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Algebraic Geometry II

8. Exercise sheet

Exercise 1 (4 points):
Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a cartesian diagram of schemes with g flat.
i) For $K \in D^+(X)$ construct a natural morphism, the "base change map",

$$\beta_K \colon g^* Rf_*(K) \to Rf'_*(g'^* K)$$

in $D^+(S')$.
Remarks: Here $D^+(Y)$ denotes the bounded below derived category of $\mathcal{O}_Y$-modules on a scheme Y. The base change map exists on $D(X)$ and without assuming g, g' flat, cf. Stacks project, Tag 08HY.
ii) Assume that f is quasi-compact and separated, and that K can be represented by a finite complex of quasi-coherent $\mathcal{O}_X$-modules. Show that β_K is an isomorphism.
Hint: Reduce to the case that S, S' are affine and that $K = \mathcal{F}$ is a quasi-coherent sheaf. Then calculate $H^\bullet(X, \mathcal{F})$ via a finite Cech complex.

Exercise 2 (4 points):
Let k be a field and let X be a proper scheme over k. For a coherent sheaf $\mathcal{F}$ on X we define the Euler characteristic

$$\chi(X, \mathcal{F}) := \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X, \mathcal{F}).$$

i) Let $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ be a short exact sequence. Prove that

$$\chi(X, \mathcal{F}') = \chi(X, \mathcal{F}) + \chi(X, \mathcal{F}'').$$

ii) Prove that for $d \in \mathbb{Z}$

$$\chi(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) = \binom{n+d}{n} := \prod_{i=1}^{n} \frac{d+i}{i}$$

iii) Assume that X is geometrically integral and $X = V(f) \subseteq \mathbb{P}^2_k$ for some non-zero $f \in H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(d))$, $d > 0$. Prove that

$$\dim_k H^1(X, \mathcal{O}_X) = \frac{(d-1)(d-2)}{2}.$$

Exercise 3 (4 points):
Let k be a field and let X be a geometrically integral proper curve over k. Assume that X is a complete intersection in $\mathbb{P}^3_k$, i.e., $X = V(f_1, f_2)$ for sections $f_i \in H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(d_i))$ such that the multiplication $\mathcal{O}_{V(f_2)} \xrightarrow{f_1} \mathcal{O}_{V(f_2)} \otimes_{\mathcal{O}_{\mathbb{P}^3_k}} \mathcal{O}_{\mathbb{P}^3_k}(d_1)$ is injective.

ii) Prove that the following sequence is exact:

$$0 \to \mathcal{O}_{\mathbb{P}^3_k}(-d_1 - d_2) \xrightarrow{(f_2, -f_1)} \mathcal{O}_{\mathbb{P}^3_k}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^3_k}(-d_2) \xrightarrow{(f_1, f_2)} \mathcal{O}_{\mathbb{P}^3_k} \to \mathcal{O}_X \to 0.$$

iii) Prove that

$$\dim_k H^1(X, \mathcal{O}_X) = \binom{3 - d_1}{3} + \binom{3 - d_2}{3} - \binom{3 - d_1 - d_2}{3}$$

and conclude that there exist proper curves which can not be embedded into the plane $\mathbb{P}^2_k$.

Exercise 4 (4 points):
Let k be an algebraically closed field and let X be a connected proper smooth curve over k. Recall that for $x \in X(k)$ the line bundle $\mathcal{O}_X(x)$ is defined as the dual of the ideal sheaf $\mathcal{O}_X(-x) \subseteq \mathcal{O}_X$ of the closed subscheme $\{x\} \subseteq X$. Prove that the following are equivalent:

i) $X \cong \mathbb{P}^1_k$

ii) $H^1(X, \mathcal{O}_X) = 0$

iii) $\mathcal{O}_X(x) \cong \mathcal{O}_X(y)$ for all closed points $x, y \in X(k)$.

iv) There exist two distinct closed points $x, y \in X(k)$ such that $\mathcal{O}_X(x) \cong \mathcal{O}_X(y)$.

Hint: For "ii) ⇒ iii)" prove that $H^1(X, \mathcal{O}_X(-x)) = 0$ for every $x \in X(k)$. Then use the exact sequence $0 \to \mathcal{O}_X(-y) \to \mathcal{O}_X(x) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-y) \to k(x) \to 0$. For "iv) ⇒ i)" find two generating sections $s_1, s_2 \in H^0(X, \mathcal{O}_X(x))$ and prove that the corresponding morphism $X \xrightarrow{(s_1, s_2)} \mathbb{P}^1_k$ is an isomorphism.

To be uploaded in eCampus: Saturday, 13.06.2020 till 23:55h.