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Algebraic Geometry II

2. Exercise sheet

Exercise 1 (4 points):
Let $\mathcal{D}$ be a triangulated category. Let

$$A \to B \to C \to A[1]$$

be a distinguished triangle in $\mathcal{D}$ and $X \in \mathcal{D}$. Show that

$$\mathrm{Hom}_{\mathcal{D}}(X, A) \to \mathrm{Hom}_{\mathcal{D}}(X, B) \to \mathrm{Hom}_{\mathcal{D}}(X, C) \to \mathrm{Hom}_{\mathcal{D}}(X, A[1])$$

is an exact sequence of abelian groups.
Remark/Hint: Show first that each two consecutive morphisms in a distinguished triangle compose to zero. The exact sequence can be extended to a long exact sequence, by rotating the distinguished triangle.

Exercise 2 (4 points):
Let $\mathcal{A}$ be an abelian category. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of complexes of objects in $\mathcal{A}$.
i) Show that there exists a natural morphism $C \to A[1]$ in $D(\mathcal{A})$ such that

$$A \to B \to C \to A[1]$$

is a distinguished triangle in $D(\mathcal{A})$, and that each distinguished triangle is of this form.
ii) Let $i \in \mathbb{Z}$ and $A \in D(\mathcal{A})$. Show that there exists an object $\tau_{\leq i}A$ with a morphism $\tau_{\leq i}A \to A$ with $H^j(\tau_{\leq i}A) = 0$ for $j > i$ and $H^j(\tau_{\leq i}A) \overset{\simeq}{\to} H^j(A)$ for $j \leq i$. Assuming that $\mathcal{A}$ has enough injectives show that $\tau_{\leq i}A$ with the morphism $\tau_{\leq i}A \to A$ is unique up to unique isomorphism and that $\tau_{\leq i}A$ is functorial in A.
Hint: Exercise 1 and the fact for any complex N and any bounded below complex I consisting of injectives the natural morphism $\mathrm{Hom}_{K(\mathcal{A})}(N, I) \to \mathrm{Hom}_{D(\mathcal{A})}(N, I)$ is an isomorphism.

Exercise 3 (4 points):
Let X be a topological space. We let $D^+(X)$ be the bounded below derived category of the category of abelian sheaves on X.
Show that for any covering $X = V_1 \cup V_2$ of X by two open subsets V_1, V_2 and any $A \in D^+(X)$ there exists a natural "Mayer-Vietoris" sequence

$$\ldots \to H^i(X, A) \to H^i(V_1, A_{|V_1}) \oplus H^i(V_2, A_{|V_2}) \to H^i(V_1 \cap V_2, A_{|V_1 \cap V_2}) \to H^{i+1}(X, A) \to \ldots$$

Hint: Let $j_{V_i} \colon V_i \to X, j_{V_1 \cap V_2} \colon V_1 \cap V_2 \to X$ be the open immersions. Construct a natural distinguished triangle

$$A \to Rj_{V_1,*}j_{V_1}^{-1}A \oplus Rj_{V_2,*}j_{V_2}^{-1}A \to Rj_{V_1 \cap V_2,*}j_{V_1 \cap V_2}^{-1}A \to A[1].$$

Exercise 4 (4 points):
Let X be a smooth projective curve over some field k and let

$$D^b(\mathcal{C}oh(X)) = D^-(\mathcal{C}oh(X)) \cap D^+(\mathcal{C}oh(X))$$

be the bounded derived category of the category of coherent sheaves on X. Show that each $A \in D^b(\mathcal{C}oh(X))$ is (quasi-)isomorphic to a bounded complex of vector bundles on X.
Hint: Recall that if $\mathcal{L}$ is an ample line bundle on X and $\mathcal{M}$ a coherent $\mathcal{O}_X$-module, then for some $n \gg 0$ the $\mathcal{O}_X$-module $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is generated by global sections.

To be uploaded in eCampus: Saturday, 02.05.2020 till 23:55h.