Dr. A. Ivanov
Dr. J. Anschütz

SS 2020

Algebraic Geometry II

Working sheet 1

Due to the situation with the Corona virus, and the shift of the start of the summer term to April 20, the lectures (as well as the exercise sessions) during the two weeks April 6-19 have to be cancelled. Instead there will appear working sheets (this is the first one), one for each cancelled lecture. Regular lectures shall begin (most probably via zoom) on April 20. More information will follow.

The goals for this term are in particular:

- Study involved properties of (morphisms of) schemes,
- Learn about the homological algebra machinery in general, and study Zariski cohomology of coherent sheaves in particular,
- Apply Zariski cohomology to prove the theorem of Riemann–Roch for curves and further results.

The rough plan for first the two weeks (April 6 - April 19) will be:

(1) Some motivation (Riemann-Roch theorem; families of schemes; flatness),

(2) General homological algebra (which is required later on to define and study the cohomology of cohoerent sheaves on schemes).

On this working sheet you will have two tasks.

Task 1

To properly explain our goals in this semester, we first look in detail at the statement of the Riemann–Roch theorem (and some simple consequences). Therefore, carefully read [1, §29]. More concretely:

- Example 29.1, Proposition 29.2 and Corollary 29.3 essentially were discussed last semester.
- Try to understand Definition 29.5 (which is actually part of Commutative Algebra) as good as possible. The last sentence means that if M is an A-module, and $\mathrm{Der}_k(A, M)$ denotes the set of derivations $A \to M$, then functorially, $\mathrm{Hom}_A(\Omega^1_{A/k}, M) = \mathrm{Der}_k(A, M)$, sending $\alpha\colon \Omega^1_{A/k} \to M$ to the derivation $\alpha \circ d_{univ}$, where $d_{univ}\colon A \to \Omega^1_{A/k}$ is the universal derivation. Show that $d(\lambda) = 0$ for $\lambda \in k$ and a k-derivation $d\colon A \to M$. It is also very helpful to work out Examples 29.6, 29.7 and 29.8 in detail.
- You can omit a proof of Proposition 29.9. Instead, recall why Corollary 29.10 follows from Proposition 29.9.
- Then we have the canonical divisor, the genus of a curve and the Riemann-Roch theorem, along with a couple of examples (29.11-15). These are the main definitions and the main result of this section.
- Then two slightly more involved examples follow (on page 109): the first (not numbered) is the proof of Theorem 29.4, and the second is the group structure on an elliptic curve. You may regard a detailed understanding of these examples as good exercises.[1]
- Finally, you might find it helpful to find some additional information (surely informal, but useful to get some ideas) on the Riemann–Roch theorem in the corresponding article on (English) Wikipedia. See in particular section “2. Statement of the theorem” there.

[1] We note that Example 29.16. contains two mistakes: 1) In line -2 on page 109, the middle term $\mathcal{O}([P] - \infty)$ is wrong and should be deleted while $[R]$ must be replaced by $[\overline{R}]$, 2) in line -1 on page 109 the term $P + Q - \overline{R}$ should be replaced by $P + Q = \overline{R}$.

Task 2

Later on, parts of our lecture will follow Scholze's lecture [2]. Read part I "Families of schemes" of [2, §1] (until the first paragraph on page 3). Some remarks:

- The important concepts are "family of schemes over a base S" and "flatness". They are connected in the following sense: a family $f\colon X \to S$ only behaves well in general if f is flat. Example 1.2, 1.7 gives an (in general) non-flat family.
- An important point is to understand, why flat families are the natural ones and the others are not. This is explained in the text. Additionally, let us add here the following (naive) explanation: if $X = \mathrm{Spec} A \to \mathrm{Spec} R = S$ is a morphism, and $s \in S$ a point corresponding to the prime ideal $\mathfrak{p} \subseteq R$, then the fiber X_s over s is equal to $X_s = \mathrm{Spec}(A \otimes_R \mathrm{Frac}(R/\mathfrak{p}))$. But $A \otimes_R -$ beheaves well, precisely because of the flatness assumption, so that all fibers of a flat family behave well. For example, the dimension of the fiber in a flat family is locally constant on S.
- Draw several pictures visualising Example 1.7, as well as the opposite situation, when you have an object "horizontal" over $\mathrm{Spec} A$, for example: $A = \mathbb{Z}_{(p)}$ and $B = \mathbb{Z}_{(p)}[t]$; there are clearly much more examples).
- Work out the proof of Proposition 1.8.
- Note: In Example 1.10 we have a family parametrized by $\mathbb{A}^1 = \mathrm{Spec} k[t]$, that is the morphism $\mathrm{Spec} k[t,x,y]/(y^2 - x^3 - x^2 - t) \to \mathrm{Spec} k[t]$.

References

[1] P. Scholze, Notes for the course Algebraic Geometry I, taught by Peter Scholze (2017), https://www.math.uni-bonn.de/people/ja/alggeoII/notes.pdf

[2] P. Scholze, Notes for the course Algebraic Geometry II, taught by Peter Scholze (2017), https://www.math.uni-bonn.de/people/ja/alggeoII/notes_II.pdf