

# The Fargues-Fontaine curve (or the fundamental curve of p-adic Hodge theory)

Fix prime  $p$ .

$K/\mathbb{Q}_p$  finite,  $C = \widehat{\bar{K}}$  completed alg. closure of  $\bar{K}$

$$G_K := \text{Gal}(\bar{K}/K) = \text{Gal}_{\text{cts}}(C/K), \quad G_K \supseteq C$$

$X \rightarrow \text{Spec } K$  proper, smooth

Thm (Fultings, Tsuji, ...) For  $n \geq 0$  ex. natural  $G_K$ -equiv. isom.

"Hodge-Tate decomposition"  $H_{\text{ét}}^n(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \simeq \bigoplus_{i+j=n} H^i(X, \mathcal{R}_{X/K}^j) \otimes_{\bar{K}} C(-j)$

Remarks: \*  $H_{\text{ét}}^n(X_{\bar{K}}, \mathbb{Q}_p) := \left( \varprojlim_k H_{\text{ét}}^n(X_{\bar{K}}, \mathbb{Z}_{p^k}) \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$

\*  $G_K$  acts diagonally on LHS, via  $C(-j)$  on RHS

Here  $M(j) := M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes j}, j \in \mathbb{Z}$ ,

where  $\mathbb{Z}_p(1) = \varprojlim_k \mu_{p^k}(C)$  "Tate twist",  $G_K \supseteq \mathbb{Z}_p(1)$

\* Analog of complex Hodge theory:

Y compact Kähler manifold  $\Rightarrow$

$$H^n(Y, \mathbb{Z}) \otimes_{\mathbb{Z}} C \simeq \bigoplus_{i+j=n} H^i(Y, \mathcal{R}_Y^j)$$

\* Does hold for proper, smooth rigid-analytic varieties (Scholze)

\* Tate twists necessary to get  $G_K$ -equivariance:

$$X = \mathbb{P}_K^1$$

$$\text{LHS : } H_{\text{ét}}^2(\mathbb{P}_{\bar{K}}^1, \mathbb{Z}_p) \simeq H_{\text{ét}}^1(\mathbb{G}_{m, \bar{K}}, \mathbb{Z}_p) \simeq \text{Hom}(\pi_1^{\text{ét}}(\mathbb{G}_{m, \bar{K}}, 1), \mathbb{Z}_p) \simeq \mathbb{Z}_p(-1)$$

$$\text{RHS : } H^1(\mathbb{P}_K^1, \mathcal{R}^1) \simeq K$$

Thm (Tate): 1)  $H_{cts}^*(G_K, C(j)) = 0$  if  $j \neq 0$

(2)

$$2) K \cong H_{cts}^0(G_K, C) \cong H_{cts}^1(G_K, C)$$

Remarks: \* Even  $C^{G_K}$  non-obvious

\*  $H_{cts}^*(G_K, C(j)) \cong H^*(R\lim_k R\Gamma(G_K, \mathcal{O}_{C/\mathbb{P}^k}(j))) \left[ \frac{1}{p} \right]$

\* In part,  $C \not\cong C(j)$ , if  $j \neq 0$   
 as  $G_K$ -repr

Cor.: For  $n \geq 0, j \geq 0$

$$H^{n-j}(X, \mathcal{R}_{X/K}^j) \cong (H^n_{\text{ét}}(X_{\bar{\eta}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C(j))^{G_K}$$

That is:

" $\mathbb{Z}_p$ -adic étale cohomology knows Hodge cohomology"

Converse not true:

\*  $X$  elliptic curve  $\Rightarrow H^1_{\text{ét}}(X_{\bar{\eta}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \cong C \oplus C(-1)$ ,

but  $H^1_{\text{ét}}(X_{\bar{\eta}}, \mathbb{Q}_p)^{G_K}$  knows if  $X$  has good/semistable reduction

\*  $X = \text{Spec } L$ ,  $L/K$  finite  $\Rightarrow$

$$H^0_{\text{ét}}(X_{\bar{\eta}}, \mathbb{Q}_p) \cong \prod_{L \subset \bar{\eta}} \mathbb{Q}_p \quad \# \cong G_K \text{ nat. permutation}$$

enough to recover  $X$

$$\text{But } H^0_{\text{ét}}(X_{\bar{\eta}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \cong C^{[L:K]} \text{ only knows } [L:K]$$

Nice application (of corollary):

$\rightsquigarrow$  smooth proj + can. bundle  
ref

Thm (Ito, Veys, Kontsevich, ...):  $\gamma, \gamma'$  smooth minimal models

If  $\gamma, \gamma'$  birational, then

$$\dim_{\mathbb{Q}} H^i(\gamma, \mathcal{R}_{\gamma}^j) = \dim_{\mathbb{Q}} H^i(\gamma', \mathcal{R}_{\gamma'}^j) \quad \forall i, j \geq 0$$

Idea of proof:  $\gamma, \gamma'$  birational + smooth minimal models

(3)

$\Rightarrow \gamma, \gamma'$  K-equivalent: ex.  $Z \curvearrowright$  proper, smooth

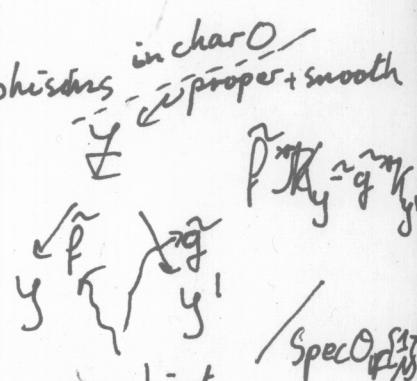
$$\begin{array}{ccc} f & \searrow g & f, g \text{ proper birational} \\ \gamma & \gamma' & f^*K_\gamma \simeq g^*K_{\gamma'} \end{array}$$

Spread out over f.g.  $\mathbb{Z}$ -alg.  $A \subseteq \mathbb{Q}$

Hodge numbers are constant for proper, smooth morphisms in char 0  
 $\Rightarrow$  can reduce to  $A = \mathcal{O}_{\mathbb{F}}[\frac{1}{N}]$ ,  $F/\mathbb{Q}$  finite.

p-adic integration implies

$$Y(\mathbb{F}_{\ell^k}) = Y'(\mathbb{F}_{\ell^k}) \quad \forall \text{ primes } \ell, (\ell, N) = 1, k \geq 0$$



proper birat.

$\text{Spec } \mathcal{O}_{\mathbb{F}}$

Fix prime  $p, (p, N) = 1$

$\Rightarrow$  If  $(\ell, Np) = 1$ , then  $H_{\text{ét}}^*(Y_{\overline{\mathbb{F}_\ell}}, \mathbb{Q}_p)^{\text{ss}} \simeq H_{\text{ét}}^*(Y'_{\overline{\mathbb{F}_\ell}}, \mathbb{Q}_p)^{\text{ss}}$  (Weil conjectures)

=) Chebotarev  $H_{\text{ét}}^*(Y_{\overline{\mathbb{F}}}, \mathbb{Q}_p)^{\text{ss}} \simeq H_{\text{ét}}^*(Y'_{\overline{\mathbb{F}}}, \mathbb{Q}_p)^{\text{ss}}$

Pick  $\mathfrak{p}/p$ ,  $K := \mathbb{F}_p$

$$=) H_{\text{ét}}^*(Y_{\overline{K}}, \mathbb{Q}_p)^{\text{ss}} \simeq H_{\text{ét}}^*(Y'_{\overline{K}}, \mathbb{Q}_p)^{\text{ss}}$$

$$\Rightarrow \dim_K H^i(Y_K, \mathcal{R}_{Y_K}^\dagger) = \dim_K H^i(Y'_K, \mathcal{R}_{Y'_K}^\dagger)$$

H.-T. comp.

+ Cor

+ E

Another application:  $\gamma/\mathbb{C}$  proper, smooth scheme

$\Rightarrow$  The Hodge-de Rham spectral sequence

$$E_1^{ij} = H^j(\gamma, \mathcal{R}^i) \Rightarrow H_{\text{dR}}^{i+j}(\gamma) = H^{i+j}(0 \rightarrow \mathcal{O}_\gamma \xrightarrow{d} \mathcal{R}_\gamma^1 \xrightarrow{d} \mathcal{R}_\gamma^2 \rightarrow \dots)$$

degenerates.

Sketch of proof: Reduce to case  $\mathcal{Y} = X \times_{\text{Spec } K} \text{Spec } \mathcal{A}$  for  $K/\mathbb{Q}_p$  finite (4)

and some  $K \hookrightarrow C$ .

Sufficient to show:  $\dim_{\mathbb{C}} H^i_{dR}(\mathcal{Y}) = \sum_{i+j=n} \dim_{\mathbb{C}} H^j(\mathcal{Y}, \mathcal{R}_{\mathcal{Y}}^i)$

$$\text{LHS} = \dim_{\mathbb{Q}_p} H^n(Y(\bar{a}), \mathbb{Q}_p) \stackrel{\text{Artin comp.}}{\underset{\text{H.-T. comp.}}{\equiv}} \dim_{\mathbb{Q}_p} \check{H}^n_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p) \stackrel{\text{H.-T. comp.}}{\underset{i+j=n}{\equiv}} \sum_{i+j=n} \dim_{\mathbb{K}} H^i(X, \mathcal{R}_X^j) = \text{RHS}$$

Q: Does the  $G_{\bar{n}}$ -repr.  $H_{\text{ét}}^n(\bar{X}, \mathbb{Q}_p)$  know  $H_{\text{dR}}^*(X)$  including the filtration?

Thm (Faltings, Tsuji,...) For  $n \geq 0$ , ex. natural  $G_K$ -equiv, filtered isom.

$$H_{\text{ét}}^n(X_{\bar{\kappa}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR} \simeq H_{dR}^n(X) \otimes_{\bar{\kappa}} B_{dR} \quad \text{"de Rham comparison"}$$

Remarks: \*  $B_{dR}$  Fontaine's field of  $p$ -adic periods,  $G_K \otimes B_{dR}$

$B_{dR} = \text{Frac}(B_{dR}^+)$ ,  $B_{dR}^+$  complete DVR with residue field  $\mathbb{C}$  ( $\sim$  abstractly)  $B_{dR}^+ \cong \mathbb{C}[[t]]$ )

$$\text{Fil}^{\oplus} B_{dR} = \{\bar{\beta}^+ B_{dR}^+, \beta \in B_{dR}^+ \text{ unif.}\}$$

not Galois  
equiv.

$$\text{gr } B_{dR} = B_{HT} := \bigoplus_{j \in \mathbb{Z}} C(j) \quad (\leadsto \text{de Rham comparison recovers Hodge-Tate decomp.})$$

\*  $G_K$ -action diag. on LHS, via  $B_{dR}$  on RHS

Filt. filtration via  $B_{dR}$  on LHS, not diagonally on RHS

$$* X = \mathbb{P}^1_K, n=2 \Rightarrow \mathbb{Z}/\mathbb{Z}_p(-1) \otimes B_{dR} \simeq B_{dR}$$

$\Rightarrow \exists$  can.  $\mathbb{Q}_p$ -line  $\mathbb{Q}_p \cdot t \subseteq B_{dR}$ , s.t.  $G_K$  acts via cyclotomic character  $\chi_{cycl}: G_K \rightarrow \mathbb{Z}_p^*$  (i.e.  $\mathbb{Q}_p \cdot t \simeq \mathbb{Q}_p(1)$ )

Fontaine's theory: for  $\varepsilon \in \mathbb{Z}_p(1)$  get  $t = \text{"log } [\varepsilon] \text{"} \in B_{dR}$   
 $= \text{"} 2\pi i \text{"}$

Assume  $X$  has good reduction, i.e.  $X = \mathbb{P}^1 \otimes_K \mathbb{X}_K$  for  $\mathbb{X} \rightarrow \text{Spec } \mathcal{O}_K$   
proper, smooth.  $\mathbb{X}_0$  = special fiber of  $\mathbb{X}$

Get ~~some~~ refinement

Thm (Faltings, Nizioł, Tsuji, ...) For  $n \geq 0$ , ex. nat.  $G_K$ -equiv, ~~not~~ filtered,  $\varphi$ -equiv isom.

$$H_{\text{ét}}^n(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}} \simeq H_{\text{cris}}^n(X_0 / \mathcal{O}_{K_0}) \otimes_{\mathcal{O}_{K_0}} B_{\text{cris}}$$

Rmk: \*  $K_0 \subseteq K$  maximal unramified subext. ( $\rightsquigarrow p$  unif. in  $\mathcal{O}_{K_0}$ ), ex. can. Frobenius

\*  $H_{\text{cris}}^n(X_0 / \mathcal{O}_{K_0})$  crystalline cohom. of  $X_0$  w.r.t.  $\mathcal{O}_{K_0}$   $\varphi \otimes \mathcal{O}_{K_0}$

$\varphi G$  ( $\hookrightarrow$  roughly, de Rham cohom. of smooth lift)

and  $(H_{\text{cris}}^n(X_0 / \mathcal{O}_{K_0})[\frac{1}{p}], \varphi)$  is filtered  $\varphi$ -module ~~for~~ <sup>$K$</sup> , i.e.

a f.d.  $K_0$ -v.s. +  $\varphi$ -semilinear isom  $\varphi_D: \varphi^* D \xrightarrow{\sim} D + \text{filtration } \text{Fil}^*(D)$

on  $D_K = D \otimes_{K_0} K$

coming from Hodge filtr.

\*  $B_{\text{cris}}$  Fontaine's ring of crystalline  $p$ -adic periods,

$\varphi G A_{\text{cris}} := H_{\text{cris}}^0(\mathcal{O}/p/\mathbb{Z}_p)$ ,  $B_{\text{cris}}^+ := A_{\text{cris}}[\frac{1}{p}]$ ,  $B_{\text{cris}}^+ \subseteq B_{\text{dR}}^+$   $G_K$ -stable

$t = \log[\varepsilon] \in B_{\text{cris}}^+$  and  $B_{\text{cris}} = B_{\text{cris}}^+[\frac{1}{t}]$ ,  $\varphi(t) = pt$

\* Inverting  $t$  is necessary:  $X = \mathbb{P}^1_K$ ,  $n=2$

\* Analog to statement in  $\ell$ -adic cohom,  $\ell \neq p$  (or complex geometry):

$\mathbb{X} \rightarrow \text{Spec } \mathcal{O}_K$  proper, smooth,  $s, \eta$  spec. resp. gen. pt of  $\text{Spec } \mathcal{O}_K$

$\Rightarrow$  ex.  $G_K$ -inv. isom.  $H_{\text{ét}}^*(\mathbb{X}_{\overline{\eta}}, \mathbb{Q}_p) \simeq H_{\text{ét}}^*(\mathbb{X}_{\overline{s}}, \mathbb{Q}_p)$

In past.,  $H_{\text{ét}}^*(\mathbb{X}_{\overline{\eta}}, \mathbb{Q}_p)$  is unramified

More mysterious if  $\ell \neq p$ ,  $H_{\text{ét}}^*(\mathbb{X}_{\overline{s}}, \mathbb{Q}_p)$  replaced by  $H_{\text{cris}}^*(\mathbb{X}_{\overline{s}} / \mathcal{O}_{K_0})$

How to pass from <sup>cont.</sup>  $G_K$ -repr. on  $\mathbb{Q}_p$ -v.s. with Frobenius and filtration over  $K$ ?

Grothendieck's mysterious functor.

question on the

Resolved by Fontaine:

$$\begin{aligned} \mathbf{Rep}_{\mathbb{Q}_p} G_K &\rightarrow \{\text{filt. } \varphi\text{-modules}\} \\ V &\mapsto D_{\text{cris}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{G_K} \\ V_{\text{cris}}(0) \text{Fil}^0(D \otimes_{K_0} B_{\text{cris}})^{\varphi=1} &\leftarrow 0 \end{aligned}$$

\* Should expect that  $H^n_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ ,  $H^n_{\text{cris}}(X_0/\mathcal{O}_{K_0})[\frac{1}{p}]$  have same information

Thm (Colmez/Fontaine) "weakly admissible implies admissible"

$D_{\text{cris}}, V_{\text{cris}}$  restrict to equivalences

$$\{\text{cryst. } G_K\text{-repr.}\} \xrightarrow{\sim} \{\text{weakly admissible filt. } \varphi\text{-modules}\}$$

Rmk: \*  $V \in \mathbf{Rep}_{\mathbb{Q}_p} G_K$  crystalline if  $\dim_{\mathbb{Q}_p} D_{\text{cris}}(V) = \dim_{\mathbb{Q}_p} V$

\* weakly admissible connected to "Newton polygon lies above Hodge polygon"

A sketch of proof of this theorem will be the aim of this course.

The essential ingredients <sup>will be the</sup> ~~with the~~ Fargues-Fontaine curve

$$X_{\text{FF}} = \mathbb{P} \text{Proj} \left( \bigoplus_{d \geq 0} (B_{\text{cris}}^+)^{\varphi=p^d} \right) \text{ over } \mathbb{Q}_p \quad (\text{a Dedekind scheme!})$$

all together with the relation of its ( $G_K$ -equivariant) vectorbundles to  $\mathbf{Rep}_{\mathbb{Q}_p} G_K$  resp.  $\{\text{filtered } \varphi\text{-modules}\}$

The rings  $B_{\text{dR}}^{(+)}, B_{\text{cris}}^{(+)}, \dots$  are closely related to functions on  $X_{\text{FF}}$ ,

e.g.  $B_{\text{dR}}^+ \simeq \text{completion of some point in } X_{\text{FF}}$