Dr. A. Ivanov **SS 2020**
Dr. J. Anschütz

Algebraic Geometry II

3. Exercise sheet

Exercise 1 (4 points):

Let $f\colon X \to S$ be a closed immersion. Then f is flat and locally of finite presentation if and only if f is an open immersion.
Hint: Use that a flat and finitely presented module over a local ring is free.

Exercise 2 (4 points):

i) Let $f\colon X \to S$ be a morphism, which is locally of finite presentation, flat and finite. Prove that the function

$$s \in S \mapsto \dim_{k(s)} \Gamma(X \times_S \mathrm{Spec}(k(s)), \mathcal{O}_{X \times_S \mathrm{Spec}(k(s))})$$

is locally constant.
Hint: If $S = \mathrm{Spec}(A)$ is affine, then $X = \mathrm{Spec}(B)$ is affine and B is a finitely presented A-module.
ii) Let X be an integral scheme of finite type over a field k. Prove that the normalization $f\colon \tilde{X} \to X$ is flat if and only if it is an isomorphism.
Hint: Use the finiteness of the normalization and part i).

Exercise 3 (4 points):

Let K be a field and let L/K be a finite Galois extension with group G.
i) Prove that the morphism

$$\begin{array}{ccc} \mathrm{Spec}(L) \times G \cong \coprod_{g \in G} \mathrm{Spec}(L) & \to & \mathrm{Spec}(L) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L) \\ (x, g) & \mapsto & (x, xg) \end{array}$$

is an isomorphism.
ii) Prove Hilbert's theorem 90: Sending a K-vector space V to $L \otimes_K V$ defines an equivalence

$$\{K\text{-vector spaces }\} \to \{L\text{-vector spaces together with a semilinear } G-\text{action}\}$$

(A G-action on an L-vector space W is called semilinear if $g(lw) = g(l)g(w)$ for all $g \in G$, $l \in L$ and $w \in W$.)
Hint: Identify L-vector spaces with a semilinar G-action with descent data for $\mathrm{Spec}(L)/\mathrm{Spec}(K)$.

Exercise 4 (4 points):

Let A be a ring and let $A \to B$ be a faithfully flat morphism.
i) Prove descent for affine schemes, i.e., prove that the category of A-algebras is equivalent to the category of B-algebras C equipped with an isomorphism

$$C \otimes_{B,\iota_1} (B \otimes_A B) \cong C \otimes_{B,\iota_2} (B \otimes_A B)$$

satisfying the cocycle condition. Here $\iota_1(b) = b \otimes 1$ and $\iota_2(b) = 1 \otimes b$ for $b \in B$.
ii) For a scheme S let $C(S)$ denote the set of closed subschemes of S. Prove the exactness of the natural sequence

$$C(\mathrm{Spec}(A)) \to C(\mathrm{Spec}(B)) \rightrightarrows C(\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)).$$

To be uploaded in eCampus: Saturday, 09.05.2020 till 23:55h.