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Algebraic Geometry II

Working sheet 2

This working sheet contains two tasks.

Task 1

Read part II "Sheaf cohomology" of [1, §1], where the cohomological approach (the one we will take) to the proof of Riemann–Roch theorem is (very roughly) explained.

Task 2

Now we want to understand how cohomology works, and see first examples of it. The most simple cohomology group on a scheme X of a quasi-coherent module (or more generally, of any sheaf of abelian groups) $\mathcal{F}$ is its 0th cohomology

$$H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}),$$

which is just the group of global sections. The first non-trivial example is the group $H^1(X, \mathcal{F})$, which has a natural interpretation in terms of torsors.

- Read the definition of a torsor, [1, Definition 7.4].
- Add to this definition, that an morphism of $\mathcal{G}$-torsors $\mathcal{P} \to \mathcal{P}'$ is a morphism of sheaves $\mathcal{P} \to \mathcal{P}'$ on X, which is equivariant for the $\mathcal{G}$-action.

Example 0.0.1. Let G be a group. Let us try to give a natural example of G-torsor P. In $\mathbb{R}^2$ (with coordinates x, y) consider the line $G = \{y = 0\}$ (regarded as a group with addition). Take $P = \{y = 1\} \subseteq \mathbb{R}^2$. Clearly, P fails to be a group itself, for example it has no neutral element. However, it looks very much like G, and this piece of "group-like" structure which P still has is precisely encoded by the fact that P is a G-torsor: indeed, G acts on P by $(x, 0), (x', 1) \mapsto (x + x', 1)$, and for any $(x_0, 1) \in P$, $G \to P$, $(x, 0) \mapsto (x_0 + x, 1)$ is a bijection.

Informally, one also could say that a G-torsor is just the group G itself, where all structures are forgotten, except how "multiplication from left by elements of G works".

A torsor for a sheaf of groups $\mathcal{G}$ on a topological space $\mathcal{X}$ is of slightly more complicated nature. Such a torsor $\mathcal{P}$ for $\mathcal{G}$ is a sheaf, which locally looks like $\mathcal{G}$, but globally might differ from $\mathcal{G}$.

Example 0.0.2. Let X be a topological space. An obvious example of a torsor is just the sheaf $\mathcal{G}$ itself, which is a $\mathcal{G}$-torsor via the action of $\mathcal{G}$ on itself by left multiplication. (This torsor is denoted by $\mathcal{G}$.) Any torsor which is isomrphic to this $\mathcal{G}$-torsor, is called *trivial*. Try to check the following on your own: a $\mathcal{G}$-torsor $\mathcal{P}$ is trivial if and only if $\mathcal{P}(X) \neq \varnothing$ (see [1, Remark 7.8]).

Example 0.0.3. A non-trivial example is given by [1, Definition 7.5]. Let us explain it in slightly more detail. A Möbius band is just a strip of paper, whose ends are glued together after half-twisting it once (a quick search through the internet gives you the right picture). Globally it is not orientable: one cannot say which side of the paper is "outer", and which is "inner". However, locally, on small pieces of the strip of paper, one surely can give such orientations. Now we can consider the constant sheaf $\mathcal{G} = \underline{\mathbb{Z}/2\mathbb{Z}}$ on the Möbius band T, and look at the sheaf $\mathcal{P}$, which to any open subset $U \subseteq T$ attaches its possible orientations (say, in the naive sense above). Then there is an action $\mathcal{G} \times \mathcal{P} \to \mathcal{P}$ ($\to$ why?), such that whenever $\mathcal{P}(U) \neq \varnothing$, $\mathcal{G}(U) \cong \mathcal{P}(U)$ (non-canonically) via this action. So $\mathcal{P}$ is a $\mathcal{G}$-torsor. On the other side $\mathcal{P}(T) = \varnothing$, so $\mathcal{P} \ncong \mathcal{G}$ is not the trivial torsor.

- Try to show that any morphism $\mathcal{P} \to \mathcal{P}'$ of $\mathcal{G}$-torsors is necessarily an isomorphism (see [1, Remark 8.3]).
- Try to explain how part 1 of [1, Definition 7.4] can be regarded as a special case of part 2 of [1, Definition 7.4].
- We now have the set of the isomorphism classes of $\mathcal{G}$-torsors on X, $H^1(X, \mathcal{G})$ [1, Definition 7.7][1]. It is a based set (also called pointed set), i.e., it has a distinguished point, given by the (isomorphism class of the) trivial torsor.

Now we can read [1, §8]:

- The central result is [1, Proposition 8.1]. It follows easily from [1, Proposition 8.2].
- Work out the proof of [1, Proposition 8.2].
- Now [1, Proposition 8.4] shows the first four terms of the *long exact cohomology sequence* attached to the short exact sequence of abelian groups on a topological space. Go through its proof.
- Finally, read the text between the end of the proof of [1, Proposition 8.4] and [1, Definition 8.5].

References

[1] P. Scholze, Notes for the course Algebraic Geometry II, taught by Peter Scholze (2017), `https://www.math.uni-bonn.de/people/ja/alggeoII/notes_II.pdf`

[1] Clearly, $H^1(X, \mathcal{G})$ will have later also another definition via the general cohomology machinery.