Dr. A. Ivanov **SS 2020**
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Algebraic Geometry II

10. Exercise sheet

Let R be a ring and $n \geq 0$. Recall that the blow-up of $\mathbb{A}_R^{n+1}$ in the zero section $0\colon \mathrm{Spec}(R) \to \mathbb{A}_R^{n+1}$ is the scheme

$$\mathrm{Bl}_0(\mathbb{A}_R^{n+1}) := \{((x_0, \ldots, x_n), (y_0 : \ldots : y_n)) \in \mathbb{A}_R^{n+1} \times_{\mathrm{Spec}(R)} \mathbb{P}_R^n \mid x_i y_j = x_j y_i \text{ for all } i, j\}.$$

The fiber $E := 0 \times_{\mathbb{A}_R^{n+1}} \mathrm{Bl}_0(\mathbb{A}_R^{n+1})$ is called the exceptional divisor of the blow-up.

Exercise 1 (4 points):

Let R be a ring.
i) Let $\pi\colon X := \mathrm{Bl}_0(\mathbb{A}_R^{n+1}) \to \mathbb{P}_R^n$ be the natural morphism. For $m \in \mathbb{Z}$ prove that

$$\pi_*(\mathcal{O}_X(mE)) \cong \bigoplus_{i \geq 0} \mathcal{O}_{\mathbb{P}_R^n}(i - m).$$

ii) Let $f\colon X \to \mathbb{A}_R^{n+1}$ be the natural morphism. For $i \geq 0$ and $m \in \mathbb{Z}$ calculate

$$R^i f_*(\mathcal{O}_X(mE)).$$

Hint: For i) show that $\mathcal{O}_X(mE) \cong \pi^*(\mathcal{O}_{\mathbb{P}_R^n}(-m))$. For ii) calculate $H^i(X, \mathcal{O}_X(mE))$ using i).

Exercise 2 (4 points):

Let k be a field and let $f\colon X := \mathrm{Bl}_0(\mathbb{A}_k^3) \to \mathbb{A}_k^3$ be the blow-up in $0 \in \mathbb{A}_k^3$. For $g \in k[x, y, z]$ let $\widetilde{V(g)} \subseteq X$ be the schematic closure of $X \times_{\mathbb{A}_k^3} (V(g) \setminus \{0\})$ in X (the "strict transform of $V(g)$"). For $g = x^2 + y^2 + z^2$ and $g = x^2 + y^2 + z^3$ calculate all singular points of $V(g)$ resp. $\widetilde{V(g)}$.

Exercise 3 (4 points):

Let X be a noetherian scheme.
i) Prove that X is affine if and only if the reduction X_{red} is affine.
ii) Prove that X is affine if and only if each irreducible component of X is affine.
Hint: In both cases express coherent $\mathcal{O}_X$-modules in terms of coherent $\mathcal{O}_{Y_i}$-modules for $Y_i \subseteq X$ closed, affine subschemes and use Serre's cohomological criterion for affineness.

Exercise 4 (4 points):

Let A be a discrete valuation ring with residue field k and fraction field K. Let $s = \mathrm{Spec}(k)$ resp. $\eta = \mathrm{Spec}(K)$ be the special resp. generic point of $S := \mathrm{Spec}(A)$. Let $f\colon X \to S$ be a proper, smooth morphism whose fibers X_s, X_η are irreducible curves of genus $g \geq 1$. Let $\sigma_1, \sigma_2\colon S \to X$ be two sections of f and assume that $\sigma_{1|s} = \sigma_{2|s}$ while $\sigma_{1|\eta} \neq \sigma_{2|\eta}$. Let $\Gamma_i = \sigma_i(X) \subseteq X = X \times_S S$ be the graph of σ_i. Finally, define the line bundle $\mathcal{L} := \mathcal{O}_X(\Gamma_1 - \Gamma_2)$ on X.

i) Prove that the base change morphism $f_*(\mathcal{L}) \otimes_A K \to H^0(X_\eta, \mathcal{L}_\eta)$ is an isomorphism, but the base change morphism $f_*(\mathcal{L}) \otimes_A k \to H^0(X_s, \mathcal{L}_s)$ not.
Hint: Use Exercise 4 from Exercise sheet 8.

ii) Construct an example for A, X, σ_i.
Hint: Find X inside $\mathbb{P}_A^2$.

To be uploaded in eCampus: Saturday, 27.06.2020 till 23:55h.