Dr. A. Ivanov **SS 2020**
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Algebraic Geometry II

9. Exercise sheet

Exercise 1 (4 points):

Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes with f a closed immersion with ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_Y$ and g, $g \circ f$ smooth of relative dimension n resp. m. Let $\omega_{X/S} := \Lambda^n \Omega^1_{X/S}$ and $\omega_{Y/S} := \Lambda^m \Omega^1_{Y/S}$ be the canonical bundles. Define the normal bundle of f as $\mathcal{N}_{X/Y} := f^*(\mathcal{I})^\vee = (\mathcal{I}/\mathcal{I}^2)^\vee$. Prove the adjunction formula

$$\omega_{X/S} \cong f^*(\omega_{Y/S}) \otimes_{\mathcal{O}_X} \Lambda^{m-n} \mathcal{N}_{X/Y}.$$

Exercise 2 (4 points):

Let k be a field and let $X \subseteq \mathbb{P}^n_k$ be a projective, geometrically connected smooth curve with $\dim_k H^1(X, \mathcal{O}_X) = g \geq 2$ which is a complete intersection, i.e., X is the vanishing locus of $n-1$ polynomials $f_i \in H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d_i))$. Prove that $\Omega^1_{X/\mathrm{Spec}(k)}$ is very ample. Using $\dim_k H^0(X, \Omega^1_{X/k}) = g$ (to be proven in the lecture) conclude that X has genus $g > 2$.
Hint: Use Exercise sheet 5, Exercise 1 and the adjunction formula from Exercise 1.

Exercise 3 (4 points):

Let k be a field. A projective, smooth, connected scheme X of dimension 2 over $\mathrm{Spec}(k)$ is called a K3-surface if $\omega_{X/k} := \Lambda^2 \Omega^1_{X/k} \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. Find all possible $n \geq 2, d_1, \ldots, d_{n-2} \in \mathbb{Z}$, such that a complete intersection $X := V(f_1, \ldots, f_{n-2}) \subseteq \mathbb{P}^n_k$ with $f_i \in H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d_i))$, and X not contained in a hyperplane, is a K3-surface.
Hint: Use Exercise 1.

Exercise 4 (4 points):

Let R be a ring. An object $K \in D(R)$ is called pseudo-coherent if $K \in D^-(R)$ and for all $i \in \mathbb{Z}$ the functor

$$\mathrm{Ext}^i_R(K, -)\colon \mathrm{Mod}_R \to \mathrm{Mod}_R$$

commutes with filtered colimits.
i) Prove that $K \in D^-(R)$ is pseudo-coherent if and only if it can be represented by a bounded above complex of finite projective R-modules.
ii) Let $\mathrm{Spec}(R_i) \subseteq \mathrm{Spec}(R), i \in I$, be an open cover of $\mathrm{Spec}(R)$ by affines. Show that $K \in D(R)$ is pseudo-coherent if and only if each $K \otimes^{\mathbb{L}}_R R_i \in D(R_i)$ is pseudo-coherent.
Hint/Remark: By exactness of $- \otimes_R R_i$ *the functor* $- \otimes_R R_i\colon \mathrm{Mod}_R \to \mathrm{Mod}_{R_i}$ *induces directly a functor* $- \otimes^{\mathbb{L}}_R R_i\colon D(R) \to D(R_i)$*. For i): use that* $M \in \mathrm{Mod}_R$ *is finitely presented if and only if the functor* $\mathrm{Hom}_R(M, -)$ *commutes with filtered colimits. Moreover, use that* $\mathrm{Hom}_{D(R)}(M, N[i]) \cong \mathrm{Ext}^i_R(M, N)$ *for* $M, N \in \mathrm{Mod}_R$ *and* $i \in \mathbb{Z}$*.*

To be uploaded in eCampus: Saturday, 20.06.2020 till 23:55h.