Prof. Dr. Y. Tian **WS 2017/18**
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Algebraic Number Theory

13. Exercise sheet

Exercise 1 (4 Points):
Let K be a number field. Let $v_1, \ldots, v_{r_2}$ be the complex places of K and for $y \in \mathbb{R}$ let $\phi_j(y) \in \mathbb{I}_K$ be the idèle with component $\exp(2\pi i y) \in K_{v_j} \cong \mathbb{C}$ at v_j and 1 at all others. Let $\varepsilon_1, \ldots, \varepsilon_t$ be a $\mathbb{Z}$-basis of the group of totally positive units of K where $t := r_1 + r_2 - 1$.

1) Prove that

$$f\colon (\hat{\mathbb{Z}} \times \mathbb{R})^t \times \mathbb{R}^{r_2} \to \mathbb{I}_K/K^\times,\ (\lambda_1, \ldots, \lambda_t, y_1, \ldots, y_{r_2}) \mapsto \varepsilon_1^{\lambda_1} \cdots \varepsilon_t^{\lambda_t} \phi_1(y_1) \cdots \phi_{r_2}(y_{r_2})$$

is a continuous group homomorphism with kernel $\mathbb{Z}^t \times \mathbb{Z}^{r_2}$ (here the exponentiation refers to Sheet 12, exercise 3).
Hint: Prove that the finite parts of $\varepsilon_1, \ldots, \varepsilon_t \in \mathbb{I}_{K,f}$ are linear independent with respect to the exponentiation by elements in $\hat{\mathbb{Z}}$.

2) Let $C \subseteq \mathbb{I}_K/K^\times$ be the image of f. Prove that

$$C \cong ((\hat{\mathbb{Z}} \times \mathbb{R})/\mathbb{Z})^t \times (\mathbb{R}/\mathbb{Z})^{r_2}$$

and that C is compact, connected and divisible.

Exercise 2 (4 Points):
Let $K = \mathbb{Q}(i) \subseteq \mathbb{C}$ and let $\mathfrak{q} := (1+i) \subseteq \mathcal{O}_K$ be the unique prime ideal above (2).

1) Prove that for every non-zero ideal $\mathfrak{a} \subseteq \mathcal{O}_K$ there exists a unique generator $\psi(\mathfrak{a}) \in \mathcal{O}_K$ such that $\psi(\mathfrak{a}) = 1 \bmod \mathfrak{q}^3$. We extend ψ to a homomorphism from fractional ideals of K which are prime to $\mathfrak{q}$ to $K^\times$.

2) For an idèle $x \in \mathbb{I}_K$ we denote by $\mathfrak{a}_x \subseteq K$ the fractional ideal defined by x. Prove that

$$\chi\colon \mathbb{I}_K \to \mathbb{C}^\times,\ x \mapsto \psi(\mathfrak{a}_{x\xi}) x_\infty^{-1} \xi^{-1}$$

with $\xi \in K^\times$ satisfying $x_\mathfrak{q}\xi \equiv 1 \bmod \mathfrak{q}^3$ yields a well-defined character satisfying $\chi(K^\times) = 1$ and $\chi(\mathcal{O}_{K_\mathfrak{p}}^\times) = 1$ for every finite prime $\mathfrak{p} \neq \mathfrak{q}$.

3) Let $\chi_\mathfrak{q}\colon K_\mathfrak{q}^\times \to \mathbb{C}^\times$ be the restriction of χ and set $U_\mathfrak{q}^{(n)} := \{y \in \mathcal{O}_{K_\mathfrak{q}}^\times \mid y \equiv 1 \bmod \mathfrak{q}^n\}$. Prove $\chi_\mathfrak{q}(U_\mathfrak{q}^{(3)}) = 1$ and $\chi_\mathfrak{q}(U_\mathfrak{q}^{(2)}) \neq 1$.

4) Compute the local ε-factor of $\chi_\mathfrak{q}$.

To be handed in: Monday, 29. January 2018.