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WS 2017/18

Algebraic Number Theory

12. Exercise sheet

Exercise 1 (4 Points):

Let R be a topological ring and $n \geq 1$. We equip $\mathrm{GL}_n(R)$ with the subspace topology coming from the embedding $\mathrm{GL}_n(R) \to R^{n\times n} \times R,\ A \mapsto (A, \det(A)^{-1})$.

1) Prove that $\mathrm{GL}_n(R)$ is a topological group.

2) If R is a local field, prove that $\mathrm{GL}_n(R) \subseteq R^{n\times n}$ carries the subspace topology.

3) Show that the conclusion in 2) is false if $R = \mathbb{A}_{\mathbb{Q}}$ is the ring of adeles of $\mathbb{Q}$.

Exercise 2 (4 Points):

Let K be a number field. Let σ be the set of finite places of K. For a function $\mathfrak{m}\colon \sigma \to \mathbb{N}$ taking the value 0 for all but finitely many places we set

$$\mathbb{I}_K^{\mathfrak{m}} := \prod_{v \text{ real}} (K_v^\times)_{>0} \times \prod_{v \text{ complex}} K_v^\times \times \prod_{v\in\sigma} U_v^{(\mathfrak{m}(v))}$$

where $U_v^{(n)}(v) := \{u \in \mathcal{O}_{K_v}^\times \mid u \equiv 1 \bmod \mathfrak{p}_v^n\}$. Prove that the closed subgroups $H \subseteq \mathbb{I}_K/K^\times$ of finite index of the idèle class group are precisely those subgroups containing the image of $\mathbb{I}_K^{\mathfrak{m}}$ in $\mathbb{I}_K/K^\times$ for some $\mathfrak{m}$ as above.

Exercise 3 (4 Points):

In the notations from Exercise 2 assume $\mathfrak{m}(v) = 0$ for every $v \in \sigma$ and decompose $\mathbb{I}_K^{\mathfrak{m}} = \mathbb{I}_{K,f}^{\mathfrak{m}} \times \mathbb{I}_{K,\infty}^{\mathfrak{m}}$ according to finite and infinite places.

1) Prove that for $\alpha \in \mathbb{I}_{K,f}^{\mathfrak{m}}$ the map $\mathbb{Z} \to \mathbb{I}_{K,f}^{\mathfrak{m}},\ n \mapsto \alpha^n$ extends by continuity to $\hat{\mathbb{Z}}$.

2) Let $\varepsilon \in \mathcal{O}_K^\times$ be totally positive. Show that $\mathbb{Z} \to \mathbb{I}_K^{\mathfrak{m}},\ n \mapsto \varepsilon^n$ extends by continuity to an exponentiation $\hat{\mathbb{Z}} \times \mathbb{R} \to \mathbb{I}_K^{\mathfrak{m}}$ such that $|\varepsilon^\lambda| = 1$ for every $\lambda \in \hat{\mathbb{Z}} \times \mathbb{R}$.

Exercise 4 (4 Points):

Let $B \subseteq \mathrm{GL}_2(\mathbb{R})$ be the subgroup of upper triangular matrices.

1) Find a Haar measure μ_B of B, i.e. a (non-zero) measure on B which is invariant under left multiplication by elements in B.

2) Determine a character $\delta\colon B \to \mathbb{R}_{>0}$ such that

$$\mu_B(U \cdot b) = \delta(b)\mu_B(U)$$

for all $b \in B$ and $U \subseteq B$ open. Here $U \cdot b := \{ub \mid u \in U\}$.

To be handed in: Monday, 22. January 2018.