Dr. A. Ivanov
Dr. J. Anschütz **SS 2020**

Algebraic Geometry II

Working sheet 3

Task 1

Work through the rest of section [2, §8], i.e., everything after [2, Definition 8.5], and the part on abelian categories in [2, §9]. Stop before [2, Definition 9.5.], but consult [2, Definition 9.7.]. Complement your study by looking up the definition of a right exact resp. exact functor (between abelian categories).

Task 2

Let us mention some examples/facts of abelian categories, and injective/projective objects.

- Let R be any ring. Then the category of R-modules Mod_R is abelian. Using the natural isomorphism
$$\mathrm{Hom}_{\mathbb{Z}}(M, N) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, N))$$
for $M \in \mathrm{Mod}_R, N \in \mathrm{Mod}_{\mathbb{Z}}$ and [2, Theorem 9.1.] one can deduce that Mod_R has enough injectives.
- Besides divisible abelian groups (cf. [2, Theorem 9.1.]) examples of injective R-modules are the following: 1) Each vector space over a field k is an injective k-module. 2) Set $R = \mathbb{Z}/n$ for $n \in \mathbb{Z} \setminus \{0\}$. Then R is an injective R-module. 3) Let R be an integral domain. Then the fraction field $\mathrm{Frac}(R)$ is an injective R-module.
- The dual notion of an injective object is a projective object. Look up the definition and convince yourself that for any ring R the category Mod_R has "enough" projectives.
- Let $\mathcal{T}or \subseteq \mathrm{Mod}_{\mathbb{Z}}$ be the category of *torsion* abelian groups, i.e., of abelian groups M such that for each $m \in M$ there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $nm = 0$. The category $\mathcal{T}or$ is abelian, has enough injectives, but not enough projectives. The last assertion can be seen as follows:[1] Each abelian category $\mathcal{A}$ with enough projectives and arbitrary colimits has a projective *generator* $G \in \mathcal{A}$. Then a sequence $0 \to A \to B \to C \to 0$ of morphisms in $\mathcal{A}$ is exact if and only if the sequence
$$0 \to \mathrm{Hom}_{\mathcal{A}}(G, A) \to \mathrm{Hom}_{\mathcal{A}}(G, B) \to \mathrm{Hom}_{\mathcal{A}}(G, C) \to 0$$
is exact. This implies by reducing to the case of abelian groups that arbitrary products in $\mathcal{A}$ are exact. However, in the category $\mathcal{A} = \mathcal{T}or$ infinite products are not necessarily exact (construct an example!).
- Let X be a topological space. Then the category $\mathrm{Sh}_{\mathrm{Ab}}(X)$ of abelian sheaves on X is abelian and according to [2, Theorem 8.8.] it has enough injectives. However, for a suitable X an infinite products of short exact sequences of *sheaves* on X need not be exact (the issue is that general infinite intersections of coverings need not be coverings). This implies as before that in general $\mathrm{Sh}_{\mathrm{Ab}}(X)$ does not contain enough projectives.

References

[1] P. Scholze, Notes for the course Algebraic Geometry I, taught by Peter Scholze (2017), `https://www.math.uni-bonn.de/people/ja/alggeoII/notes.pdf`

[2] P. Scholze, Notes for the course Algebraic Geometry II, taught by Peter Scholze (2017), `https://www.math.uni-bonn.de/people/ja/alggeoII/notes_II.pdf`

[1] Up to set-theoretic issues.