Dr. A. Ivanov **SS 2020**
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Algebraic Geometry II

7. Exercise sheet

Let R be a ring. A coalgebra A over R is an R-module A together with a comultiplication $\mu\colon A \to A \otimes_R A$, and a counit $e\colon A \to R$, satisfying coassociativity and counitality, i.e.,

$$(\mu \otimes \mathrm{Id}_A) \circ \mu = (\mathrm{Id}_A \otimes \mu) \circ \mu \colon A \to A \otimes_R A \otimes_R A$$

and

$$(\mathrm{Id}_A \otimes e) \circ \mu = (e \otimes \mathrm{Id}_A) \circ \mu = \mathrm{Id}_A.$$

A comodule M over a coalgebra R is an R-module M together with a coaction $m\colon M \to M \otimes_R A$ such that

$$(m \otimes \mathrm{Id}_A) \circ m = (\mathrm{Id}_M \otimes \mu) \circ m \colon M \to M \otimes_R A \otimes_R A$$

resp.

$$(\mathrm{Id}_M \otimes e) \circ m = \mathrm{Id}_M.$$

A natural source of examples of coalgebras arise from affine group schemes. Set $Y = \mathrm{Spec}(R)$ and let $G \to Y$ be an affine group scheme over Y, cf. Algebraic Geometry I, Sheet 5, Exercise 4. Then $G = \mathrm{Spec}(A)$ for an R-algebra A and the multiplication $G \times_Y G \to G$ and the unit $Y \to G$ yield morphisms

$$\mu\colon A \to A \otimes_R A$$
$$e\colon A \to R$$

making A into a coalgebra over R. Even more, the inversion on G yields an "antipode" map $\iota\colon A \to A$ on A.
In this example, the category of comodules for A is reasonably called the category of representations of G.

Exercise 1 (4 points):
Let R be a ring and set $A := R[T, T^{-1}]$, i.e., $\mathrm{Spec}(A) = \mathbb{G}_{m,R}$. Write down the comultiplication/counit/antipode for A and show that the category of comodules for A is equivalent to the category of graded R-modules, i.e., R-modules M together with a grading $M = \bigoplus\limits_{n\in\mathbb{Z}} M_n$.
Hint: Associated to a graded R-module $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is the A-comodule M with comultiplication $\Sigma_{n\in\mathbb{Z}} a_n \mapsto \Sigma_{n\in\mathbb{Z}} a_n \otimes T^n$ for $a_n \in M_n$. Conversely, set M_n as the equalizer of $M \to M \otimes_R A,\ a \mapsto a \otimes T^n$ and the comultiplication $m\colon M \to M \otimes_R A$.

Exercise 2 (4 points):
Let $f\colon X \to Y$ be a morphism of schemes, which is a $\mathbb{G}_{m,Y} := \mathrm{Spec}(\mathbb{Z}[T, T^{-1}]) \times_{\mathrm{Spec}(\mathbb{Z})} Y$-torsor, i.e., f is quasi-compact and faithfully flat, $\mathbb{G}_{m,Y}$ acts on X (cf. Algebraic Geometry I, Sheet 10), and the morphism

$$X \times_Y \mathbb{G}_{m,Y} \to X \times_Y X,\ (x, g) \mapsto (x, gx)$$

is an isomorphism.
1) Show that $X \cong \mathbb{V}(\mathcal{L}) \setminus \{0\}$ for a line bundle $\mathcal{L}$ on Y.
2) Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Show that

$$R\Gamma(Y, \mathcal{F}) \cong R\Gamma(X, f^*\mathcal{F})_0,$$

where the subscript 0 indicates the degree 0 part in $R\Gamma(X, f^*\mathcal{F}) \cong R\Gamma(Y, f_*f^*\mathcal{F})$ with grading induced from the grading of $f_*f^*\mathcal{F}$.
Hint: Show first that f is affine. For 1): Write $\mathcal{A} := f_*(\mathcal{O}_X)$. By exercise 1 the $\mathcal{O}_Y$-algebra $\mathcal{A}$ is naturally graded, i.e., $\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}_n$ for $\mathcal{O}_Y$-modules $\mathcal{A}_n$. Now prove that the R-module $\mathcal{L} := \mathcal{A}_1$ is invertible. For 2): Use that f is affine and Exercise 3.ii) of Sheet 6.

Exercise 3 (4 points):
Let $n \geq 1$. Let R be a ring and consider $j\colon U := \mathbb{A}^{n+1}_R \setminus \{0\} \to X := \mathbb{A}^{n+1}_R$. Show that

$$R^a j_*\mathcal{O}_U = 0$$

if $a \notin \{0, n\}$ and that it is $\mathcal{O}_X$ if $a = 0$, and it is the sheaf associated to the $A := R[x_0, x_1, \ldots, x_n]$-module $M := \frac{1}{x_0\cdots x_n} R[x_0^{-1}, x_1^{-1}, \ldots, x_n^{-1}]$ if $a = n$.
Hint/Remark: The precise A-module structure on M is part of the exercise. Set $U_i := D(x_i)$. Show that $\mathcal{O}_U$ has a resolution of the form

$$C\colon \quad 0 \to \mathcal{O}_U \to \prod_{i=0}^{n} j_{i,*}\mathcal{O}_{U_i} \to \prod_{i_0<i_1} j_{i_0,i_1,*}\mathcal{O}_{U_{i_0}\cap U_{i_1}} \to \ldots \to j_{0,1,\ldots,n,*}\mathcal{O}_{U_0\cap\ldots\cap U_n} \to 0$$

by restricting the resolution to U_i and using that $\check{H}^a(U_i, \mathcal{O}_{U_i}) = 0$ for $a > 0$. Then use Exercise 3.ii) of Sheet 6 to obtain a complex computing $Rj_*\mathcal{O}_U$. Finish the argument by writing the global sections of C as a colimit over $b \geq 0$ of Koszul complexes for the regular sequence $(x_0^b, \ldots, x_n^b)$.

Exercise 4 (4 points):
Let R be a ring and set $Y := \mathbb{P}^n_R$. Using exercise 2 and exercise 3 redo the calculation of the cohomology on the sheaves $\mathcal{O}_Y(d)$, $d \in \mathbb{Z}$, i.e., prove that

$$H^i(Y, \mathcal{O}_Y(d)) \cong \begin{cases} R[x_0, \ldots, x_n]_d & \text{if } i = 0, \\ 0 & \text{if } i \notin \{0, n\}, \\ (\frac{1}{x_0\ldots x_n} R[x_0^{-1}, \ldots, x_n^{-1}])_d & \text{if } i = n. \end{cases}$$

Here the subscript d denotes the space of homogenous polynomials of degree d. Thus in particular, $H^*(Y, \mathcal{O}_Y(d)) = 0$ if $-n-1 < d < 0$.
Hint: Use the natural $\mathbb{G}_{m,Y}$-torsor $\mathbb{A}^{n+1}_R \setminus \{0\} \to Y,\ (x_0, \ldots, x_n) \mapsto [x_0\colon \ldots : x_n]$.

To be uploaded in eCampus: Saturday, 06.06.2020 till 23:55h.