Dr. A. Ivanov **SS 2020**
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Algebraic Geometry II

5. Exercise sheet

A morphism $f\colon X \to Y$ of schemes is called (locally) quasi-finite if it is (locally of) finite type and each $x \in X$ is discrete in its fiber $\mathrm{Spec}(k(f(x))) \times_Y X$.

Exercise 1 (4 points):

Let S be a scheme. Prove that there exists an exact sequence

$$0 \to \Omega^1_{\mathbb{P}^n_S/S} \to \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^n_S/S}(-1) \xrightarrow{(x_0,\ldots,x_n)} \mathcal{O}_{\mathbb{P}^n_S/S} \to 0$$

where $x_0, \ldots, x_n \in H^0(X, \mathcal{O}_{\mathbb{P}^n_S/S}(1)) \cong \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n_S/S}(-1), \mathcal{O}_{\mathbb{P}^n_S/S})$ are the canonical sections.
Hint: Either try to construct the sequence by glueing or use Exercise 4 from Exercise sheet 4. One can reduce to $S = \mathrm{Spec}(\mathbb{Z})$.

Exercise 2 (4 points):

Let k be a field and let $f\colon X \to \mathrm{Spec}(k)$ be a morphism locally of finite type. Prove that the following assertions are equivalent:
i) f is étale.
ii) f is unramified.
iii) f is smooth and locally quasi-finite.
iv) X is the disjoint union of schemes $\mathrm{Spec}(l)$ where l/k is a finite separable field extension.
Hint: Let K be an algebraic closure of k. Check that each statement is satisfied for f if and only if it is satisfied for $f_K\colon X_K := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \to \mathrm{Spec}(K)$ (e.g., that smoothness of f_K implies smoothness of f was proven in the lecture). If $K = k$ use that $\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x} \cong \Omega^1_{X/k} \otimes k(x)$ for $x \in X$ closed.

Exercise 3 (4 points):

Let A be a ring and let $a \in A^\times$ be a unit. Let $n \in \mathbb{Z}$ such that n is invertible in A. Prove that $\mathrm{Spec}(A[X]/(X^n - a)) \to \mathrm{Spec}(A)$ is finite and étale.

Exercise 4 (4 points):

Let $f\colon Z \to S, g\colon X \to S$ be smooth morphisms of schemes and let $i\colon Z \to X$ be a closed immersion over S. Prove that for all $z \in Z$ there exists a neighborhood $U \subseteq X$ of $i(z)$ and sections $f_1, \ldots, f_n \in \mathcal{O}_X(U)$ such that $Z \cap U = V(f_1, \ldots, f_d)$ for some $d \leq n$ and the induced diagram

$$\begin{array}{ccc} Z \cap U & \longrightarrow & U \\ \downarrow{\scriptstyle (f_{d+1},\ldots,f_n)} & & \downarrow{\scriptstyle (f_1,\ldots,f_n)} \\ \mathbb{A}^{n-d}_S & \xrightarrow{\alpha} & \mathbb{A}^n_S \end{array}$$

with $\alpha((x_{d+1}, \ldots, x_n)) = (0, \ldots, 0, x_{d+1}, \ldots, x_n)$ is cartesian with both vertical arrows étale.
Hint: Let $\mathcal{I}$ be the ideal sheaf of Z in X. Prove that the sequence $0 \to \mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{X/S} \to \Omega^1_{Z/S} \to 0$ is an exact sequence of vector bundles and choose, locally around z, sections $f_1, \ldots, f_n \in \mathcal{O}_X$ such that the differentials $df_1, \ldots, df_n$ form an adapted basis of $i^*\Omega^1_{X/S}$.

To be uploaded in eCampus: Saturday, 23.05.2020 till 23:55h.