Dr. A. Ivanov **SS 2020**
Dr. J. Anschütz

Algebraic Geometry II

1. Exercise sheet

Exercise 1 (4 points):
Let $\mathcal{A}$ be an abelian category and let

$$\begin{array}{ccccccccc} & & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_2} & & \downarrow{\scriptstyle d_3} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & & \end{array}$$

be a commutative diagram with exact rows. Without embedding $\mathcal{A}$ into a category of modules prove the snake lemma, i.e., that there exists a natural exact sequence

$$\ker(d_1) \to \ker(d_2) \to \ker(d_3) \xrightarrow{\delta} \operatorname{coker}(d_1) \to \operatorname{coker}(d_2) \to \operatorname{coker}(d_3),$$

and deduce that a short exact sequence of complexes in $\mathcal{A}$ induces a long exact sequence in cohomology.
Hint: To construct δ let $K \subseteq A_2$ be the preimage $\alpha_2^{-1}(\ker(d_3))$. Then d_2 factors over a morphism $d_2' \colon K \to B_1$. Deduce that the composition $K \to B_1 \to \operatorname{coker}(d_1)$ factors over $K/\alpha_1(A_1) \cong \ker(d_3)$.

Exercise 2 (4 points):
i) Let $\mathcal{A} = (\mathrm{Ab})$ be the category of abelian groups. Prove that every bounded above complex $A^\bullet \in \mathcal{C}^-(\mathcal{A})$ is quasi-isomorphic to the sum of its cohomology groups.
Hint: Take a projective resolution of $A^\bullet$ and use that for abelian groups submodules of free modules are again free.
ii) Construct an example of an abelian category $\mathcal{A}$ and two complexes $A^\bullet, B^\bullet \in \mathcal{C}(\mathcal{A})$ having isomorphic cohomology in each degree, but which are not quasi-isomorphic.
Hint: Set $\mathcal{A}$ for example as the category of R-modules with $R = k[x,y]$ or $k[x]/(x^2)$.

Exercise 3 (4 points):
Let $\mathcal{A}$ be an abelian category and let $f \colon A^\bullet \to B^\bullet$ be a morphism of complexes of $\mathcal{A}$. We define the mapping cone $C(f)$ of f as $C(f)^i := B^i \oplus A^{i+1}$ with differential given by

$$C(f)^i \to C(f)^{i+1},\ (b,a) \mapsto (d_{B^\bullet}(b) + f(a), -d_{A^\bullet}(a))$$

i) Prove that there exists a short exact sequence

$$0 \to B^\bullet \xrightarrow{\iota} C(f) \to A^\bullet[1] \to 0$$

where $A^\bullet[1]$ denotes the shifted complex with $(A^\bullet[1])^i = A^{i+1}$ and differential $d_{A^\bullet[1]} = -d_{A^\bullet}$. Prove that the associated connecting morphism $\delta \colon H^i(A^\bullet[1]) = H^{i+1}(A^\bullet) \to H^{i+1}(B^\bullet)$ is given by $H^{i+1}(f)$.
ii) Construct a canonical null homotopy h_0 of $\iota \circ f$. Let $g \colon B^\bullet \to C^\bullet$ be a second morphism of complexes and let h be a null homotopy of the composition $g \circ f$. Construct a canonical morphism $k \colon C(f) \to C^\bullet$ such that $k \circ \iota = g$ and $k \circ h_0 = h$.

Exercise 4 (4 points):

Let X be a topological space and let $\mathcal{F}$ be a sheaf of abelian groups on X. Let $\mathcal{T}$ be an $\mathcal{F}$-torsor with corresponding extension

$$0 \to \mathcal{F} \to \widetilde{\mathcal{T}} \to \mathbb{Z} \to 0$$

constructed similarly to the lecture (for $\mathcal{O}_X$-modules on a scheme). We set $\Phi(\mathcal{T}) := \delta_{\widetilde{\mathcal{T}}}(1)$ where $\delta_{\widetilde{\mathcal{T}}} \colon \mathbb{Z}(X) \to H^1(X, \mathcal{F})$ is the connecting morphism (for the δ-functor $H^i(X, -)$ constructed in the lecture using injective resolutions).

i) Prove that sending $\mathcal{T}$ to $\Phi(\mathcal{T})$ defines a bijection

$$\{\mathcal{F} - \text{torsors}\}/\text{isom.} \cong H^1(X, \mathcal{F}).$$

ii) For two $\mathcal{F}$-torsors $\mathcal{T}, \mathcal{T}'$ construct naturally an $\mathcal{F}$-torsor $\mathcal{T}''$ such that $\Phi(\mathcal{T}'') = \Phi(\mathcal{T}) + \Phi(\mathcal{T}')$.

Hint: For surjectivity in i) take a short exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0$ *with* $\mathcal{I}$ *injective and use it to construct, for a given* $a \in H^1(X, \mathcal{F})$*, a suitable extension* $0 \to \mathcal{F} \to \widetilde{\mathcal{T}} \to \mathbb{Z} \to 0$.

Additional Exercise (this exercise does not give any points and will not corrected):

Let $(X, \mathcal{O}_X)$ be a ringed space, let $V \subseteq X$ be an open subset and set $\mathcal{O}_V := \mathcal{O}_{X|V}$. Let $j \colon (V, \mathcal{O}_V) \to (X, \mathcal{O}_X)$ be the canonical morphism of ringed spaces.

i) Let $\mathcal{F}$ be an $\mathcal{O}_V$-module and set as in the previous semester $j_!\mathcal{F}$ as the sheafification of the presheaf

$$U \subseteq X \mapsto \begin{cases} \mathcal{F}(U) & \text{if } U \subseteq V \\ 0 & \text{otherwise.} \end{cases}$$

Prove that the functor $j_!$ from $\mathcal{O}_V$-modules to $\mathcal{O}_X$-modules is left adjoint to the restriction functor j^*.

ii) Let $\mathcal{F}$ be an injective $\mathcal{O}_X$-module. Prove that the restriction $\mathcal{F}_{|V}$ is an injective $\mathcal{O}_V$-module.

iii) Give an example of a scheme X, an open subset $U \subseteq X$ and a quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}$, s.t. $j_!\mathcal{F}$ is not quasi-coherent.

To be uploaded in eCampus: Saturday, 25.04.2020 till 23:25h.