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Algebraic Geometry II

6. Exercise sheet

Exercise 1 (4 points):
Let $A \to B$ be a morphism of rings such that B is free of finite rank over A. For $b \in B$ let $\mathrm{tr}_{B/A}(b) \in A$ be the trace of the endomorphism

$$b\cdot : B \to B,\ b' \to bb'.$$

Prove that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is (finite) étale if and only if the trace bilinear form

$$\mathrm{Tr}_{B/A} : B \otimes_A B \to A, (b, b') \mapsto \mathrm{tr}_{B/A}(bb')$$

is non-degenerate, i.e., it induces an isomorphism $B \cong \mathrm{Hom}_A(B, A),\ b \mapsto \mathrm{Tr}_{B/A}(b, -)$.
Remark/Hint: Reduce to the case that A is an algebraically closed field. The same statement, with an appropriate definition of the trace pairing, also holds if B is only assumed to be locally free of finite rank over A.

Exercise 2 (4 points):
Let k be an algebraically closed field. Prove that every finite étale morphism $f : X \to Y := \mathbb{P}^1_k$ is trivial, i.e., X is isomorphic to a disjoint union of copies of Y.
Hint: Consider the quasi-coherent $\mathcal{O}_Y$-algebra $\mathcal{A} := f_*(\mathcal{O}_X)$ with its multiplication $\mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{A} \to \mathcal{A}$. Use the description of vector bundles on $\mathbb{P}^1_k$, i.e., Sheet 8, Exercise 4 of Algebraic Geometry I, to write $\mathcal{A} = \bigoplus \mathcal{A}_i$ with $\mathcal{A}_i \cong \mathcal{O}_Y(i)^{\oplus k_i}, k_i \geq 0$. Use your knowledge about the global sections $\mathcal{O}_Y(i)(Y)$ to prove that each element in $\mathcal{A}_i$ is nilpotent for $i > 0$. Use exercise 1 to conclude that $\mathcal{A} = \mathcal{A}_0$, i.e., $\mathcal{A} \cong \mathcal{O}_Y \otimes_k H^0(X, \mathcal{O}_X)$ with $H^0(X, \mathcal{O}_X)$ a finite dimensional étale k-algebra.

Exercise 3* (4 bonus points):
Let $f : Y \to X$ be a morphism of schemes and let $\mathcal{F}$ be an abelian sheaf on Y. Prove that the canonical morphism

$$H^n(X, f_*(\mathcal{F})) \to H^n(Y, \mathcal{F})$$

is an isomorphism for any $n \geq 0$ if

i) f is a closed immersion, or

ii) f is affine and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module.

Exercise 4* (4 bonus points):
Let A be a ring and let $d < 0$ (the case $d \geq 0$ has been handled in the lecture). Prove that

$$H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(d)) \cong \begin{cases} 0 & \text{if } i < n \\ (\frac{1}{x_0 \cdots x_n} A[x_0^{-1}, \ldots, x_n^{-1}])_d & \text{if } i = n \end{cases}$$

for every $n, i \geq 0$. Here the subscript d denotes the space of homogenous polynomials of degree d. Thus in particular, $H^*(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(d)) = 0$ if $-n - 1 < d < 0$.

Hint: Use Čech cohomology for the standard covering of $\mathbb{P}^n_A$. To prove the vanishing statement use induction on n and the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n_A}(d-1) \xrightarrow{x_0} \mathcal{O}_{\mathbb{P}^n_A}(d) \to i_*(\mathcal{O}_{\mathbb{P}^{n-1}_A}(d)) \to 0$$

where $i\colon \mathbb{P}^{n-1}_A \to \mathbb{P}^n_A,\ (x_1 : \ldots : x_n) \mapsto (0 : x_1 : \ldots : x_n)$.

To be uploaded in eCampus: Saturday, 30.05.2020 till 23:55h.