Dr. A. Ivanov **SS 2020**
Dr. J. Anschütz

Algebraic Geometry II

12. Exercise sheet

Exercise 1 (4 points):

Let k be a field and let X, Y be two quasi-compact and separated schemes over k. Let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module of finite rank and let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Let $p\colon X \times_k Y \to X$ resp. $q\colon X \times_k Y \to Y$ be the projections. Prove the Künneth formula

$$H^n(X \times_k Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_k Y}} q^*\mathcal{G}) \cong \bigoplus_{i+j=n} H^i(X, \mathcal{F}) \otimes_k H^j(Y, \mathcal{G})$$

for $n \geq 0$.
Hint: Compute $R\Gamma(X \times_k Y, -) \cong R\Gamma(X, Rp_*(-))$ using the projection formula from Exercise sheet 11, Exercise 3 and flat base change. Then use or prove that every complex of k-vector spaces is quasi-isomorphic to its cohomology groups.

Exercise 2 (4 points):

Let k be a field and let X be a proper smooth geometrically connected curve over k of genus g. Let $\mathcal{L}$ be a line bundle on X of degree $> 2g$. Prove that $\mathcal{L}$ is very ample, i.e., $\mathcal{L} \cong i^*\mathcal{O}_{\mathbb{P}^n_k}(1)$ for some closed immersion $i\colon X \to \mathbb{P}^n_k, n \geq 0$. Conclude that a line bundle on X is ample if and only if it has positive degree.
Hint: Use faithfully flat descent to reduce to the case that k is algebraically closed. Now use Serre duality to show that $H^1(X, \mathcal{L}(-x-y)) = 0$ for all closed points $x, y \in X$, and conclude.

Exercise 3 (4 points):

Let k be an algebraically closed field and let X be an elliptic curve over k, i.e., X is a proper smooth curve over k of genus 1 together with a distinguished base point $x_0 \in X(k)$. Prove that X can be embedded into $\mathbb{P}^2_k$ as a plane curve defined by the affine Weierstraß equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

such that x_0 maps to the point $(x : y : z) = (0 : 1 : 0)$.
Hint: Using Riemann-Roch and Serre duality prove that $\dim_k H^0(X, \mathcal{O}_X(nx_0)) = n$ for $n \geq 1$. Then pick $x \in H^0(X, \mathcal{O}_X(2x_0)) \setminus H^0(X, \mathcal{O}_X(x_0)) \subseteq H^0(X, \mathcal{O}_X(3x_0))$ and $y \in H^0(X, \mathcal{O}_X(3x_0)) \setminus H^0(X, \mathcal{O}_X(2x_0))$.

Exercise 4 (4 points):

Let k be a field and let X be a proper smooth geometrically connected curve over k. Let $U \subseteq X$ be an open subset with $U \neq X$. Show that U is affine.
Hint: Use Exercise 2.

To be uploaded in eCampus: Saturday, 11.07.2020 till 23:55h.