq A_{inf} / \alpha_{r'}, n \gg 0$

and $I = \varprojlim_{r' \geq 0} I_{r'} \ (\sim I + \alpha_{r'} / \alpha_{r'})$

Claim: I gen. by prim. elt. of degree 1, ~~is not~~ & $|\mathcal{P}_{\gamma_n} + \alpha_{r'}|, n \rightarrow \infty$

Indeed, fix $r' > r$ and n , s.t.

(\sim autom. $I \in |Y_r|$)

$$\mathcal{P}_{\gamma_n} + \alpha_{r'} = I + \alpha_{r'}$$

Write $\mathcal{P}_{\gamma_n} = (\xi_{\gamma_n}) \sim$ ex. $x \in \alpha_{r'}$, s.t. $a := \xi_{\gamma_n} + \alpha_{r'} \in I$

~~Then~~ Then $a \in \text{Prim}_1$ (as $r' > r$)

~~Now~~ Now, (a) = I ~~is~~ $A_{inf}/(a)$ val. ring and if (b) $\neq I$
Namely,

$\Rightarrow \exists r_0, s.t. (b) + \alpha_{r_0} \subset I$. Let $r'' > \sup\{r_0, r\}$ and $\left. \begin{matrix} \text{use } \ominus_{\gamma_m} \\ \{ \end{matrix} \right\}$
 m s.t. $I + \alpha_{r''} = \mathcal{P}_{\gamma_m} + \alpha_{r''} \Rightarrow \alpha_{r_0} \subseteq I \subseteq \mathcal{P}_{\gamma_m} + \alpha_{r''} \Rightarrow r_0 \geq r'' \checkmark$

Clear, $P_n \rightarrow I, n \rightarrow \infty$ as $P_n + a_{r'} = I + a_{r'}$ for $r' > 0, n > 0$ (5) \square

Thm (Fargues-Fontaine): $f \in A_{\text{inf}}, \lambda \neq 0$ slope of $\text{Newt}(f)$

$$= 1 \exists \gamma \in |\mathcal{Y}|, v(\pi(\gamma)) = -\lambda, f(\gamma) = 0$$

Sketch of proof:

1) Reduction to case $f \in A_{\text{inf}}$ primitive of some degree d

~~Note~~ wlog $f \in A_{\text{inf}}$

$$\text{Write } f = \sum_{n=0}^{\infty} [x_n] \pi^n, x_0 \neq 0 = \lim_{d \rightarrow \infty} P_d, P_d = \sum_{n=0}^d [x_n] \pi^n$$

\uparrow up to mult. by some Teichmüller lift primitive of degree d

$\exists D \geq 0$, s.t. λ appears in $\text{Newt}(P_d)$ for $d \geq D$

~~with~~ (with mult. bdd. independent of d)

$$\text{Set } X_d := \{ \gamma \in |\mathcal{Y}| \mid P_d(\gamma) = 0, v(\pi(\gamma)) = -\lambda \}$$

Check: Can choose $\{ \gamma_d \}_{d \geq 0}, \gamma_d \in X_d$
Cauchy sequence

Prop. $\Rightarrow \{ \gamma_d \}_{d \geq 0}$ converges against $\gamma \in |\mathcal{Y}|, v(\pi(\gamma)) = -\lambda, f(\gamma) = 0$

2) Assume $f \in A_{\text{inf}}$ primitive of degree d

~~show $f \in A_{\text{inf}}$ with a primitive of degree d~~

wlog $\lambda < 0$ maximal slope of f

Claim: ex. sequence $\gamma_n \in |\mathcal{Y}|$, s.t.

a) $v(f(\gamma_n)) \geq -(d+n)\lambda$

b) $d(\gamma_n, \gamma_{n+1}) \geq \frac{d+n}{d} \lambda$

c) $v(\pi(\gamma_n)) = -\lambda$

(\Rightarrow Thm using Prop. as $\{ \gamma_n \}_{n \geq 0}$ will converge)

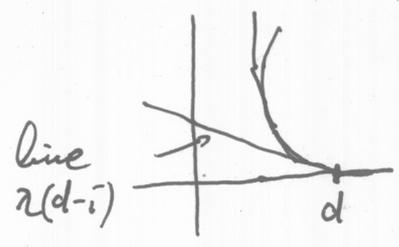
Prf (only γ_1): Write $f = \sum_{n=0}^{\infty} [x_n] \pi^n$ ($-1 \leq x_d \in \mathcal{O}_F^\times$)

Let $\bar{z} \in \mathcal{O}_F$ be a zero of $\sum_{i=0}^d x_i T^i \in \mathcal{O}_F[T]$ (note $x_d \in \mathcal{O}_F^\times$)

of ~~deg~~ $v(\bar{z}) = -\alpha$ (use Newton polygons for polynomials)

Set $\gamma_1 := (\pi - [\bar{z}])$

As $\alpha < 0$ maximal slope, $v(x_i) \geq \alpha(d-i)$
 $\forall 0 \leq i \leq d$



$\Rightarrow x_i \bar{z}^i = w_i \bar{z}^d$ with $w_i \in \mathcal{O}_F$

Then $f(\gamma_1) = \Theta_{\gamma_1}(f) = \Theta_{\gamma_1}\left(\sum_{n=0}^{\infty} [x_n] \pi^n\right) = \sum_{i=0}^d \Theta_{\gamma_1}([x_i \bar{z}^i]) + \pi^{d+1} \sum_{i=d+1}^{\infty} \Theta_{\gamma_1}([x_i]) \pi^{d-i-1}$
 $= \pi^d \sum_{i=0}^d \Theta_{\gamma_1}(w_i)$

But $\sum_{i=0}^d w_i = 0$, i.e. $\sum_{i=0}^d [w_i] \in \pi A_{inf}$

$\Rightarrow f(\gamma_1) \in \pi^{d+1} \mathcal{O}_{C_{\gamma_1}}$, i.e. $v(f(\gamma_1)) \geq (d+1)\alpha$

Assume γ_n constructed.

Write $f = \sum_{r=0}^{\infty} [a_r] \zeta_{\gamma_n}^r$

Let $\bar{z} \in \mathcal{O}_F$ be a zero of $\sum_{i=0}^d a_i T^i$ of ~~max slope~~ minimal ^{valuation} slope.

Then $\bar{z} \in \mathcal{O}_F$ (check $a_d \in \mathcal{O}_F^\times$ using $A_{inf} \rightarrow W_{\mathcal{O}_E}(k)$, $k = \mathcal{O}_F/\mathfrak{m}_F$)

Check $\gamma_{n+1} = (\zeta_{\gamma_n} - [\bar{z}])$ works. □

Rem: $\ast \mathfrak{m}_F \rightarrow |\gamma|, a \mapsto (\pi - [a])$ surj. But can give better description ($E = \mathcal{O}_p$)

- Exercise:
- ~~\mathcal{O}_p alg. fld, non-arch, $\mathcal{O}_C^b \cong \mathcal{O}_F, \varepsilon = (1, \zeta_p, \dots) \in \mathcal{O}_C^b$~~
 $\Rightarrow u_\varepsilon = 1 + [\varepsilon^{\frac{1}{p}}] + \dots + [\varepsilon^{\frac{p-1}{p}}] = \frac{[\varepsilon] - 1}{[\varepsilon^{\frac{1}{p}}] - 1}$ dist. elt & $u_\varepsilon \in \ker \Theta$
 - $a, b \in A_{inf}$ dist., $(a) \subseteq (b) \iff (b) = (a)$ ($\neq \emptyset$ in part, $(u_\varepsilon) = \ker \Theta$)
 - ~~Let $\varepsilon \in 1 + \mathfrak{m}_F$~~ $u_\varepsilon := 1 + [\varepsilon^{\frac{1}{p}}] + \dots + [\varepsilon^{\frac{p-1}{p}}]$ ($\sim u_\varepsilon$ primitive of degree 1)
 $\Rightarrow (u_\varepsilon) = (u_{\varepsilon'})$ if $\varepsilon \equiv \varepsilon'$ in $(1 + \mathfrak{m}_F) / \mathcal{Z}_p^\times$
 where \mathcal{Z}_p^\times acts on $1 + \mathfrak{m}_F$ via $a + \varepsilon := \sum_{i=0}^{\infty} \binom{d}{i} (\varepsilon - 1)^i$

Hint: C/\mathcal{O}_p alg. fld, non-arch
 $\Rightarrow T_p C^\times$ is free of rank one over \mathcal{Z}_p