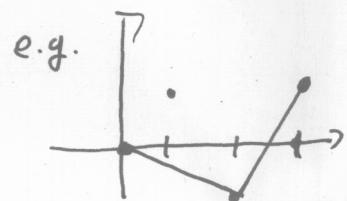


Newton polygons & factorizations

K non-arch. field, $v: K \rightarrow \mathbb{R} \cup \{\infty\}$ valuation

$$f(T) = \sum_{i=0}^n a_i T^i \in K[T]$$



Def: $\text{Nent}_{\text{poly}}(f) = \text{largest convex polygon below } \{(i, v(a_i))\}_{i=0}^n$

Better descr. via Legendre transform

$$\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}, \mathcal{F} := \{\mathbb{R} \rightarrow \bar{\mathbb{R}} \text{ arb. maps}\}$$

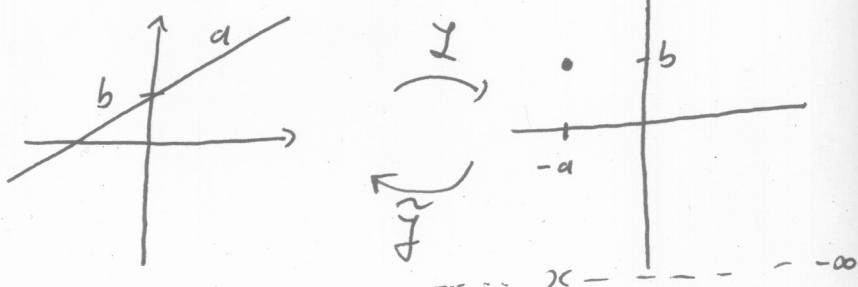
Def: $\mathcal{L}: \mathcal{F} \rightarrow \mathcal{F}, \varphi \mapsto (\lambda \mapsto \inf_{x \in \mathbb{R}} \{\varphi(x) + \lambda x\})$ "Legendre transform"

$\tilde{\mathcal{L}}: \mathcal{F} \rightarrow \mathcal{F}, \varphi \mapsto (x \mapsto \sup_{\lambda \in \mathbb{R}} \{\varphi(\lambda) - \lambda x\})$ "inverse Legendre transform"
 $(\tilde{\mathcal{L}}(\varphi) = -\mathcal{L}(-\varphi))$

Slogan: $\mathcal{L}, \tilde{\mathcal{L}}$ interchange x -coord. and slopes

$$\text{Ex: } \varphi(x) = ax + b$$

$$\Rightarrow \mathcal{L}(\varphi)(\lambda) = \begin{cases} b, & \lambda = -a \\ -\infty, & \lambda \neq -a \end{cases}$$



$$\tilde{\mathcal{L}}\mathcal{L}(\varphi) = \varphi$$

La: 1) $\varphi \in \mathcal{F} \Rightarrow \mathcal{L}(\varphi)$ concave (i.e. $\mathcal{L}(\varphi)(a\lambda + b\mu) \geq a\mathcal{L}(\varphi)(\lambda) + b\mathcal{L}(\varphi)(\mu)$)
 $\tilde{\mathcal{L}}(\varphi)$ convex for $a, b \geq 0, a+b=1$

2) $\varphi \leq \psi \Rightarrow \mathcal{L}(\varphi) \leq \mathcal{L}(\psi), \tilde{\mathcal{L}}(\varphi) \leq \tilde{\mathcal{L}}(\psi)$



3) $\tilde{\mathcal{L}}\mathcal{L}(\varphi) \leq \varphi, \varphi \leq \mathcal{L}\tilde{\mathcal{L}}(\varphi)$

4) φ admits a supporting line at x of slope λ (i.e. $\varphi(y) \geq \varphi(x) + \lambda(y-x) \forall y$),
then $\mathcal{L}(\varphi)$ admits a capping line at $-\lambda$ of slope x

5) $\tilde{\mathcal{L}}\mathcal{L}(\varphi)$ is the largest convex fct. below φ

6) $\mathcal{L}, \tilde{\mathcal{L}}$ define inverse bijections $\{\varphi: \mathbb{R} \rightarrow \bar{\mathbb{R}} \text{ convex}\} \xleftrightarrow{\mathcal{L}} \{\varphi: \mathbb{R} \rightarrow \bar{\mathbb{R}} \text{ concave}\}$

Pf: Only 5), 6): For $a, b \in \mathbb{R}$ set $\varphi_{a,b}(x) = ax + b$, and $M := \{(a, b) \mid \varphi_{a,b} \leq \varphi\} \quad (2)$

Then $\sup \{\varphi_{a,b}\}$ is largest convex fct. below φ

$$\stackrel{1), 3)}{\Rightarrow} \tilde{\mathcal{I}}\mathcal{L}(\varphi) \subseteq M. \text{ Moreover, } \varphi_{a,b} \stackrel{M}{\in} \tilde{\mathcal{I}}\mathcal{L}(\varphi_{a,b}) \subseteq \tilde{\mathcal{I}}\mathcal{L}(\varphi) \quad \forall (a, b) \in M$$

Ex. 2)

$$\Rightarrow \tilde{\mathcal{I}}\mathcal{L}(\varphi) = M.$$

6) follows from 3), 5):

$(\mathcal{F}, \leq) \xrightarrow{\tilde{\mathcal{I}}, \tilde{\mathcal{L}}} (\mathcal{G}, \leq)$ define adjoint pair of functors

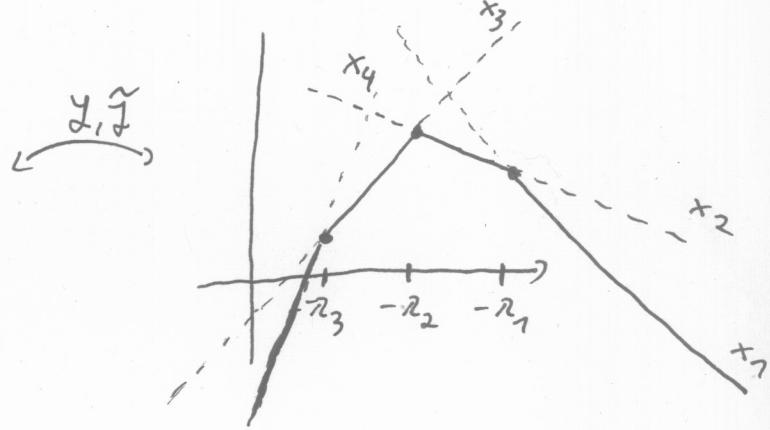
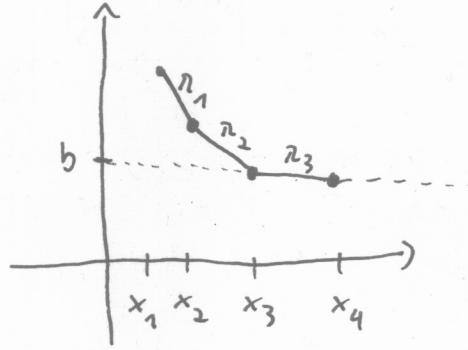
$$\Rightarrow \mathcal{I}, \mathcal{L} \text{ induce inverse bijections } \{\varphi \mid \mathcal{I}\mathcal{L}\varphi = \varphi\} \xleftrightarrow{1:1} \{\psi \mid \mathcal{L}\mathcal{I}\psi = \psi\}$$

(15) (15)

$\{\varphi \text{ convex}\}$

$\{\psi \text{ concave}\}$

Ex:



La: $\mathcal{I}, \tilde{\mathcal{I}}$ preserve piecewise linear fcts.

$$\text{Back to: } f(T) = \sum_{i=0}^n a_i T^i \in K[T]$$

$\Rightarrow \text{Nev}_\text{poly}(f) = \text{largest convex fct. below } \{(i, v(u_i))\}_{i \in \mathbb{Z}}$,
 $(a_i := 0 \text{ for } i \notin \{0, \dots, n\})$

$$\text{i.e. } \mathcal{I}(\text{Nev}_\text{poly}(f))(r) = \inf_{i \in \mathbb{Z}} \{v(u_i) + ri\} = v_r(f)$$

Geometric interpretation (if $r \in v(K^\bullet)$)

$$v_r(f) = \inf \{v(f(x)) \mid x \in K, v(x) = r\}$$

Exercise: $v_r(f \cdot g) = v_r(f) + v_r(g)$ $\forall r \in \mathbb{R}$ (3)

$\Rightarrow \mathcal{L}(\varphi_{f \cdot g}) = \mathcal{L}(\varphi_f) + \mathcal{L}(\varphi_g)$ where φ_f = piecewise linear fit connecting the $\{(i, v(a_i))\}_{i \in \mathbb{Z}}$, if $f = \sum_{i=0}^n a_i T^i$
 "concatenate the piecewise linear fit's to a new fit"

Def: $\varphi, \psi \in \mathcal{F}, -\infty \notin \text{Im } \varphi \cup \text{Im } \psi$

=> "convolution" $\varphi * \psi: \mathbb{R} \rightarrow \overline{\mathbb{R}}, x \mapsto \inf_{a+b=x} \{\varphi(a) + \psi(b)\}$

La: 1) φ, ψ convex, $-\infty \notin \text{Im } \varphi \cup \text{Im } \psi \Rightarrow \varphi * \psi$ convex
 2) $\mathcal{L}(\varphi * \psi) = \mathcal{L}(\varphi) + \mathcal{L}(\psi)$

Prf: Exercise.

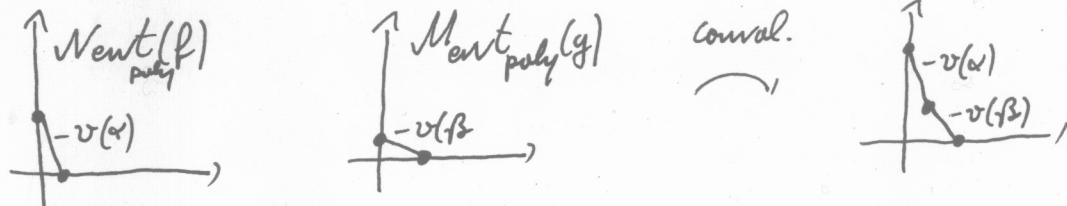
Cor: $f, g \in K[T]$, then

$$\mathcal{A} \quad \text{Neut}_{\text{poly}}(f \cdot g) = \text{Neut}_{\text{poly}}(f) * \text{Neut}_{\text{poly}}(g)$$

Prf: Both sides are convex and

$$\begin{aligned} \mathcal{L}(\text{Neut}_{\text{poly}}(f) * \text{Neut}_{\text{poly}}(g)) &= \mathcal{L}(\text{Neut}_{\text{poly}}(f)) + \mathcal{L}(\text{Neut}_{\text{poly}}(g)) \\ &= \mathcal{L}(\varphi_f) + \mathcal{L}(\varphi_g) = \mathcal{L}(\varphi_{f \cdot g}) = \mathcal{L}(\text{Neut}(f \cdot g)) \end{aligned}$$

Ex: $f = T - \alpha, g = T - \beta$



$(\Rightarrow f \in K[T] \text{ polynomial}, \alpha_1, \dots, \alpha_n \in \bar{K} \text{ all zeros of } f$
 Then $\text{Neut}_{\text{poly}}(f)$ has exactly the $-v(\alpha_1), \dots, -v(\alpha_n)$ as its slopes with same multiplicities)

Now, $f \in \mathcal{O}_n[[T]], f(T) = \sum_{i=0}^n a_i T^i$

Def: $\text{Neut}(f) :=$ largest, decreasing convex fct. below $\{(i, v(a_i))\}_{i \in \mathbb{Z}}$, i.e.

$$\mathcal{L}(\text{Neut}(f)) = \begin{cases} v_r(f) := \inf_{r \geq 0} \{v(a_i) + ri\}, & r \geq 0 \\ -\infty, & r < 0 \end{cases}$$

Rmk: * f defines a fct on "open rigid-analytic unit disc" $\text{ID} := \{x \mid \|x\| < 1\}$ (can ~~only~~ handle general $f \in \mathcal{O}_K[[T]]$ as functions if one is careful with convergence)

- * $f \in \mathcal{O}_K[T] = \text{Nent}(f)$ omits positive slopes from $\text{Nent}_{\text{poly}}(f)$ (these don't correspond to zeros in ID)

Thm (Lazard): $\exists \neq 0$ slope of $\text{Nent}(f)$,

$$\Rightarrow \exists \text{ zero } \alpha \in \widehat{K} \text{ of } f \text{ with } v(\alpha) = -\lambda, \text{ i.e. } f = (T-\alpha)g, g \in \mathcal{O}_K[[T]]$$

Rmk: * If $\text{Nent}(f)$ eventually constant (e.g. if K disc. valued), i.e. $f = af$, $a \in \mathcal{O}_K$, f prim. of some degree d

$$\Rightarrow f = a \cdot P \cdot q$$

Weierstraß preparation $\left\{ \begin{array}{l} \text{unit} \\ \text{monic poly.} \\ \text{of degree } d \end{array} \right.$

$$*\log_p(1-x) = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{x^i}{i} \text{ has zeros precisely at } \mu_{p^\infty}(\bar{K})$$

Exercise: Draw the Newton polygon of \log_p and prove this

Back to A_{inf} : p prime, E/\mathcal{O}_p finite, $\pi \in \mathcal{O}_E \rightarrow \mathbb{F}_q, F/\mathbb{F}_q$ non-arch, alg. cld, $A_{\text{inf}} = W_{\mathcal{O}_E}(\mathcal{O}_F)$

$v: F \rightarrow \mathbb{R} \cup \{-\infty\}$ valuation

$$\text{Def: } f = \sum_{i=0}^{\infty} [a_i] \pi^i \in A_{\text{inf}}$$

$\text{Nent}(f) = \text{largest decreasing convex fct. below } \{(i, v(a_i))\}_{i \in \mathbb{Z}}$

$$\text{i.e. } \llbracket \text{Nent}(f) \rrbracket = \begin{cases} v_r(f) := \inf_i \{v(a_i) + ri\}, & r \geq 0 \\ -\infty, & r < 0 \end{cases}$$

\triangleleft La: v_r are valuations on A_{inf} ($\Rightarrow \text{Nent}(f \cdot g) = \text{Nent}(f) + \text{Nent}(g)$)

Prf: Show $v_r(f \cdot g) = v_r(f) + v_r(g)$. Clear $v_r(f \cdot g) \geq v_r(f) + v_r(g)$.

$r=0 \sim$ Exercise

Fix $r > 0$. Write $f = \sum_{i=0}^{\infty} [a_i] \pi^i, g = \sum_{i=0}^{\infty} [b_i] \pi^i$. Then

Ex. n, m least integers, s.t. $v_r(f) = v_r([a_n] \cdot \pi^n)$

$$v_r(g) = v_r([b_m] \pi^m)$$

Write $f = x' + [a_n] \pi^n + \pi^{n+1} x'', v_r(x'') > v_r(f), v_r(\pi^{n+1} x'') \geq v_r(f)$

$g = y' + [b_m] \pi^m + \pi^{m+1} y'', v_r(y'') > v_r(g), v_r(\pi^{m+1} y'') \geq v_r(g)$

$$\Rightarrow f \cdot g = z + [a_n \cdot b_m] \pi^{n+m} + \pi^{n+m+1} w \text{ with } v_r(z) > v_r(f) + v_r(g)$$

Introduce $\tilde{v}: A_{\text{inf}} \rightarrow \mathbb{R} \cup \{\infty\}$, $h = \sum_{i=0}^{\infty} [c_i] \pi^i \mapsto \inf \{v(c_i)\} + r(n+m)$

Then 1) $\tilde{v}(0) = \infty$

2) $\tilde{v}(h_1 + h_2) \geq \inf \{v(h_1), v(h_2)\}$ "semi valuation"

~~if $h_1, h_2 \neq 0$~~

3) $\tilde{v}(h) = v_r(h)$

$$\begin{aligned} \Rightarrow v_r(f \cdot g) &= v_r(z + [a_n \cdot b_m] \pi^{n+m} + \pi^{n+m+1} w) & v_r(z) > v_r([a_n \cdot b_m] \cdot \pi^{n+m}) \\ &\leq \tilde{v}(z + [a_n \cdot b_m] \cdot \pi^{n+m} + \pi^{n+m+1} \cdot w) = \tilde{v}(z + [a_n \cdot b_m] \pi^{n+m}) \stackrel{?}{=} \tilde{v}([a_n \cdot b_m] \cdot \pi^{n+m}) \\ &= v(a_n \cdot b_m) + r(n+m) = v_r(f) + v_r(g) \end{aligned}$$

Let now $a = u \cdot \pi - [a_0] \in \text{Prim}_1$, $D := A_{\text{inf}}/(a)$, $\mathcal{O}: A_{\text{inf}} \rightarrow D$

Prop: 1) D π -compl., π -tors. free

2) $D^b \cong \mathcal{O}_F$

3) $D \rightarrow D, x \mapsto x^p$ is surj. (\Rightarrow each d can be written as $\mathcal{O}[[x]]$, $x \in \mathcal{O}_F$)

Prf: ~~if $p \mid n$~~ For 3) note $u \pi \equiv [a_0] \pmod{a}$ admits arbitrary p -th roots

Then calculate in truncated Witt vectors (this yields alg. eq. in F , which admit solutions)

(Cor: 1) D is a complete val. ring with valuation $v: D \rightarrow \mathbb{R} \cup \{\infty\}$, $d = \mathcal{O}[[x]] \mapsto v(x)$
 2) $\text{Frac}(D)$ is alg. closed well-defined

Prf: 1) Suffice D is an integral domain, because then can use ~~messy calculations~~
~~messy calculations~~
 $\mathcal{O}_F \xrightarrow{G \#} D$ and

(6)

La: R int. domain with \mathfrak{m} . Then R valuation ring iff

for all $x \in \text{Frac}(R)$ either $x^{\epsilon R} \in R$.

Assume $d \cdot e = 0$, $d, e \in D$. Write $d = \mathcal{O}([x])$, $e = \mathcal{O}([y])$

$$\Rightarrow \mathcal{O}([x \cdot y]) = \mathcal{O}, \text{ i.e. } [x \cdot y] = a \cdot \pi, a \in A_{\text{inf}}$$

But $\text{Nent}([x \cdot y])$ is a horizontal line, while

$\text{Nent}(a \cdot \pi) = \text{Nent}(a) + \text{Nent}(\pi)$ has $-\nu(a_0)$ as a slope if $a \neq 0$

2) Let $P(T) = T^d + b_{d-1}T^{d-1} + \dots + b_0 \in \mathcal{O}_D[T]$ irred, $d > 0$

Replace P by $c^{-d}P(cT)$ for ~~suitable~~ c with $d\nu(c) = \nu(b_0)$

$$\Rightarrow \nu(b_0) = 0$$

Let $Q(T) \in \mathcal{O}_F[T]$, s.t. $Q(T) \equiv P(T)$ in $\mathcal{O}_{D/\pi}[T] \cong \mathcal{O}_F/a_0[T]$

Let $y \in \mathcal{O}_F$ be a zero of Q

Then $P(T+y^\#)$ has const. term div. by π and is irred.

Consider $P_1(T) := c^{-d}P(cT+y^\#)$ where $d\nu(c) = \nu(P(y^\#)) \geq \nu(\pi)$

~~so~~ $P_1(T) \in \mathcal{O}_D[T]$ and there ex. $y_1 \in \mathcal{O}_F$, s.t. $\nu(P_1(y_1)) \geq \nu(\pi)$, i.e.

$$\nu(P(cy_1 + y^\#)) \stackrel{?}{\neq} \nu(c) + \nu(\pi)$$

\Rightarrow Iterate to find a ~~no~~ zero of P

Finished. $\{C/F \text{ non-arch., alg. closed, } c: \mathcal{O}_C^\times \cong \mathcal{O}_F^\times\} \cong \text{Prim}_1/A_{\text{inf}}^\times$

~~so~~

Thm (Fargues-Fontaine) $f \in A_{\text{inf}}$, $\pi \neq 0$ slope of $\text{Nent}(f)$

$\Rightarrow \exists a \in \mathcal{O}_F$ with $\nu(a) = -\pi$, s.t. $f = (\pi - [a]) \cdot g$ with $g \in A_{\text{inf}}$

(Without proof)

~~representation~~

Set $|Y| := \text{Prim}_1/A_{\text{inf}}^\times$ = Analogy of \mathbb{P}^1_F underlying top. space of open unit disc D_F

$y \in |Y| \Rightarrow C_y := A_{\text{inf}}/(a)[\frac{1}{\pi}]$, $a \in Y$, $\mathcal{O}_y: A_{\text{inf}} \rightarrow C_y$
 Each $f \in A_{\text{inf}}$ defines a "function" on $|Y|$ ~~which is not a function~~

$$f(y) := \mathcal{O}(f) \in C_y$$

and the above thm yields a zero of f on $|Y|$