Recall that last time we stated our second main theorem

Theorem 0.1. *The functor $\mathcal{E}$ defines a bijection of isomorphism classes*

$$\varphi\text{–}\mathrm{Mod}_{\breve{E}}/\sim \longrightarrow \mathrm{Bun}_{X_{\mathrm{FF}}}/\sim$$

Proof: next time. Today, we discuss a few things we need for the proof. On the way, we will see some applications of X_{FF} to p-divisible groups and p-adic Hodge theory.

For this we will need some background that strictly speaking isn't in the prerequisites of this course (so don't worry if there are bits you don't understand, this lecture is a bit of a "survey" anyway). We therefore start with

1 p-divisible groups (a crash course)

Definition 1.1. Let R be a ring. A p-divisible group G over R of height h is a collection $(G_n, i_n)_{n\in\mathbb{N}}$ of finite flat group schemes G_n of order p^{hn} over R together with closed immersions $i_n : G_n \to G_{n+1}$ such that the following sequence is left exact:

$$0 \to G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{[p^n]} G_{n+1}$$

We also write $G[p^n] := G_n$. It follows from the axioms that we have a short exact sequence for any n, m:

$$0 \to G[p^n] \xrightarrow{i_{n,m}} G[p^{n+m}] \xrightarrow{j_{n,m}} G[p^m] \to 0.$$

where j is induced from $[p^n] : G[p^{n+m}] \to G[p^{n+m}]$. Surjectivity on the right is the reason for the name "p-divisible".

Example 1.2.

- Let $G_n = \underline{\frac{1}{p^n}\mathbb{Z}/\mathbb{Z}}$ be the constant group scheme, and i_n the natural inclusion. This defines a p-divisible group called $\mathbb{Q}_p/\mathbb{Z}_p$ of height 1.
- Let $G_n = \mu_{p^n} := \mathrm{Spec}(R[X]/(X^{p^n} - 1))$ be the group scheme of p^n-th unit roots. This defines a p-divisible group μ_{p^∞} of height 1.
- Let A be an abelian scheme over R of dimension d. Then the p^n-torsion $G_n := A[p^n]$ defines a p-divisible group $A[p^\infty]$ of height $2d$.

Definition 1.3. For any p-divisible group G, we obtain its dual p-divisible group $G^\vee$ by letting $(G^\vee)_n := (G_n)^\vee = \underline{\mathrm{Hom}}(G_n, \mathbb{G}_m)$ be its Cartier dual, and $i_n^\vee := j_{1,n}^\vee$.

The natural evaluation isomorphisms $G_n \to (G_n^\vee)^\vee$ are compatible and define an isomorphism of p-divisible groups

$$G \to (G^\vee)^\vee$$

The functor $G \mapsto G^\vee$ is thus a (contravariant) auto-duality.

Example 1.4.

- $(\mathbb{Q}_p/\mathbb{Z}_p)^\vee = \mu_{p^\infty}$ and $(\mu_{p^\infty})^\vee = \mathbb{Q}_p/\mathbb{Z}_p$.
- $A[p^\infty]^\vee = A^\vee[p^\infty]$ where $A^\vee$ is the dual abelian variety.

1.1 Tate modules

We now specialise to the case of C a complete algebraically closed extension of $\mathbb{Q}_p$ and $R = \mathcal{O}_C$. This is the case we shall focus on today.

Definition 1.5. For a p-divisible group G over $\mathcal{O}_C$, we define its Tate module to be

$$T_pG := \varprojlim \Big(\xrightarrow{[p]} G[p^2](C) \xrightarrow{[p]} G[p](C) \to 1 \Big).$$

Example 1.6.
- We have $T_p(\mathbb{Q}_p/\mathbb{Z}_p) = \mathbb{Z}_p$.
- We write $T_p\mu_{p^\infty} =: \mathbb{Z}_p(1)$ for the Tate module of μ_{p^∞} over $\mathcal{O}_C$. It is isomorphic to $\mathbb{Z}_p$, but the isomorphism depends on a choice of compatible ζ_{p^n}. For any $\mathbb{Z}_p$-module M and $n \in \mathbb{Z}$ we set $M(n) := M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes n}$ (interpreted as the module dual for $n < 1$)
- $T_p(A[p^\infty]) = T_pA$. This can be canonically identified with the dual of $H^1_{\text{ét}}(A, \mathbb{Z}_p)$ (and when we work over any field K, this identification is Galois equivariant).

Lemma 1.7. {l:p-div-groups-T
1. *There is a natural isomorphism of $\mathbb{Z}_p$-modules $T_pG = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G)$.*
2. *The natural map $T_pG = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G) \xrightarrow{-^\vee} \mathrm{Hom}(G^\vee, \mu_{p^\infty})$ defines a perfect pairing*

$$T_pG \times T_pG^\vee \to T_p\mu_{p^\infty} = \mathbb{Z}_p(1).$$

Proof. Exercise. Use $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G) \xrightarrow{T_p()} \mathrm{Hom}(\mathbb{Z}_p, T_pG) \xrightarrow{ev(1)} T_pG$ □

For $G = A[p^\infty]$, this duality pairing can be identified with the Weil pairing.

1.2 Dieudonne modules

It is an important task in arithmetics to classify all p-divisible groups over a given ring. This was first known for perfect fields k of characteristic p:

Theorem 1.8 (Dieudonné, Cartier (60's))**.** *Let k be a perfect field of characteristic p. Then there is an exact equivalence of categories*

$$M : \{p\text{-divisible groups over } k\} \to \{\text{Dieudonné modules over } W(k)\}.$$

Here a Dieudonné module is a finite free $W(k)$-module together with a φ-linear action of an operator F and a φ^{-1}-linear action of an operator V such that $FV = p = VF$.

Actually, they prove this for finite flat group schemes, the case of p-divisible groups follows. In fact, p-divisible groups where historically introduced after this.

Remark 1.9. In this setting, one often instead uses a *contra*-variant version of Dieudonné modules, making the above an anti-equivalence. We will today use the covariant version. The two can be translated into each other via the auto-duality $G \mapsto G^\vee$. Note: Going from contra to co does not affect the linearity properties of F and V, but it means that now F on M corresponds to V on G and vice versa.

Nowadays, one can classify p-divisible groups in many more cases e.g. over perfectoid bases (see ARGOS). Grothendieck–Messing and Berthelot–Breen–Messing (70') extended the definition of Dieudonné-modules to any ring R on which p is nilpotent, using the formalism of the crystalline site. Important for us is the following case:

Let C be as before and consider the semi-perfect ring $\mathcal{O}_C/p$. Recall that we had defined a ring $\mathbb{A}_{\text{inf}} \to A_{\text{crys}}$ ("adjoin things of the form $p^n/n!$ so you can do log")

Definition 1.10. A Dieudonné-module over $\mathcal{O}_C/p$ is a finite free A_{crys}-module M together with linear operators

$$F : M \otimes_{A_{\text{crys}},\varphi} A_{\text{crys}} \to M$$
$$V : M \to M \otimes_{A_{\text{crys}},\varphi} A_{\text{crys}}$$

such that $FV = p = VF$.

Proposition 1.11 (Grothendieck–Messing, Scholze–Weinstein)**.** *There is a fully faithful covariant functor*

$$M_{\text{crys}}(-) : \{\textit{p-divisible groups over } \mathcal{O}_C/p\} \to \{\textit{Dieudonné modules over } A_{\text{crys}}\}.$$

We have $\operatorname{rk}(M_{\text{crys}}(G)) = \operatorname{ht}(G)$ *and* $M_{\text{crys}}(G^\vee) = M_{\text{crys}}(G)^\vee$*, the* A_{crys}*-module dual.*

Example 1.12.

- $M(\mathbb{Q}_p/\mathbb{Z}_p) = A_{\text{crys}}$ with $F = p$, $V = 1$.
- $M(\mu_{p^\infty}) = A_{\text{crys}}$ with $F = 1$, $V = p$.
- $M(A[p^\infty])^\vee = H^1_{\text{crys}}(A_{\mathcal{O}_C/p}|A_{\text{crys}})$ can be naturally identified with the crystalline cohomology, in a way that identifies F with the Frobenius φ.

Explain why $F = p$ for the étale group scheme: This is because we dualised.

2 connection to p-adic Hodge theory

Question: How can one compare T_pG and $M(G_{\mathcal{O}_C/p})$, can one perhaps recover one from the other? In the case of $G = A[p^\infty]$ for an abelian variety A over $\mathcal{O}_C$, this is essentially asking how to compare $H^1_{\text{ét}}(A_C, \mathbb{Z}_p)$ and $H^1_{\text{crys}}(A_{\mathcal{O}_C/p}|A_{\text{crys}})$.

The mathematical field studying such comparison isomorphisms between p-adic cohomology theories is p-adic Hodge theory.

Theorem 2.1 ([BMS, Theorem 14.5.(i)], '16)**.** *Let X be a smooth proper formal scheme over $\mathcal{O}_C$. Then for any $i \geq 0$, we have an étale-crystalline comparison isomorphism*

$$H^i_{\text{ét}}(X_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{crys}} = H^i_{\text{crys}}(X_{\mathcal{O}_C/p}|A_{\text{crys}}) \otimes_{A_{\text{crys}}} B_{\text{crys}}.$$

Remark 2.2.

- We think of this as a p-adic analogue of the following complex comparison isomorphism: Let Y be a smooth variety over $\mathbb{C}$, then

  $$H^i_{\text{sing}}(Y(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^i_{\text{dR}}(Y|\mathbb{C}).$$

  In the p-adic version, the role of the ring of periods $\mathbb{C}$ (that one needs for the Poincaré-Lemma) is played by the ring B_{crys}.

- If X comes via base-change from $\mathcal{O}_K$ for $K|\mathbb{Q}_p$ finite, this is already due to Tsuji, after previous work by Fontaine–Messing, Bloch–Kato and Faltings). One can then also identify the Galois actions on both sides. In particular, this shows that the p-adic Galois representation $V_p(G) := T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is "crystalline", i.e. we have

  $$\dim_{\mathbb{Q}_p} V_p(G) = \dim_{\mathbb{Q}_p}(V_p(G) \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{G_K}$$

  is an isomorphism.

Back to p-divisible groups. It turns out that such a comparison holds more generally:

{prop:T_pG vs M(G

Proposition 2.3. *Let G be a p-divisible group over $\mathcal{O}_C$. We write $M(G)$ for the Dieudonné-module associated to $G_{\mathcal{O}_C/p}$. Then there is a natural φ-equivariant isomorphism*

$$\beta_G : T_pG \otimes_{\mathbb{Z}_p} B_{\mathrm{crys}} = M(G) \otimes_{A_{\mathrm{crys}}} B_{\mathrm{crys}}.$$

After tensoring up to B_{dR}^+, the respective B_{dR}^+-sublattices satisfy

$$T_pG \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+ \subseteq \Xi := M(G) \otimes_{A_{\mathrm{crys}}} B_{\mathrm{dR}}^+ \subseteq t^{-1}(M(G) \otimes_{A_{\mathrm{crys}}} B_{\mathrm{dR}}^+).$$

So we are essentially proving the crystalline comparison Theorem for abelian varieties!

Sketch of proof. We'll construct β_G and an inverse: Recall that $T_pG = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G)$. Given any $\alpha : \mathbb{Q}_p/\mathbb{Z}_p \to G$, we can apply $M(-)$ to get a φ-equivariant map

$$M(\alpha) \colon M(\mathbb{Q}_p/\mathbb{Z}_p) \to M(G).$$

We define β_G by sending α to the image of 1. Extending $\mathbb{Z}_p$-linearly, this defines a map

$$T_pG \otimes_{\mathbb{Z}_p} A_{\mathrm{crys}} \to M(G).$$

To show that this is an isomorphism after inverting p and t, we construct a (generic) inverse mapping using the dual: Applying the discussion so far to $G^\vee$, we obtain a map

$$\beta_{G^\vee} : T_p(G^\vee) \otimes_{\mathbb{Z}_p} A_{\mathrm{crys}} \to M(G^\vee). \tag{1}$$ {beta_G^vee}

The inverse will be induced by its A_{crys}-module dual $(\beta_{G^\vee})^\vee$ (this is a common trick).

Using $M(G^\vee) = M(G)^\vee$, we have a natural isomorphism

$$M(G) = M(G^\vee)^\vee$$

(this is an isomorphism by full faithfulness of M but we don't need this). On the other hand, we have by Lemma 1.7.2

$$T_p(G^\vee) = \mathrm{Hom}(T_pG, T_p\mu_{p^\infty}) = (T_pG)^\vee \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1).$$

Recall $A_{\mathrm{crys}}(-1) := (A_{\mathrm{crys}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1))^\vee$. Upon applying duals to (1), we then obtain a map

$$M(G) = M(G^\vee)^\vee \xrightarrow{(\beta_{G^\vee})^\vee} (T_p(G^\vee) \otimes_{\mathbb{Z}_p} A_{\mathrm{crys}})^\vee = T_pG \otimes_{\mathbb{Z}_p} A_{\mathrm{crys}}(-1).$$

Using the natural B_{crys}^+-linear isomorphism

$$\mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} B_{\mathrm{crys}}^+ \to tB_{\mathrm{crys}}^+, \quad \epsilon^a \otimes x \mapsto \log(\epsilon^a) \cdot x = t \cdot a \cdot x,$$

we thus get a map

$$M(G) \otimes B_{\mathrm{crys}}^+ \to T_pG \otimes_{\mathbb{Z}_p} B_{\mathrm{crys}}^+(-1) \to T_pG \otimes_{\mathbb{Z}_p} t^{-1}B_{\mathrm{crys}}^+.$$

One can check that after passing to $B_{\mathrm{crys}} = B_{\mathrm{crys}}^+[\frac{1}{t}]$, this defines an inverse to β_G. □

3 vector bundles on the Fargues–Fontaine curve

Today $E = \mathbb{Q}_p$, $\pi = p$. Recall that $X_{\mathrm{FF}} = \mathrm{Proj}(P) = \mathrm{Proj}(\oplus_{d\geq 0}(B^+_{\mathrm{crys}})^{\varphi=p^d})$ is the Fargues–Fontaine curve, a Dedekind scheme over $\mathbb{Q}_p$. Recall also that the natural morphism

$$\theta : A_{\mathrm{inf}} \to \mathcal{O}_C$$

defines a point $\infty \in X_{\mathrm{FF}}$ with associated closed immersion

$$i_\infty : \mathrm{Spec}(C) \to X_{\mathrm{FF}}.$$

By definition, the completion of X_{FF} at this point is given by

$$\mathrm{Spec}(\mathbb{B}^+_{\mathrm{dR}}) \to X_{\mathrm{FF}}$$

where $\mathbb{B}^+_{\mathrm{dR}}$ is a DVR with pseudo-uniformiser $t = \log([\epsilon])$, $\epsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$.
Here the element t is already defined in the smaller ring $\mathbb{A}_{\mathrm{inf}} \to A_{\mathrm{crys}} \to \mathbb{B}^+_{\mathrm{dR}}$. Moreover,

- $B^+_{\mathrm{crys}} = \mathbb{A}_{\mathrm{crys}}[\frac{1}{p}]$,
- $B_{\mathrm{crys}} = B^+_{\mathrm{crys}}[\frac{1}{t}]$,
- $B_{\mathrm{dR}} = B^+_{\mathrm{dR}}[\frac{1}{t}]$, a discretely valued field.

The Fargues–Fontaine curve organises all these p-adic period rings in a nice, geometric way.

Definition 3.1. Let G be a p-divisible group over $\mathcal{O}_C/p$. We can associate to G a quasi-coherent sheaf on X_{FF} by setting

$$\mathcal{E}(G) := \left(\bigoplus_{d\geq 0} M_{\mathrm{crys}}(G)[\tfrac{1}{p}]^{\varphi=p^d}\right)^{\sim}.$$

We will soon see that this is in fact a vector bundle of rank $\mathrm{ht}G$.

By Proposition 2.3, there is a natural morphism

$$\beta_G : T_pG \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathrm{FF}}} \to \mathcal{E}(G)$$

that is an isomorphism over the locus $X_{\mathrm{FF}} \setminus \{\infty\}$ where t is invertible:

$$\mathrm{Spec}(C) \xhookrightarrow{\iota_\infty} X_{\mathrm{FF}} \hookleftarrow X_{\mathrm{FF}} \setminus \{\infty\}$$

Corollary 3.2. *Let $\mathcal{F} := T_p(G) \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathrm{FF}}}$. Then there is a natural short exact sequence of sheaves on X_{FF}*

$$0 \to \mathcal{F} \xrightarrow{\beta_G} \mathcal{E}(G) \to i_{\infty *}W \to 0$$

where W is the fin. dim. C-vector space given by the image of $M(G) \otimes_{A_{\mathrm{crys}}} B^+_{\mathrm{dR}}$ under

$$T_pG \otimes_{\mathbb{Z}_p} t^{-1}B^+_{\mathrm{dR}} \xrightarrow{\mathrm{id}\otimes\theta(-1)} T_pG \otimes_{\mathbb{Z}_p} C(-1).$$

Proof. It is clear that the cokernel of $\mathcal{F} \to \mathcal{E}(G)$ is supported at ∞. To calculate the stalk at ∞, we need to reduce mod C, i.e. tensor with $B^+_{\mathrm{crys}} \to \mathbb{B}^+_{\mathrm{dR}} \to C$. We then use the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_pG \otimes \mathbb{B}_{\mathrm{dR}}^+ & \longrightarrow & M(G) \otimes \mathbb{B}_{\mathrm{dR}}^+ & \longrightarrow & \mathrm{coker} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \wr & & \\ 0 & \longrightarrow & T_pG \otimes \mathbb{B}_{\mathrm{dR}}^+ & \longrightarrow & T_pG \otimes t^{-1}\mathbb{B}_{\mathrm{dR}}^+ & \xrightarrow{\mathrm{id}\otimes\theta(-1)} & T_pG \otimes C(-1) & \longrightarrow & 0. \end{array}$$

□

Definition 3.3. A short exact sequence on X_{FF} of the form

$$0 \to \mathcal{F} \xrightarrow{\beta} \mathcal{F}' \to i_{\infty *}W \to 0$$

where $\mathcal{F}, \mathcal{F}'$ are vector bundles and W is a finite dimensional C-vector space is called a **minuscule modification**. We have thus defined a functor (important for next time)

$$\boxed{\{p\text{-divisible groups over } \mathcal{O}_C\} \to \{\text{minuscule modifications on } X_{\mathrm{FF}}\}.}$$

Sending a minuscule modification to its cokernel defines a forgetful morphism

$$\{\text{minuscule modifications on } X_{\mathrm{FF}}\} \to \{W \subseteq T_pG \otimes_{\mathbb{Z}_p} C(-1)\}.$$

The following amazing Theorem says that the data of T_pG together with W is equivalent to the datum of G:

{SW-p-div-over-OC

Theorem 3.4 ([SW13], Scholze–Weinstein '12)**.** *The functor defined above*

$$\{p\text{-divisible groups over } \mathcal{O}_C\} \to \left\{ \begin{array}{l} \text{pairs } (T, W) \text{ consisting of} \\ \bullet\, T \text{ finite free } \mathbb{Z}_p\text{-module,} \\ \bullet\, W \subseteq T \otimes_{\mathbb{Z}_p} C(-1) \end{array} \right\}$$

is an equivalence of categories.

Proof. No way. □

Remark 3.5.
- This is in stark contrast to the usual classifications of p-divisible groups in terms of *semi*-liner algebra data.
- We think of this as an analogue to Riemann's Theorem: complex abelian varieties A are equivalent to pairs (Λ, W) of a finite free $\mathbb{Z}$-module Λ and $W \subseteq \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ such that (Λ, W) is a polarisable Hodge structure of weight -1. For more on this: [Sch13, §4].
- There is a second, equivalent, way to characterise W, namely as the Hodge–Tate filtration $\mathrm{Lie}(G) \subseteq T_pG \otimes_{\mathbb{Z}_p} C(-1)$ of G (this is the definition in [SW13]). In particular,
  - For $G = \mathbb{Q}_p/\mathbb{Z}_p$, we have $(T, W) = (\mathbb{Z}_p, C(-1) \subseteq C(-1))$.
  - For $G = \mu_{p^\infty}$, we have $(T, W) = (\mathbb{Z}_p(1), 0 \subset C)$.
  - For $G = A[p^\infty]$, the Hodge–Tate filtration has $\dim W = \dim A$ and is of the form

$$0 \to W = \mathrm{Lie}(A) \to T_pA \otimes C(-1) \to \omega_A(-1) \to 0.$$

4 modifications of vector bundles

Back to X_{FF}. Our next goal is to see that one can in fact reconstruct $\mathcal{E}(G)$ from the trivial vector bundle $T_pG \otimes \mathcal{O}_{X_{\mathrm{FF}}}$ when given the data of the B_{dR}^+-lattice $\Xi \subseteq T_pG \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$ from Prop. 2.3. By the second part of the proposition, for this to work, the lattice must satisfy

$$T_pG \otimes_{\mathbb{Z}_p} \mathbb{B}_{\mathrm{dR}}^+ \subseteq \Xi \subseteq t^{-1}(T_pG \otimes_{\mathbb{Z}_p} \mathbb{B}_{\mathrm{dR}}^+).$$

Definition 4.1. Such a lattice is called a **minuscule lattice**.

We note that taking the image under θ defines an equivalence

$$\{\text{minuscule lattices in } T_pG \otimes_{\mathbb{Z}_p} \mathbb{B}_{\mathrm{dR}}^+\} \to \{\text{sub-}C\text{-vector spaces of } T_pG \otimes_{\mathbb{Z}_p} C(-1)\}$$

(and in particular, one can also state Theorem 3.4 in terms of minuscule B_{dR}^+-lattices).

The idea is that this gives "infinitesimal information at ∞" that one use to extend $\mathcal{F}$ to a new vector bundle over all of X_{FF}. This hinges on the following algebra fact:

Lemma 4.2 (Beauville–Laszlo). *Let A be a Noetherian ring and let $f \in A$. Consider the completion $\widehat{A} := \varprojlim_{n\in\mathbb{N}} A/f^n$. Then the natural functor*

$$\begin{aligned} A\text{−Mod} \to & A[\tfrac{1}{f}]\text{−Mod} \times_{\widehat{A}[\frac{1}{f}]\text{−Mod}} \widehat{A}\text{−Mod} \\ M \mapsto \quad & (M[\tfrac{1}{f}], \widehat{M}, \mathrm{can}) \end{aligned}$$

is an equivalence of categories. Moreoever, it restricts to an equivalence of categories on finite projective modules.

Here the category on the right is given by triples (N_1, N_2, α) where N_1 is an $A[\frac{1}{f}]$-module, N_2 is an $\widehat{A}$-module, and α is an $\widehat{A}[\frac{1}{f}]$-linear isomorphism

$$\alpha : N_1 \otimes_{A[\frac{1}{f}]} \widehat{A}[\tfrac{1}{f}] \to N_2 \otimes_{\widehat{A}} \widehat{A}[\tfrac{1}{f}].$$

The functor is given by sending $M \mapsto (M[\frac{1}{f}], \widehat{M}, \mathrm{can})$.

Proof. We explain how the inverse is constructed: Let (N_1, N_2, α) be an object in the RHS. We define M as the kernel of the natural map

$$0 \to M \to N_1 \oplus N_2 \xrightarrow{\alpha, -\mathrm{id}} N_2 \otimes_{\widehat{A}} \widehat{A}[\tfrac{1}{f}].$$

Exercise in commutative algebra: This defines an inverse. Hint: Use the exact sequence

$$0 \to A \to A[\tfrac{1}{t}] \times \widehat{A} \to \widehat{A}[\tfrac{1}{f}] \to 0$$

and the fact that $A \to A[\frac{1}{t}] \times \widehat{A}$ is faithfully flat (flatness uses A Noetherian).

The statement about finite proj. modules follows by fpqc-descent along $A \to A[\frac{1}{f}] \times \hat{A}$. □

The following Corollary explains what the Lemma means geometrically

Corollary 4.3. *Let X be a Dedekind scheme and let $x \in X$ be a closed point. Let $\widehat{X} :=$ $\mathrm{Spec}(\hat{\mathcal{O}}_{X,x}) \to X$ be the completion at x. Then the natural functor*

$$\mathrm{Bun}_X \to \mathrm{Bun}_{X\setminus\{x\}} \times_{\mathrm{Bun}_{\widehat{X}\setminus\{x\}}} \mathrm{Bun}_{\widehat{X}}$$

is an equivalence of categories.

[draw some picture]

Proof. By passing to an open neighbourhood of x, we can without loss of generality assume that X is affine. After shrinking $X = \mathrm{Spec}(A)$ if necessary, x is cut out by a single $f \in A$ since X is Dedekind, and we can apply Beauville–Laszlo. □

Definition 4.4. A (finite free) Breuil–Kisin–Fargues module (BKF-module) is a finite free $\mathbb{A}_{\mathrm{inf}}$-module M together with an $\mathbb{A}_{\mathrm{inf}}$-linear isomorphism

$$\varphi_M : \varphi^* M[\tfrac{1}{\varphi(\xi)}] \xrightarrow{\sim} M[\tfrac{1}{\varphi(\xi)}]$$

Theorem 4.5 (Fargues, [SW, Thm 14.1.1])**.** *The following categories are equivalent:*

1. *Breuil–Kisin–Fargues modules,*
2. *Quadruples $(\mathcal{F}, \mathcal{F}', \beta, T)$ consisting of vector bundles $\mathcal{F}, \mathcal{F}'$ on X_{FF} such that $\mathcal{F}$ is trivial, $\beta : \mathcal{F}_{|X_{FF} \setminus \{\infty\}} \to \mathcal{F}'_{|X_{FF} \setminus \{\infty\}}$, and $T \subseteq H^0(X_{\mathrm{FF}}, \mathcal{F})$ is a $\mathbb{Z}_p$-lattice,*
3. *Pairs (T, Ξ) where T is a finite free $\mathbb{Z}_p$-module and $\Xi \subseteq T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$ is a B^+_{dR}-lattice.*

This restricts to an equivalence of categories

1. *BKF-modules such that $M \subseteq \varphi_M(M) \subseteq \frac{1}{\varphi(\xi)} M$,*
2. *Quadruples for which β extends to a minuscule modification,*
3. *Pairs where Ξ is minuscule,*
4. *p-divisible groups over $\mathcal{O}_C$.*

Proof. We first show that 2 and 3 are equivalent:

Recall that $H^0(X_{FF}, \mathcal{O}_{X_{\mathrm{FF}}}) = \mathbb{Q}_p$. Consequently, the category of trivial vector bundles on X_{FF} is equivalent to the category of finite dimensional $\mathbb{Q}_p$ vector space via the functors

$$\mathcal{F} \mapsto H^0(X_{\mathrm{FF}}, \mathcal{F}) \tag{2}$$

$$V \otimes_{\mathbb{Q}_p} \mathcal{O}_{X_{\mathrm{FF}}} \leftarrow\!\shortmid V. \tag{3}$$

Under this equivalence, the datum of $T \subseteq H^0(X_{\mathrm{FF}}, \mathcal{F})$ corresponds to a $\mathbb{Z}_p$-lattice $T \subseteq V$.

We now apply Beauville–Laszlo glueing to the diagram

$$\begin{array}{ccc} \mathrm{Spec}(B^+_{\mathrm{dR}}) & \xrightarrow{\iota_\infty} & X_{\mathrm{FF}} \\ \uparrow & & \uparrow \\ \mathrm{Spec}(B_{\mathrm{dR}}) & \xrightarrow{\iota_{\infty,\eta}} & X_{\mathrm{FF}} \setminus \{\infty\} \end{array}$$

Starting with $(\mathcal{F}, \mathcal{F}', \beta, T)$, we now have an isomorphism

$$\iota^*_{\infty,\eta}\beta : T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} = i^*_{\infty,\eta}(T \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathrm{FF}} \setminus \{\infty\}}) = \iota^*_{\infty,\eta}\mathcal{F}_{|X_{\mathrm{FF}} \setminus \{\infty\}} \xrightarrow{\beta} i^*_{\infty,\eta}\mathcal{F}'_{|X_{\mathrm{FF}} \setminus \{\infty\}}$$

and we can define Ξ as the preimage of $H^0(\mathrm{Spec}(B^+_{\mathrm{dR}}), \iota^*_\infty \mathcal{F}')$.

Conversely, given (T, Ξ), we obtain a vector bundle $\mathcal{F}'$ on X_{FF} by extending

$$\iota^*_{\infty,\eta}\mathcal{F}_{|X_{\mathrm{FF}} \setminus \{\infty\}} = T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$$

according to the B^+_{dR}-sublattice $\Xi \subseteq T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$.

$$\begin{array}{ccc} \Xi =: \iota_\infty^* \mathcal{F}' & \leftarrow\cdots & \exists \mathcal{F}' \\ \downarrow & & \vdots \\ T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} & \longleftarrow & T \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathrm{FF}} \setminus \{\infty\}} \end{array} \qquad \text{via } \beta.$$

These two constructions are mutually inverse by Beauville–Laszlo.

1)⇒ 3) Given a BKF-module, we can associate to it a pair (T, Ξ) by sending

$$M \mapsto \big(T = (M \otimes_{A_{\mathrm{inf}}} W(C^\flat))^{\varphi_M \otimes \varphi = 1}, \Xi = M \otimes_{A_{\mathrm{inf}}} \mathbb{B}_{\mathrm{dR}}^+\big).$$

One can check that $T \otimes_{\mathbb{Z}_p} \mathbb{B}_{\mathrm{dR}} = M \otimes_{A_{\mathrm{inf}}} \mathbb{B}_{\mathrm{dR}}$, so Ξ really defines a lattice (in fact, this is already true over the much smaller ring $\mathbb{A}_{\mathrm{inf}}[\frac{1}{\mu}]$ where $\mu = [\epsilon] - 1$) see [BMS, Lemma 4.26].

2) ⇒ 1) (sketch) Conversely, given $\mathcal{F}'$ we get the associated BKF-module using the adic Fargues–Fontaine curve (which we didn't discuss). There is an analytic adic space

$$\mathcal{Y} = \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})(p[p^\flat] \neq 0)$$

such that $\mathcal{X}_{FF} = \mathcal{Y}/\varphi^{\mathbb{Z}}$. There is also a map $\mathcal{X}_{FF} \to X_{\mathrm{FF}}$. Pulling back along

$$\mathcal{Y} \to \mathcal{X}_{FF} \to X_{\mathrm{FF}}$$

we pick up a φ-action. By a Theorem of Kedlaya, any such a vector bundle on $\mathcal{Y}$ comes from an A_{inf}-module M. The descent datum along $\mathcal{Y} \to \mathcal{X}_{FF}$ is precisely the map φ_M.

The second part follows from the Theorem of Scholze–Weinstein. In fact, it turns out that one has $M \otimes A_{\mathrm{crys}} = M_{\mathrm{crys}}(G)$. □

In summary, we have discussed functors

$$\left\{\begin{matrix} p\text{-divisible groups} \\ \text{over } \mathcal{O}_C \end{matrix}\right\} \to \left\{\begin{matrix} \text{minuscule modifications} \\ \text{of vector bundes on } X_{\mathrm{FF}} \end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix} \text{pairs } (T,\Xi) \text{ consisting of} \\ \bullet\, T \text{ finite free } \mathbb{Z}_p\text{-module,} \\ \bullet\, \Xi \subseteq T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+ \} \end{matrix}\right\}$$

$$G \mapsto (T_pG \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_{\mathrm{FF}}}, \mathcal{E}(G), \beta_G, T_pG) \mapsto (T_pG, \Xi).$$

What is the Breuil–Kisin–Fargues module good for? In the case of $G = A[p^\infty]$, it is this thing that sees the crystalline realisation of G, namely $H^1_{\mathrm{crys}}(A_{\mathcal{O}_C/p}|A_{\mathrm{crys}})$. In this sense, the lattice in part 3 is from the Hodge-de Rham comparison.

In general, we can use BKF to recover the Dieudonné-module of the special fibre G_k:

Proposition 4.6 ([SW, Cor 14.4.4.]). *Assume that $k = \mathcal{O}_C/\mathfrak{m}$ admits a natural splitting $k \to \mathcal{O}_C^\flat$ (for example if $C = \mathbb{C}_p$, as then $k = \overline{\mathbb{F}}_p$). This gives rise to a map $\mathbb{A}_{\mathrm{inf}} \to W(k)$. The Dieudonné module associated to G_k is then $M \otimes_{\mathbb{A}_{\mathrm{inf}}} W(k)$.*

(Without the intermediate step of BKF-modules, it's not clear how to associate a Dieudonne module to (T, Ξ).)

5 A glimpse on BMS1

In the case of $G = A[p^\infty]$ coming from an abelian variety, the Breuil–Kisin–Fargues module M is an $\mathbb{A}_{\mathrm{inf}}$-module that recovers both the crystalline cohomology

$$H^1_{\mathrm{crys}}(A_{\mathcal{O}_C/p}|A_{\mathrm{crys}}) = M \otimes_{\mathbb{A}_{\mathrm{inf}}} A_{\mathrm{crys}}$$

As well as the étale cohomology, via

$$H^1_{\text{ét}}(A_C, \mathbb{Z}_p) = (M \otimes_{\mathbb{A}_{\inf}} W(C^\flat))^\varphi.$$

Moreover, it allows for an *integral* comparison between the two.

It is natural to wonder whether a similar thing is possible for A replaced by any proper smooth formal scheme over $\mathcal{O}_C$. Amazingly, this turns out to be possible – it is the starting point of the work of Bhatt–Morrow–Scholze, [BMS]. The basic idea is to define a cohomology theory that takes values in Breuil–Kisin–Fargues modules.

This later culminated in the prismatic cohomology of Bhatt–Scholze.

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