

Algebraic Geometry II

4. Exercise sheet

For a ring A we set $A[\varepsilon] := A[t]/t^2$.

Exercise 1 (4 points):

Let $A \rightarrow B \rightarrow C$ be morphisms of rings.

i) Show that there exists an exact sequence

$$C \otimes_B \Omega_{B/A}^1 \rightarrow \Omega_{C/A}^1 \rightarrow \Omega_{C/B}^1 \rightarrow 0$$

of C -modules.

ii) Assume that $B \rightarrow C$ is surjective with kernel $I := \ker(B \rightarrow C)$. Prove that there exists an exact sequence (of C -modules)

$$C \otimes_B I \cong I/I^2 \xrightarrow{\alpha} C \otimes_B \Omega_{B/A}^1 \rightarrow \Omega_{C/A}^1 \rightarrow 0$$

where $\alpha(g) := dg$ for $g \in I$.

Exercise 2 (4 points):

Compute $\Omega_{B/A}^1$ for

i) $B = A[X]/(f(X))$ with $f(X) \in A[X]$ a polynomial.

ii) $B = \mathbb{Z}[i]$, $A = \mathbb{Z}$.

iii) $B = k[x, y]/(y^2 - x^3 - x)$ with $A = k$ not of characteristic 2.

iv) $B = k[x, y]/(xy)$, $A = k$.

Hint: Use exercise 1.ii).

Exercise 3 (4 points):

Let A be a perfect \mathbb{F}_p -algebra, i.e., the Frobenius $\text{Fr}_A: A \rightarrow A$, $x \mapsto x^p$ of A is bijective. Prove that $\text{Spec}(A) \rightarrow \text{Spec}(\mathbb{F}_p)$ is formally étale.

Hint: If R is a ring of characteristic p , $I \subseteq R$ an ideal with $I^2 = 0$ and $R_0 = R/I$, then for $x \in R$ the element x^p only depends on the image $x_0 \in R_0$ of x .

Exercise 4 (4 points):

Let $X \rightarrow S$ be a morphism of schemes. We define a functor, the “tangent bundle” $\mathcal{T}_{X/S}$ of X/S , by sending an affine scheme $\text{Spec}(A)$ over S to the set $X(A[\varepsilon])$. Prove that $\mathcal{T}_{X/S}$ is representable by the relative spectrum $\underline{\text{Spec}}_{\mathcal{O}_X}(\text{Sym}^\bullet(\Omega_{X/S}^1))$.

Hint: Note that there is a natural morphism $X(A[\varepsilon]) \rightarrow X(A)$. Use this to reduce to the case that $X = \text{Spec}(R)$ and S are affine.

To be uploaded in eCampus: Saturday, 16.05.2020 till 23:55h.