Prof. Dr. Y. Tian
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Algebraic Number Theory

11. Exercise sheet

Exercise 1 (4 Points):
We put $\mathbb{S} := \varprojlim_n \mathbb{R}/n\mathbb{Z}$, where for integers $n, m \geq 1$ with $n|m$, the transition map $\mathbb{R}/m\mathbb{Z} \to \mathbb{R}/n\mathbb{Z}$ is the natural projection.

1) Show that $\mathbb{S}$ is connected.
2) Show that for each integer $n \geq 1$, $\mathbb{S}$ is uniquely divisible by n.
3) Show that there exists an isomorphism of topological groups $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \cong \mathbb{S}$, where $\mathbb{A}_{\mathbb{Q}}$ is the ring of adeles of $\mathbb{Q}$.

Exercise 2 (4 Points):
Let $\mathbb{A}_{\mathbb{Q}}$ be the adele ring of $\mathbb{Q}$ and let M be a finite free $\mathbb{A}_{\mathbb{Q}}$-module of rank n. Let $R \subseteq M$ be a $\mathbb{Q}$-vector space. Prove that $R \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}} \cong M$ if and only if R is discrete in M and M/R is compact.

Exercise 3 (4 Points):
Let G_i, $i \in I$, be a family of topological groups and let

$$\chi\colon \prod_i G_i \to \mathbb{C}^\times$$

be a character, i.e., a continuous group homomorphism. For $i \in I$ let $\chi_i := \chi_{|G_i}\colon G_i \to \mathbb{C}^\times$ be the restriction of χ to G_i. Prove that

$$\chi((g_i)_{i\in I}) = \prod_i \chi_i(g_i)$$

for every $(g_i)_{i\in I} \in \prod\limits_i G_i$.

Exercise 4 (4 Points):
Let L/K be a finite extension of number fields with rings of adeles $\mathbb{A}_L$ and $\mathbb{A}_K$.

1) For $(\alpha_w)_w \in \mathbb{A}_L$ show $\mathrm{Tr}_{\mathbb{A}_L/\mathbb{A}_K}(\alpha) = (\sum_{w|v} \mathrm{Tr}_{L_w/K_v}\alpha_w)_v \in \mathbb{A}_K$ and similarly for the norm $N_{\mathbb{A}_L/\mathbb{A}_K}$.
2) Let $\iota_L\colon L \to \mathbb{A}_L$ resp. $\iota_K\colon K \to \mathbb{A}_K$ be the respective embeddings of principal adeles. Prove that $\mathrm{Tr}_{\mathbb{A}_L/\mathbb{A}_K}(\iota_L(l)) = \iota_K(\mathrm{Tr}_{L/K}(l))$ for every $l \in L$ and similarly for the norm $N_{\mathbb{A}_L/\mathbb{A}_K}$.

To be handed in: Monday, 15. January 2018.